Electronic Transactions on Numerical Analysis.
Volume 26, pp. 320-329, 2007.

ISSN 1068-9613.

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SINGULAR VALUE DECOMPOSITION NORMALLY ESTIMATED GERŠGORIN SETS*

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Abstract. Let $B \in \mathbb{C}^{N\times N}$ denote a finite-dimensional square complex matrix, and let $V\Sigma W^*$ denote a fixed singular value decomposition (SVD) of B. In this note, we follow up work from Smithies and Varga [Linear Algebra Appl., 417 (2006), pp. 370–380], by defining the SV-normal estimator $\epsilon_{V\Sigma W^*}$, (which satisfies $0 \le \epsilon_{V\Sigma W^*} \le 1$), and showing how it defines an upper bound on the norm, $\|B^*B - BB^*\|_2$, of the commutant of B and its adjoint, $B^* = \bar{B}^T$. We also introduce the SV-normally estimated Geršgorin set, $\Gamma^{\mathrm{NSV}}(V\Sigma W^*)$, of B, defined by this SVD. Like the Geršgorin set for B, the set $\Gamma^{\mathrm{NSV}}(V\Sigma W^*)$ is a union of N closed discs which contains the eigenvalues of B. When $\epsilon_{V\Sigma W^*}$ is zero, $\Gamma^{\mathrm{NSV}}(V\Sigma W^*)$ is exactly the set of eigenvalues of B; when $\epsilon_{V\Sigma W^*}$ is small, the set $\Gamma^{\mathrm{NSV}}(V\Sigma W^*)$ provides a good estimate of the spectrum of B. We end this note by expanding on an example from Smithies and Varga [Linear Algebra Appl., 417 (2006), pp. 370–380], and giving some examples which were generated using Matlab of the sets $\Gamma^{\mathrm{NSV}}(V\Sigma W^*)$ and $\Gamma^{\mathrm{RNSV}}(V\Sigma W^*)$, the reduced SV-normally estimated Geršgorin set.

Key words. Geršgorin type sets, normal matrices, eigenvalue estimates

AMS subject classifications. 15A18, 47A07

1. Introduction. In [6], we developed a theoretical analysis of how the set of all singular value decompositions of an $N\times N$ complex matrix can be used to estimate the eigenvalues of the matrix. Because our methods were motivated by functional analysis, that work allowed N to be countably infinite, but throughout this note, N is finite and B is a fixed $N\times N$ complex matrix. The purpose of this note is to introduce the *SV-normal estimator* $\epsilon_{V\Sigma W^*}$, of B and to define an associated *Geršgorin-type set* of B, $\Gamma^{\mathrm{NSV}}(V\Sigma W^*)$. The set $\Gamma^{\mathrm{NSV}}(V\Sigma W^*)$, which contains the eigenvalues of B, is the union of N closed discs in the complex plane. The SV-normal estimator is a parameter between 0 and 1; it is 0 when $V\Sigma W^*$ diagonalizes B in some orthonormal basis of $\mathbb{C}^N$, i.e., for each $j = 1, \cdots, \mathrm{Rank}(B)$, the j-th column of V is a unit length complex multiple of the j-th column of W. This happens for some SVD of B if and only if B is normal. In this case, our SV-normally estimated Geršgorin set is exactly the set of eigenvalues of B. Of course, the more computationally useful observation is that when $\epsilon_{V\Sigma W^*}$ is small, the centers of the discs, comprising $\Gamma^{\mathrm{NSV}}(V\Sigma W^*)$, are good estimates of the eigenvalues of B.

We begin by recalling some standard notation and definitions. Throughout this note, elements of $\mathbb{C}^N$ are $N\times 1$ matrices, and the inner product of $x, y \in \mathbb{C}^N$ is denoted by $< x, y >:= \sum_{l=1}^N x_l \bar{y}_l$. The norm on $\mathbb{C}^N$, induced by the inner product, is $\|x\|_2 := \sqrt{\sum_{l=1}^N |x|_l^2}$, and the corresponding operator norm on $B \in \mathbb{C}^{N\times N}$ is defined as $\|B\|_2 = \sup_{z\neq 0} \frac{\|Bz\|_2}{\|z\|_2}$. Finally, the Frobenius inner product of the matrices of $A, B \in \mathbb{C}^{N\times N}$ is denote here as $(A,B)_F := \mathrm{Trace}(A^*B)$, and the corresponding operator norm is denoted as $\|B\|_F := \sqrt{(B,B)_F}$.

A singular value decomposition of a non-zero complex $N\times N$ matrix B is an expression of B as a product $B = V\Sigma W^*$, where V and W are unitary matrices in $\mathbb{C}^{N\times N}$, and Σ

*Received August 17, 2006. Accepted for publication March 26, 2007. Recommended by Z. Strakos.

