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AN ADDITIVE SCHWARZ METHOD FOR MORTAR MORLEY FINITE ELEMENT DISCRETIZATIONS OF 4TH ORDER ELLIPTIC PROBLEM IN 2D*

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Abstract. In this paper we introduce and analyze a parallel ASM preconditioner for the system of equations arising from the finite element discretizations of a fourth order elliptic problem with large jumps in coefficients on nonconforming meshes. Locally Morley nonconforming element is used. The condition number estimate proved here is almost optimal, i.e., it grows polylogarithmically as the sizes of the meshes decrease.

Key words. Plate problem, mortar finite element method, Morley nonconforming plate element, domain decomposition, preconditioner, additive Schwarz method.

AMS subject classifications. 65N55, 65N30, 65N22, 74S05.

1. Introduction. Domain decomposition methods, which form a group of effective parallel solvers for the system of algebraic equations arising from the discretization of partial differential equations, e.g. by finite element method, have been studied and used for many years, e.g., cf. [29, 30, 32].

In this paper we introduce and analyze a domain decomposition method based on Additive Schwarz Method abstract scheme for solving a system of algebraic equations arising from Morley mortar element discretization of a fourth order model elliptic problem in two dimensions.

The mortar technique which was introduced by Bernardi, Maday and Patera [6] allows us to construct discretization methods on nonconforming grids. The meshes in subdomains can be nonmatching across interfaces, i.e., the common edges to adjacent subdomains and thus can be generated independently, e.g. one local subdomain mesh in each processor. The mortar technique imposes on the finite element functions special weak continuity integral coupling conditions on the interfaces of adjacent subdomains. The mortar method applied to many types of problems has been intensively studied recently; see [4, 5, 6] for second order elliptic problems and [22, 3, 26] for mortar discretization of fourth order problems.

In this paper locally in subdomains we utilize the well known Morley nonconforming finite element which can be described as one of the simplest finite elements for fourth order problems such as plate problems. The mortar method which utilizes Morley discretization was developed and analyzed in [26] and [21]. (There is a slightly different approach in the latter paper.)

Additive Schwarz Method (ASM) framework is a very powerful tool of constructing parallel preconditioners, e.g., cf. [32]. It is defined in terms of a decomposition of finite element space into subspaces and local bilinear forms defined onto these subspaces.

ASM type solvers for Morley finite element discretization built on one conforming triangulation were developed and analyzed, e.g., cf. [10, 9, 13], and references therein.

Many solvers including ASM type methods for problems arising from mortar discretizations were developed recently (see, e.g., [1, 2, 7, 8, 14, 15, 17, 20, 19, 25, 31, 33, 34, 35] and the references therein), but there are not many domain decomposition methods for solving systems of equations arising from the mortar element type discretization of fourth order problems; see [25] for ASM preconditioners for conforming HCT element and [36] for multigrid

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algorithm. The results of these two papers concern fourth order problems with continuous coefficients, and if the ideas contained there are followed in order to construct and analyze analogous algorithms for an elliptic fourth order problem with discontinuous coefficients, then some constants in the estimates would be dependent on jumps of the coefficients, e.g., the coarse grid used in [25] is not suitable for the problems with jumps in coefficients even for the case of one conforming mesh.

To our knowledge there are no parallel ASM type methods for mortar Morley discretization or any other mortar discretization with locally nonconforming elements of a fourth order elliptic problem and there are also no algorithms for any mortar discretizations with locally conforming or nonconforming elements for the case of fourth order elliptic problem with discontinuities of coefficients in literature.

The nonconformity of the Morley element leads to lines of proof that are usually much more technical. Simultaneously, the Morley finite element has fewer degrees of freedom than other finite elements. It is important that there is only one degree of freedom related to a crosspoint, i.e., a common vertex to some substructures. This allows us to construct a special coarse space in our ASM method which is suitable for problems with discontinuous coefficients, which is not the case, e.g., in [25] where conforming HCT element is considered.

In this paper our domain decomposition method is presented in terms of ASM abstract framework, i.e., we introduce space decomposition into a coarse space and two types of local spaces and bilinear forms defined on these spaces. Then we show an almost optimal condition number estimate which is independent of the jumps of the coefficients, i.e., the number of conjugate gradient iterations is proportional only to $(1 + \log(H/h))$, where H and h denote the subdomain sizes and mesh sizes, respectively.

The paper is organized as follows. We introduce the mortar Morley discretization of our model problem in Section 2. Section 3 is devoted to definition of our ASM method. In Section 4 we introduce and analyze a few necessary technical lemmas and finally in Section 5 we prove the condition estimate.

In the paper the following notation is used: $u \asymp v$, $x \succeq y$ and $w \preceq z$ mean that there exist positive constants c and C independent of the parameter of the fine triangulation of any substructure, and the number of subdomains such that

$$c\,u \le \; v \; \le \; C\,u, \quad x \ge c\,y \quad \text{and} \quad w \le C\,z, \quad \text{respectively.}$$

2. Discrete problem. Let assume that Ω is a polygonal domain in $\mathbf{R}^2$ and that there exists a decomposition of Ω into non-overlapping polygonal subdomains Ω_k such that

$$\overline{\Omega} = \bigcup_{k=1}^{N} \overline{\Omega}_k \ \text{ with } \ \Omega_k \cap \Omega_l = \emptyset, \quad k \neq l.$$

The intersection of boundaries of two different subdomains $\partial\Omega_k \cap \partial\Omega_l, k \neq l$, is either the empty set, a vertex or a common edge. Thus $\{\Omega_k\}_{k=1}^N$ forms a coarse triangulation of Ω with a parameter $H = \max_k H_k$, where $H_k = \text{diam}\ \Omega_k$. Let assume the shape regularity of that decomposition in the sense of Section 2, p. 5 in [11].

In this paper we consider the following model differential problem:
Find $u^* \in H_0^2(\Omega)$ such that

$$a(u^*, v) = \int_\Omega f v\, dx \quad \forall v \in H_0^2(\Omega),$$

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where $f \in L^2(\Omega)$ and

$$a(u,v) = \sum_{k=1}^{N} \int_{\Omega_k} \rho_k \left[u_{x_1x_1} v_{x_1x_1} + 2\, u_{x_1x_2} v_{x_1x_2} + u_{x_2x_2} v_{x_2x_2} \right] \, dx.$$

Here

$$H_0^2(\Omega) = \{ v \in H^2(\Omega) : \ v = \partial_n v = 0 \ \text{ on } \ \partial\Omega \},$$

∂_n is the normal unit derivative outward to $\partial\Omega$, ρ_k are positive coefficients (arbitrarily large), and $u_{x_ix_j} := \frac{\partial^2 u}{\partial x_i \partial x_j}$ for $i,j = 1,2$. From the Lax-Milgram theorem, the continuity and ellipticity of the bilinear form $a(\cdot,\cdot)$ it follows that there exists a unique solution of the problem; see, e.g., [12] or [16].

An important role in mortar discretizations is played by the interface, defined by $\overline{\Gamma} = \bigcup_{k=1}^{N} \overline{\partial\Omega_k \setminus \partial\Omega}$.

For each subdomain Ω_k we introduce $T_h(\Omega_k)$ a quasiuniform triangulation with parameter $h_k = \max(\text{diam } \tau)$ for $\tau \in T_h(\Omega_k)$ which consists of non-overlapping triangles, cf. [12]. Note that each common edge to two subdomains Ω_k and Ω_l called below an interface Γ_{kl} inherits two independent 1-D triangulations: $T_h^k(\Gamma_{kl})$ - the h_k -one from $T_h(\Omega_k)$ and $T_h^l(\Gamma_{kl})$ - the h_l-one introduced by $T_h(\Omega_l)$.

An important role is also played by two types of nodal points, the vertices of triangles and midpoints of edges of triangles. Let $\mathcal{A} \subset \Omega_i$ e.g. $\mathcal{A}$ may be an edge, the boundary or the whole Ω_i, then let $\mathcal{A}_{i,h}^V, \mathcal{A}_{i,h}^M$ be sets of all vertices and midpoints, respectively, of elements of $T_h(\Omega_i)$ which are in $\mathcal{A}$. Of course we drop subscript i whenever it does not cause any confusion. E.g., we write $\Omega_{i,h}^V$ for the set of all vertices of $T_h(\Omega_i)$.

We now introduce the mortar method that locally uses the nonconforming Morley element, cf. [16]. Let the local Morley finite element space $X_h(\Omega_k)$ be defined as follows, cf. Figure 2.1:

$$\begin{aligned} X_h(\Omega_k) = \{ v \in L_2(\Omega_k) : \ & v_{|\tau} \in P_2(\tau), \ v \text{ is continuous at vertices of } \tau \in T_h(\Omega_k) \\ & \text{and } \partial_n v \text{ is continuous at midpoints of edges of } \tau \text{ and} \\ & v(p) = \partial_n v(m) = 0 \text{ for a vertex } p \in \partial\Omega \text{ and a midpoint } m \in \partial\Omega \}. \end{aligned}$$

Here and below $P_k(G), k = 0,1,2,\ldots$ is the space of polynomials of degree up to k defined over a domain G.

The degrees of freedom of the Morley element are given by

$$\{ v(p_1), v(p_2), v(p_3), \partial_n v(m_1), \partial_n v(m_2), \partial_n v(m_3) \},$$

where p_k is a vertex of an element and m_l is the midpoint of an edge of an element, cf. Figure 2.1.

Next we can introduce an auxiliary global space $X_h(\Omega) = \prod_{k=1}^{N} X_h(\Omega_k)$ with so called broken norms and seminorms: $\|u\|^2_{H_h^s(\Omega)} = \sum_{k=1}^{N} \|u\|^2_{H_h^s(\Omega_k)}$, $|u|^2_{H_h^s(\Omega)} = \sum_{k=1}^{N} |u|^2_{H_h^s(\Omega_k)}$, where we have $\|u\|^2_{H_h^s(\Omega_k)} = \sum_{\tau \in T_h(\Omega_k)} \|u\|^2_{H^s(\tau)}$ and $|u|^2_{H_h^s(\Omega_k)} = \sum_{\tau \in T_h(\Omega_k)} |u|^2_{H^s(\tau)}$, $s = 0,1,2$.

We now choose an open disjoint side of $\Gamma_{kl} \subset \Gamma \cap \partial\Omega_k$ denoted by $\gamma_{m,k}$ and name it master (mortar). The side of $\Gamma_{kl} \subset \partial\Omega_l$ is called slave (nonmortar) and is denoted by $\delta_{m,l}$.
Assumptions For the subdomains Ω_k and Ω_l related to the master $\gamma_{m,k} = \Gamma_{kl}$ and the slave $\delta_{m,l}$, it holds that:

A.1 The coefficients satisfy $\rho_k \geq \rho_l$.

