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POLYNOMIAL BEST CONSTRAINED DEGREE REDUCTION IN STRAIN ENERGY*

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Abstract. We exhibit the best degree reduction of a given degree n polynomial by minimizing the strain energy of the error with the constraint that continuity of a prescribed order is preserved at the two endpoints. It is shown that a multidegree reduction is equivalent to a step-by-step reduction of one degree at a time by using the Fourier coefficients with respect to Jacobi orthogonal polynomials. Then we give explicitly the optimal constrained one degree reduction in Bézier form, by perturbing the Bézier coefficients.

Key words. reduction, polynomials, approximation, Bézier curves

AMS subject classifications. 41A10, 65D05, 65D17

1. Introduction.

Degree reduction of polynomials consists in approximating a given polynomial $f(t)$ by a lower degree polynomial $g(t)$ by minimizing the error $f(t) - g(t)$ with respect to a certain measure. The most often used measure in degree reduction is the L_p-norm of the error for $p = 1, 2, \infty$; see [1, 2, 3, 4, 6, 7, 8, 9, 10, 12]. In this paper we are concerned with the following problem: Given a degree n polynomial $f(t)$, find a degree $m(< n)$ polynomial $g(t)$, $t \in [0, 1]$, such that

- $g(t)$ and $f(t)$ have the same first $k + 1$ derivatives at $t = 0$ and the same first $l + 1$ derivatives at $t = 1$, i.e.,

$$\begin{aligned} g^{(i)}(0) &= f^{(i)}(0), \quad i = 0, \cdots, k+1; \\ g^{(j)}(1) &= f^{(j)}(1), \quad j = 0, \cdots, l+1; \end{aligned} \qquad l + k \le m - 4 \tag{1.1}$$

- $g(t)$ minimizes the error strain energy $E = \displaystyle\int_0^1 \left(f''(t) - g''(t)\right)^2 dt$ for all such possible polynomials of degree $\le m$ that satisfy the endpoint constraints (1.1).

This process is useful for many tasks in geometric modeling, such as data compression, data comparison, rendering. Degree reduction is also needed to simplify some geometric or graphical algorithms for intersection calculation of two polynomial curves or surfaces.

There have been many methods developed for degree reduction. As this is essentially a problem of approximation, methods from classical approximation theory can be employed. Watkins and Worsey [12] used the Chebyshev economization to produce the best L_∞-approximation of degree $n - 1$ without constraint to a given degree n polynomial. Later, Bogacki et al. [1] achieve the best uniform approximation with endpoint interpolation by modifying the economization procedure. The endpoints constraints that guarantee a prescribed order of continuity are often required in many applications and especially when degree reduction is combined with subdivision to generate continuous piecewise approximation. Lachance [7] and Eck [3] made deeper the Chebyshev economization procedure for the best L_∞-approximation with prescribed order of continuity at the endpoints, but it seems that there was no explicit formula for the degree reduced polynomial as pointed out in [4]. These difficulties can be avoided by using the L_2-norm. The degree reduction with endpoint interpolation that minimizes the L_2-norm has been studied by Eck [4]. His method used the inverse of a polynomial degree elevation process in Bézier form [5] and obtained two sets of control points, then

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considered a simple convex combination of these two sets of control points to generate the control points for the degree reduced polynomial. Recently, Lutterkort et al. [8] discovered a surprising result: finding the best L_2- approximation in Bézier form without constraint is equivalent to finding the best Euclidean approximation of Bézier coefficients. This result can be extended to the multivariate case [9]. The best degree reduction of Bézier curves in L_1-norm with endpoint interpolation has been solved by Kim and Moon [6]. The optimal degree reduction with respect to various norms was studied by Brunnett et al. [2] who have also shown the separability of degree reduction into the different components of a parametric curve.