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is a non-negative diagonal matrix in $\mathbb{C}^{N\times N}$. Because we will need to use the specifics of this development, we will recall a construction of a singular value decomposition for B in detail. The square of the absolute value of B is $|B|^2 := B^*B$. If $B = V\Sigma W^*$ then $|B|^2 = W\Sigma V^* V\Sigma W^* = W\Sigma^2 W^*$. Since W is assumed to be unitary (i.e., $W^*W = WW^* = I_n \in \mathbb{C}^{N\times N}$), the columns of W must form an orthonormal basis of $\mathbb{C}^N$, consisting of eigenvectors of $|B|^2$. We assume, without loss of generality, that the columns of W are arranged so that the corresponding eigenvalues $\{\sigma_j^2\}_{j=1}^N$ of $|B|^2$ are in non-increasing order. Thus,

$$\Sigma := \mathrm{Diag}(\sigma_1, \sigma_2, \cdots, \sigma_N),$$

where $\sigma_1 \geq \sigma_2 \geq, \cdots, \geq \sigma_N \geq 0$. Similarly, the columns of V form an orthonormal basis of $\mathbb{C}^N$ consisting of eigenvectors of $|B^*|^2 = BB^*$.

The singular value decomposition of B is never unique. Clearly, any unitary matrix U which satisfies $U\Sigma = \Sigma U$ defines another SVD of B, namely, $B = (VU)\Sigma(WU)^*$. The conditions, $U\Sigma = \Sigma U$ and $U^*U = UU^* = I_n \in \mathbb{C}^{N\times N}$ are not only sufficient to define another SVD, but also necessary. This was first pointed out to the authors by Roger Horn. Professor Horn directed our attention to the discussion of [3, Theorem 3.1.1]. One of the referees of this note pointed out that the following lemma is also developed (for non-square matrices) in [1] and [4].

LEMMA 1.1. *Let A be a non-zero $N \times N$ matrix, and let $V\Sigma W^*$ and $X\Sigma Y^*$ be two SVDs of A, where*

$$\Sigma = \mathrm{Diag}(\sigma_1, \cdots, \sigma_1, \sigma_2, \cdots, \sigma_2, \cdots, \sigma_r, \cdots, \sigma_r, 0, \cdots, 0),$$

and σ_k has multiplicity n_k, for $k = 1, \cdots, r+1$. Then, there exists a collection of matrices $\{U_k\}_{k=1}^r$ where each U_k is an $n_k \times n_k$ unitary matrix, and $n_{r+1} \times n_{r+1}$ unitary matrices, S_0 and T_0 such that for

$$S := \mathrm{Diag}(U_1, \cdots, U_r, S_0), \text{ and } T := \mathrm{Diag}(U_1, \cdots, U_r, T_0),$$

the relations $X = VS$ and $Y = WT$ hold. Moreover, if A is non-singular, then $V^*W = X^*Y$.

Given an $N\times N$ matrix B, one can construct an SVD of B as follows. Let $|B|^2 := B^*B$. Then, $|B|^2$ is a self-adjoint matrix which is non-negative, in the sense that for all column vectors $x \in \mathbb{C}^N$, $< |B|^2 x, x >$ is a non-negative number. Hence, there are known robust numerical packages which, for N not too large, can provide us with an accurate orthonormal basis $\{\psi_j\}_{j=1}^N$ for $\mathbb{C}^N$, which are eigenvectors of $|B|^2$. Furthermore, we can, without loss of generality, assume that these eigenvectors satisfy $|B|^2\psi_j = \sigma_j^2\psi_j$, and that the associated eigenvalues are ordered so that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_N \geq 0$. The collection of numbers $\{\sigma_j\}_{j=1}^N$ is called the set of *singular values* of B. They are the eigenvalues of $|B| = \sqrt{|B|^2}$.

Let K index the smallest non-zero singular value σ_K, i.e., $\sigma_l = 0$ for all $l > K$. Then, K is the rank of B. Both kernels $\mathrm{Ker}(B)$ and $\mathrm{Ker}(B^*)$ have dimension $L := N - K$. Fix any orthonormal basis $\{\gamma_k\}_{k=1}^L$ of $\mathrm{Ker}(B^*)$, and define the vectors ϕ_j by

$$\phi_j := \begin{cases} \frac{1}{\sigma_j} B\psi_j & \text{if } \sigma_j > 0 \qquad (\text{i.e., if } j \leq K), \\ \gamma_j & \text{if } \sigma_j = 0 \qquad (\text{i.e., if } j > K). \end{cases}$$

The vectors $\{\phi_j\}_{j=1}^N$, defined above, form an orthonormal basis of $\mathbb{C}^N$, consisting of eigenvectors of $|B^*|^2$. Let V be the $N \times N$ matrix whose j-th column is the vector ϕ_j, and let

$\Sigma = \mathrm{Diag}(\sigma_1, \cdots, \sigma_N)$. Define W to be the $N \times N$ matrix whose j-th column is the vectors ψ_j. By checking its action on the basis $\{\psi_j\}_{j=1}^N$, it is easy to see that

$$B = V\Sigma W^* = \begin{bmatrix} | & \cdots & | \\ \phi_1 & \cdots & \phi_N \\ | & \cdots & | \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_N \end{bmatrix} \begin{bmatrix} - & \bar{\psi}_1^T & - \\ \vdots & \vdots & \vdots \\ - & \bar{\psi}_N^T & - \end{bmatrix}. \tag{1.1}$$

