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NON-MATCHING MORTAR DISCRETIZATION ANALYSIS FOR THE COUPLING STOKES-DARCY EQUATIONS*

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Abstract. We consider the coupling across an interface of fluid and porous media flows with Beavers-Joseph-Saffman transmission conditions. Under an adequate choice of Lagrange multipliers on the interface we analyze inf-sup conditions and optimal a priori error estimates associated with the continuous and discrete formulations of this Stokes-Darcy system. We allow the meshes of the two regions to be non-matching across the interface. Using mortar finite element analysis and appropriate scaled norms we show that the constants that appear on the a priori error bounds do not depend on the viscosity, permeability and ratio of mesh parameters. Numerical experiments are presented.

Key words. inf-sup condition, error estimates, mortar finite elements, multiphysics, porous media flow, incompressible fluid flow, Lagrange multipliers, saddle point problems, non-matching grids, discontinuous coefficients

AMS subject classifications. 65N30, 65N15, 65N12, 35Q30, 35Q35, 76D033, 76D07

1. Introduction.

We analyze the coupling across an interface of fluid and porous media flows. This problem appears in several applications like well-reservoir coupling in petroleum engineering, transport of substances across groundwater and surface water, and (bio)fluid-organ interactions. More precisely, we consider the following situation: an incompressible fluid in a region Ω_f can flow both ways across an interface Γ into a saturated porous medium domain Ω_p. The model studied here consists of *Stokes equations* in the fluid region Ω_f and *Darcy law* for the filtration velocity in the porous medium region Ω_p. The transmission conditions we consider on the interface Γ are the Beavers-Joseph-Saffman conditions [3, 19, 27] which are widely accepted by the scientific community. In this paper we study inf-sup conditions and a priori error estimates associated with the continuous and discrete formulations of this Stokes-Darcy system. There are previous works addressing such issues [8, 13, 20, 26] as well as related problems such as Stokes-Laplacian systems [10, 11, 25], Stokes-Navier Stokes [16, 24], and preconditioned iterative methods [10, 12, 13, 14], among others [2, 21].

This paper is organized as follows: in Section 2 we discuss norms and seminorms of dual spaces on subsets. The differential systems are introduced in Section 3, where velocity and normal flux are considered as the boundary data for the Stokes part $\Gamma_f = \partial\Omega_f\backslash\Gamma$ and the Darcy part $\Gamma_p = \partial\Omega_p\backslash\Gamma$, respectively; for other formulations and boundary data see [11, 12]. The transmission conditions on the interface Γ, known as Beavers-Joseph-Saffman conditions, are then introduced. In Section 4 we analyze weak formulations of the continuous model and we discuss the choice $H^{1/2}(\Gamma)$ as the space for Lagrange multipliers in order to couple these two systems of partial differential equations. In [20], Layton, Schieweck, and Yotov developed existence and uniqueness of the weak solution for this problem. They were able to show the inf-sup condition on the smaller space $H_{00}^{1/2}(\Gamma)$. Recall that $H_{00}^{1/2}(\Gamma)$ is the subspace of functions in $H^{1/2}(\partial\Omega_p)$ that vanish on $\partial\Omega_p \setminus \Gamma$. In this paper, we use tools developed in Section 2 and in [20] to present a complete analysis for the inf-sup condition with Lagrange multipliers on the space $H^{1/2}(\Gamma)$. We note that from the physical point of view the space $H^{1/2}(\Gamma)$ is the correct choice since the Lagrange multipliers are related to the Darcy pressure on the interface Γ and the value of the Darcy pressure at $\Gamma \cap \partial(\Omega_f \cup \Omega_p)$ is

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not prescribed when flux boundary condition is imposed on the porous side exterior boundary Γ_p. We note however that in the case where the pressure is imposed as the boundary condition on the Darcy exterior boundary Γ_p, the space $H_{00}^{1/2}(\Gamma)$ would be the correct choice; see [11]. In Section 5 we derive the discrete inf-sup conditions and in Section 6 the a priori error estimates. We consider the triangular $P2\backslash P1$ Taylor Hood elements space for the free flow region Ω_f and the lowest order Raviart-Thomas for the Darcy region Ω_p. In [20], Layton, Schieweck and Yotov developed a priori error estimates for the matching case, while in [26] Riviere and Yotov, and also in [8] Burman and Hansbo, considered the non-matching case using discontinuous Galerkin finite element discretizations. In this paper we consider the coupling via Lagrange multipliers and we develop an analysis based on mortar finite elements techniques [4, 29] and scaled norms in order to obtain constants independent of the permeability, viscosity and ratio of mesh parameters. We pay special attention to the constants appearing in the a priori error estimates. In Appendix B we provide the construction of the Fortin interpolation for $P2\backslash P1$ Taylor Hood elements. In Section 7 we test numerically the algorithms and in Section 8 we make some conclusions.

2. Preliminaries and notations. Let Ω be a bounded Lipschitz continuous domain and let $\Gamma \subset \partial\Omega$ and $\Gamma^c := \partial\Omega \setminus \Gamma$ be of non-vanishing $(n-1)$-dimensional measure with respect to $\partial\Omega$. Here $n = 2$ or 3. To avoid the proliferation of constants, we will use the notation $A \preceq B$ to represent the inequality $A \leq (constant) \cdot B$.

LEMMA 2.1. *Given $\mu \in H^{1/2}(\Gamma)$, define $E_\Gamma^{1/2}\mu := \gamma_0\varphi$ where γ_0 is the trace on $\partial\Omega$ and φ is the weak solution of*

$$\begin{cases} -\Delta\varphi = 0 & in\ \Omega \\ \varphi = \mu & on\ \Gamma \\ \partial_{\boldsymbol{\eta}}\varphi = 0 & on\ \Gamma^c. \end{cases}$$

Then $E_\Gamma^{1/2}\mu \in H^{1/2}(\partial\Omega)$ and $\|E_\Gamma^{1/2}\mu\|_{H^{1/2}(\partial\Omega)} \preceq \|\mu\|_{H^{1/2}(\Gamma)}$.

For $\mu \in H^{1/2}(\Gamma)$ let $E_{00,\Gamma}^{1/2}\mu$ denote the extension by zero on Γ^c. Remember that $E_{00,\Gamma}^{1/2}\mu \in H^{1/2}(\partial\Omega)$ if and only if $\mu \in H_{00}^{1/2}(\Gamma)$. We have the following result.

LEMMA 2.2. *For all $\mu \in H^{1/2}(\partial\Omega)$ there exist $\mu_\Gamma \in H^{1/2}(\Gamma)$ and $\mu_{\Gamma^c} \in H_{00}^{1/2}(\Gamma^c)$ such that $\mu = E_\Gamma^{1/2}\mu_\Gamma + E_{00,\Gamma^c}^{1/2}\mu_{\Gamma^c}$. This decomposition is unique.*

Proof. Let $\mu \in H^{1/2}(\partial\Omega)$. Take $\mu_\Gamma = \mu|_\Gamma$ and $\mu_{\Gamma^c} = \varphi|_{\Gamma^c}$ where $\varphi = \mu - E_\Gamma^{1/2}\mu_\Gamma$. Observe that $\mu_\Gamma \in H^{1/2}(\Gamma)$ and

$$\|E_\Gamma^{1/2}\mu_\Gamma\|_{H^{1/2}(\partial\Omega)} \preceq \|\mu_\Gamma\|_{H^{1/2}(\Gamma)} \leq \|\mu\|_{H^{1/2}(\partial\Omega)},$$

therefore, $\varphi \in H^{1/2}(\partial\Omega)$. Observe also that $E_{00,\Gamma^c}^{1/2}\mu_{\Gamma^c} = \varphi$ because μ and $E_\Gamma^{1/2}\mu_\Gamma$ coincide on Γ. For the uniqueness, if $0 = E_\Gamma^{1/2}\mu_\Gamma + E_{00,\Gamma^c}^{1/2}\mu_{\Gamma^c}$ then $E_\Gamma^{1/2}\mu_\Gamma$ is the trace of the weak solution of the problem: $-\Delta\varphi = 0$ in Ω, $\varphi = 0$ on Γ, $\partial_{\boldsymbol{\eta}}\varphi = 0$ on Γ^c. Then $\mu_\Gamma = 0$. ∎

We have two dual spaces associated with Γ, the space $H_{00}^{-1/2}(\Gamma)$ (the dual of $H_{00}^{1/2}(\Gamma)$) and $H^{-1/2}(\Gamma)$ (the dual space of $H^{1/2}(\Gamma)$). The first space is larger than the second one.