2. Degree reduction method. The degree reduction is accomplished through two stages. In the first stage, we construct a degree $p = k+l+1$ polynomial $\phi(t)$ interpolating the second derivative $f''(t)$ at $t = 0$ up to the $(k + 1)$th order continuity and at $t = 1$ up to the $(l + 1)$th order of continuity as follows

$$\phi(t) = \sum_{i=0}^{k+1} f^{(i)}(0)\frac{d^2 H_i^{k,l}(t)}{dt^2} + \sum_{j=0}^{l+1} f^{(j)}(1)\frac{d^2 M_j^{k,l}(t)}{dt^2},$$

where $H_i^{k,l}(t)$ and $M_j^{k,l}(t)$ are the so called Hermite polynomials of degree $p = k + l + 3$ defined by

$$\frac{d^j H_i^{k,l}(t)}{dt^j}|_{t=0} = \begin{cases} 1, & j = i \\ 0, & \text{otherwise} \end{cases}, \qquad \frac{d^h H_i^{k,l}(t)}{dt^h}|_{t=1} = 0,$$

$$i, j = 0, \cdots, k + 1, \quad h = 0, \cdots, l + 1,$$

and

$$\frac{d^j M_i^{k,l}(t)}{dt^j}|_{t=1} = \begin{cases} 1, & j = i \\ 0, & \text{otherwise} \end{cases}, \qquad \frac{d^h M_i^{k,l}(t)}{dt^h}|_{t=0} = 0,$$

$$i, j = 0, \cdots, l + 1, \quad h = 0, \cdots, k + 1.$$

Both $H_i^{k,l}(t)$ and $M_j^{k,l}(t)$ are uniquely defined since they have $k+l+4$ degrees of freedom, and $k + l + 4$ constraints as well.

The second stage is to determine $g''(t) - \phi(t)$ by minimizing the error strain energy $E = \int_0^1 \left(f''(t) - g''(t)\right)^2 dt$, and then deduce the degree reduced polynomial $g(t)$.

Note that the error strain energy can be expressed in terms of the L_2-norm as

$$E = \|f'' - g''\|_2^2,$$

in the Hilbert space $C([0, 1])$ with the inner product $< f, g >= \int_0^1 f(t)g(t)d\mu(t)$, for a convenient Borel positive measure $\mu(t)$. For such a problem, choosing proper basis functions often simplifies the computation. In our case, in order to allow a direct determination of the polynomial $g(t)$ without solving a linear system, the appropriate basis functions should be orthogonal with respect the above inner product. Let us consider the Jacobi polynomials $J_i(t)$ of degree i that are orthogonal with respect to the inner product

$$\langle f, g\rangle = \int_0^1 t^{2k}(1 - t)^{2l} f(t)g(t)dt, \quad k, l \geq 0.$$

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They are defined by Rodrigues' formula [11]

$$J_i(t) = \frac{(-1)}{i!} t^{-2k}(1-t)^{-2l} \frac{d^i}{dt^i}\left[t^{2k+i}(1-t)^{2l+i}\right], \quad i \geq 1, \tag{2.1}$$

with $J_0(t) = 1$ and satisfy the orthogonality relation

$$\int_0^1 t^{2k}(1-t)^{2l} J_i(t)J_j(t)dt = \begin{cases} \alpha_{ikl} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{2.2}$$

with $\alpha_{ikl} = \dfrac{1}{2i+2k+2l+1}\dbinom{i+2k}{2k}\Big/\dbinom{i+2k+2l}{2k}$. From the identity (2.1), the first few polynomials $J_i(t)$ are $J_0(t) = 1$, $J_1(t) = 2(k+l+1)t - (2k+l)$, $J_2(t) = (k+l+2)(2k+2l+3)t^2 - 2(k+1)(2k+2l+3)t + (k+1)(2k+1)$.

LEMMA 2.1. *The functions $f''(t) - \phi(t)$ and $g''(t) - \phi(t)$ can be expressed as*

$$f''(t) - \phi(t) = t^k(1-t)^l\left(a_0J_0(t) + \cdots + a_{n-k-l-2}J_{n-k-l-2}(t)\right), \tag{2.3}$$

$$g''(t) - \phi(t) = t^k(1-t)^l\left(b_0J_0(t) + \cdots + b_{m-k-l-2}J_{m-k-l-2}(t)\right), \tag{2.4}$$

where a_i and b_i are the Fourier coefficients defined by

$$a_i = (\alpha_{ikl})^{-1}\int_0^1 t^k(1-t)^l(f''(t) - \phi(t))J_i(t)dt, \tag{2.5}$$

$$b_i = (\alpha_{ikl})^{-1}\int_0^1 t^k(1-t)^l(g''(t) - \phi(t))J_i(t)dt. \tag{2.6}$$