As Lemma 1.1 above suggests, the choice of the vectors $\{\phi_j\}_{j=1}^N$ and $\{\psi_j\}_{j=1}^N$, in the above construction of an SVD of B, is somewhat arbitrary for $j > K$. Even if the kernel of B is trivial (i.e., $K = N$), the singular value decomposition, given by the above development, is not uniquely determined. Specifically, the vectors $\{\psi_j\}_{j=1}^N$ can be replaced with any orthonormal basis of eigenvectors for $|B|$. However, the matrix Σ is the same for *every* singular value decomposition of B. A consequence of the non-uniqueness of the singular value decomposition is that the SV-normal estimator $\epsilon_{V\Sigma W^*}$ and the corresponding Geršgorin type set $\Gamma^{\mathrm{NSV}}(V\Sigma W^*)$, which we define below, *depend* on the choice of SVD of B.

Let B be a non-zero $N \times N$ matrix of rank K and fix a singular value decomposition of B, as in (1.1). The components of this SVD of B can be used to express B as a sum of rank one operators in $\mathbb{C}^{N\times N}$. Specifically, for any fixed j, $\sigma_j \phi_j \psi_j^*$ is the rank one $N \times N$ matrix $\sigma_j\, \phi_j (\bar{\psi}_j)^T$. It takes any $x \in \mathbb{C}^N$ to the complex multiple $\sigma_j < x, \psi_j >$ of the vector ϕ_j. Thus,

$$B = \sum_{l=1}^{N} \sigma_l \phi_l \psi_l^* = \sum_{l=1}^{K} \sigma_l \phi_l \psi_l^*.$$

2. Lemmas. Using an expression of B as a sum of rank one operators, given by the SVD, $V\Sigma W^*$, we now define the parameter $\epsilon_{V\Sigma W^*}$, called *the SV-normal estimator*, which, in essence, measures how close the given SVD is to directly defining a spectral decomposition.

DEFINITION 2.1. *Let B be a non-zero $N \times N$ matrix. Let K index the smallest positive singular value of B. That is, K is the rank of B. Let $B = V\Sigma W^* = \sum_{l=1}^K \sigma_l \phi_l \psi_l^*$ denote a fixed SVD of B. Define the SV-compatibility index of this SVD, K' as follows: if $< \phi_1, \psi_1 >= 0$, set $K' = 0$. Otherwise, let K' be the maximal number which is less than or equal to K such that*

$$< \phi_l, \psi_l > \neq 0 \quad \textit{for all} \quad l = 1, \cdots, K'.$$

We say the SVD is fully compatible (with our methods) if $K' = K$. We remark that full compatibility is only needed for the construction of the reduced normally estimated Geršgorin set, $\Gamma^{\mathrm{RNSV}}(V\Sigma W^*)$, defined below. Set

$$\epsilon_l := \sqrt{1 - | < \phi_l, \psi_l > |^2}, \quad l = 1, \cdots, K,$$

and define

$$\epsilon_{V\Sigma W^*} := \max\{\epsilon_l\}_{l=1}^K.$$

We call $\epsilon_{V\Sigma W^*}$ the SV-normal estimator of B, defined by this SVD.

By the Cauchy-Schwarz inequality, each ϵ_l satisfies $0 \le \epsilon_l \le 1$. Moreover, if $K' \neq K$, then $\epsilon_{V\Sigma W^*} = 1$. The next lemma sheds some light on our motivation for defining the

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parameters ϵ_l above. We would like to thank the referees for pointing out a simplified proof of this lemma.

LEMMA 2.2. *Let x and y be unit vectors in $\mathbb{C}^N$, and define the $N \times N$ matrix A by $A = xx^* - yy^*$. Then, the norm of A, $\|A\|_2$, is given by*

$$\|A\|_2 = \sqrt{1 - |<x, y>|^2}.$$

Proof. If x and y are linearly dependent, the lemma is clearly true. In this case, A is the zero matrix and $|<x, y>| = 1$. Otherwise, $N \geq 2$ and A is an $N \times N$ matrix of rank 2. Since A is self-adjoint, $\|A\|_2^2$ is the largest eigenvalue of the $N \times N$ matrix

$$|A|^2 = A^*A = xx^* - <x, y> yx^* - <y, x> xy^* + yy^*.$$

It is easy to calculate that $|A|^2x = (1 - |<x, y>|^2)x$ and $|A|^2y = (1 - |<x, y>|^2)y$. That is, when x and y are linearly independent, the two non-zero eigenvalues of $|A|^2$ are $1 - |<x, y>|^2$. □

The SV-normal estimator $\epsilon_{V\Sigma W^*}$, defined by the SVD of B, $V\Sigma W^*$, provides an estimate on the commutant of B and B^*, as described in the following lemma.