DEFINITION 2.3. *If $f \in H^{-1/2}(\partial\Omega)$, then $f|_{\Gamma^c} = 0$ means by definition that*

$$\langle f, E_{00,\Gamma^c}^{1/2}\mu\rangle_{\partial\Omega} = 0 \quad for\ all\ \mu \in H_{00}^{1/2}(\Gamma^c).$$

A useful result related with this definition is the following:

LEMMA 2.4. *If $f \in H^{-1/2}(\partial\Omega)$, there are $f_\Gamma \in H^{-1/2}(\Gamma)$ and $f_{\Gamma^c} \in H_{00}^{-1/2}(\Gamma^c)$ such*

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that, for all $\mu \in H^{1/2}(\partial\Omega)$*, let* $\mu = E_{\Gamma}^{1/2}\mu_{\Gamma} + E_{00,\Gamma^c}^{1/2}\mu_{\Gamma^c}$ *as defined in Lemma 2.2, we have*

$$\langle f, \mu\rangle_{\partial\Omega} = \langle f_{\Gamma}, \mu_{\Gamma}\rangle_{\Gamma} + \langle f_{\Gamma^c}, \mu_{\Gamma^c}\rangle_{\Gamma^c}. \tag{2.1}$$

Proof. For $\mu_{\Gamma} \in H^{1/2}(\Gamma)$ and $\mu_{\Gamma^c} \in H_{00}^{1/2}(\Gamma^c)$ define

$$\langle f_{\Gamma}, \mu_{\Gamma}\rangle_{\Gamma} := \langle f, E_{\Gamma}^{1/2}\mu_{\Gamma}\rangle_{\partial\Omega} \qquad \langle f_{\Gamma^c}, \mu_{\Gamma^c}\rangle_{\Gamma^c} := \langle f, E_{00,\Gamma^c}^{1/2}\mu_{\Gamma^c}\rangle_{\partial\Omega}.$$

We obtain

$$\langle f_{\Gamma}, \mu_{\Gamma}\rangle_{\Gamma} \le \|f\|_{H^{-1/2}(\partial\Omega)}\|E_{\Gamma}^{1/2}\mu_{\Gamma}\|_{H^{1/2}(\partial\Omega)} \preceq \|f\|_{H^{-1/2}(\partial\Omega)}\|\mu_{\Gamma}\|_{H^{1/2}(\Gamma)},$$

and so $f_{\Gamma} \in H^{-1/2}(\Gamma)$. Analogously, $f_{\Gamma^c} \in H_{00}^{-1/2}(\Gamma^c)$. Moreover,

$$\langle f_{\Gamma}, \mu_{\Gamma}\rangle_{\Gamma} + \langle f_{\Gamma^c}, \mu_{\Gamma^c}\rangle_{\Gamma^c} = \langle f, E_{\Gamma}^{1/2}\mu_{\Gamma} + E_{00,\Gamma^c}^{1/2}\mu_{\Gamma^c}\rangle_{\partial\Omega} = \langle f, \mu\rangle_{\partial\Omega}. \qquad \square$$

REMARK 2.5. In particular, if $f \in H^{-1/2}(\partial\Omega)$ and $f|_{\Gamma^c} = 0$ (see Definition 2.3 above), we have from (2.1) that

$$\langle f, \mu\rangle_{\partial\Omega} = \langle f_{\Gamma}, \mu_{\Gamma}\rangle_{\Gamma}.$$

Hence, functionals in $H^{-1/2}(\partial\Omega)$ which are zero when restricted to $\partial\Omega \setminus \Gamma$ can be identified with functionals in $H^{-1/2}(\Gamma)$.

REMARK 2.6. Given $f_{\Gamma} \in H^{-1/2}(\Gamma)$ we can define $f \in H^{-1/2}(\partial\Omega)$ by $\langle f, \mu\rangle_{\partial\Omega} := \langle f_{\Gamma}, \mu_{\Gamma}\rangle_{\Gamma}$, where $\mu = E_{\Gamma}^{1/2}\mu_{\Gamma} + E_{00,\Gamma^c}^{1/2}\mu_{\Gamma^c}$ as defined in Lemma 2.2. We have a similar result for $f_{\Gamma} \in H_{00}^{-1/2}(\Gamma^c)$.

Define the space $\boldsymbol{H}(\mathrm{div}, \Omega)$ by

$$\boldsymbol{H}(\mathrm{div}, \Omega) := \left\{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \ \nabla\cdot\boldsymbol{v} \in L^2(\Omega)\right\},$$

with the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathrm{div},\Omega)}^2 := \|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\nabla\cdot\boldsymbol{v}\|_{L^2(\Omega)}^2. \tag{2.2}$$

Recall that if $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ then $\boldsymbol{v}\cdot\boldsymbol{\eta} \in H^{-1/2}(\partial\Omega)$. For the next result, see [30].

LEMMA 2.7. *For each* $\boldsymbol{u} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ *with* $\int_{\partial\Omega} \boldsymbol{u}\cdot\boldsymbol{\eta} = \langle \boldsymbol{u}\cdot\boldsymbol{\eta}, 1\rangle_{\partial\Omega} = 0$ *we have*

$$\sup_{\substack{\phi \in H^{1/2}(\partial\Omega)\\ \phi \neq \text{ constant}}} \frac{\langle \boldsymbol{u}\cdot\boldsymbol{\eta}, \phi\rangle_{\partial\Omega}}{|\phi|_{H^{1/2}(\partial\Omega)}} \preceq \|\boldsymbol{u}\cdot\boldsymbol{\eta}\|_{H^{-1/2}(\partial\Omega)} \le \sup_{\substack{\phi \in H^{1/2}(\partial\Omega)\\ \phi \neq \text{ constant}}} \frac{\langle \boldsymbol{u}\cdot\boldsymbol{\eta}, \phi\rangle_{\partial\Omega}}{|\phi|_{H^{1/2}(\partial\Omega)}}.$$

with a constant which depends only on Ω*.*

Using an argument similar to the one given in [30], we have

LEMMA 2.8. *For each* $f \in H^{-1/2}(\Gamma)$ *with* $\int_{\Gamma} f = \langle f, 1\rangle_{\Gamma} = 0$*, we have*

$$\sup_{\substack{\phi \in H^{1/2}(\Gamma)\\ \phi \neq \text{ constant}}} \frac{\langle f, \phi\rangle_{\Gamma}}{|\phi|_{H^{1/2}(\Gamma)}} \preceq \|f\|_{H^{-1/2}(\Gamma)} \le \sup_{\substack{\phi \in H^{1/2}(\Gamma)\\ \phi \neq \text{ constant}}} \frac{\langle f, \phi\rangle_{\Gamma}}{|\phi|_{H^{1/2}(\Gamma)}},$$

with a constant which depends only on Γ*.*

Proof. Observe that if α is a constant then $\langle f, \alpha\rangle_\Gamma = \alpha\langle f, 1\rangle_\Gamma = 0$ and for $\phi \in H^{1/2}(\Gamma)$ non-constant we have

$$\frac{\langle f, \phi\rangle_\Gamma}{\|\phi\|_{H^{1/2}(\Gamma)}} \leq \frac{\langle f, \phi\rangle_\Gamma}{|\phi|_{H^{1/2}(\Gamma)}},$$