Proof. By construction the polynomial $\phi(t)$ interpolates the second derivative $f''(t)$ at $t = 0$ up to the $(k+1)$th order continuity and at $t = 1$ up to the $(l+1)$th order continuity. Then the polynomials $f''(t) - \phi(t)$ and $g''(t) - \phi(t)$ have k-fold zeros at $t = 0$ and l-fold zeros at $t = 1$. A common factor $t^k(1-t)^l$ can be factored out from $f''(t)-\phi(t)$ and $g''(t)-\phi(t)$, thus we can set $f''(t)-\phi(t) = t^k(1-t)^l F_{n-k-l-2}(t)$ and $g''(t)-\phi(t) = t^k(1-t)^l G_{m-k-l-2}(t)$. Now we express the polynomials $F_{n-k-l-2}(t)$ and $G_{m-k-l-2}(t)$ in terms of the Jacobi polynomials $J_i(t)$, then using the orthogonality relation (2.2), we obtain the expressions (2.5) and (2.6) for the Fourier coefficients a_i and b_i. □

THEOREM 2.2. *The best degree reduced polynomial $g(t)$ is such that*

$$\begin{aligned} &g''(t) = \phi(t) + t^k(1-t)^l\left(a_0J_0(t) + \cdots + a_{m-k-l-2}J_{m-k-l-2}(t)\right) \\ &g(0) = f(0), \quad g(1) = f(1), \end{aligned} \tag{2.7}$$

and $a_0, a_1, \cdots, a_{m-k-l-2}$ are given by (2.5).

Proof. Consider the error strain energy

$$\begin{aligned} E &= \int_0^1 \left[(f''(t) - \phi(t)) - (g''(t) - \phi(t))\right]^2 dt \\ &= \int_0^1 t^{2k}(1-t)^{2l}\left(\sum_{i=0}^{m-k-l-2}(a_i - b_i)J_i(t) + \sum_{i=m-k-l-1}^{n-k-l-2} a_iJ_i(t)\right)^2 dt. \end{aligned}$$

Differentiating it with respect to the coefficient b_j gives

$$\frac{\partial E}{\partial b_j} = -2\int_0^1 t^{2k}(1-t)^{2l}\left(\sum_{i=0}^{m-k-l-2}(a_i - b_i)J_i(t)J_j(t) + \sum_{i=m-k-l-1}^{n-k-l-2} a_iJ_i(t)J_j(t)\right) dt.$$

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Taking into account the orthogonality relation (2.2), we obtain

$$\frac{\partial E}{\partial b_j} = -2(a_j - b_j)\alpha_{jkl}, \quad j = 0, \cdots, m-k-l-2.$$

To minimize E, we equate this derivative to zero and obtain $b_i = a_i$ for $i = 0, \cdots, m-k-l-2$. From (2.4), we deduce the expression of $g''(t)$ in (2.7). □

Equation (2.7) shows that if the decomposition (2.3) is available for the second derivative $f''(t)$ of the given polynomial, then the second derivative $g''(t)$ of the constrained approximation in strain energy can be immediately obtained by just removing the last $n-m$ terms in the square bracket of (2.3). This means that a multidegree reduction is equivalent to a step-by-step reduction of one degree at a time. In the next section we give explicitly the optimal constrained one degree reduction in Bézier form [5], by perturbing the Bézier coefficients.

3. Coefficients perturbation in Bézier form. To obtain the coefficients of the reduced degree polynomial without computing the Fourier coefficients we give a direct method based on perturbing the coefficients of the initial polynomial.