LEMMA 2.3. *Let $B \in \mathbb{C}^{N\times N}$ and let $\epsilon_{V\Sigma W^*}$ denote the SV-normal estimator, defined from the SVD $V\Sigma W^*$ of B. Then,*

$$\|B^*B - BB^*\|_2 \leq \|B\|_F^2 \ \epsilon_{V\Sigma W^*}.$$

Proof. Let B be an $N \times N$ matrix and let $K \leq N$ index the smallest non-zero singular value of B. Let $B = V\Sigma W^* = \sum_{l=1}^K \sigma_l \phi_l \psi_l^*$ denote a fixed SVD of B. Then, $B^* = W\Sigma V^* = \sum_{l=1}^K \sigma_l \psi_l (\phi_l)^*$. Hence,

$$\begin{aligned}\|B^*B - BB^*\|_2 &= \|\sum_{l=1}^K \sigma_l^2 [\psi_l\psi_l^* - \phi_l\phi_l^*]\|_2 \leq \sum_{l=1}^K \sigma_l^2 \|\psi_l\psi_l^* - \phi_l\phi_l^*\|_2 \\ &= \sum_{l=1}^K \sigma_l^2 \epsilon_l \leq \epsilon_{V\Sigma W^*} \ \mathrm{Trace}(|B|^2) = \epsilon_{V\Sigma W^*} \|B\|_F^2. \qquad \square\end{aligned}$$

Note that the factor $\|B\|_F^2 = \sum_{l=1}^K \sigma_l^2$ is independent of the choice of SVD of B. It is a measure of the scaling of B. The other factor, $\epsilon_{V\Sigma W^*}$, *depends* on the specific SVD, $V\Sigma W^*$ of B. It is a number which measures how close a given SVD is to directly defining a spectral decomposition. More precisely, $\epsilon_{V\Sigma W^*} = 0$ when the columns of W comprise an orthonormal basis of $\mathbb{C}^N$, consisting of eigenvectors of B. When the parameter $\epsilon_{V\Sigma W^*}$ is small, the SVD can be expressed as the sum of a normal matrix and a matrix whose norm is small. In this case, the given SVD yields an inexpensive and accurate estimate of eigenvalues, as described below. This estimate can be far more accurate than the usual Geršgorin estimates of the eigenvalues; see Section 4, below.

3. Theorems. In this section we define the Geršgorin-type sets $\Gamma^{\rm NSV}(V\Sigma W^*)$ and $\Gamma^{\rm RNSV}(V\Sigma W^*)$, corresponding to a fixed SVD $V\Sigma W^*$ of an $N \times N$ matrix B. We also show that each of these sets contains all of the eigenvalues of B.

DEFINITION 3.1. *Let B be an $N \times N$ matrix and let $K \leq N$ index the smallest non-zero singular value of B. Let*

$$B = V\Sigma W^* = \sum_{l=1}^K \sigma_l \phi_l \psi_l^*$$

denote a fixed singular value decomposition of B, and let K', from Definition 2.1, denote the compatibility index of this SVD. If $K' = 0$, define the SV-normally estimated Geršgorin set to be

$$\Gamma^{\mathrm{NSV}}(V\Sigma W^*) := \bar{D}(0, \|B\|_2) := \{z \in \mathbb{C} \ : \ |z| \leq \|B\|_2\}.$$

Now, assume $1 \leq K' \leq K$ and set

$$R := \sqrt{2}\sqrt{\sum_{l=1}^{K'} \sigma_l^2 \, \epsilon_l^2}. \tag{3.1}$$

For each $l = 1, \cdots, K'$, set $e^{\imath\theta_l} := \frac{<\phi_l,\psi_l>}{|<\phi_l,\psi_l>|}$ and set $\mu_l := \sigma_l e^{\imath\theta_l}$. For consistency of notation, if $K' < N$ define $\mu_{K'+1} := 0$, and if $K' = N$, define $\sigma_{K'+1} = 0$. Set $M :=$ $\min\{N, K'+1\}$. For $K' \neq 0$, we define the SV-normally estimated Geršgorin set as

$$\Gamma^{\mathrm{NSV}}(V\Sigma W^*) := \cup_{l=1}^{M} \bar{D}(\mu_l, R+\sigma_{K'+1}) := \cup_{l=1}^{M}\{z \in \mathbb{C} \ : \ |z-\mu_l| \leq R+\sigma_{K'+1}\}. \tag{3.2}$$

Next, we show that for each SVD of any fixed matrix $B \in \mathbb{C}^{N\times N}$, its SV-normally estimated Geršgorin set $\Gamma^{\mathrm{NSV}}(V\Sigma W^*)$ necessarily contains the eigenvalues of B.

THEOREM 3.2. *Let $B \in \mathbb{C}^{N\times N}$ and let $V\Sigma W^*$ denote a fixed SVD of B. Then, the set of eigenvalues of B is contained in the SV-normal Geršgorin set $\Gamma^{\mathrm{NSV}}(V\Sigma W^*)$.*