then

$$\|f\|_{H^{-1/2}(\Gamma)} = \sup_{\substack{\phi\in H^{1/2}(\Gamma)\\ \phi\neq\text{constant}}} \frac{\langle f, \phi\rangle_\Gamma}{\|\phi\|_{H^{1/2}(\Gamma)}} \leq \sup_{\substack{\phi\in H^{1/2}(\Gamma)\\ \phi\neq\text{constant}}} \frac{\langle f, \phi\rangle_\Gamma}{|\phi|_{H^{1/2}(\Gamma)}},$$

which gives the right inequality. Using a Poincaré inequality, there exists a positive constant which depends only on Γ, such that

$$\|\psi\|^2_{H^{1/2}(\Gamma)} \preceq |\psi|^2_{H^{1/2}(\Gamma)}$$

holds for all $\psi \in H^{1/2}(\Gamma)$ with $\int_\Gamma \psi = 0$. For $\phi \in H^{1/2}(\Gamma)$ non-constant, we have

$$\psi := \phi - \int_\Gamma \phi \neq 0$$

and

$$\frac{\langle f, \psi\rangle_\Gamma}{\|\psi\|_{H^{1/2}(\Gamma)}} = \frac{\langle f, \phi\rangle_\Gamma}{\|\psi\|_{H^{1/2}(\partial\Omega)}} \succeq \frac{\langle f, \phi\rangle}{|\psi|_{H^{1/2}(\Gamma)}} = \frac{\langle f, \phi\rangle}{|\phi|_{H^{1/2}(\Gamma)}}. \quad \square$$

This gives an equivalent norm in the subspace of $H^{-1/2}(\Gamma)$ of zero average functionals.

DEFINITION 2.9. *For $f \in H^{-1/2}(\Gamma)$, f with zero average ($\langle f, 1\rangle_\Gamma = 0$), define*

$$|f|_{H^{-1/2}(\Gamma)} := \sup_{\substack{\phi\in H^{1/2}(\Gamma)\\ \phi\neq \textit{ constant}}} \frac{\langle f, \phi\rangle_\Gamma}{|\phi|_{H^{1/2}(\Gamma)}} = \sup_{\substack{\phi\in H^{1/2}(\Gamma)\\ \int_\Gamma \phi=0,\phi\neq0,}} \frac{\langle f, \phi\rangle_\Gamma}{|\phi|_{H^{1/2}(\Gamma)}}.$$

We have the following result.

LEMMA 2.10. *For $\mu \in H^{1/2}(\Gamma)$ with $\int_\Gamma \mu = 0$ we have*

$$|\mu|_{H^{1/2}(\Gamma)} = \sup_{\substack{f\in H^{-1/2}(\Gamma)\\ \langle f,1\rangle=0}} \frac{\langle f, \mu\rangle_\Gamma}{|f|_{H^{-1/2}(\Gamma)}}.$$

Proof. Consider $(H^{1/2}(\Gamma) \cap L^2_0(\Gamma))^*$, the dual space of $H^{1/2}(\Gamma) \cap L^2_0(\Gamma)$, and observe that a functional $f_0 \in (H^{1/2}(\Gamma) \cap L^2_0(\Gamma))^*$ can be extended to one in $H^{1/2}(\Gamma)^*$, say f, by the following formula: $\langle f, \phi\rangle := \langle f_0, \phi_0\rangle$ where $\phi \in H^{1/2}(\Gamma)$ and $\phi_0 := \phi - \int_\Gamma \phi$. $\square$

3. P.D.E model. In general, $\Omega_f,\ \Omega_p \subset \mathbb{R}^n$, $\Gamma = \overline{\Omega}_f \cap \overline{\Omega}_p$, $\Omega = \text{int}(\overline{\Omega}_f \cup \overline{\Omega}_p)$, Ω_f and Ω_p are Lipschitz, so it is possible to define outward unit normal vectors, denoted by $\boldsymbol{\eta}_j$, $j = f, p$. The tangent vectors on Γ are denoted by $\boldsymbol{\tau}_1$ ($n = 2$), or $\boldsymbol{\tau}_l$, $l = 1, 2$ ($n = 3$). In order to avoid a setting that is too general, when $n = 2$ we consider $\Omega_f = (1, 2) \times (0, 1)$ and $\Omega_p = (0, 1) \times (0, 1)$ or a regular Lipschitz perturbation of this configuration. Analogous conditions are consider for the case $n = 3$.

Define $\Gamma_j := \partial\Omega_j \setminus \Gamma$, $j = f, p$. Velocities are denoted by $\boldsymbol{u}_j : \Omega_j \to \mathbb{R}^n$, $j = f, p$. Pressures are $p_j : \Omega_j \to \mathbb{R}$, $j = f, p$.

As was mentioned previously, Stokes equations are the model for the fluid region. The model basically consists of conservation of mass and conservation of momentum, and we have

$$\left\{\begin{array}{rll} -\nabla\cdot T(\boldsymbol{u}_f, p_f) & = \boldsymbol{f}_f & \text{in } \Omega_f \\ \nabla\cdot \boldsymbol{u}_f & = g_f & \text{in } \Omega_f \\ \boldsymbol{u}_f & = \boldsymbol{h}_f & \text{on } \Gamma_f. \end{array}\right. \tag{3.1}$$

Here $T(\boldsymbol{v}, p) := -pI + 2\nu \boldsymbol{D}\boldsymbol{v}$ where ν is the fluid viscosity and $\boldsymbol{D}\boldsymbol{v} := \frac{1}{2}(\nabla \boldsymbol{v} + \nabla^T \boldsymbol{v})$ is the linearized strain tensor.

For the porous domain Ω_p, Darcy's law is used, i.e., $(\boldsymbol{u}_p, p_p)$ satisfies on Ω_p

$$\left\{\begin{array}{rlll} \boldsymbol{u}_p & = -\frac{\kappa}{\nu}\nabla p_p + \boldsymbol{f}_p & \text{in } \Omega_p & \text{(Darcy's law)} \\ \nabla\cdot \boldsymbol{u}_p & = g_p & \text{in } \Omega_p & \\ \boldsymbol{u}_p\cdot\boldsymbol{\eta}_p & = h_p & \text{on } \Gamma_p. & \end{array}\right. \tag{3.2}$$

In general κ is a symmetric and a uniformly positive definite tensor that represents the rock permeability. For simplicity of the analysis, we assume that κ is a real positive constant. Recall that ν is the fluid viscosity.

We also impose the compatibility condition

$$\int_{\Omega_f} g_f + \int_{\Omega_p} g_p - \int_{\Gamma_f} \boldsymbol{h}_f\cdot\boldsymbol{\eta}_f - \int_{\Gamma_p} h_p = 0. \tag{3.3}$$

The systems presented above must be coupled across the interface Γ. The following conditions are imposed (see [11, 12, 13, 20] and references therein):

Conservation of mass across Γ*:* It is expressed by:

$$\boldsymbol{u}_f\cdot\boldsymbol{\eta}_f + \boldsymbol{u}_p\cdot\boldsymbol{\eta}_p = 0 \text{ on } \Gamma. \tag{3.4}$$

This means that the fluid that is leaving a region enters in the other one.

Balance of normal forces across Γ*:* From Cauchy formula we see that

$$\Sigma(\boldsymbol{u}_f, p_f) := T(\boldsymbol{u}_f, p_f)\boldsymbol{\eta}_f$$

is the force on $\partial\Omega_f$ acting on the fluid volume inside Ω_f, i.e., Σ is the Cauchy stress (or traction) vector. The force on Γ from Ω_f side is then $\Sigma(\boldsymbol{u}_f, p_f)$. The only force acting on the interface from Ω_p side is the one given by p_p in the direction of $\boldsymbol{\eta}_p$ and must be equal to the component of Σ in this direction. We get

$$p_f - 2\nu\boldsymbol{\eta}_f^T \boldsymbol{D}(\boldsymbol{u}_f)\boldsymbol{\eta}_f = p_p \quad \text{on } \Gamma. \tag{3.5}$$

The other components of Σ, i.e., $\Sigma\cdot\boldsymbol{\tau}_l$, $l = 1, n-1$, are more delicate and treated below.