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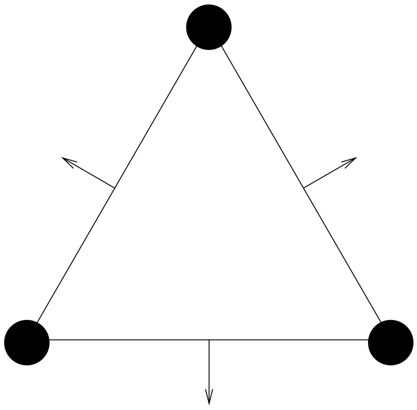

FIG. 2.1. *Morley element.*

A.2 The mesh parameters of $T_h(\Omega_k)$ and $T_h(\Omega_l)$ satisfy $h_k \leq h_l$.

A.3 The two side elements of the slave triangulation $T_h^l(\delta_{m,l})$, i.e., the ones that touch the ends of $\delta_{m,j}$, are longer than the respective elements of the master (mortar) triangulation $T_h^k(\gamma_{m,k})$.

Assumption A.2 is necessary for the proofs of some technical results and is due to the fact that any local finite element function is not sufficiently regular. It is worth noting that a good practical and meaningful alternative could be (cf. [34]):

$$\frac{h_i}{h_j} \asymp \left(\frac{\rho_i}{\rho_j}\right)^{\lambda}, \qquad 0 \leq \lambda \leq 1.$$

See also Remark 4.2, below.

The assumption A.3 is technical. If one considers an interface Γ_{ij} with the master $\gamma_{m,i}$ and the slave $\delta_{m,j}$ then it is easily seen that if a function $u \in X_h(\Omega)$ satisfies (2.2), see below, then the values of normal derivatives of u at the midpoints of 1D elements of $T_h^j(\delta_{m,j})$ depend not only on degrees of freedom (dofs) at nodes of $T_h^i(\gamma_{m,i})$ but also on some dofs related to the nodes which are not on $\overline{\gamma}_{m,i}$. It may happen in general that the values of $\partial_n u_j$ at a midpoint of an end element of $\delta_{m,j}$ are dependent on values of dofs associated with nodal points on an interface $\Gamma_{ki} \subset \partial\Omega_i$ such that $\overline{\Gamma}_{ki} \cap \overline{\Gamma}_{ij} = \{c_r\}$ a vertex, which is obviously a bad property. Assumption A.3 yields that it is not possible at all.

We also need two auxiliary spaces on each slave (nonmortar) $\delta_{m,l}$. Let the first one denoted by $M_n^{h_l}(\delta_{m,l})$, be the space formed by piecewise constant functions on the h_l-triangulation of $\delta_{m,l}$.

For simplicity of presentation, we also assume that the both 1-D triangulations of the interface Γ_{kl}, $T_h^k(\gamma_{m,k})$ - the h_k one of its master $\gamma_{m,k}$ and $T_h^l(\delta_{m,l})$ - the h_l one of its slave $\delta_{m,l}$, have even numbers of the elements. Let consider $\delta_{m,l}$ and let $\overline{\delta}_{m,l,h}^V = \{p_0, p_1, \ldots, p_{N_{m,l}}\}$ be a set of vertices of the h_l triangulation of this slave, ($N_{m,l}$ is even). Then we introduce an operator $I_{2h_l,2} : C^0(\Gamma_{kl}) \rightarrow L^2(\Gamma_{kl})$:

DEFINITION 2.1. *Let $I_{2h_l,2} : C^0(\Gamma_{kl}) \rightarrow L^2(\Gamma_{kl})$ and $I_{2h_l,2}u$ be defined by the values of u at all points of $\overline{\delta}_{m,l,h}$ as follows:*

1. *$I_{2h_l,2}u$ is a quadratic polynomial on each $[p_i, p_{i+2}]$ for even i,*
2. *$I_{2h_l,2}u(p_i) = u(p_i) \quad for \quad p_i \in \overline{\delta}_{m,l,h}^V$.*

In other words, $I_{2h_l,2}u$ is the piecewise quadratic interpolant defined over the $2h_l$ triangulation of $\delta_{m,l}$ that is made of elements $[p_i, p_{i+2}], i = 0, 2, \ldots, N_{m,l} - 2$. The operator

$I_{2h_k,2}$ that corresponds to the h_k mesh of master $\overline{\gamma}^V_{m,k,h}$ we define in the same way.

We next define an auxiliary space $M_t^{2h_l}(\delta_{m,l})$ as follows:

$$M_t^{2h_l}(\delta_{m,l}) = \{v \in C(\delta_{m,l}) : \ v \in P_2([p_i, p_{i+2}]) \text{ for even } i \neq 0, N_{m,l} - 2, \\ \text{and } v \in \ P_1([p_i, p_{i+2}]) \text{ for } i = 0, N_{m,l} - 2\}.$$

Then a mortar finite element space is introduced:

$$V^h = \{u \in X_h(\Omega) : u \text{ continuous at the crosspoints and} \\ u \text{ on each } \Gamma_{kl} = \gamma_{m,k} = \delta_{m,l} \text{ satisfies mortar conditions :}$$

$$\int_{\delta_m} (I_{2h_k,2}u_k - I_{2h_l,2}u_l)\psi \, ds = 0, \qquad \forall \psi \in M_t^{2h_l}(\delta_{m,l}) \tag{2.1}$$

$$\int_{\delta_m} (\partial_n u_k - \partial_n u_l)\phi \, ds = 0, \qquad \forall \phi \in M_n^{h_l}(\delta_{m,l}). \tag{2.2}$$

Next we define our discrete problem: Find $u_h^* \in V^h$ such that

$$a_H(u_h^*, v) = f(v), \qquad \forall v \in V^h, \tag{2.3}$$

for $a_H(u,v) = \sum_{k=1}^N a_{h,k}(u,v)$ and

$$a_{h,k}(u,v) = \sum_{\tau \in T_h(\Omega_k)} \int_\tau \rho_k \left[u_{x_1x_1}v_{x_1x_1} + 2\, u_{x_1x_2}v_{x_1x_2} + u_{x_2x_2}v_{x_2x_2}\right] dx.$$

This problem has a unique solution and some error estimates are established [26].

3. Additive Schwarz algorithm. In this section we describe the parallel algorithm for solving the problem (2.3).

3.1. An equivalent formulation. In this section we reformulate the problem (2.3) into a spectrally equivalent one. Then we will be able to construct a preconditioner for this equivalent problem using ASM abstract scheme, i.e., we replace (2.2) by (3.1), see below.

We formulate the new problem because of the nonconformity of Morley finite element space and (2.2), namely a function $u \in X_h(\Omega_i)$ which is zero at all vertices and midpoints of a master $\gamma_{m,i} = \delta_{m,j} = \Gamma_{ij} \subset \partial\Omega_i$ can have zero values of normal derivatives at all midpoints of $\gamma^M_{m,i}$ and can have nonzero trace of the normal derivative onto $\gamma_{m,i}$, i.e., in particular $\int_{\delta_{m,j}} u_i \psi \, ds$ can be nonzero for a test function $\psi \in M_n^{h_j}(\delta_{m,j})$. This could yield many technical problems in the construction and analysis of the ASM algorithm, cf. also [24] and [28].

DEFINITION 3.1. *For each $u \in V^h$ we define a function $\tilde{u} \in X_h(\Omega)$ as follows:*

1. *$\tilde{u}(x) = u(x)$ for all vertices x in $\bigcup_{k=1}^N \overline{\Omega}^V_{k,h}$.*
2. *$\partial_n \tilde{u}(m) = \partial_n u(m)$ for all midpoints m in $\bigcup_{k=1}^N \overline{\Omega}^M_{k,h} \setminus \bigcup_{\delta_{s,l} \subset \Gamma} \overline{\delta}^M_{s,l,h}$, ($\delta_{s,l}$ is a slave), i.e., midpoints that are not on a slave,*
3. *on any slave (nonmortar) $\delta_{s,l} = \Gamma_{kl} \subset \partial\Omega_l$ with the associated mortar (master) $\gamma_{s,k} = \Gamma_{kl} \subset \partial\Omega_k$ the values of normal derivative at the midpoints of $\delta^M_{s,l,h}$ are determined by the following condition:*

$$\int_{\delta_s} (\partial_n \tilde{u}_l - \psi_k(u_k))\phi \, ds = 0 \quad \forall \phi \in M_n^{h_j}(\delta_{m,l}) \tag{3.1}$$

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where $\psi_k(u_k)$ is piecewise constant function on the h_k triangulation of $\gamma_{s,k} = \delta_{s,l} = \Gamma_{kl}$, i.e., $T_h^k(\gamma_{s,k})$ such that

$$\psi_k(u_k)(m) = \partial_n u_k(m) = \frac{1}{e}\int_e \partial_n u_k \, ds \tag{3.2}$$

for each midpoint $m \in \gamma_{s,k,h}^M$ of an element $e \in T_h^k(\gamma_{s,k})$.

From (3.1) it follows that the value $\tilde{u}_l(m)$ for a midpoint $m \in \delta_{m,l,h}^M$ of an 1-D element $e \in T_h^l(\delta_{m,l})$ is computed by

$$\tilde{u}_l(m) = |e|^{-1}\int_e \partial_n \tilde{u}_l(s)\, ds = |e|^{-1}\int_e \psi_k(s)\, ds, \tag{3.3}$$

where ψ_k is defined as in (3.2).

Note that the values of respective degrees of freedom of $\tilde{u}$ and u differ only at the midpoints on slaves. Then following the lines of proof of Proposition 3.1, p. 7, [27], we get

PROPOSITION 3.2. *For all $u \in V^h$ and $\tilde{u} \in X_h(\Omega_k)$ from Definition 3.1 it holds that*

$$a_H(u,u) \asymp a_H(\tilde{u},\tilde{u}).$$

We now introduce a new space

$$\tilde{V}^h = \{v \in X_h(\Omega) : \exists u \in V^h \quad v = \tilde{u}\},$$

i.e., $\tilde{V}^h$ is the image of V^h of the mapping defined in Definition 3.1.

And we can formulate a new problem:

$$a_H(\tilde{u}_h^*, v) = f(v), \quad \forall v \in \tilde{V}^h. \tag{3.4}$$

We now introduce nodal bases for V^h and then $\tilde{V}^h$.