Let $B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$, $i = 0, \cdots, n$, be the degree n Bernstein polynomials basis. A degree $n-1$ Bézier polynomial $g(t) = \sum_{i=0}^{n-1} q_i B_i^{n-1}(t)$, can be expressed in terms of Bernstein polynomials of higher degree $m (> n-1)$. In particular we can write $g(t) = \sum_{i=0}^{n} p_i B_i^n(t)$, with the new Bézier coefficients $p_0 = q_0; p_i = (i/n)q_{i-1} + ((n-i)/n)q_i$, $i = 1, \cdots, n-1; p_n = q_{n-1}$. However, the converse is generally not true unless the coefficient of the nth degree term of $g(t)$ vanishes, which implies

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} p_i = 0. \tag{3.1}$$

It has been proved (see [3]) that under the condition (3.1), we have

$$q_i = \frac{(-1)^i}{\binom{n-1}{i}} \sum_{j=0}^{i} \binom{n}{j} p_j, \quad i = 0, \cdots, n-1, \tag{3.2}$$

Now, the coefficients perturbation method consists in finding a perturbation vector $(\epsilon_0, \epsilon_1, \cdots, \epsilon_n)$, such that given a degree n polynomial $f(t) = \sum_{i=0}^{n} p_i B_i^n(t)$, the perturbed polynomial

$$f_\epsilon(t) = \sum_{i=0}^{n} (p_i + \epsilon_i) B_i^n(t),$$

satisfies the condition (3.1), i.e., $\sum_{i=0}^{n} (-1)^i \binom{n}{i} (p_i + \epsilon_i) = 0$, and minimizes the error strain energy

$$E = \int_0^1 (f''(t) - f''_\epsilon(t))^2 dt = \int_0^1 \left(\sum_{i=0}^{n-2} \Delta^2 \epsilon_i B_i^{n-2}(t) \right)^2 dt,$$

with $\Delta^2 \epsilon_i = \epsilon_{i+2} - 2\epsilon_{i+1} + \epsilon_i$.

The continuity constraints (1.1) at the endpoints $f(0) = p_0$ and $f(1) = p_n$ have a very simple formulation,

LEMMA 3.1. *If $f_\epsilon(t)$ is required to match $f(t)$ up to the $(k+1)-$th derivative at $t = 0$, then $\epsilon_0 = \cdots = \epsilon_{k+1} = 0$. Similarly, $\epsilon_{n-l-1} = \cdots = \epsilon_n = 0$ guarantees C^{l+1} continuity between $f_\epsilon(t)$ and $f(t)$ at $t = 1$.*

Proof. The derivatives of order r of a degree n polynomial $h(t) = \sum_{i=0}^{n} h_i B_i^n(t)$ at the endpoints h_0 and h_n are given by [5]

$$\frac{d^r}{dt^r}h(0) = \frac{n!}{(n-r)!}\Delta^r h_0,$$
$$\frac{d^r}{dt^r}h(1) = \frac{n!}{(n-r)!}\Delta^r h_{n-r},$$

where Δ^r is the iterated forward difference operator defined by

$$\Delta^0 h_i = h_i, \quad \Delta^r h_i = \Delta^{r-1}h_{i+1} - \Delta^{r-1}h_i, \quad r = 1, 2, \cdots,$$

we list a few examples $\Delta^1 h_0 = h_1 - h_0$, $\Delta^2 h_0 = h_2 - 2h_1 + h_0$, $\Delta^3 h_0 = h_3 - 3h_2 + 3h_1 - h_0$. Thus the r-th derivative of a Bézier curve at an endpoint depends only on the $r + 1$ Bézier coefficients near (and including) that endpoint. □

Note that, if $\sum_{i=0}^{n}(-1)\binom{n}{i}p_i = 0$, then the given polynomial $f(t)$ is of degree less than $n - 1$. Otherwise, we proceed with the degree reduction, by introducing a Lagrange's multiplier λ and then, including the constraints, our problem is equivalent to

$$\min_{(\epsilon_{k+2},\cdots,\epsilon_{n-l-2},\lambda)} L(\epsilon_{k+2}, \cdots, \epsilon_{n-l-2}, \lambda), \tag{3.3}$$

with

$$L(\epsilon_{k+2}, \cdots, \epsilon_{n-l-2}, \lambda) = \int_0^1 \left(\sum_{i=k+2}^{n-l-2} \Delta^2 \epsilon_i B_i^{n-2}(t)\right)^2 dt - \lambda \sum_{i=0}^{n}(-1)\binom{n}{i}(p_i + \epsilon_i). \tag{3.4}$$

The right hand side of $L(\epsilon_{k+2}, \cdots, \epsilon_{n-l-2}, \lambda)$ takes into account the values of the ϵ_i's given by Lemma 3.1.