Beavers-Joseph-Saffman condition: This condition is a kind of empirical law that gives an expression for the tangential component of Σ. It is expressed by:

$$\boldsymbol{u}_f\cdot\boldsymbol{\tau}_l = -\frac{\sqrt{\kappa}}{\alpha_f} 2\boldsymbol{\eta}_f^T \boldsymbol{D}(\boldsymbol{u}_f)\boldsymbol{\tau}_l \quad \text{on } \Gamma,\ l = 1, n-1. \tag{3.6}$$

In the general case, κ is a symmetric and uniformly positive definite tensor, and κ in (3.6) is replaced by $\tau_l \cdot \kappa \cdot \tau_l$.

A related condition is

$$(\boldsymbol{u}_f - \boldsymbol{u}_p)\cdot\boldsymbol{\tau}_l = -\frac{\sqrt{\kappa}}{\alpha_f} 2\boldsymbol{\eta}_f^T \boldsymbol{D}(\boldsymbol{u}_f)\boldsymbol{\tau}_l \qquad \text{on } \Gamma, l = 1, n-1,$$

which is known as the Beavers-Joseph condition. But it turns out in practice that the component of $\boldsymbol{u}_p$ in $\boldsymbol{\tau}_l$ direction is small compared with that of $\boldsymbol{u}_f$. When more general cases are considered, suitable interface conditions have to be imposed. An analytical way to find the right interface conditions is via homogenization; see [18].

4. Weak formulations and inf-sup analysis. In this section we derive and analyze several weak formulations associated with the Stokes-Darcy system presented in Section 3.

4.1. Weak formulations. According to Appendix A, it is enough to consider the case $g_f = 0$ and $\boldsymbol{h}_f = \boldsymbol{0}$ in (3.1) and $g_p = 0$ and $h_p = 0$ in (3.2).

For Ω_f define

$$\boldsymbol{X}_f := H_0^1(\Omega_f, \Gamma_f)^n \text{ and } M_f := L^2(\Omega_f),$$

where $H_0^1(\Omega_f, \Gamma_f)^n$ means by definition the subspace of functions $\boldsymbol{v}_f$ such that each component of $\boldsymbol{v}_f$ belongs to $H^1(\Omega_f)$ and vanishes on Γ_f.

For Ω_p we introduce the following spaces:

$$\boldsymbol{X}_p := \boldsymbol{H}_0(\text{div}, \Omega_p, \Gamma_p) \text{ and } M_p := L^2(\Omega_p),$$

where $\boldsymbol{H}_0(\text{div}, \Omega_p, \Gamma_p)$ is defined as the subspace of $\boldsymbol{H}(\text{div}, \Omega_p)$ of functions with vanishing normal component on Γ_p in the sense of Definition 2.3. Recall that if $\boldsymbol{u}_p \in \boldsymbol{H}(\text{div}, \Omega_p)$ then $\boldsymbol{u}_p\cdot\boldsymbol{\eta}_p \in H^{-1/2}(\partial\Omega_p)$; see (2.2).

Define $\boldsymbol{X} := \boldsymbol{X}_f \times \boldsymbol{X}_p$ with the usual norm, i.e., given $\boldsymbol{v} = (\boldsymbol{v}_f, \boldsymbol{v}_p) \in \boldsymbol{X}$,

$$\|\boldsymbol{v}\|_{\boldsymbol{X}}^2 := |\boldsymbol{v}_f|_{H^1(\Omega_f)^n}^2 + \|\boldsymbol{v}_p\|_{\boldsymbol{H}(\text{div}, \Omega_p)}^2.$$

We also set $M := M_f \times M_p$ with the norm $\|q\|_M^2 := \|q_f\|_{L^2(\Omega_f)}^2 + \|q_f\|_{L^2(\Omega_p)}^2$.

In order to derive a weak formulation we first proceed formally and then we introduce the adequate rigorous framework.

We start with the Stokes equation (3.1). For all $\boldsymbol{v}_f \in \boldsymbol{X}_f$ we have

$$(-2\nu\nabla\cdot\boldsymbol{D}\boldsymbol{u}_f, \boldsymbol{v}_f)_{\Omega_f} + (\nabla p_f, \boldsymbol{v}_f)_{\Omega_f} = (\boldsymbol{f}_f, \boldsymbol{v}_f)_{\Omega_f}. \tag{4.1}$$

From the Green formula we have

$$\begin{aligned} -(\Delta\boldsymbol{u}_f, \boldsymbol{v}_f)_{\Omega_f} &= (\nabla\boldsymbol{u}_f, \nabla\boldsymbol{v}_f)_{\Omega_f} - (\nabla\boldsymbol{u}_f\boldsymbol{\eta}_f, \boldsymbol{v}_f)_\Gamma \\ &= (\nabla\boldsymbol{u}_f, \nabla\boldsymbol{v}_f)_{\Omega_f} - \langle\boldsymbol{\eta}_f^T\nabla\boldsymbol{u}_f\boldsymbol{\eta}_f, \boldsymbol{v}_f\cdot\boldsymbol{\eta}_f\rangle_\Gamma - \sum_{l=1}^{n-1}\langle\boldsymbol{\tau}_l^T\nabla\boldsymbol{u}_f\boldsymbol{\eta}_f, \boldsymbol{v}_f\cdot\boldsymbol{\tau}_l\rangle_\Gamma, \end{aligned}$$

and

$$\begin{aligned} -(\nabla\cdot\nabla\boldsymbol{u}_f^T, \boldsymbol{v}_f)_{\Omega_f} &= (\nabla\boldsymbol{u}_f^T, \nabla\boldsymbol{v}_f)_{\Omega_f} - \langle\nabla\boldsymbol{u}_f^T\boldsymbol{\eta}_f, \boldsymbol{v}_f\rangle_\Gamma \\ &= (\nabla\boldsymbol{u}_f^T, \nabla\boldsymbol{v}_f)_{\Omega_f} - \langle\boldsymbol{\eta}_f^T\nabla\boldsymbol{u}_f^T\boldsymbol{\eta}_f, \boldsymbol{v}_f\cdot\boldsymbol{\eta}_f\rangle_\Gamma - \sum_{l=1}^{n-1}\langle\boldsymbol{\tau}_l^T\nabla\boldsymbol{u}_f^T\boldsymbol{\eta}_f, \boldsymbol{v}_f\cdot\boldsymbol{\tau}_l\rangle_\Gamma, \end{aligned}$$

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then

$$-(2\nabla\cdot \boldsymbol{D}\boldsymbol{u}_f, \boldsymbol{v}_f)_{\Omega_f} = 2(\boldsymbol{D}\boldsymbol{u}_f, \boldsymbol{D}\boldsymbol{v}_f)_{\Omega_f} - 2\langle \boldsymbol{\eta}_f^T \boldsymbol{D}\boldsymbol{u}_f \boldsymbol{\eta}_f, \boldsymbol{v}_f\cdot \boldsymbol{\eta}_f\rangle_\Gamma$$
$$-2\sum_{l=1}^{n-1} \langle \boldsymbol{\tau}_l^T \boldsymbol{D}\boldsymbol{u}_f \boldsymbol{\eta}_f, \boldsymbol{v}_f\cdot \boldsymbol{\tau}_l\rangle_\Gamma.$$

For the second term on (4.1) we have

$$(\nabla p_f, \boldsymbol{v}_f)_{\Omega_f} = \langle p_f, \boldsymbol{v}_f\cdot \boldsymbol{\eta}_f\rangle_\Gamma - (p_f, \nabla\cdot \boldsymbol{v}_f)_{\Omega_f}. \tag{4.2}$$

For $\boldsymbol{u}_f, \boldsymbol{v}_f \in \boldsymbol{X}_f$ and $q_f \in M_f$ define

$$a_f(\boldsymbol{u}_f, \boldsymbol{v}_f) := 2\nu(\boldsymbol{D}\boldsymbol{u}_f, \boldsymbol{D}\boldsymbol{v}_f)_{\Omega_f} + \sum_{l=1}^{n-1} \frac{\nu\alpha_f}{\sqrt{\kappa}} \langle \boldsymbol{u}_f\cdot \boldsymbol{\tau}_l, \boldsymbol{v}_f\cdot \boldsymbol{\tau}_l\rangle_\Gamma, \tag{4.3}$$

$$b_f(\boldsymbol{v}_f, q_f) := -(q_f, \nabla\cdot \boldsymbol{v}_f)_{\Omega_f}. \tag{4.4}$$