For each degree of freedom (a vertex or a midpoint) that is **not on a slave** we associate a nodal basis function in V^h. Let ϕ_x^V the nodal basis function related to a vertex x be defined as follows

$$\begin{cases} \phi_x^V(x) &= 1 \\ \phi_x^V(y) &= 0 \qquad \forall y \in \bigcup_k \overline{\Omega}_{k,h}^V \setminus \bigcup_{\delta_s \subset \Gamma} \delta_{s,h}^V \qquad y \neq x \\ \partial_n \phi_x^V(y) &= 0 \qquad \forall m \in \bigcup_k \overline{\Omega}_{k,h}^M \setminus \bigcup_{\delta_s \subset \Gamma} \delta_{s,h}^M. \end{cases}$$

The function ϕ_m^M related to a midpoint m **not on a slave** we set by

$$\begin{cases} \phi_m^M(y) &= 0 \qquad \forall y \in \bigcup_k \overline{\Omega}_{k,h}^V \setminus \bigcup_{\delta_s \subset \Gamma} \delta_{s,h}^V \\ \partial_n \phi_m^M(m) &= 1 \\ \partial_n \phi_m^M(p) &= 0 \qquad \forall p \in \bigcup_k \overline{\Omega}_{k,h}^M \setminus \bigcup_{\delta_s \subset \Gamma} \delta_{s,h}^M \quad p \neq m. \end{cases}$$

The functions are uniquely defined by these conditions as the values of degrees of freedom related to both types of nodal points on slaves are determined by (2.1) and (2.2), respectively.

In the same way we can define nodal basis functions of $\tilde{V}^h$: $\tilde{\phi}_x^V$ and $\tilde{\phi}_m^M$ the nodal basis functions related to a vertex x and a midpoint m which are not on any slave.

Note that the the values of degrees of freedom related to both types of nodal points on slaves are determined by (2.1) and (3.1), respectively. Thus, e.g., $\phi_x^V \in V^h$ and $\tilde{\phi}_x^V \in$

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$\tilde{V}^h$ which are both elements of $X_h(\Omega)$ may have different values of normal derivatives at midpoints on a slave.

Note also that $u \in V^h$ and $\tilde{u} \in \tilde{V}^h$ defined in Definition 3.1 have the same vector representation $\vec{u} \in R^n$ (n is the dimension of V^h) in the respective nodal bases and if we introduce the matrix representations of $a_H(u,v)$ in the both nodal bases, i.e., let A denote the one for V^h and $\tilde{A}$ the one for $\tilde{V}^h$, then Proposition 3.2 yields, (cf. [27])

COROLLARY 3.3. *The matrices A and $\tilde{A}$ are spectrally equivalent with constants independent of the mesh parameters, jumps of the coefficients and the number of subdomains.*

Thus further we construct a preconditioner for $\tilde{A}$ instead for A or in other words for the problem (3.4) instead for (2.3).

3.2. Additive Schwarz method. Now we construct a parallel method for solving (3.4). The construction and analysis of our method is based on Additive Schwarz Method scheme, cf. [30], i.e., it is defined in terms of decomposition of $\tilde{V}^h$ into subspaces, local bilinear forms defined over these subspaces and special projections onto these subspaces. Using the abstract Additive Schwarz Method (ASM) framework we will then get a preconditioner for the problem (3.4) and then (2.3), cf. Corollary 3.3.

We first introduce a decomposition of any $u \in X_h(\Omega_i)$ into $u = \mathcal{H}_i u + \mathcal{P}_i u$ where

$$a_{h,i}(\mathcal{P}_i u, v) = a_{h,i}(u,v) \qquad \forall v \in X_{h,0}(\Omega_i), \tag{3.5}$$

where

$$X_{h,0}(\Omega_i) = \{u \in X_h(\Omega_i)\ :\ u(p) = \partial_n u(m) = 0 \\ \text{for all midpoints } m \text{ and vertices } p \text{ on } \partial\Omega_i\ \}$$

Thus $\mathcal{H}_i u = u - \mathcal{P}_i u$ and it is easy to see that equivalently we can define $\mathcal{H}_i u$ as follows

$$\begin{cases} a_{h,i}(\mathcal{H}_i u, v) = 0 & \forall\, v \in X_{h,0}(\Omega_i) \\ \mathcal{H}_i u(p) = u(p) & \forall\, p \ \text{ vertex on } \partial\Omega_i \\ \partial_n \mathcal{H}_i u(m) = \partial_n u(m) & \forall\, m \text{ midpoint on } \partial\Omega_i \end{cases} \tag{3.6}$$

which gives us another characterization: $\mathcal{H}_i u$ is a unique function such that

$$a_{h,i}(\mathcal{H}_i u, \mathcal{H}_i u) = \min\{a_{h,i}(v,v) : v \in X_h(\Omega_i) \text{ such that } v(p) = u(p) \text{ and} \\ \partial_n v(m) = \partial_n u(m) \text{ for all midpoints } m \text{ and all vertices } p \text{ on } \partial\Omega_i\}. \tag{3.7}$$

Then we can decompose any $u = (u_1, \ldots, u_N) \in V^h$ as $u = \mathcal{P}u + \mathcal{H}u$, with $\mathcal{P}u = (\mathcal{P}_1 u_1, \ldots, \mathcal{P}_N u_N)$ and $\mathcal{H}u = (\mathcal{H}_1 u_1, \ldots, \mathcal{H}_N u_N)$. It is worth to note that $\mathcal{P}u$ and $\mathcal{H}v$ are orthogonal with respect to $a_H(\cdot,\cdot)$ which follows directly from the definition of $\mathcal{P}_i$ and $\mathcal{H}_i$. $\mathcal{H}u$ (and $\mathcal{H}_i u_i$) is called discrete biharmonic part of u (u_i respectively).

We now introduce the space decomposition.

For the simplicity of presentation we assume that all Ω_i are triangles.

We first define a coarse space $V_0 \subset \tilde{V}^h$. Let $u \in V_0$ be a discrete biharmonic function such that on any master $\gamma_{m,i}$ we have $u(p) = I_{1,\gamma_m} u(p)$ at $p \in \gamma_{m,i,h}$ and its normal derivative takes a constant value at all midpoints of this mortar, i.e., $\partial_n u(m) = Const$ for a m a midpoint on $\gamma_{m,i}$. Here $I_{1,\gamma_m} u \in L^2(\gamma_{m,i})$ is a linear function such that $I_{1,\gamma_m} u(a_k) = u(a_k)$, $k = 1,2$ for a_k an end of this interface. Note that as $u \in V_0$ is an element of $\tilde{V}^h$. Thus it is continuous at crosspoints and this gives that its values at vertices of any mortar are solely determined by linear interpolation between the ends of that master. It is obvious that dim V_0

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equals the number of crosspoints plus the number of edges because degrees of freedom are associated with a constant per edge and values at crosspoints.

REMARK 3.1. The values of degrees of freedom $u \in V_0$ at nodal points on a slave are determined by the mortar conditions. However it directly follows that they are determined in the same way as on the mortar side, i.e., at the vertices the function takes the values of the same linear function as on the mortar side, and at the midpoints on this slave the normal derivative takes the same constant value as on the mortar side, which follows from (3.3).

The bilinear form $b_0(u,v)$ is defined as equal to the original $a(u,v)$.

The next subspaces are associated with domains, let $V_k \subset \tilde{V}^h$ be formed by functions which are in $X_{h,0}(\Omega_k)$ and are zero over all other substructures, equivalently we can write $V_k = \{u \in \tilde{V}^h : u = (0,\ldots,0,u_k,0,\ldots,0), \quad u_k \in X_{h,0}(\Omega_k)\}$. We take $b_k(u,v) = a_H(u_{|\Omega_k}, v_{|\Omega_k})$ as the bilinear form corresponding to the subspace V_k.

Finally we define a family of spaces corresponding to the mortars.

Let $\gamma_{s,k} \subset \partial\Omega_k$ be a master (mortar) and let $\delta_{s,l}$ be its associated slave. Then let $V_{\gamma_{s,k}} \subset \tilde{V}^h$ be a space of piecewise discrete biharmonic functions that may have nonzero degrees of freedom only these which are associated with $\gamma_{s,i}$, i.e., values at vertices and values of normal derivative at midpoints which are on this open master. The bilinear form associated with $V_{\gamma_{s,k}}$ is defined as the restriction of $a_H(u,v)$ to Ω_k, i.e.,

$$b_s(u,v) = a_{h,k}(u_k, v_k) \qquad \forall u, v \in V_{\gamma_{s,k}}.$$

We have

$$\tilde{V}^h = V_0 + \sum_{k=1}^{N} V_k + \sum_{\gamma_s \subset \Gamma} V_{\gamma_s}.$$

Next we introduce $P_0 : \tilde{V}^h \to V_0$, $P_k : \tilde{V}^h \to V_k$, $k = 1,\ldots,N$ and $T_s : \tilde{V}^h \to V_{\gamma_s}$, $\forall \gamma_s \subset \Gamma$ defined by

$$\begin{aligned} b_0(P_0 u, v) &= a_H(u,v) \quad \forall v \in V_0 \\ a_k(P_k u, v) &= a_H(u,v) \quad \forall v \in V_k,\ k = 1,\ldots,N \\ b_s(T_s u, v) &= a_H(u,v) \quad \forall v \in V_{\gamma_s},\ \forall \gamma_s \subset \Gamma. \end{aligned}$$

Let an operator $T : \tilde{V}^h \to \tilde{V}^h$ be defined by

$$T = P_0 + \sum_{k=1}^{N} P_k + \sum_{\gamma_s \subset \Gamma} T_s.$$

Then, the main result of this section is the following:

THEOREM 3.4. *For any $u \in \tilde{V}^h$ it holds that*

$$\left(1 + \log\left(\frac{H}{\underline{h}}\right)\right)^{-2} a_H(u,u) \preceq a_H(Tu,u) \preceq a_H(u,u),$$

where $H = \max_k H_k$ and $\underline{h} = \min_k h_k$.

It follows directly that for the matrix representation in the nodal basis presented in § 3.1 of T it holds that $T = B^{-1}\tilde{A}$, for a parallel preconditioner $B = \tilde{A}T^{-1}$. (Here T denotes both the operator and its matrix representation.)

Thus we can rewrite Theorem 3.4 as, (e.g., cf. [18]),

$$\left(1+\log\left(\frac{H}{\underline{h}}\right)\right)^{-2} \vec{u}^T B\vec{u} \preceq \vec{u}^T \tilde{A}\vec{u} \preceq \vec{u}^T B\vec{u} \quad \forall \vec{u} \in R^n,$$

where $\underline{h} = \min_k h_k$ and $H = \max_k H_k$.