THEOREM 3.2. *The minimization problem (3.3) has a unique solution given by the system of linear equations*

$$\left\{ \begin{array}{rl} \sum_{i=k+2}^{n-l-2} \epsilon_i \Delta^2 \alpha_i^j - \lambda(-1)^j\binom{n}{j} & = 0, \quad j = k+2, \cdots, n-l-2, \\ \sum_{i=k+2}^{n-l-2}(-1)^i\binom{n}{i}\epsilon_i + \sum_{i=0}^{n}(-1)^{i+1}\binom{n}{i}p_i & = 0 \end{array} \right\},$$

where $\alpha_i^j = \frac{2}{2n-3}\binom{n-2}{i}\left[\binom{n-2}{j}/\binom{2n-4}{i+j} - 2\binom{n-2}{i+j}/\binom{2n-4}{i+j-1} + \binom{n-2}{j-2}/\binom{2n-4}{i+j-2}\right]$, $i, j = k+2, \cdots, n-l-2$, $\alpha_i^j = 0$ *for* $i \leq k+1$ *or* $i \geq n-l-1$ *and* $\Delta^2\alpha_i^j = \alpha_{i+2}^j - 2\alpha_{i+1} + \alpha_i$.

Proof. By Theorem 2.2, the constrained degree reduction problem has a unique solution that implies the same property for (3.3). Taking the partial derivative of $L = L(\epsilon_{k+2}, \cdots, \epsilon_{n-l-2}, \lambda)$ defined by (3.4), with respect to $\epsilon_{k+2}, \cdots, \epsilon_{n-l-2}, \lambda$, and setting the derivatives equal to zero leads to

$$\frac{\partial L}{\partial \epsilon_j} = 2\int_0^1 \left(\sum_{i=k+2}^{n-l-2} \Delta^2 \epsilon_i B_i^{n-2}(t)(B_j^{n-2}(t) - 2B_{j-1}^{n-2}(t) + B_{j-2}^{n-2}(t))\right) dt - \lambda(-1)^j\binom{n}{j} = 0,$$

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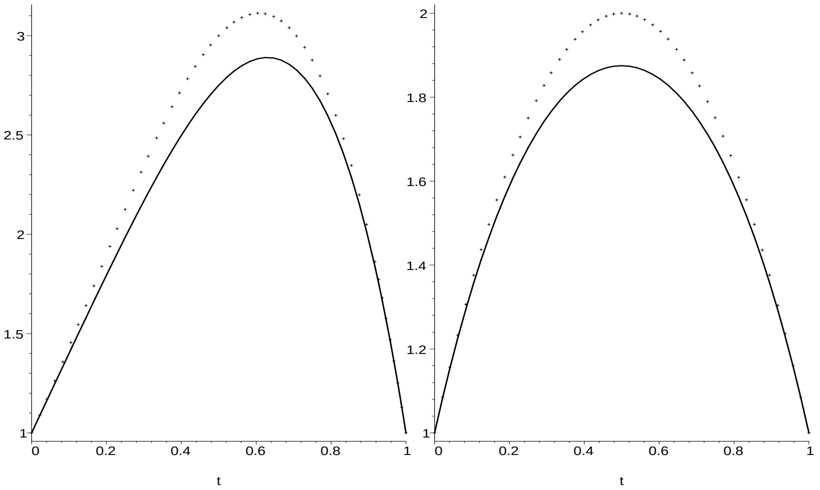

FIG. 3.1. *(left, $k = l = 0$), (solid) degree 4 Bézier curve, (cross) reduced degree 3 curve, (right,$k = 0, l = 0$), (solid) degree 4 Bézier curve, (cross) reduced degree 2 curve.*

for $j = k+2, \cdots, n-l-2$,

$$\frac{\partial L}{\partial \lambda} = \sum_{i=k+2}^{n-l-2} (-1)^i \binom{n}{i} \epsilon_i + \sum_{i=0}^{n} (-1)^{i+1} \binom{n}{i} p_i = 0.$$