Proof. Let $B = V\Sigma W^* = \sum_{l=1}^{K} \sigma_l \phi_l \psi_l^*$ denote a fixed SVD of B. Let K' be the compatibility index of this SVD, and let λ be an eigenvalue of B, where $Bx = \lambda x$ with $\|x\|_2 = 1$. We show that λ is in the set $\Gamma^{\mathrm{NSV}}(V\Sigma W^*)$. This containment is trivial if $K' = 0$ since $|\lambda| = \|Bx\|_2 \leq \|B\|_2$. Now suppose $1 \leq K' \leq N$. In this case, the set $\Gamma^{\mathrm{NSV}}(V\Sigma W^*)$ is defined as in (3.2). Recall that $M = \min\{N, K'+1\}$, and for all $l = 1, \cdots, K'$, $e^{\imath\theta_l}$ denotes the complex rotation $\frac{<\phi_l,\psi_l>}{|<\phi_l,\psi_l>|}$ and $\mu_l := \sigma_l e^{\imath\theta_l}$. For consistency of notation, let $\sigma_{N+1} := 0$ and $\mu_{K'+1} := 0$. For $1 \leq l \leq K'$, define $\beta_l \in \mathbb{C}^N$ by $\beta_l := \phi_l - e^{\imath\theta_l}\psi_l$. Then

$$\begin{aligned} \|\beta_l\|_2^2 \ &= < \phi_l - e^{\imath\theta_l}\psi_l, \phi_l - e^{\imath\theta_l}\psi_l > = 2 - 2\,\mathrm{Re}(e^{-\imath\theta_l} < \phi_l, \psi_l >) \\ &= 2(1 - | < \phi_l, \psi_l > |) \leq 2(1 - | < \phi_l, \psi_l > |^2) = 2\epsilon_l^2. \end{aligned} \tag{3.3}$$

Thus, $\|\beta_l\|_2 \leq \sqrt{2}\epsilon_l$, for all $l = 1, \cdots K'$. We can use the vectors β_l to rewrite B as the sum of a normal matrix and two matrices which have small norms when $\epsilon_{V\Sigma W^*}$ is small. Specifically, let

$$A := \sum_{l=1}^{K'} \mu_l \phi_l (\phi_l)^*, \quad E := \sum_{l=1}^{K'} \mu_l \phi_l (\beta_l)^*, \quad \text{and} \quad C := \sum_{l=K'+1}^{K} \sigma_l \phi_l \psi_l^*. \tag{3.4}$$

Then, $B = A - E + C$ since

$$\begin{aligned} A - E + C &= \sum_{l=1}^{K'} \mu_l \phi_l (\phi_l)^* - \sum_{l=1}^{K'} \mu_l \phi_l (\beta_l)^* + \sum_{l=K'+1}^{K} \sigma_l \phi_l \psi_l^* \\ &= \sum_{l=1}^{K'} \mu_l \phi_l (\phi_l - \beta_l)^* + \sum_{l=K'+1}^{K} \sigma_l \phi_l \psi_l^* = \sum_{l=1}^{K} \sigma_l \phi_l \psi_l^* = B. \end{aligned}$$

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The matrix A in (3.4) is normal. Its spectrum consists of 0, with multiplicity $N - K'$, and the elements of $\{\mu_l\}_{l=1}^{K'}$. For each complex number ν, which is outside the spectrum A, we have the following formula for its distance from the spectrum of A:

$$d(\nu, \sigma(A)) := \min_{b\in\sigma(A)} \{|\nu - b|\} = \frac{1}{||(A - \nu I)^{-1}||_2}.$$

This distance formula holds since A is normal; see [2] or [5].

Recall that $Bx = \lambda x$ with $||x||_2 = 1$. We will show that $d(\lambda, \sigma(A)) \leq \|C\|_2 + \|E\|_2$, where E and C are defined in (3.4). If $\lambda \in \sigma(A)$, then we are done. Otherwise, we have the above equality to calculate the distance of λ from the spectrum of A. Now

$$1 = ||x||_2 = \|(A - \lambda I)^{-1}(A - \lambda I)x\|_2 \leq \|(A - \lambda I)^{-1}\|_2\|Ax - \lambda x\|_2.$$

This gives us $d(\lambda, \sigma(A)) \leq ||Ax - \lambda x||_2 = ||Ax - Bx||_2 = ||(C - E)x||_2 \leq ||C - E||_2 \leq ||C||_2 + ||E||_2$.

It remains to show that $\|C\|_2 + \|E\|_2 \leq R + \sigma_{K'+1}$ where R is defined in (3.1) and E and C are defined in (3.4).

Let y be a fixed unit vector in $\mathbb{C}^N$. Parseval's theorem implies that

$$\|Ey\|_2^2 = \sum_{l=1}^{K'} |\mu_l|^2| < y, \beta_l > |^2 \leq \sum_{l=1}^{K'} \sigma_l^2\|\beta_l\|_2^2 = 2\sum_{l=1}^{K'} \sigma_l^2\epsilon_l^2.$$

Thus, $\|E\|_2 \leq R$. Moreover, $\|C\|_2 \leq \sigma_{K'+1}$. If $K' = K$, then C is the zero matrix and $\sigma_{K'+1} = 0$. Otherwise, for any unit vector y,

$$\|Cy\|_2^2 = \sum_{l=K'+1}^{K} \sigma_l^2| < y, \psi_l > |^2 \leq \sum_{l=K'+1}^{K} \sigma_{K'+1}^2| < y, \psi_l > |^2 \leq \sigma_{K'+1}^2\|y\|_2^2.$$