From this and Corollary 3.3 it follows in a standard way, see, e.g., [18], that

COROLLARY 3.5. *It holds that*

$$\left(1+\log\left(\frac{H}{\underline{h}}\right)\right)^{-2} \vec{u}^T B\vec{u} \preceq \vec{u}^T A\vec{u} \preceq \vec{u}^T B\vec{u} \quad \forall \vec{u} \in R^n,$$

where $\underline{h} = \min_k h_k$ *and* $H = \max_k H_k$. Thus we can use the preconditioner B^{-1} in conjugate gradient method for solving a following problem: find $\vec{u}^* \in R^n$ such that:

$$A\vec{u}^* = \vec{f}$$

which is the matrix form of (2.3). Here $\vec{u}^*$ is a vector representation of u_h^* a solution of (2.3).

The number of conjugate gradient iterations of the preconditioned problem will grow only logarithmically with the ratio $H/\underline{h}$ and is independent of mesh sizes and the jump of coefficients ρ_k.

4. Technical tools. In this section we define several auxiliary operators and prove some useful results which are necessary for the proofs of our theorems.

First we introduce the so called mortar projection: $\pi^2_{m,j} : L^2(\delta_{m,j}) \to H^1_0(\delta_{m,j})$, an operator corresponding to $\delta_{m,j}$, a slave defined as follows, cf. [4].

DEFINITION 4.1. *Let* $\pi^2_{m,j}u$ *for* $u \in L^2(\delta_{m,j})$ *be a continuous function which is a polynomial of second degree over all elements of the* $2h_j$ *triangulation of* $\delta_{m,j}$ *vanishing at ends of this slave and satisfying*

$$\int_{\delta_{m,j}} (I - \pi^2_{m,j})\, u\, \psi\, ds = 0 \qquad \forall \psi \in M_t^{2h_j}(\delta_{m,j}). \tag{4.1}$$

(Note that $\pi^2_{m,j}u = I_{2h_j,2}\pi^2_{m,j}u$*).*

The next lemma is a special case of Lemma 2.2 in [4], (the L^2 stability is given in the proof of this lemma; see (2.19) in [4]):

LEMMA 4.2. *The operator* $\pi^2_{m,j}$ *defined in (4.1) is stable in* L^2 *and* $H^{1/2}_{00}$ *norms, i.e., it holds that*

$$\begin{aligned} \|\pi^2_{m,j}u\|_{L^2(\delta_{m,j})} &\preceq \|u\|_{L^2(\delta_{m,j})} \;\; \forall u \in L^2(\delta_{m,j}) \\ \|\pi^2_{m,j}u\|_{H^{1/2}_{00}(\delta_{m,j})} &\preceq \|u\|_{H^{1/2}_{00}(\delta_{m,j})} \;\; \forall u \in H^{1/2}_{00}(\delta_{m,j}). \end{aligned} \tag{4.2}$$

We next introduce a locally continuous finite element space which is a counter-partner of the Morley local finite element space: $X_h^{HCT}(\Omega_k)$ - the Hsieh-Clough-Tocher (HCT) macro finite element local space, cf. [16], and a local equivalence mapping; see Section 3, (3.2) in [13], $\mathcal{M}_k : X_h(\Omega_k) \to X_h^{HCT}(\Omega_k)$. This will allow us to utilize all technical results known for HCT spaces to Morley finite element local space.

The local finite element space $X_h^{HCT}(\Omega_k)$ for HCT element is defined by, cf. Figure 4.1,

$$\begin{aligned} X_h^{HCT}(\Omega_k) = \{v \in C^1(\Omega_k) : \;\; & v_{|\tau} \in P_3(\tau_i), \text{ for triangles } \tau_i, \;\; i = 1,2,3, \\ & \text{formed by connecting the vertices of } \tau \in T_h(\Omega_k) \\ & \text{to its centroid, } v = \partial_n v = 0 \text{ on } \partial\Omega_k \cap \partial\Omega\}. \end{aligned}$$

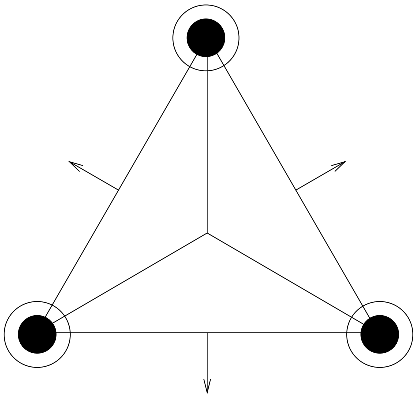

FIG. 4.1. *HCT macro element.*

The degrees of freedom of HCT element are given by

$$\{v(p), v_{x_1}(p), v_{x_2}(p), \partial_n v(m)\},$$

cf. Figure 4.1.

In the next definition, we use the fact that for quadratic polynomial $q \in P_2([a,b])$ we have

$$q'((a+b)/2) = \frac{q(b)-q(a)}{b-a}.$$

Thus ∇u for $u \in X_h(\Omega_k)$ is well defined at all midpoints. Let m_p be an adjacent midpoint of the vertex p if both points belong to the same edge in $T_h(\Omega_k)$. The choice of the midpoint is not unique and this fact will be used below.

DEFINITION 4.3. *([13]) We define $\mathcal{M}_k : X_h(\Omega_k) \to X_h^{HCT}(\Omega_k)$ by setting its degrees of freedom at all vertices and midpoints of Ω_k, i.e., let p be a vertex and m the midpoint of an edge of an element of $T_h(\Omega_k)$, then*

$$\begin{aligned} \mathcal{M}_k u\,(p) &= u(p) \quad \forall \text{ vertices } \ p, \\ \partial_n \mathcal{M}_k u\,(m) &= \partial_n u(m) \quad \forall \text{ midpoints } \ m, \\ \nabla \mathcal{M}_k u\,(p) &= \nabla u(m_p) \ \ \forall \text{ vertices } \ \mathrm{p}, \ \text{ where } \ \mathrm{m_p} \ \text{ is an adjacent midpoint.} \end{aligned}$$

Note that the choice of the adjacent midpoint m_p in the above definition may be arbitrary and it will further be of a great use.

In the following lemma, we state some properties of the local equivalence mapping defined above.

LEMMA 4.4. *For all $u \in X_h(\Omega_k)$, it holds that*

$$\begin{aligned} |\mathcal{M}_k u|_{H^s(\Omega_k)} &\asymp |u|_{H_h^s(\Omega_k)} \quad s = 0,1,2, \\ \|u - \mathcal{M}_k u\|_{L^2(\Omega_k)} + h_k |u - \mathcal{M}_k u|_{H_h^1(\Omega_k)} &\preceq h_k^2 |u|_{H_h^2(\Omega_k)}, \\ \|u - \mathcal{M}_k u\|_{L^2(\Gamma_{kl})} + h_k \|\partial_n u - \partial_n \mathcal{M}_k u\|_{L^2(\Gamma_{kl})} &\preceq h_k^{3/2} |u|_{H_h^2(\Omega_k)}. \end{aligned}$$

Here Γ_{kl} is an edge of Ω_k.

This is Lemma 10, in § 4.2 of [26], the proof of the first two statements were given earlier in [13].

We also have a simple but important remark which follows form the flexibility of choosing the adjacent midpoint in the definition of $\mathcal{M}$:

REMARK 4.1. Consider an edge $\Gamma_{kl} \subset \partial\Omega_k$ and let $u \in X_h(\Omega_k)$ be a function which has all degree of freedom associated with nodes that are on $\partial\Omega_k \setminus \Gamma_{kl}$ equal to zero. Then we can choose $\mathcal{M}_k u$ such that $\mathcal{M}_k u = 0$ and $\nabla\mathcal{M}_k u = 0$ on $\partial\Omega_k \setminus \Gamma_{kl}$.

The next lemma shows an important property of discrete biharmonic functions:

LEMMA 4.5. *Let $u \in X_h(\Omega_i)$ be a discrete biharmonic, i.e., $\mathcal{H}_i u = u$, cf. (3.6). Then we have*

$$|u|_{H_h^2(\Omega_i)} \preceq |\nabla\mathcal{M}_i u|_{H^{1/2}(\partial\Omega_i)}.$$

Proof. Let $\hat{u} \in X_h^{HCT}(\Omega_i)$ be a function defined as follows: $Tr\,\hat{u} = Tr\,\mathcal{M}_i u$ on $\partial\Omega_i$, here $Tr\,w = (w, \nabla w)$ on $\partial\Omega$, and

$$a_{h,i}(\hat{u}, v) = 0 \qquad \forall v \in X_{h,0}^{HCT}(\Omega_i),$$

where $X_{h,0}^{HCT}(\Omega_i) = X_h^{HCT}(\Omega_i) \cap H_0^2(\Omega_i)$, i.e., $\hat{u}$ is a discrete biharmonic in HCT sense extension of $\mathcal{M}_i u_{|\partial\Omega_i}$. Then by the discrete analog of the extension theorem for Sobolev spaces, cf. Theorem 4.4 in [23], and the fact that discrete biharmonic functions have a minimal energy property, we have

$$|\hat{u}|_{H_h^2(\Omega_i)} \preceq |\nabla\,\hat{u}|_{H^{1/2}(\partial\Omega_i)} = |\nabla\mathcal{M}_i u|_{H^{1/2}(\partial\Omega_i)}.$$

Next we introduce a function $w \in X_h(\Omega_i)$ by setting its all degrees of freedom related to vertices and midpoints of $T_h(\Omega_i)$ to the respective ones of $\hat{u}$. The next estimate is obtained by a standard argument following from the fact that all degrees of freedom of w are included in the set of degrees of freedom of $\hat{u}$:

$$|w|_{H_h^2(\Omega_i)} \preceq |\hat{u}|_{H^2(\Omega_i)}.$$

Note that from Definition 4.3 follows that $w(p) = \hat{u}(p) = u(p)$ for all vertices $p \in \partial\Omega_i$ and $\partial_n w(m) = \partial\hat{u}(m) = \partial u(m)$ for all midpoints $m \in \partial\Omega_i$.

Hence by (3.7) we get

$$|u|_{H_h^2(\Omega_i)} \preceq |w|_{H_h^2(\Omega_i)} \preceq |\hat{u}|_{H^2(\Omega_i)} \preceq |\nabla\mathcal{M}_i u|_{H^{1/2}(\partial\Omega_i)}$$

what ends the proof of this lemma. □

The next lemma is Lemma 4.1 in [23].

LEMMA 4.6. *Let $\mathcal{E}$ be an edge of Ω_k. If $u \in H^{1/2}(\partial\Omega_k)$ is equal to zero on $\partial\Omega_k \setminus \mathcal{E}$ and $\|\partial_t u\|_{L^\infty(\mathcal{E})} \preceq h_k^{-1}\|u\|_{L^\infty(\mathcal{E})}$, then it holds that*

$$|u|^2_{H^{1/2}(\partial\Omega_k)} \preceq |u|^2_{H^{1/2}(\mathcal{E})} + (1 + \log(|\mathcal{E}|/h_k))\|u\|^2_{L^\infty(\mathcal{E})},$$

where ∂_t denotes the tangential derivative on $\mathcal{E}$.