Then by the identity $B_i^n(t) B_j^n(t) = \binom{n}{i}\binom{n}{j} / \binom{2n}{i+j} B_{i+j}^{2n}(t)$, and the formula [5], $\int_0^1 B_i^n(t)dt = \frac{1}{n+1}$, we get

$$\frac{\partial L}{\partial \epsilon_j} = \sum_{i=k+2}^{n-l-2} \alpha_i^j \Delta^2 \epsilon_i - \lambda(-1)^j \binom{n}{j} = 0, \quad j = k+2, \cdots, n-l-2, \tag{3.5}$$

with $\alpha_i^j = \frac{2}{2n-3}\binom{n-2}{i}\left[\binom{n-2}{j}/\binom{2n-4}{i+j} - 2\binom{n-2}{i+j}/\binom{2n-4}{i+j-1} + \binom{n-2}{j-2}/\binom{2n-4}{i+j-2}\right]$, $i, j = k+2, \cdots, n-l-2$.

Rearranging the terms of the sum in (3.5), we can write

$$\frac{\partial L}{\partial \epsilon_j} = \sum_{i=k+2}^{n-l-2} \epsilon_i \Delta^2 \alpha_{i-2}^j - \lambda(-1)^j \binom{n}{j} = 0, \quad j = k+2, \cdots, n-l-2.$$

where $\alpha_i^j = 0$ for $i \le k+1$ or $i \ge n-l-1$ and $\Delta^2 \alpha_i^j = \alpha_{i+2}^j - 2\alpha_{i+1} + \alpha_i$.

Finally, the following linear system of $n-(k+l+2)$ equations gives the unknowns $\epsilon_{k+2}, \cdots, \epsilon_{n-l-2}, \lambda$,

$$\begin{cases} \sum_{i=k+2}^{n-l-2} \epsilon_i \Delta^2 \alpha_i^j - \lambda(-1)^j \binom{n}{j} &= 0, \quad j = k+2, \cdots, n-l-2, \\ \sum_{i=k+2}^{n-l-2} (-1)^i \binom{n}{i} \epsilon_i + \sum_{i=0}^{n} (-1)^{i+1} \binom{n}{i} p_i &= 0. \qquad \square \end{cases} \tag{3.6}$$

REMARK 3.3. When the polynomial $f(t)$ is of degree $< n$, (i.e. $\sum_{i=0}^{n}(-1)\binom{n}{i}p_i = 0$), system (3.6) has the unique solution $\epsilon_{k+2} =, \cdots, = \epsilon_{n-l-2} = \lambda = 0$, which implies

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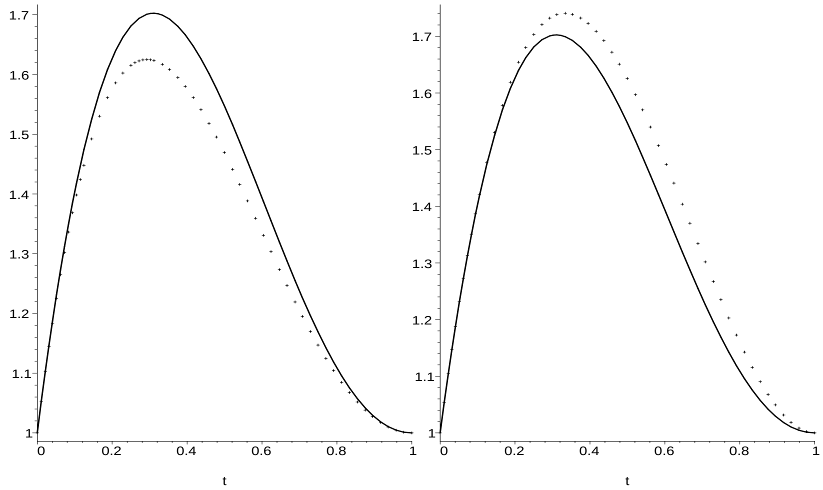

FIG. 3.2. *(left, $k = 0, l = 1$), (solid) degree 5 Bézier curve, (cross) reduced degree 4 curve, (right,$k = 1, l = 0$), (solid) degree 5 Bézier curve, (cross) reduced degree 3 curve.*

$f_\epsilon(t) = f(t)$. Thereby, the method of coefficient perturbation reproduces the initial polynomial whenever the exact degree reduction exists. On the other hand, the solution of the system (3.6) produces, in general, a polynomial of reduced degree $\leq n-1$.