By replacing (4.2) in (4.1), and using the condition (3.6), we obtain for all $\boldsymbol{v}_f \in \boldsymbol{X}_f$ and $q_f \in M_f$

$$\left\{ \begin{array}{ll} a_f(\boldsymbol{u}_f, \boldsymbol{v}_f) + b_f(\boldsymbol{v}_f, p_f) + \langle p_f - 2\nu \boldsymbol{\eta}_f^T \boldsymbol{D}(\boldsymbol{u}_f)\boldsymbol{\eta}_f, \boldsymbol{v}_f\cdot \boldsymbol{\eta}_f\rangle_\Gamma & = (\boldsymbol{f}_f, \boldsymbol{v}_f)_{\Omega_f} \\ b_f(\boldsymbol{u}_f, q_f) & = 0. \end{array} \right. \tag{4.5}$$

Analogously, defining

$$a_p(\boldsymbol{u}_p, \boldsymbol{v}_p) := (\frac{\nu}{\kappa}\boldsymbol{u}_p, \boldsymbol{v}_p)_{\Omega_p} \quad \text{for all } \boldsymbol{u}_p, \boldsymbol{v}_p \in \boldsymbol{X}_p, \tag{4.6}$$

$$b_p(\boldsymbol{v}_p, q_p) := -(q_p, \nabla\cdot \boldsymbol{v}_p)_{\Omega_p} \text{ for all } \boldsymbol{v}_p \in \boldsymbol{X}_p \text{ and } q_p \in M_p,$$

we have for all $\boldsymbol{v}_p \in \boldsymbol{X}_p$ and $q_p \in M_p$

$$\left\{ \begin{array}{ll} a_p(\boldsymbol{u}_p, \boldsymbol{v}_p) + b_p(\boldsymbol{v}_p, p_p) + \langle p_p, \boldsymbol{v}_p\cdot \boldsymbol{\eta}_p\rangle_\Gamma & = (\boldsymbol{f}_p, \boldsymbol{v}_p)_{\Omega_p} \\ b_p(\boldsymbol{u}_p, q_p) & = 0. \end{array} \right. \tag{4.7}$$

To couple the two subproblems (4.5) and (4.7) we use balance of normal forces (3.5) and a Lagrange multiplier which also approximate the Darcy pressure on the interface Γ. Introduce the Lagrange multiplier,

$$\lambda = p_p = p_f - 2\nu \boldsymbol{\eta}_f^T \boldsymbol{D}(\boldsymbol{u}_f)\boldsymbol{\eta}_f = p_f - 2\nu \boldsymbol{\eta}_f^T \nabla \boldsymbol{u}\boldsymbol{\eta}_f. \tag{4.8}$$

Then we get

$$\left\{ \begin{array}{lll} a_f(\boldsymbol{u}_f, \boldsymbol{v}_f) + b_f(\boldsymbol{v}_f, p_f) + \langle \boldsymbol{v}_f\cdot \boldsymbol{\eta}_f, \lambda\rangle_\Gamma & = (\boldsymbol{f}_f, \boldsymbol{v}_f)_{\Omega_f} & \text{for all } \boldsymbol{v}_f \in \boldsymbol{X}_f \\ a_p(\boldsymbol{u}_p, \boldsymbol{v}_p) + b_p(\boldsymbol{v}_p, p_p) + \langle \boldsymbol{v}_p\cdot \boldsymbol{\eta}_p, \lambda\rangle_\Gamma & = (\boldsymbol{f}_p, \boldsymbol{v}_p)_{\Omega_p} & \text{for all } \boldsymbol{v}_p \in \boldsymbol{X}_p \\ b_f(\boldsymbol{u}_f, q_f) & = 0 & \text{for all } q_f \in M_f \\ b_p(\boldsymbol{u}_p, q_p) & = 0 & \text{for all } q_f \in M_p \\ \langle \boldsymbol{u}_f\cdot \boldsymbol{\eta}_f + \boldsymbol{u}_p\cdot \boldsymbol{\eta}_p, \mu\rangle_\Gamma & = 0 & \text{for all } \mu \in \Lambda, \end{array} \right. \tag{4.9}$$

where the space Λ is defined below.

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Define $a : \boldsymbol{X} \times \boldsymbol{X} \to \mathbb{R}$ and $b : \boldsymbol{X} \times M \to \mathbb{R}$ by:

$$a(\boldsymbol{u}, \boldsymbol{v}) := a_f(\boldsymbol{u}_f, \boldsymbol{v}_f) + a_p(\boldsymbol{u}_p, \boldsymbol{v}_p), \tag{4.10}$$

$$b(\boldsymbol{v}, q) := b_f(\boldsymbol{v}_f, q_f) + b_p(\boldsymbol{v}_p, q_p). \tag{4.11}$$

Using (3.4), we obtain

$$\begin{cases} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) + \langle \boldsymbol{v}_f \cdot \boldsymbol{\eta}_f + \boldsymbol{v}_p \cdot \boldsymbol{\eta}_p, \lambda \rangle_\Gamma & = (\boldsymbol{f}_f, \boldsymbol{v}_f)_{\Omega_f} + (\boldsymbol{f}_p, \boldsymbol{v}_p)_{\Omega_p} \\ b(\boldsymbol{u}, q) & = 0 \\ \langle \boldsymbol{u}_f \cdot \boldsymbol{\eta}_f + \boldsymbol{u}_p \cdot \boldsymbol{\eta}_p, \mu \rangle_\Gamma & = 0. \end{cases} \tag{4.12}$$

Note that if p is a solution of (4.12), then p plus any constant is also a solution of (4.12); this follows directly from applying the divergence theorem on the first equation of (4.12) and using (4.8). In addition, using the the divergence theorem on the second equation of (4.12) and the compatibility condition (3.3) we have that the equation (4.12) is automatically satisfied for constant test functions $q \in M$. Therefore, we can replace the space M in (4.9) by the following subspace of M

$$M^\circ := \left\{ q = (q_f, q_p) \in M \ : \ \int_{\Omega_f} q_f + \int_{\Omega_p} q_p = 0 \right\}. \tag{4.13}$$

We have to choose a suitable function space Λ for λ. Observe that on the porous exterior boundary Γ_p we consider zero flux as boundary condition, i.e., $\boldsymbol{v}_p \cdot \boldsymbol{\eta}_p = 0$ on Γ_p. Recalling Definition 2.3, this means that

$$\langle \boldsymbol{v}_p \cdot \boldsymbol{\eta}_p, E^{1/2}_{00,\Gamma_p} \phi \rangle_{\partial\Omega_p} = 0 \quad \text{for all } \phi \in H^{1/2}_{00}(\Gamma_p),$$

where $E^{1/2}_{00,\Gamma_p}$ denotes the extension by zero on $\Gamma_p^c = \Gamma$. Then, according to Lemma 2.4 and Remark 2.5 we can think of $\boldsymbol{v}_p \cdot \boldsymbol{\eta}_p$ as a distribution in $H^{-1/2}(\Gamma)$, more precisely, we can define $\boldsymbol{v}_p \cdot \boldsymbol{\eta}_p|_\Gamma \in H^{-1/2}(\Gamma)$ as

$$\langle \boldsymbol{v}_p \cdot \boldsymbol{\eta}_p|_\Gamma, \phi \rangle_\Gamma := \langle \boldsymbol{v}_p \cdot \boldsymbol{\eta}_p, E^{1/2}_\Gamma \phi \rangle_{\partial\Omega_p}, \quad \phi \in H^{1/2}(\Gamma), \tag{4.14}$$