Combining these estimates for the norms of E and C, we have shown that any eigenvalue of B is a most the distance $R+\sigma_{K'+1}$ from an element of the spectrum of A. This exactly says the eigenvalues of B are contained in the SV-normally estimated set $\Gamma^{\mathrm{NSV}}(V\Sigma W^*)$. $\square$

The above development has a useful reformulation for the case when B has sufficiently small rank K and the SVD, $V\Sigma W^*$, is fully compatible (i.e., $< \phi_l, \psi_l > \neq 0$ for all $l = 1, \cdots, K$). Notice that all of the circles in the set $\Gamma^{\mathrm{NSV}}(V\Sigma W^*)$ have radius $R + \sigma_{K'+1}$. A more detailed examination of the above proof allows us to replace $R + \sigma_{K'+1}$ with a family of radii $R_l = \sqrt{2K}\sigma_l\epsilon_l$, for $l = 1, \cdots, K$. Since each radius has a factor of $\sqrt{K}$, this reformulation is only useful when the rank of B is small.

DEFINITION 3.3. *Let $V\Sigma W^* = \sum_{l=1}^{K} \sigma_l\phi_l\psi_l^*$ denote a fixed, fully compatible, singular value decomposition of a non-zero $N \times N$ matrix B. For each $l = 1, \cdots, K$, let $e^{\imath\theta_l} = \frac{<\phi_l,\psi_l>}{|<\phi_l,\psi_l>|}$, and define the the l-th small rank SV-normally estimated Geršgorin disc to be the set*

$$\Gamma_l^{\mathrm{RNSV}}(V\Sigma W^*) := \{z \in \mathbb{C} \, : \, |z - \sigma_l e^{\imath\theta_l}| \leq \sqrt{2K}\sigma_l\epsilon_l\}.$$

Define the small rank SV-normally estimated Geršgorin set of B which is given by this SVD to be the set

$$\Gamma^{\mathrm{RNSV}}(V\Sigma W^*) := \cup_{l=1}^{K}\Gamma_l^{\mathrm{RNSV}}(V\Sigma W^*) \cup \mathcal{O},$$

where $\mathcal{O}$ is the empty set if $K = N$ and $\{0\}$ otherwise.

Next, we show that the small rank SV-normally estimated Geršgorin set contains the eigenvalues of the matrix. Of course, unless the rank, K, of B is small, the factor $\sqrt{K}$ will tend to make the set $\Gamma^{\mathrm{RNSV}}(V\Sigma W^*)$ too large to be useful. More precisely, this estimate is useful only when the SV-normal estimator, $\epsilon_{V\Sigma W^*}$, is smaller than $\frac{1}{\sqrt{2K}}$.

THEOREM 3.4. *Let $B \in \mathbb{C}^{N\times N}$ and let $V\Sigma W^*$ denote a fixed, fully compatible SVD of B. Then, the set of eigenvalues of B is contained in the small rank SV-normally estimated Geršgorin set $\Gamma^{\mathrm{RNSV}}(V\Sigma W^*)$.*

Proof. Let $B = V\Sigma W^* = \sum_{l=1}^K \sigma_l \phi_l \psi_l^*$ denote a fixed, fully compatible SVD of B. Let $Bx = \lambda x$ where $\|x\|_2 = 1$. Since the SVD is fully compatible, $<\phi_l, \psi_l> \neq 0$ for all $l = 1, \cdots, K$. For each $l = 1, \cdots, K$, let $e^{\imath\theta_l}$ denote the complex rotation $\frac{<\phi_l,\psi_l>}{|<\phi_l,\psi_l>|}$. Define $\beta_l \in \mathbb{C}^N$ by $\beta_l := \phi_l - e^{\imath\theta_l}\psi_l$. Recall from (3.3) that $\|\beta_l\|_2 \leq \sqrt{2}\epsilon_l$, for $1 \leq l \leq K$. We can use the vectors β_l to write B as

$$B = \sum_{l=1}^K \sigma_l \phi_l (\psi_l)^* = \sum_{l=1}^K \sigma_l e^{\imath\theta_l} \phi_l (\phi_l)^* - \sum_{l=1}^K \sigma_l e^{\imath\theta_l} \phi_l (\beta_l)^*.$$

The equation $Bx = \lambda x$ for $\lambda \neq 0$ implies that x is in the range of B and hence in the span of the vectors $\{\phi_l\}_{l=1}^K$. Thus,

$$\lambda x = \lambda \sum_{l=1}^K < x, \phi_l > \phi_l.$$

On the other hand,

$$Bx = \sum_{l=1}^K \sigma_l e^{\imath\theta_l} < x, \phi_l > \phi_l - \sum_{l=1}^K \sigma_l e^{\imath\theta_l} < x, \beta_l > \phi_l.$$

Equating coefficients of the orthonormal vectors ϕ_l gives us

$$(\sigma_l e^{\imath\theta_l} - \lambda) < x, \phi_l >= \sigma_l e^{\imath\theta_l} < x, \beta_l > \quad \text{for all} \quad l = 1, \cdots, K.$$

By the Cauchy-Schwarz inequality, $|\sigma_l e^{\imath\theta_l} < x, \beta_l > | \leq \sigma_l \|\beta_l\|_2 \leq \sqrt{2}\sigma_l \epsilon_l$. Moreover, since $\|x\|_2^2 = 1 = \sum_{l=1}^K | < x, \phi_l > |^2$, there must exist an index L between 1 and K such that $| < x, \phi_L > | \geq \frac{1}{\sqrt{K}}$. For this index, the above calculations yield

$$|\lambda - \sigma_L e^{\imath\theta_L}| \leq \sqrt{2K}\sigma_L \epsilon_L.$$

This exactly says λ is in the L-th disc defining the set $\Gamma^{\mathrm{RNSV}}(V\Sigma W^*)$. □

4. Examples.

We end this note with some examples which illustrate how our methods work.