We also need to introduce trace spaces defined over an interface Γ_{ij}:

$$W^h(\Gamma_{ij}) = \{u_{|\Gamma_{ij}} : u \in X_h^{HCT}(\Omega_j)\}, \qquad W_0^h(\Gamma_{ij}) = H_0^2(\Gamma_{ij}) \cap W^h(\Gamma_{ij}),$$

and an auxiliary operator defined on the trace space.

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DEFINITION 4.7. *We now define an auxiliary operator defined on an edge* Γ_{ij}: $\pi^t_{m,j} : H^2(\Gamma_{ij}) \to W^h_0(\delta_{m,j})$. *Let* $w = \pi^2_{m,j} u$ *then for any* $p \in \delta_{m,j,h}$

$$\pi^t_{m,j} u(p) = w(p)$$
$$\frac{d}{dt}\pi^t_{m,j} u(p) = w'(m_p) = (w(p) - w(q))/|p-q|$$

where $q \in \overline{\delta}_{m,j,h}$ *is a neighboring nodal point of* p *and* $m_p = \frac{p+q}{2}$ *is an adjacent midpoint.* Note that for each p the selection of q can be decided in conformity with the similar choice in definition of $\mathcal{M}_j$, which is further used in the proofs.

LEMMA 4.8. *The operator defined in Definition 4.7 satisfies:*

$$\|\pi^t_{m,j} u\|_{L^2(\delta_{m,j})} \preceq \|u\|_{L^2(\delta_{m,j})} \quad \forall u \in L^2(\delta_{m,j})$$
$$\|\pi^t_{m,j} u\|_{H^{3/2}_{00}(\delta_{m,j})} \preceq \|u\|_{H^{3/2}_{00}(\delta_{m,j})} \quad \forall u \in H^{3/2}_{00}(\delta_{m,j}),$$

where $H^{3/2}_{00}(\delta_{m,j}) = [L^2(\delta_{m,j}), H^2_0(\delta_{m,j})]_{3/2}$.

Proof. First we prove stabilities in L^2 -norm and H^2-seminorm and then the second statement of the lemma is obtained by a standard Hilbert space interpolation argument, cf. [12].

Take any $u \in L^2(\delta_{m,j})$ and let $\hat{w} = \pi^t_{m,j} u$ and $w = \pi^2_{m,j} u$, cf. Definitions 4.1 and 4.7. Then, by a standard scaling argument, Definition 4.7, and (4.2), we get

$$\begin{aligned}\|\hat{w}\|^2_{L^2(\delta_{m,j})} &\asymp \sum_{p\in\delta_{m,j,h}} h_j \left(|\hat{w}(p)|^2 + h_j^2 |\hat{w}'(p)|^2\right)\\ &\asymp \sum_{p\in\delta_{m,j,h}} h_j \left(|w(p)|^2 + h_j^2\left(\frac{w(p)-w(q)}{h_j}\right)^2\right) \preceq \sum_p h_j |w(p)|^2\\ &\preceq \|w\|_{L^2(\delta_{m,j})} \preceq \|u\|_{L^2(\delta_{m,j})}.\end{aligned}$$

Here q is a neighboring nodal point of p.

Now consider $u \in H^2_0(\Gamma_{ij})$. Let again $\hat{w} = \pi^t_{m,j} u$ and $w = \pi^2_{m,j} u$.

We also need an additional function $v \in W^h_0(\delta_{m,j})$ which is defined as follows:

$$v(p) = u(p)$$
$$\frac{d}{dt} v(p) = (u(p) - u(q))/|p-q|$$

where $q \in \overline{\delta}_{m,j,h}$ is a neighboring nodal point of p chosen in exactly same way as in the definition of π^t_m.

Take an element $e = (p_1, p_2) \in T^j_h(\delta_{m,j})$, and let l_1 be a linear polynomial interpolating u (and v) at $p_k,\ k = 1, 2$, i.e., such that $l_1(p_k) = u(p_k) = v(p_k)$, $k = 1, 2$, then we get by standard arguments

$$\|u - v\|^2_{L^2(e)} \preceq \|u - l_1\|^2_{L^2(e)} + \|v - l_1\|^2_{L^2(e)} \preceq h_j^4 |u|^2_{H^2(e)} + \|v - l_1\|^2_{L^2(e)}.$$

The second term is estimated as follows:

$$\begin{aligned}\|v - l_1\|^2_{L^2(e)} &\preceq \sum_{k=1,2} h_j^3 |v'(p_k) - l_1'|^2\\ &= \sum_{k=1,2} h_j^3 \left(\frac{u(p_k)-u(q_k)}{p_k - q_k} - \frac{u(p_2)-u(p_1)}{p_2-p_1}\right)^2\\ &= \sum_{k=1,2} h_j^3 |u'(\xi_1) - u'(\xi_2)|^2 \preceq h_j^3 \|u' + c\|^2_{L^\infty(\hat{e})},\end{aligned}$$

where $\hat{e}$ is a sum of e and its all neighboring elements, $\xi_1 \in e$, $\xi_2 \in \hat{e}$, c is any constant, and q_k are the respective nodal points (as in the definition of v).

Then, by a Sobolev embedding $C^0 \hookrightarrow H^1$ in 1-D, a quotient space argument, and a scaling argument we get

$$\|u - v\|^2_{L^2(e)} \preceq h_j^4 \|u\|^2_{H^2(\hat{e})}. \tag{4.3}$$

Now we estimate

$$|\hat{w}|_{H^2(\delta_m)} \preceq |\hat{w} - v|_{H^2(\delta_m)} + |v|_{H^2(\delta_m)}. \tag{4.4}$$

For the second term we get

$$|v|^2_{H^2(\delta_m)} = \sum_{e \in T_h^j(\delta_{m,j})} |v - I_{1,h} u|^2_{H^2(e)},$$

where $I_{1,h}$ is a piecewise linear interpolant defined on $T_h^j(\delta_{m,j})$. If we consider an element $e \in T_h^j(\delta_{m,j})$ then by an inverse inequality, (4.3), and the well known properties of $I_{1,h}$ we have

$$|v - I_{1,h} u|^2_{H^2(e)} \preceq h_j^{-4} \|v - I_{1,h} u\|^2_{L^2(e)} \preceq h_j^{-4} (\|v - u\|^2_{L^2(e)} + \|u - I_{1,h} u\|^2_{L^2(e)}) \preceq |u|^2_{H^2(\hat{e})},$$

where $\hat{e}$ is a sum of e and its all neighboring elements. Summing over all $e \in T_h^j(\delta_{m,j})$ ends the estimate of the second term because every element is counted three times at most.

Thus to get the stability of π_m^t in H^2 seminorm it suffices to estimate the first term in (4.4). By an inverse inequality, and again the fact that $u(p) = v(p)$ and $w(p) = \hat{w}(p)$ we have

$$\begin{aligned}
|\hat{w} - v|^2_{H^2(\delta_m)} &\preceq h_j^{-4} \|\hat{w} - v\|^2_{L^2(\delta_m)} \asymp h_j^{-3} \sum_{p \in \delta^V_{m,j,h}} |\hat{w}(p) - v(p)|^2 + h_j^2 |\hat{w}'(p) - v'(p)|^2 \\
&\preceq h_j^{-3} \sum_{p \in \delta^V_{m,j,h}} |w(p) - u(p)|^2 = h_j^{-3} \sum_{p \in \delta^V_{m,j,h}} |\pi^2_{m,j} u(p) - I_{2h_j,2} u(p)|^2 \\
&\asymp h_j^{-4} \|\pi^2_{m,j} u - I_{2h_j,2} u\|^2_{L^2(\delta_m)},
\end{aligned}$$

where $I_{2h_j,2}$ is a piecewise quadratic interpolant defined over the $2h_j$-mesh of $\delta_{m,j}$. (Note that $I_{2h_j,2} u(p) = u(p)$).

Now note that from Definition 4.1 it holds that $I_{h_j,2} u = \pi^2_{m,j} I_{h_j,2} u$, and thus by (4.2) and well-known properties of $I_{h_j,2}$ we get

$$\begin{aligned}
\|\pi^2_{m,j} u - I_{h_j,2} u\|_{L^2(\delta_m)} &= \|\pi^2_{m,j}(u - I_{h_j,2} u)\|_{L^2(\delta_m)} \\
&\preceq \|u - I_{h_j,2} u\|_{L^2(\delta_m)} \preceq h_j^2 \|u\|_{H^2(\delta_{m,j})}.
\end{aligned}$$

The proof of the stability in H^2 norm is completed.

The $H_{00}^{3/2}$ case follows by an interpolation argument, e.g. cf. [12]. □

We also need an auxiliary operator defined on the trace space.

DEFINITION 4.9. *We now define an auxiliary operator $\pi^n_{s,j} : H^1(\Gamma_{ij}) \to L^2(\delta_{s,j})$. Let $w = Q_{s,j} u$ for $Q_{s,j}$ the L^2 orthogonal projection onto piecewise constant functions on $T_h^j(\delta_{s,j})$, then let $\pi^n_{s,j} u$ be a piecewise quadratic continuous function defined over the h_j-mesh of $\delta_{s,j}$ defined by its values at interior nodal points p and midpoints m.*

$$\begin{aligned}
\pi^n_{s,j} u(p) &= w(m_p) \qquad \forall p - \text{nodal point} \\
\pi^n_{s,j} u(m) &= w(m) \qquad \forall m - \text{midpoint},
\end{aligned}$$

where m_p is any midpoint of an element of $T_h^j(\delta_{s,j})$ of which p is an end.
The choice of m_p for each p is decided in conformity with the similar situation in definition of $\mathcal{M}_j$ and π^t.