REMARK 3.4. If we would like $f_\epsilon(t)$ to be of degree m $(< n-1)$, the conditions (3.1) must be replaced by $n-m$ constraints

$$T_j = \sum_{i=0}^{j} (-1)^i \binom{j}{i} p_i + \sum_{i=k+2}^{\min(n-l-2,j)} (-1)^i \binom{n}{j} \epsilon_i = 0, \quad j = m+1, \cdots, n.$$

Introducing $n-m$ Lagrange's multipliers $\lambda_1, \cdots, \lambda_{n-m}$, we construct the fonctional

$$L(\epsilon_{k+2}, \cdots, \epsilon_{n-l-2}; \lambda_1, \cdots, \lambda_{n-m}) = \int_0^1 \left(\sum_{i=k+2}^{n-l-2} \epsilon_i B_i^n(t) \right)^2 dt - \sum_{j=m+1}^{n} \lambda_{j-m} T_j$$

then solving the minimization problem

$$\min_{(\epsilon_{k+2}, \cdots, \epsilon_{n-l-2}, \lambda_1, \cdots, \lambda_{n-m})} L(\epsilon_{k+2}, \cdots, \epsilon_{n-l-2}; \lambda_1, \cdots, \lambda_{n-m})$$

gives the perturbation coefficients ϵ_i with obviously more complicated expressions.

REMARK 3.5. It is worth mentioning that the techniques developed in this paper can be applied to compute the degree reduction of parametric Bézier curves.

3.1. Particular case. For the one degree reduction, if $k+2 = n-l-2$, then all the ϵ_i's are equal to zero, except ϵ_{k+2} that is given by the last equation in (3.6)

$$\epsilon_{k+2} = \frac{1}{\binom{n}{k+2}} \sum_{i=0}^{n} (-1)^{i+k+3} \binom{n}{i} p_i.$$

The reduced degree polynomial $g(t) = \sum_{i=0}^{n-1} q_i B_i^{n-1}(t)$, according to (3.2), is such that

$$q_i = \frac{(-1)^i}{\binom{n-1}{i}} \sum_{j=0}^{i} \binom{n}{j} (p_j + \epsilon_j), \quad i = 0, \cdots, n-1.$$

For $n = 4$, $(k = l = 0)$, the constraints (3.1) correspond to the endpoints and endtangent vectors interpolation and we have $\epsilon_0 = \epsilon_1 = 0, \epsilon_2 = \sum_{i=0}^{4}(-1)^{i+3}\binom{4}{i}p_i, \epsilon_3 = \epsilon_4 = 0$. Fig. 3.1 shows two numerical examples where the reduced degree polynomials $g(t)$ are of degree 3 and 2. For $n = 5$, we have $k+l = 1$, which implies $(k, l) = (0, 1)$ or $(k, l) = (1, 0)$. Fig. 3.2 shows two examples where for $(k, l) = (0, 1)$ the reduced degree curve is of degree 4, and for $(k, l) = (1, 0)$ the reduced degree curve is of degree 3. Note that the relative position of the reduced degree curve with respect to the initial curve depends on the sign of ϵ_{k+2}; indeed we have $(f(t) - f_\epsilon(t)) = \epsilon_{k+2}B^n_{k+2}(t)$.

4. Error estimation. Using the convex hull property of Bernstein polynomials, the rate of approximation for the one degree reduction, is given by

$$\sup_{t\in[0,1]} \|f(t) - f_\epsilon(t)\| \leq \sup_{k+2\leq i\leq n-l-2} \|\epsilon_i\| \tag{4.1}$$

When the approximation error between $f(t)$ and $f_\epsilon(t)$ is larger than the prescribed tolerance, we can subdivide the interval $[0, 1]$ and perform constrained degree reduction on each subinterval. For in the particular case $k + 2 = n - l - 2$, if we subdivide the given curve $f(t)$ at parameter values $0 = t_0 < t_1 < \cdots < t_h = 1$, then the error estimation (4.1) is decreased by a factor $1/\delta^n$ where $\delta = \max_i(t_{i+1} - t_i)$. We get finally a continuous, piecewise approximation of $f(t)$ of lower degree.

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