EXAMPLE 4.1. As a simple illustration of our methods, we will calculate all possible SV-normally estimated Geršgorin sets of the matrix $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then, $|B|^2 = B^*B$ is the identity matrix on $\mathbb{C}^2$. The most natural singular value decomposition of B is given by the choice of the standard basis for $\mathbb{C}^2$ as the eigenvector basis for $|B|$. That is,

$$B = V\Sigma W^* = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

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Any other singular value decomposition $B = V_1 \Sigma W_1^*$ must have the form $V_1 = VU$ and $W_1 = WU$ where $UU^* = U^*U = I_2$. That is,

$$U = \begin{bmatrix} ae^{\imath\lambda} & -be^{\imath\mu} \\ be^{-\imath\mu} & ae^{-\imath\lambda} \end{bmatrix} \qquad 0 \le a,\, b \le 1;\; a^2 + b^2 = 1;\; 0 \le \lambda,\, \mu < 2\pi.$$

This choice of U gives us the SVD factors

$$V_1 = VU = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} ae^{\imath\lambda} & -be^{\imath\mu} \\ be^{-\imath\mu} & ae^{-\imath\lambda} \end{bmatrix} = \begin{bmatrix} ae^{\imath\lambda} & -be^{\imath\mu} \\ -be^{-\imath\mu} & -ae^{-\imath\lambda} \end{bmatrix} := \begin{bmatrix} | & | \\ \phi_1 & \phi_2 \\ | & | \end{bmatrix},$$

and

$$W_1^* = U^*W^* = U^* = \begin{bmatrix} ae^{-\imath\lambda} & be^{\imath\mu} \\ -be^{-\imath\mu} & ae^{\imath\lambda} \end{bmatrix} := \begin{bmatrix} -\bar{\psi}_1^T- \\ -\bar{\psi}_2^T- \end{bmatrix}.$$

For this SVD of B, $< \phi_1, \psi_1 >= a^2 - b^2$ and $< \phi_2, \psi_2 >= b^2 - a^2$. First, assume that $a \neq b$. The centers for the normally estimated SV Geršgorin discs from this SVD are $\mu_l = \frac{<\phi_l,\psi_l>}{|<\phi_l,\psi_l>|}$ for $l = 1, 2$. Thus, the centers are 1 and -1, the eigenvalues of B. For $l = 1$ and 2, the constant ϵ_l^2 is $1 - | < \phi_l, \psi_l > |^2 = 4a^2b^2$. Both of the SV-normally estimated Geršgorin discs constructed from this SVD have radius $R = \sqrt{2(\sigma_1^2\epsilon_1^2 + \sigma_2^2\epsilon_2^2)} = 4ab$. For example, if $a = .01$, then $b = \sqrt{.9999} = .999949987\cdots$ and $4ab = .0399979\cdots$. The SV-normally estimated Geršgorin set $\Gamma^{\text{NSV}}(V_1\Sigma W_1^*)$ is the union of the closed discs with centers -1 and 1, each with radius nearly .04. As a approaches but does not equal b, the centers of the discs in $\Gamma^{\text{NSV}}(V_1\Sigma W_1^*)$ remain 1 and -1 and the radius of each disc approaches 2, from below. In the extreme case, $a = b = \frac{1}{\sqrt{2}}$, the calculations defining our set would involve a division by zero. In this case we have defined the SV-normally estimated Geršgorin set to be the closed disc centered at the 0 of radius 1 (i.e., radius $\|B\|_2$).

EXAMPLE 4.2. This example was generated using version 6.1 of Matlab on the matrix

$$B = \begin{bmatrix} 1 & 20 & 30 \\ 21 & 4 & 51 \\ 31 & 50 & 8 \end{bmatrix}.$$

The asterisks in Figure 4.1 denote the eigenvalues of B. The union of the very small (magenta) discs around these asterisks is the SV-normally estimated Geršgorin $\Gamma^{\text{NSV}}(V\Sigma W^*)$. The large (green) discs are the Geršgorin circles in the Geršgorin set of B, $\Gamma(B)$. This example shows that when our method is applied to a matrix which is normal, or almost normal, the set of the centers of the discs in $\Gamma^{\text{NSV}}(V\Sigma W^*)$ can be a much better estimate of the set of eigenvalues of B, than the eigenvalue estimates given by the usual Geršgorin set, $\Gamma(B)$.