where $E^{1/2}_\Gamma$ is the extension operator defined in Lemma 2.1. This is the main mathematical motivation for choosing Λ as $H^{1/2}(\Gamma)$ rather than $H^{1/2}_{00}(\Gamma)$. On the fluid exterior boundary Γ_f we are using Dirichlet boundary condition, i.e., $\boldsymbol{v}_f = \boldsymbol{0}$ on Γ_f. Then $\boldsymbol{v}_f \cdot \boldsymbol{\eta}_f|_\Gamma \in H^{1/2}_{00}(\Gamma)$ relatively to $\partial\Omega_f$. Then $\boldsymbol{v}_f \cdot \boldsymbol{\eta}_f|_\Gamma \in H^{1/2}_{00}(\Gamma)$ relatively to $\partial\Omega_p$. Here we use the fact that $H^{1/2}_{00}(\Gamma)$, which is the trace of $H^1_0(\Omega_f, \Gamma_f)$, is equivalent to the trace of $H^1_0(\Omega_p, \Gamma_p)$ if the shape and measure of Ω_f are of the similar size of those of Ω_p; see [17, 23]. Since $H^{1/2}_{00}(\Gamma) \subset H^{-1/2}(\Gamma)$ we conclude that $\boldsymbol{v}_f \cdot \boldsymbol{\eta}_f|_\Gamma \in H^{-1/2}(\Gamma)$. In what follows we denote $\boldsymbol{v}_p \cdot \boldsymbol{\eta}_p|_\Gamma$ simply by $\boldsymbol{v}_p \cdot \boldsymbol{\eta}_p$ and $\boldsymbol{v}_f \cdot \boldsymbol{\eta}_f|_\Gamma$ by $\boldsymbol{v}_f \cdot \boldsymbol{\eta}_f$.

From the previous discussion we conclude that $\boldsymbol{v}_f \cdot \boldsymbol{\eta}_f + \boldsymbol{v}_p \cdot \boldsymbol{\eta}_p \in H^{-1/2}(\Gamma)$ and so we choose for λ the space

$$\Lambda := H^{1/2}(\Gamma) \quad \text{with} \quad \| \cdot \|^2_\Lambda := \| \cdot \|^2_{H^{1/2}(\Gamma)} = \| \cdot \|^2_{L^2(\Gamma)} + | \cdot |^2_{H^{1/2}(\Gamma)} \tag{4.15}$$

and define $b_\Gamma : \boldsymbol{X} \times \Lambda \to \mathbb{R}$ by

$$b_\Gamma(\boldsymbol{v}, \mu) := \langle \boldsymbol{v}_f \cdot \boldsymbol{\eta}_f, \mu \rangle_\Gamma + \langle \boldsymbol{v}_p \cdot \boldsymbol{\eta}_p, \mu \rangle_\Gamma, \quad \boldsymbol{v} = (\boldsymbol{v}_f, \boldsymbol{v}_p) \in \boldsymbol{X}, \quad \mu \in \Lambda, \tag{4.16}$$

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with the second duality pairing as in (4.14).

From Lemma 2.4, we obtain

LEMMA 4.1. *$b_\Gamma : \boldsymbol{X} \times \Lambda \to \mathbb{R}$ defined in (4.16) and (4.14) is continuous.*

Another reason for choosing $H^{1/2}(\Gamma)$ instead of $H^{1/2}_{00}(\Gamma)$ is because the Lagrange multiplier represents the porous pressure on Γ, see (4.8), and hence there is no physical reason for the pressure p_p to vanish on $\Gamma \cap \partial\Omega$ when flux boundary conditions are imposed on the porous side exterior boundary Γ_p. The space we choose for Λ is richer than $H^{1/2}_{00}(\Gamma)$, therefore the equation

$$b_\Gamma(\boldsymbol{u}, \mu) = 0 \quad \text{for all } \mu \in \Lambda = H^{1/2}(\Gamma)$$

applied to $\boldsymbol{u}$ is a stronger condition than considering μ on the space $H^{1/2}_{00}(\Gamma)$. As a result, better mass conservation near $\Gamma \cap \partial\Omega$ is achieved. On the other hand, choosing $H^{1/2}_{00}(\Gamma)$ as the spaces of Lagrange multipliers associated to the porous pressure would be more appropriate if zero pressure was imposed on $\partial\Omega$; see [11].

4.1.1. First weak formulation. We finally arrive to the weak formulation of the problem: Find $(\boldsymbol{u}^1, p^1, \lambda^1) \in \boldsymbol{X} \times M^\circ \times \Lambda$ such that

$$\left\{ \begin{array}{lll} a(\boldsymbol{u}^1, \boldsymbol{v}) + b(\boldsymbol{v}, p^1) + b_\Gamma(\boldsymbol{v}, \lambda^1) & = \ell(\boldsymbol{v}) & \text{for all } \boldsymbol{v} \in \boldsymbol{X} \\ b(\boldsymbol{u}^1, q) & = 0 & \text{for all } q \in M^\circ \\ b_\Gamma(\boldsymbol{u}^1, \mu) & = 0 & \text{for all } \mu \in \Lambda, \end{array} \right. \tag{4.17}$$

where

$$\ell(\boldsymbol{v}) := (\boldsymbol{f}_f, \boldsymbol{v}_f)_{\Omega_f} + (\boldsymbol{f}_p, \boldsymbol{v}_p)_{\Omega_p} \text{ for all } \boldsymbol{v} \in \boldsymbol{X}.$$

and the bilinear forms a, b and b_Γ are defined in (4.10), (4.11) and, (4.16) and (4.14), respectively.

Next we introduce two other weak formulations and we refer to them as the second and the third weak formulations; see (4.20) and (4.23). The second weak formulation is an intermediate step for deriving the third weak formulation. The third formulation is the most fundamental one among the three formulations and it is where most of the analysis is carried on. Once the inf-sup condition is established for the third weak formulation, the inf-sup for the other two formulations follow straightforwardly; see Remark 4.8. The analysis of the third weak formulation is based on seminorms and on the theoretical tools developed in Section 2. The three weak formulations are all *equivalent* in the following sense (see Remarks 4.2 and 4.3):

1. If we know a solution $(\hat{\boldsymbol{u}}, \hat{p}, \hat{\lambda})$ for one weak formulation, then we can construct a solution for the other two weak formulations. This construction is done by removing or by recovering the mean value of the fluid and porous pressure solutions and the mean value of the Lagrange multiplier solution.
2. All three weak formulations have the same velocity solutions.

The Proposition 5.7 establishes the inf-sup condition for the third weak formulation, therefore, the existence and uniqueness of the solution follow; see Subsection 4.1.3. Hence, existence of a solution for the first and second weak formulations follows from Remarks 4.2 and 4.3. Finally, the Remark 4.8 establishes the inf-sup conditions for the first and second weak formulations and therefore, the uniqueness of their solution.

4.1.2. Second weak formulation. We now introduce an equivalent weak formulation for (4.17) by eliminating the velocities with non-zero mean normal jump across Γ and also

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the Lagrange multipliers that are constants; see Remark 4.2 below. Define

$$\boldsymbol{X}^\circ = \left\{ \boldsymbol{v} = (\boldsymbol{v}_f, \boldsymbol{v}_p) \in \boldsymbol{X} \;:\; b_\Gamma(\boldsymbol{v}, 1) = \int_\Gamma \boldsymbol{v}_f \cdot \boldsymbol{\eta}_f + \boldsymbol{v}_p \cdot \boldsymbol{\eta}_p = 0 \right\} \tag{4.18}$$

and

$$\Lambda^\circ := H^{1/2}(\Gamma) \cap L^2_0(\Gamma) \quad \text{with norm } |\cdot|_{\Lambda^\circ} := |\cdot|_{H^{1/2}(\Gamma)}. \tag{4.19}$$