LEMMA 4.10. *The operator defined in Definition 4.9 satisfies:*

$$\|\pi^n_{s,j}u\|_{L^2(\delta_{s,j})} \preceq \|u\|_{L^2(\delta_{s,j})} \quad \forall u \in L^2(\delta_{m,j})$$
$$\|\pi^n_{s,j}u\|_{H^{1/2}_{00}(\delta_{s,j})} \preceq \|u\|_{H^{1/2}_{00}(\delta_{s,j})} \quad \forall u \in H^{1/2}_{00}(\delta_{s,j}).$$

Proof. The lines of the proof basically have the same structure as those of the proof of Lemma 4.8; see also the proof of Lemma 3.5 in [24]. $\square$

LEMMA 4.11. *Let $u \in \tilde{V}^h$ be a discrete biharmonic function which is zero at all degrees of freedom on the boundary nodes except on the open mortar $\gamma_{m,i} = \Gamma_{ij}$ and its corresponding slave $\delta_{m,j} = \Gamma_{ij}$. Then we have*

$$\rho_j|u_j|^2_{H^2_h(\Omega_j)} \preceq \rho_i|u_i|^2_{H^2_h(\Omega_i)}.$$

Proof. Let $\mathcal{M}_k u, k = i,j$ be defined in such a way that $Tr\ \nabla\mathcal{M}_k u = 0$ on $\partial\Omega_k \setminus \Gamma_{ij}$; cf. Remark 4.1. Then we get by Lemma 4.5

$$|u|^2_{H^2_h(\Omega_j)} \preceq |\mathcal{M}_j u|^2_{H^2(\Omega_j)} \preceq \|\mathcal{M}_j u\|^2_{H^{3/2}_{00}(\delta_{m,j})} + \|\partial_n\mathcal{M}_j u\|^2_{H^{1/2}_{00}(\delta_{m,j})}. \tag{4.5}$$

We estimate both terms independently.

First we get

$$\|\mathcal{M}_j u\|^2_{H^{3/2}_{00}(\delta_{m,j})} \preceq \|\mathcal{M}_j u_j - \pi^t_{m,j}\mathcal{M}_i u_i\|^2_{H^{3/2}_{00}(\delta_{m,j})} + \|\pi^t_{m,j}\mathcal{M}_i u_i\|^2_{H^{3/2}_{00}(\delta_{m,j})}. \tag{4.6}$$

The second term can be estimated using Lemma 4.8, the trace theorem and Lemma 4.4 as follows:

$$\|\pi^t_{m,j}\mathcal{M}_i u_i\|_{H^{3/2}_{00}(\delta_{m,j})} \preceq \|\mathcal{M}_i u_i\|_{H^{3/2}_{00}(\delta_{m,j})} \preceq |\mathcal{M}_i u_i|_{H^2(\Omega_i)} \asymp |u_i|_{H^2_h(\Omega_i)}. \tag{4.7}$$

Note that the tangential (normal) trace of $\mathcal{M}_j u_j$ on each edge is solely defined by the values of u_j at vertices of $T_h^j(\delta_{m,j})$. Thus we can properly define $\mathcal{M}_j w$ for any $w \in C(\overline{\delta}_{m,j})$ (or any $w \in L^2(\delta_{m,j})$ which has uniquely defined values at vertices on this slave). Using this we see that the tangential trace $\mathcal{M}_j u_j$ on $\delta_{m,j}$ equals $\mathcal{M}_j u_j = \mathcal{M}_j \pi^2_{m,j} I_{2h_i,2} u_i$ by the mortar condition (2.1) and Definition 4.3.

Let $w = \mathcal{M}_j u_j - \pi^t_{m,j}\mathcal{M}_i u_i = \mathcal{M}_j(\pi^2_{m,j} I_{2h_i,2}u_i - \pi^t_{m,j}\mathcal{M}_i u_i) \in L^2(\delta_{m,j})$. Now we recall Definitions 4.3 and 4.7 and have

$$w(p) = [\pi^2_{m,j}(I_{2h_i,2} - \mathcal{M}_i)u_i](p)$$
$$\frac{d}{dt}w(p) = [\frac{d}{dt}\pi^2_{m,j}(I_{2h_i,2} - \mathcal{M}_i)u_i](m_p)$$

where m_p is an adjacent midpoint. (Here we have the same choices of this midpoint in the both definitions!). This yields $w = \pi^t_{m,j}(I_{2h_i,2} - \mathcal{M}_i)u_i$. Additionally we have $I_{2h_i,2}\mathcal{M}_i u_i = I_{2h_i,2}u_i$.

Thus by Lemma 4.8 and an inverse inequality we conclude that

$$\begin{aligned}\|w\|_{H^{3/2}_{00}(\delta_{m,j})} &\preceq \|\pi^t_{m,j}(I_{2h_i,2} - \mathcal{M}_i)u_i\|_{H^{3/2}_{00}(\delta_{m,j})}\\ &\preceq \|I_{2h_i,2}u_i - \mathcal{M}_i u_i\|_{H^{3/2}_{00}(\delta_{m,j})}\\ &\preceq h_i^{-3/2}\|I_{2h_i,2}\mathcal{M}_i u_i - \mathcal{M}_i u_i\|_{L^2(\Gamma_{ij})}.\end{aligned} \tag{4.8}$$

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Using standard properties of a piecewise quadratic interpolant, the trace theorem and Lemma 4.4 we conclude that

$$\begin{aligned}\|I_{2h_i,2}\mathcal{M}_iu_i - \mathcal{M}_iu_i\|_{L^2(\Gamma_{ij})} &\preceq h_i^{3/2}\|\mathcal{M}_iu_i\|_{H^{3/2}_{00}(\Gamma_{ij})}\\ &\preceq h_i^{3/2}|\mathcal{M}_iu_i|_{H^2(\Omega_i)} \asymp h_i^{3/2}|u_i|_{H^2_h(\Omega_i)},\end{aligned}$$

which together with (4.6), (4.7), and (4.8) gives us the bound of $\|\mathcal{M}_ju\|^2_{H^{3/2}_{00}(\delta_{m,j})}$.

Now we have to estimate the second term in (4.5).

$$\begin{aligned}\|\partial_n\mathcal{M}_ju_j\|_{H^{1/2}_{00}(\delta_{m,j})} \le \|\partial_n\mathcal{M}_ju_j - \pi^n_{m,j}\partial_n\mathcal{M}_iu_i\|_{H^{1/2}_{00}(\delta_{m,j})} +\\ + \|\pi^n_{m,j}\partial_n\mathcal{M}_iu_i\|_{H^{1/2}_{00}(\delta_{m,j})}.\end{aligned}\tag{4.9}$$

By Lemma 4.10, a trace theorem and Lemma 4.4 we get the estimate of the second term:

$$\|\pi^n_{m,j}\partial_n\mathcal{M}_iu_i\|_{H^{1/2}_{00}(\delta_{m,j})} \preceq \|\partial_n\mathcal{M}_iu_i\|_{H^{1/2}_{00}(\delta_{m,j})} \preceq |\mathcal{M}_iu_i|_{H^2(\Omega_i)} \asymp |u_i|_{H^2_h(\Omega_i)}.\tag{4.10}$$

We now estimate the first term in (4.9), i.e., $\|\partial_n\mathcal{M}_ju_j - \pi^n_{m,j}\partial_n\mathcal{M}_iu_i\|_{H^{1/2}_{00}(\delta_{m,j})}$. Let $z = \partial_n\mathcal{M}_ju_j - \pi^n_{m,j}\partial_n\mathcal{M}_iu_i$. We have by an inverse inequality

$$\|z\|^2_{H^{1/2}_{00}(\delta_{m,j})} \preceq h_j^{-1}\|z\|^2_{L^2(\delta_{m,j})} \asymp \sum_{x\in\delta^M_{m,j,h}} |z(x)|^2 + \sum_{p\in\delta^V_{m,j,h}} |z(p)|^2.$$

By Definitions 4.3 and 4.9 we have

$$\partial_n\mathcal{M}_ju_j(p) = \partial_nu_j(x_p) \qquad \text{and} \qquad \pi^n_{m,j}\partial_n\mathcal{M}_iu_i(p) = Q_{m,j}(\partial_n\mathcal{M}_iu_i)(x_p),$$

where x_p is an adjacent midpoint for a vertex $p \in \delta^V_{m,j,h}$.

Note that ∂_nu is linear over e a 1-D element of $T^j_h(\delta_{m,j})$ with x as the midpoint, hence, cf. (3.3),

$$\partial_nu_j(x) = |e|^{-1}\int_e \partial_nu_j(s)\,ds = Q_{m,j}\partial_nu_j(x) = Q_{m,j}\psi_i(x),$$

where ψ_i is a piecewise constant functions on $T^i_h(\gamma_{m,i})$ as in (3.1). The last equality follows from the mortar condition (3.1).

Thus we can conclude that

$$\begin{array}{rcll} z(p) &=& z(x_p) = Q_{m,j}(\psi_i)(x_p) - Q_{m,j}\partial_n\mathcal{M}_iu_i(x_p), & p\in\delta^V_{m,j,h},\\ z(x) &=& Q_{m,j}(\psi_i)(x) - Q_{m,j}\partial_n\mathcal{M}_iu_i(x), & x\in\delta^M_{m,j,h},\end{array}$$

where x_p is an adjacent midpoint of a vertex p, and thus

$$\begin{aligned}\|z\|^2_{H^{1/2}_{00}(\delta_{m,j})} &\preceq \sum_{x\in\delta^M_{m,j,h}} |z(x)|^2 = \sum_{x\in\delta^M_{m,j,h}} |Q_{m,j}(\psi_i - \partial_n\mathcal{M}_iu_i)(x)|^2\\ &\asymp h_j^{-1}\|Q_{m,j}(\psi_i - \partial_n\mathcal{M}_iu_i)\|^2_{L^2(\delta_{m,j})} \le h_j^{-1}\|\psi_i - \partial_n\mathcal{M}_iu_i\|^2_{L^2(\delta_{m,j})}\\ &\preceq h_j^{-1}\|\psi_i - \partial_nu_i\|^2_{L^2(\delta_{m,j})} + h_j^{-1}\|\partial_nu_i - \partial_n\mathcal{M}_iu_i\|^2_{L^2(\delta_{m,j})}\end{aligned}$$

Because ψ_i is the orthogonal L^2 projection of ∂_nu_i onto the space of piecewise constant functions on $T^i_h(\gamma_{m,i})$, by a trace theorem, a scaling argument, and a quotient space argument, we get in a standard way that $\|\psi_i - \partial_nu_i\|^2_{L^2(\delta_{m,j})} \preceq h_i|u_i|^2_{H^2_h(\Omega_i)}$.

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Thus, Lemma 4.4 applied to the second term, and $h_i \preceq h_j$ yields

$$\|\partial_n \mathcal{M}_j u_j - \pi^n_{m,j}\partial_n \mathcal{M}_i u_i\|_{H^{1/2}_{00}(\delta_{m,j})} \preceq |u_i|_{H^2_h(\Omega_i)}. \tag{4.11}$$

Finally, (4.9), (4.10), and (4.11) yield the proper bound of $\|\partial_n \mathcal{M}_j u\|_{H^{1/2}_{00}(\delta_{m,j})}$, what together with the assumption on the coefficients on each interface $\Gamma_{ij} = \gamma_{m,i} = \delta_{m,j}$, i.e., $\rho_i \geq \rho_j$, ends the proof. □

REMARK 4.2. In the end of proof of Lemma 4.11 Assumptions A.1 and A.2, cf. page 36 in Section 2, are used to bound the factor

$$\frac{h_i}{h_j}\frac{\rho_j}{\rho_i},$$

by a constant. Thus a good practical alternative to A.2 seems (cf. [34]) to be

$$\frac{h_i}{h_j} \asymp \left(\frac{\rho_i}{\rho_j}\right)^{\lambda}, \qquad 0 \leq \lambda \leq 1,$$

in the case when A.2 does not hold.