EXAMPLE 4.3. Our last example was generated using version 6.1 of Matlab and the 10×10 matrix of rank 3, $B =$

$$\begin{bmatrix}
1.761 & 0.309 & 0.701 & -0.07 & 0.682 & 1.166 & -0.17 & 0.549 & 0.195 & -0.50 \\
0.306 & 0.560 & 0.722 & 0.395 & 0.552 & 0.037 & 0.544 & 0.739 & -0.03 & 0.611 \\
0.697 & 0.725 & 0.992 & 0.458 & 0.786 & 0.266 & 0.614 & 0.981 & 0.002 & 0.635 \\
-0.07 & 0.394 & 0.458 & 0.907 & 0.538 & 0.539 & 0.408 & 0.020 & 0.932 & 1.066 \\
0.677 & 0.553 & 0.786 & 0.538 & 0.716 & 0.577 & 0.402 & 0.585 & 0.391 & 0.588 \\
1.162 & 0.032 & 0.266 & 0.539 & 0.577 & 1.731 & -0.39 & -0.45 & 1.397 & 0.042 \\
-0.18 & 0.547 & 0.614 & 0.408 & 0.402 & -0.385 & 0.680 & 0.730 & -0.20 & 0.794 \\
0.547 & 0.744 & 0.981 & 0.020 & 0.585 & -0.45 & 0.730 & 1.386 & -0.85 & 0.336 \\
0.118 & -0.04 & -0.01 & 0.964 & 0.381 & 1.368 & -0.182 & -0.87 & 1.770 & 0.739 \\
-0.49 & 0.605 & 0.630 & 1.051 & 0.583 & 0.045 & 0.783 & 0.334 & 0.676 & 1.489
\end{bmatrix}.$$

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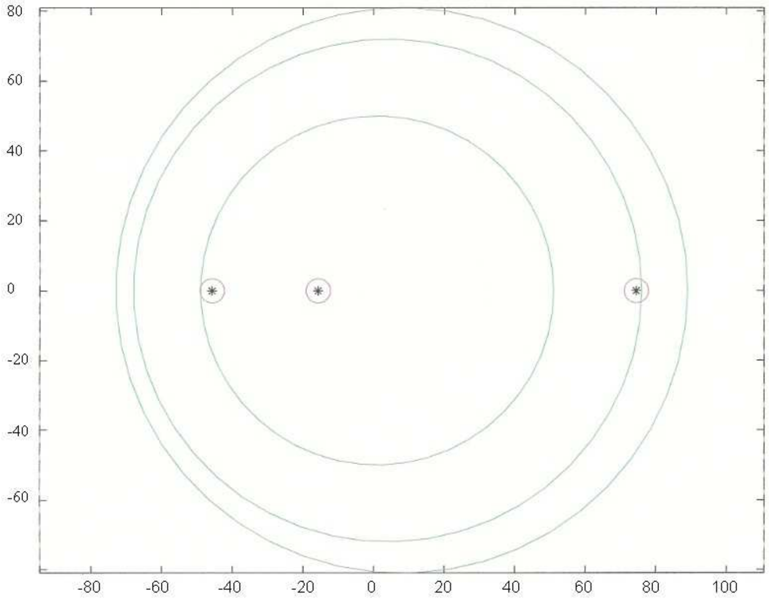

FIG. 4.1. $\Gamma^{\mathrm{NSV}}(V\Sigma W^*)$ *is the small discs (magenta);* $\Gamma(B)$ *is large discs (green).*

In Figure 4.2, the set $\{0\}$ and the union of the three smallest (blue) discs is the reduced SV-

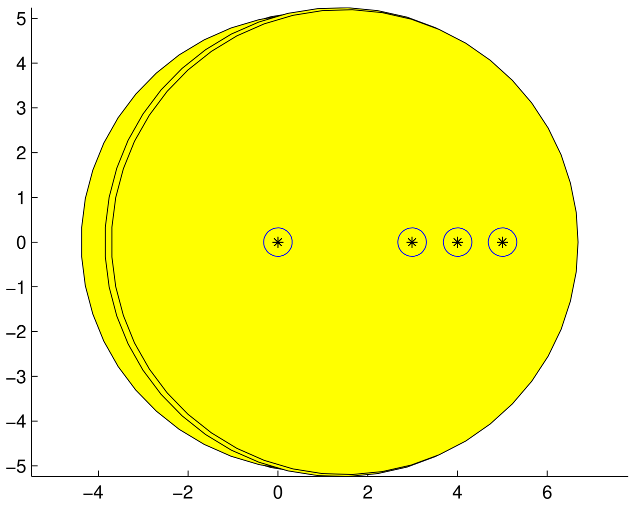

FIG. 4.2. $\Gamma^{\mathrm{RNSV}}(V\Sigma W^*)$ *is small discs (blue);* $\Gamma(B)$ *is the large discs (yellow).*

normally estimated Geršgorin set, $\Gamma^{\mathrm{RNSV}}(V\Sigma W^*)$. The union of the larger circles (yellow) is the standard Geršgorin circles, $\Gamma(B)$. This example shows that the eigenvalue estimates given by $\Gamma^{\mathrm{RNSV}}(V\Sigma W^*)$ can be much better than the estimate given by the usual Geršgorin sets.

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