The second weak formulation is : Find $(\boldsymbol{u}^2, p^2, \lambda^2) \in \boldsymbol{X}^\circ \times M^\circ \times \Lambda^\circ$ such that

$$\left\{ \begin{array}{lll} a(\boldsymbol{u}^2, \boldsymbol{v}) + b(\boldsymbol{v}, p^2) + b_\Gamma(\boldsymbol{v}, \lambda^2) & = \ell(\boldsymbol{v}) & \text{for all } \boldsymbol{v} \in \boldsymbol{X}^\circ \\ b(\boldsymbol{u}^2, q) & = 0 & \text{for all } q \in M^\circ \\ b_\Gamma(\boldsymbol{u}^2, \mu) & = 0 & \text{for all } \mu \in \Lambda^\circ. \end{array} \right. \tag{4.20}$$

REMARK 4.2. It is easy to see that if $(\boldsymbol{u}^1, p^1, \lambda^1) \in \boldsymbol{X} \times M^\circ \times \Lambda$ solves the weak formulation (4.17) then $\boldsymbol{u}^1 \in \boldsymbol{X}^\circ$ and $(\boldsymbol{u}^1, p^1, \lambda^2)$ solves (4.20) with $\lambda^2 = \lambda^1 - \frac{1}{|\Gamma|}\int_\Gamma \lambda^1$. To see the converse, let $(\boldsymbol{u}^2, p^2, \lambda^2) \in \boldsymbol{X}^\circ \times M^\circ \times \Lambda^\circ$ be a solution of (4.20). Construct $\boldsymbol{w} = (0, \boldsymbol{w}_p) \in \boldsymbol{X}$ such that

$$\boldsymbol{w}_p \cdot \boldsymbol{\eta}_p = \frac{1}{|\Gamma|} \text{ on } \Gamma, \quad \boldsymbol{w}_p \cdot \boldsymbol{\eta}_p = 0 \text{ on } \Gamma_p,$$

define

$$\bar\lambda := \ell(\boldsymbol{w}) - a(\boldsymbol{u}^2, \boldsymbol{w}) - b(\boldsymbol{w}, p^2),$$

and set $\lambda^1 := \lambda^2 + \bar\lambda$. Then $(\boldsymbol{u}^2, p^2, \lambda^1)$ solves (4.17). Indeed, observe that $b_\Gamma(\boldsymbol{w}, \lambda^1) = \bar\lambda$ and that for $\boldsymbol{v} = (\boldsymbol{v}_f, \boldsymbol{v}_p) \in \boldsymbol{X}$ we can find α such that $\boldsymbol{v}^2 := \boldsymbol{v} + \alpha \boldsymbol{w} \in \boldsymbol{X}^\circ$. Hence, we obtain

$$\begin{aligned} a(\boldsymbol{u}^2, \boldsymbol{v}) + b(\boldsymbol{v}, p^2) + b_\Gamma(\boldsymbol{v}, \lambda^1) &= \{a(\boldsymbol{u}^2, \boldsymbol{v}^2) + b(\boldsymbol{v}^2, p^2) + b_\Gamma(\boldsymbol{v}^2, \lambda^2)\} \\ &\quad -\alpha\{a(\boldsymbol{u}^2, \boldsymbol{w}) + b(\boldsymbol{w}, p^2) + b_\Gamma(\boldsymbol{w}, \lambda^1)\} \\ &= \ell(\boldsymbol{v}^2) - \alpha\{a(\boldsymbol{u}^2, \boldsymbol{w}) + b(\boldsymbol{w}, p^2) + \bar\lambda\} \\ &= \ell(\boldsymbol{v}^2) - \alpha \ell(\boldsymbol{w}) = \ell(\boldsymbol{v}). \end{aligned}$$

The second and third equations of (4.17) are also easily verified.

4.1.3. Third weak formulation. We can continue with the elimination of piecewise constant pressures on each subdomain together with velocities with non-zero mean normal component on Γ. Define

$$\boldsymbol{X}^{\circ\circ} = \left\{ \boldsymbol{v} = (\boldsymbol{v}_f, \boldsymbol{v}_p) \in \boldsymbol{X}^\circ \;:\; \int_\Gamma \boldsymbol{v}_f \cdot \boldsymbol{\eta}_f = 0 \text{ and } \int_\Gamma \boldsymbol{v}_p \cdot \boldsymbol{\eta}_p = 0 \right\} \tag{4.21}$$

and

$$M^{\circ\circ} := \left\{ q = (q_f, q_p) \in M_f \times M_p \;:\; \int_{\Omega_f} q_f = 0 \text{ and } \int_{\Omega_p} q_p = 0 \right\}, \tag{4.22}$$

and consider the following formulation: Find $(\boldsymbol{u}^3, p^3, \lambda^3) \in \boldsymbol{X}^{\circ\circ} \times M^{\circ\circ} \times \Lambda^\circ$ such that

$$\left\{ \begin{array}{lll} a(\boldsymbol{u}^3, \boldsymbol{v}) + b(\boldsymbol{v}, p^3) + b_\Gamma(\boldsymbol{v}, \lambda^3) & = \ell(\boldsymbol{v}) & \text{for all } \boldsymbol{v} \in \boldsymbol{X}^{\circ\circ} \\ b(\boldsymbol{u}^3, q) & = 0 & \text{for all } q \in M^{\circ\circ} \\ b_\Gamma(\boldsymbol{u}^3, \mu) & = 0 & \text{for all } \mu \in \Lambda^\circ. \end{array} \right. \tag{4.23}$$

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REMARK 4.3. Let $(\boldsymbol{u}^2, p^2, \lambda^2) \in \boldsymbol{X}^\circ \times M^\circ \times \Lambda^\circ$ be a solution of (4.20). We next show that $\boldsymbol{u}^2 \in \boldsymbol{X}^{\circ\circ}$. Consider the following piecewise constant pressure $p^c = (1, -\frac{|\Omega_f|}{|\Omega_p|}) \in M^\circ$. From the second equation in (4.20) we have

$$0 = \int_{\Omega_f} \nabla\cdot \boldsymbol{u}_f^2 - \frac{|\Omega_f|}{|\Omega_p|}\int_{\Omega_p} \nabla\cdot \boldsymbol{u}_p^2 = \int_\Gamma \boldsymbol{u}_f^2\cdot \boldsymbol{\eta}_f - \frac{|\Omega_f|}{|\Omega_p|}\int_\Gamma \boldsymbol{u}_p^2\cdot \boldsymbol{\eta}_p,$$

and since $\boldsymbol{u}^2 \in \boldsymbol{X}^\circ$, i.e.,

$$\int_\Gamma \boldsymbol{u}_f^2\cdot \boldsymbol{\eta}_f + \int_\Gamma \boldsymbol{u}_p^2\cdot \boldsymbol{\eta}_p = 0,$$

we obtain $\int_\Gamma \boldsymbol{u}_f^2\cdot \boldsymbol{\eta}_f = \int_\Gamma \boldsymbol{u}_p^2\cdot \boldsymbol{\eta}_p = 0$, therefore, $\boldsymbol{u}^2 \in \boldsymbol{X}^{\circ\circ}$. Now set

$$p^3 := \left(p_f^2 - \frac{1}{|\Omega_f|}\int_{\Omega_f} p_f^2, p_p^2 - \frac{1}{|\Omega_p|}\int_{\Omega_p} p_p^2 \right) \in M^{\circ\circ}.$$

Then $b(\boldsymbol{v}, p^3) = b(\boldsymbol{v}, p^2)$ for all $\boldsymbol{v} \in \boldsymbol{X}^{\circ\circ}$ and we conclude that $(\boldsymbol{u}^2, p^3, \lambda^2)$ solves (4.23). Now for the converse, suppose $(\boldsymbol{u}^3, p^3, \lambda^3) \in \boldsymbol{X}^{\circ\circ} \times M^{\circ\circ} \times \Lambda^\circ$ solves (4.23). Let $\boldsymbol{z} = (\boldsymbol{z}_f, \boldsymbol{z}_p) \in \boldsymbol{X}^\circ$ be any function such that $\int_\Gamma \boldsymbol{z}_f\cdot \boldsymbol{\eta}_f = -\int_\Gamma \boldsymbol{z}_p\cdot \boldsymbol{\eta}_p = \frac{|\Omega_p|}{|\Omega_f|+|\Omega_p|}$. Then

$$b(\boldsymbol{z}, p^c) = \int_{\Omega_f} \nabla\cdot \boldsymbol{z}_f - \frac{|\Omega_f|}{|\Omega_p|}\int_{\Omega_p} \nabla\cdot \boldsymbol{z}_p = \int_\Gamma \boldsymbol{z}_f\cdot \boldsymbol{\eta}_f - \frac{|\Omega_f|}{|\Omega_p|}\int_\Gamma \boldsymbol{z}_p\cdot \boldsymbol{\eta}_p = 1.$$