Here the master side of an edge Γ_{ij} is $\gamma_{m,i} \subset \partial\Omega_i$ with the slave $\delta_{m,j} \subset \partial\Omega_j$.

LEMMA 4.12. *Let $u_0 \in V_0$, then it holds that*

$$|u_0|^2_{H^2_h(\Omega_i)} \preceq \left(1 + \log\left(\frac{H_i}{h_i}\right)\right) \sum_{\Gamma_{ij}\subset\partial\Omega_i} |\overline{\psi}_{\Gamma_{ij}} - \partial_n I_H u_{0\,|\Gamma_{ij}}|^2,$$

where $\overline{\psi}_{\Gamma_{ij}}$ is a constant value taken by the normal derivative of u_0 at all midpoints of elements of $T^i_h(\Gamma_{ij})$ and $\partial_n I_H u_{0\,|\Gamma_{ij}}$ is the trace of the normal derivative of the linear interpolant of u_0 defined by the values of u_0 at the vertices of Ω_i (also taking constant value on a given edge).

Proof. First note that $|u_0|_{H^2_h(\Omega_i)} = |u_0 - I_H u_0|_{H^2_h(\Omega_i)}$. Let $w = u_0 - I_H u_0$ and we see that $w(x) = 0$ for all vertices $x \in \partial\Omega^V_{i,h}$ and $\partial_n w(m) = \overline{\psi}_{\Gamma_{ij}} - \partial_n I_H u_{0\,|\Gamma_{ij}} = Const$ for midpoints on an edge Γ_{ij}, what follows from the definition of V_0. Let further $c_{ij} = \overline{\psi}_{\Gamma_{ij}} - \partial_n I_H u_{0\,|\Gamma_{ij}} = Const$.

We can split w as follows $w = \sum_{\Gamma_{ij}\subset\partial\Omega_i} w_{ij}$, where w_{ij} is discrete biharmonic function defined by the values of respective degrees of freedom on $\partial\Omega_i$:

$$w_{ij}(x) = 0 \quad x \in \partial\Omega^V_{i,h}$$
$$\partial_n w_{ij}(m) = \begin{cases} \partial_n w(m) & m \quad \text{midpoint on } \Gamma_{ij}, \\ 0 & m \quad \text{midpoint on } \partial\Omega_i \setminus \Gamma_{ij}. \end{cases}$$

The interior degrees of freedom are set by (3.6).

We have $|u_0|_{H^2_h(\Omega_i)} = |u_0 - I_H u_0|_{H^2_h(\Omega_i)} \leq \sum_{\Gamma_{ij}\subset\partial\Omega_i} |w_{ij}|_{H^2_h(\Omega_i)}$. We estimate each term separately. We can now choose $\mathcal{M}_i w_{ij}$ in such a way that $\partial_n \mathcal{M}_i w_{ij} \in H^{1/2}_{00}(\Gamma_{ij})$; cf. Remark 4.1.

Note that $\partial_t \mathcal{M}_i w_{ij}(m) = (w_{ij}(a) - w_{ij}(b))/(b-a) = 0$ for any midpoint $m \in \Gamma^M_{ij,h}$ of a 1-D edge element $(a,b) \in T^i_h(\Gamma_{ij})$ for any edge $\Gamma_{ij} \subset \partial\Omega_i$. Thus from Definition 4.3 it follows that $w_{ij}(x) = \partial_t \mathcal{M}_i w_{ij}(x) = 0$ for any x which is a midpoint or a vertex in $\overline{\Gamma}_{ij,h}$. As the trace of $\mathcal{M}_i w_{ij}$ onto Γ_{ij} is a C^1 continuous piecewise cubic function, we can conclude that $\mathcal{M}_i w_{ij}$ is zero on Γ_{ij}, i.e., in particular $\partial_t \mathcal{M}_i w_{ij}$ is also zero on Γ_{ij}.

By this and Lemma 4.5 we get

$$|w_{ij}|^2_{H^2_h(\Omega_i)} \preceq \|\partial_t \mathcal{M}_i w_{ij}\|^2_{H^{1/2}_{00}(\Gamma_{ij})} + \|\partial_n \mathcal{M}_i w_{ij}\|^2_{H^{1/2}_{00}(\Gamma_{ij})} = \|\partial_n \mathcal{M}_i w_{ij}\|^2_{H^{1/2}_{00}(\Gamma_{ij})}.$$

Next Lemma 4.6 yields that

$$\|\partial_n \mathcal{M}_i w_{ij}\|^2_{H^{1/2}_{00}(\Gamma_{ij})} \preceq |\partial_n \mathcal{M}_i w_{ij}|^2_{H^{1/2}(\Gamma_{ij})} + \left(1 + \log\left(\frac{H_i}{h_i}\right)\right) \|\partial_n \mathcal{M}_i w_{ij}\|^2_{L^\infty(\Gamma_{ij})}.$$

By Definition 4.3 and the definition of w_{ij} we see that

$$\partial_n \mathcal{M}_i w_{ij}(p) = \partial_n w(m_p) = c_{ij} \qquad \text{and} \qquad \partial_n \mathcal{M}_i w_{ij}(m) = \partial_n w(m) = c_{ij}$$

for any $p \in \Gamma^V_{ij,h}$ and $m \in \Gamma^M_{ij,h}$, where m_p is an adjacent midpoint of p. Thus $\partial_n \mathcal{M}_i w_{ij}$ is a continuous piecewise quadratic function on the 1-D mesh of Γ_{ij} which equals zero at the ends of this edge and takes the constant value c_{ij} at all remaining nodal points of $T^i_h(\Gamma_{ij})$, in particular it equals the constant c_{ij} on all elements except two end elements of $T^i_h(\Gamma_{ij})$. Hence $\partial_n \mathcal{M}_i w_{ij} - c_{ij}$ can be nonzero only on the two end elements of $T^i_h(\Gamma_{ij})$ and $\|\partial_n \mathcal{M}_i w_{ij}\|^2_{L^\infty(\Gamma_{ij})} = |c_{ij}|^2$. This together with an inverse inequality gives us that

$$|\partial_n \mathcal{M}_i w_{ij}|^2_{H^{1/2}(\Gamma_{ij})} = |\partial_n \mathcal{M}_i w_{ij} - c_{ij}|^2_{H^{1/2}(\Gamma_{ij})} \preceq h_i^{-1} \|\partial_n \mathcal{M}_i w_{ij} - c_{ij}\|^2_{L^2(\Gamma_{ij})} \preceq |c_{ij}|^2.$$

Summing all these estimates yields that

$$|w_{ij}|^2_{H^2_h(\Omega_i)} \preceq \left(1 + \log\left(\frac{H_i}{h_i}\right)\right) |c_{ij}|^2 = \left(1 + \log\left(\frac{H_i}{h_i}\right)\right) |\overline{\psi}_{\Gamma_{ij}} - \partial_n I_H u_{0\,|\Gamma_{ij}}|^2,$$

which concludes the proof. ☐

5. The proof of ASM theorem. The proof is based on the abstract scheme of the ASM method; cf. [30]. It consists of checking three assumptions. Assumption II is satisfied by the standard coloring argument. Assumption III is satisfied for V_0 and V_k with $\omega = 1$ as local bilinear forms equal to the original one and Lemma 4.11 gives the estimate of ω for the remaining interface subspaces V_{γ_m}.

It remains to check Assumption I which we formulate as a lemma:

LEMMA 5.1. *For any $u \in \tilde{V}^H$ there exist $u_0 \in V_0$, $u_k \in V_k$ and $u_m \in V_{\gamma_m}$ such that*

$$u = u_0 + \sum_{k=1}^{N} u_k + \sum_{\gamma_m \subset \Gamma} u_m \tag{5.1}$$

and

$$b_0(u_0, u_0) + \sum_{k=1}^{N} a_H(u_k, u_k) + \sum_{\gamma_m \subset \Gamma} b_m(u_m, u_m) \preceq \left(1 + \log\left(\frac{H}{\underline{h}}\right)\right)^2 a_H(u, u). \tag{5.2}$$

Then, Lemma 3, Section 5.2 in [30] ends the proof of Theorem 3.4.

Proof. We define $u_0 \in V_0$ by setting its values at vertices and constant values taken by the normal derivative of this function on masters:

$$\begin{aligned} u_0(c_r) &= u(c_r) \quad \text{for } c_r \text{ a crosspoint} \\ u_{0|\gamma_{s,i}} &= \overline{\partial_n u}_{\gamma_{s,i}} \quad \forall \gamma_{s,i} = \Gamma_{ij} \subset \Gamma, \end{aligned}$$

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where $\overline{\partial_n u}_{\gamma_{s,i}} = \frac{1}{N_{s,i}} \sum_{m \in \gamma^M_{s,i,h}} \partial_n u(m)$ for $N_{s,i} = \#\gamma^M_{s,i,h}$ (the number of midpoints of elements of $T^i_h(\gamma_{s,i})$).

Let $w = u - u_0$ and let $u_k = (0, \dots, 0, \mathcal{P}_k w, 0, \dots, 0) \in V_k$ for $k = 1, \dots, N$, (i.e., $u_k = \mathcal{P}_k w$ in Ω_k and it is extended as zero onto the other subdomains). Note that $\mathcal{H}(w - \sum_k u_k) = \mathcal{H} w$. Finally we define $u_s \in V_{\gamma_s}$ by setting

$$u_s(p) = w(p) \quad p \in \gamma^V_{s,i,h} \quad \text{and} \quad \partial_n u_s(m) = \partial_n w(m) \quad m \in \gamma^M_{s,i,h}.$$

Note that these functions sum to u, i.e., this decomposition satisfies (5.1).

We now estimate $b_0(u_0, u_0)$. Lemma 4.12 yields

$$\begin{aligned}\left(1 + \log\left(\frac{H_i}{h_i}\right)\right)^{-1} a_{h,i}(u_0, u_0) &\preceq \rho_i \sum_{\Gamma_{ij} \subset \partial\Omega_i} |\overline{\psi}_{\Gamma_{ij}} - \partial_n I_H u_0|^2 \\ &= \rho_i \sum_{\gamma_{s,i} = \Gamma_{ij} \subset \partial\Omega_i} |\overline{\partial_n u}_{\gamma_{s,i}} - \partial_n I_H u_{|\Gamma_{ij}}|^2 + \rho_i \sum_{\delta_{r,i} = \Gamma_{ik} \subset \partial\Omega_i} |\overline{\partial_n u}_{\gamma_{r,k}} - \partial_n I_H u_{|\Gamma_{ik}}|^2,\end{aligned}$$

where the first sum is taken over all mortars (masters) $\gamma_{s,i} = \Gamma_{ij}$ of $\partial\Omega_i$ and the second over all slaves $\delta_{r,i} = \gamma_{r,k} = \Gamma_{ik}$. Note also that $I_H u_{|\Gamma_{ik}}$ is defined uniquely by the values of u at the ends of this edge.

Let us first consider a master $\gamma_{s,i} \subset \partial\Omega_i$. Let $z = u - I_H u$. Then by Schwarz inequality and standard techniques we get

$$\sum_{x \in \gamma^M_{s,i,h}} (\partial_n z)(x) \le \sqrt{N_{s,i}} \left(\sum_{x \in \gamma^M_{s,i,h}} (\partial_n z(x))^2 \right)^{1/2} \preceq N_{s,i} H_i^{-1/2} \|\partial_n z\|_{L^2(\Gamma_{ij})}. \tag{5.3}$$

Here $N_{s,i}$ is the number of h_i-elements of $T^i_h(\gamma_{s,i})$. Finally we have

$$\begin{aligned}|\overline{\partial_n u}_{\gamma_{s,i}} - \partial_n I_H u| = |\overline{\partial_n z}_{\gamma_{s,i}}| &\preceq H_i^{-1/2} \|\partial_n z\|_{L^2(\Gamma_{ij})} \\ &\le H_i^{-1/2} \left(\|\partial_n \mathcal{M}_i u - \partial_n I_H u\|_{L^2(\Gamma_{ij})} + \|\partial_n \mathcal{M}_i u - \partial_n u\|_{L^2(\Gamma_{ij})} \right).\end{aligned}$$

Note that

$$I_H u = I_H \mathcal{M}_i u \tag{5.4}$$

and thus a trace theorem, Lemma 4.4, a scaling argument and a quotient space argument yield that

$$H_i^{-1/2} \|\partial_n \mathcal{M}_i u - \partial_n I_H u\|_{L^2(\Gamma_{ij})} \preceq |\mathcal{M}_i u|_{H^2(\Omega_i)} \asymp |u|_{H^2_h(\Omega_i)}.$$

Using again Lemma 4.4 directly for the second term we get

$$H_i^{-1/2} \|\partial_n \mathcal{M}_i u - \partial_n u\|_{L^2(\Gamma_{ij})} \preceq |u|_{H^2_h(\Omega_i)}.$$

Thus we conclude that

$$\rho_i |\overline{\partial_n u}_{\gamma_{s,i}} - \partial_n I_H u|^2 \preceq \rho_i |u|^2_{H^2_h(\Omega_i)}.$$

Proceeding analogously in the case when $\Gamma_{ik} = \delta_{r,i}$ is a slave of $\partial\Omega_i$ with associated mortar $\gamma_{r,k} \subset \partial\Omega_k$ we have

$$\rho_i |\overline{\partial_n u}_{\gamma_{r,k}} - \partial_n I_H u|^2 \preceq \rho_i |u|^2_{H^2_h(\Omega_k)} \le \rho_k |u|^2_{H^2_h(\Omega_k)}.$$

We have also used the assumption $\rho_k \geq \rho_i$.

Combining these estimates and summing over all mortars and slaves of Ω_i and then all domains Ω_i gives

$$b_0(u_0, u_0) = a_H(u_0, u_0) \preceq \left(1 + \log\left(\frac{H}{\underline{h}}\right)\right) a_H(u, u). \tag{5.5}$$

Next we estimate the terms associated with $u_k \in V_k$. By (3.5) we have

$$\sum_{k=1}^{N} a_H(u_k, u_k) = \sum_{k=1}^{N} a_{k,h}(\mathcal{P}_k w, \mathcal{P}_k w) \leq a_H(w, w) \leq 2\, a_H(u, u) + 2a_H(u_0, u_0).$$

This and (5.5) yields

$$\sum_{k=1}^{N} a_H(u_k, u_k) \preceq \left(1 + \log\left(\frac{H}{\underline{h}}\right)\right) a_H(u, u). \tag{5.6}$$

Now let consider a mortar $\gamma_{s,i}$ which geometrically occupies the place of the interface Γ_{ij}. We have to estimate $b_s(u_s, u_s) = a_{h,i}(u_s, u_s) = \rho_i |u_s|^2_{H^2_h(\Omega_i)}$. Because u_s is discrete biharmonic we can utilize Lemma 4.5 and get

$$b_s(u_s, u_s) \preceq \rho_i |\nabla \mathcal{M}_i u_s|^2_{H^{1/2}(\partial\Omega_i)}.$$

Note that the tangential and normal traces of $\mathcal{M}_i$ are equal zero on $\partial\Omega_i \setminus \gamma_{s,i}$ thus we can conclude that

$$|u_s|^2_{H^2_h(\Omega_i)} \preceq \|\partial_t \mathcal{M}_i u_s\|^2_{H^{1/2}_{00}(\gamma_{s,i})} + \|\partial_n \mathcal{M}_i u_s\|^2_{H^{1/2}_{00}(\gamma_{s,i})}.$$

We estimate both terms using Lemma 4.6 and get

$$\begin{aligned} |u_s|^2_{H^2_h(\Omega_i)} &\preceq |\partial_t \mathcal{M}_i u_s|^2_{H^{1/2}(\gamma_{s,i})} + |\partial_n \mathcal{M}_i u_s|^2_{H^{1/2}(\gamma_{s,i})} \\ &\quad + (1 + \log(H_i/h_i)) \left(\|\partial_t \mathcal{M}_i u_s\|^2_{L^\infty(\gamma_{s,i})} + \|\partial_n \mathcal{M}_i u_s\|^2_{L^\infty(\gamma_{s,i})}\right). \end{aligned} \tag{5.7}$$

We show the bounds for all these terms successively. Note that the tangential trace of u_s equals to $u - I_H u$. Thus the trace theorem, Lemma 4.4, (5.4), a quotient space argument, and a scaling argument yield that

$$|\partial_t \mathcal{M}_i u_s|^2_{H^{1/2}(\gamma_{s,i})} \preceq \sum_{k=1}^{2} H_i^{2(k-2)} |\mathcal{M}_i(u - I_H u)|^2_{H^k(\Omega_i)} \preceq |u|^2_{H^2_h(\Omega_i)}. \tag{5.8}$$

Note that $\partial_n u_s(m) = \partial_n u(m) - \overline{\partial_n u}_{\gamma_{s,i}}$ for a midpoint $m \in \gamma^M_{s,i,h}$ and then from Definition 4.3 it follows that $\partial_n \mathcal{M}_i u_s(x) = \partial_n \mathcal{M}_i u(x) - \overline{\partial_n u}_{\gamma_{s,i}}$ for any nodal point $x \in \gamma_{s,i,h}$ and $\partial_n \mathcal{M}_i u_s(a_k) = 0,\ k = 1, 2$, where a_k is an end of $\gamma_{s,i}$.

Hence we get

$$\partial_n \mathcal{M}_i u_s = \partial_n \mathcal{M}_i u_{|\gamma_{s,i}} - \overline{\partial_n u}_{\gamma_{s,i}} - \sum_{k=1,2} (\partial_n \mathcal{M}_i u(a_k) - \overline{\partial_n u}_{\gamma_{s,i}}) \psi_{a_k} \quad \text{on} \quad \gamma_{s,i},$$

where ψ_{a_k} is a nodal basis function of the finite element space of continuous piecewise quadratic functions defined on $T^i_h(\gamma_{s,i})$ associated with a_k - one of the ends of $\gamma_{s,i}$, i.e.,

ψ_{a_k} is equal to one at a_k and is equal to zero at all remaining vertices and midpoints of $\overline{\gamma}_{s,i,h}$. Note that $|\psi_{a_k}|_{H^{1/2}(\gamma_{s,i})} \preceq 1$. Thus we get

$$(5.9)\qquad |\partial_n \mathcal{M}_i u_s|_{H^{1/2}(\gamma_{s,i})} \preceq |\partial_n \mathcal{M}_i u|_{H^{1/2}(\gamma_{s,i})} + \|\partial_n \mathcal{M}_i u - \overline{\partial_n u}_{\gamma_{s,i}}\|_{L^\infty(\Omega_i)}.$$

The second term is estimated together with $\|\partial_n \mathcal{M}_i u_s\|^2_{L^\infty(\gamma_{s,i})}$, cf. (5.11) below, and the first term can be estimated by a trace theorem, and Lemma 4.4 as follows:

$$(5.10)\qquad |\partial_n \mathcal{M}_i u|_{H^{1/2}(\gamma_{s,i})} \preceq |\mathcal{M}_i u|_{H^2(\Omega_i)} \asymp |u|_{H^2_h(\Omega_i)}.$$

The last two terms in (5.7) can be estimate using (5.3) and standard arguments like discrete Sobolev like inequality, cf. Lemma 4.15 in [32]:

$$\|\partial_t \mathcal{M}_i u_s\|^2_{L^\infty(\gamma_{s,i})} \le |\mathcal{M}_i(u - I_H u)|^2_{W^{1,\infty}(\Omega_i)} \preceq (1 + \log(H_i/h_i))|u|^2_{H^2_h(\Omega_i)},$$

$$(5.11)\qquad \|\partial_n \mathcal{M}_i u_s\|^2_{L^\infty(\gamma_{s,i})} \le \|\partial_n \mathcal{M}_i u - \overline{\partial_n u}_{\gamma_{s,i}})\|^2_{L^\infty(\Omega_i)} \preceq (1 + \log(H_i/h_i))|u|^2_{H^2_h(\Omega_i)}.$$

These estimates together with (5.7), (5.8), (5.9), and (5.10) give

$$b_s(u_s, u_s) \preceq (1 + \log(H_i/h_i))^2 \rho_i |u|^2_{H^2_h(\Omega_i)}.$$

Summing by all mortars we get

$$\sum_{\gamma_s \subset \Gamma} b_s(u_s, u_s) \preceq \sum_{i=1}^{N} (1 + \log(H_i/h_i))^2 \rho_i |u|^2_{H^2_h(\Omega_i)} \le (1 + \log(H/\underline{h}))^2 a_H(u, u).$$

Finally, (5.5), (5.6) and the last bound yield (5.2) what ends the proof. $\square$

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