Define

$$\gamma := \ell(\boldsymbol{z}) - a(\boldsymbol{u}^3, \boldsymbol{z}) - b(\boldsymbol{z}, p^3) - b_\Gamma(\boldsymbol{z}, \lambda^3)$$

and $p^2 := p^3 + \gamma p^c$ where, as before, $p^c = (1, -\frac{|\Omega_f|}{|\Omega_p|})$. Next we show that $(\boldsymbol{u}^3, p^2, \lambda^3)$ solves (4.20). Indeed, if $(\boldsymbol{v}, q, \mu) \in \boldsymbol{X}^\circ \times M^\circ \times \Lambda^\circ$, we can find ϵ such that $\boldsymbol{v}^3 := \boldsymbol{v} + \epsilon \boldsymbol{z} \in \boldsymbol{X}^{\circ\circ}$. Then we have

$$\begin{aligned} a(\boldsymbol{u}^3, \boldsymbol{v}) + b(\boldsymbol{v}, p^2) + b_\Gamma(\boldsymbol{v}, \lambda^3) &= \{a(\boldsymbol{u}^3, \boldsymbol{v}^3) + b(\boldsymbol{v}^3, p^3) + b_\Gamma(\boldsymbol{v}^3, \lambda^3)\} + \gamma b(\boldsymbol{v}^3, p^c) \\ &\quad -\epsilon\{a(\boldsymbol{u}^3, \boldsymbol{z}) + b(\boldsymbol{z}, p^3) + b_\Gamma(\boldsymbol{z}, \lambda^3) + \gamma b(\boldsymbol{z}, p^c)\} \\ &= \ell(\boldsymbol{v}^3) - \epsilon \ell(\boldsymbol{z}) = \ell(\boldsymbol{v}). \end{aligned}$$

Here we have used the fact that $b(\boldsymbol{v}^3, p^c) = 0$ for all $\boldsymbol{v}^3 \in \boldsymbol{X}^{\circ\circ}$. The second and third equation of (4.20) are also easily verified.

4.2. Inf-sup analysis. In the subsequent sections, we consider only the formulation (4.23), and we abandon the super-index 3 to avoid proliferation of indexes. In particular we establish the inf-sup associated to this formulation, see Proposition 4.7. See also Remark 4.8 for the inf-sup of the first and second weak formulations.

Define

$$\boldsymbol{V} = (\boldsymbol{V}_f, \boldsymbol{V}_p) := \{\boldsymbol{v} \in \boldsymbol{X}^{\circ\circ} \ :\ b_\Gamma(\boldsymbol{v}, \mu) = 0 \text{ for all } \mu \in \Lambda^\circ\}$$

with $\boldsymbol{X}^{\circ\circ}$ and Λ° defined in (4.21) and (4.19), respectively. The space $\boldsymbol{V}$ is closed because the linear map $B_\Gamma : \boldsymbol{X} \to \Lambda'$ defined by $B_\Gamma(\boldsymbol{v})\mu := b_\Gamma(\boldsymbol{v}, \mu)$ is continuous and $\boldsymbol{V} = \text{Ker } B_\Gamma$. It

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is easy to see that for $\boldsymbol{v} \in \boldsymbol{V}$ we have $\boldsymbol{v}_p \cdot \boldsymbol{\eta}_p = \boldsymbol{v}_f \cdot \boldsymbol{\eta}_p \in H_{00}^{1/2}(\Gamma)$. Then we can formulate the problem (4.23) as

$$\left\{ \begin{array}{lll} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) & = \ell(\boldsymbol{v}) & \text{for all } \boldsymbol{v} \in \boldsymbol{V} \\ b(\boldsymbol{u}, q) & = 0 & \text{for all } q \in M^{\circ\circ}, \end{array} \right. \tag{4.24}$$

with $M^{\circ\circ}$ defined in (4.22). Since $\boldsymbol{u}_p \cdot \boldsymbol{\eta}_p = \boldsymbol{u}_f \cdot \boldsymbol{\eta}_f \in H_{00}^{1/2}(\Gamma)$, some regularity results on $\boldsymbol{u}_p$ and p_p can be derived which depends on smoothness and convexity properties of $\partial\Omega_p$. We note however that no regularity is used to establish the continuous and discrete inf-sup conditions. Regularity is assumed only in the Section 6 where a priori error estimates are established.

Now, define

$$\boldsymbol{Z} = (\boldsymbol{Z}_f, \boldsymbol{Z}_p) := \{\boldsymbol{v} \in \boldsymbol{X}^{\circ\circ} \ : \ b(\boldsymbol{v}, q) = 0 \quad \text{for all } \ q \in M^{\circ\circ}\}\,. \tag{4.25}$$

Then we can also formulate problem (4.23) as:

$$\left\{ \begin{array}{lll} a(\boldsymbol{u}, \boldsymbol{v}) + b_\Gamma(\boldsymbol{v}, \lambda) & = \ell(\boldsymbol{v}) & \text{for all } \boldsymbol{v} \in \boldsymbol{Z} \\ b_\Gamma(\boldsymbol{u}, \mu) & = 0 & \text{for all } \mu \in \Lambda^\circ. \end{array} \right. \tag{4.26}$$

REMARK 4.4. The Korn inequality implies that the bilinear form a_f defined in (4.3) is $\boldsymbol{X}_f$-*elliptic*; see [5, 23]. The bilinear for a_p defined in (4.6) is $\boldsymbol{H}(\text{div}^0, \Omega_p)$-*elliptic*, here $\boldsymbol{H}(\text{div}^0, \Omega_p)$ consists of functions in $\boldsymbol{H}(\text{div}, \Omega_\text{p})$ with vanishing divergence, i.e., the kernel of bilinear form b_p. Then the bilinear form "a" defined in (4.10) is $\boldsymbol{X}_f \times \boldsymbol{H}(\text{div}^0, \Omega_p)$-*elliptic*.

Define

$$\boldsymbol{W}_p := \boldsymbol{X}_p \cap H^1(\Omega_p)^2 \quad \text{and } \boldsymbol{W} = (\boldsymbol{X}_f, \boldsymbol{W}_p) \tag{4.27}$$

with

$$\|\boldsymbol{v}\|_{\boldsymbol{W}_p} := \|\boldsymbol{v}_p\|_{H^1(\Omega_p)^2} \quad \text{and} \quad \|\boldsymbol{v}\|^2_{\boldsymbol{W}} := \|\boldsymbol{v}_f\|^2_{\boldsymbol{X}_f} + \|\boldsymbol{v}_p\|^2_{\boldsymbol{W}_p}. \tag{4.28}$$

The use of a subspace $\boldsymbol{W} \cap \boldsymbol{X}^{\circ\circ}$ with a stronger norm $\|\cdot\|_{\boldsymbol{X}} \leq \|\cdot\|_{\boldsymbol{W}}$ is a common strategy in showing continuous and discrete inf-sup conditions without assuming any regularity on the solution of the associated problem [7, 15]; see also Lemmas 4.5 and 5.5 and Proposition 5.3.

From the usual inf-sup condition for the Stokes problem on the whole domain Ω and since $M^{\circ\circ} \subset M^\circ$, we easily derive the inf-sup condition associated to the formulation (4.24).

LEMMA 4.5. *There is a constant $\rho > 0$ such that*

$$\inf_{\substack{q \in M^{\circ\circ} \\ q \neq 0}} \sup_{\substack{\boldsymbol{v} \in \boldsymbol{V} \\ \boldsymbol{v} \neq 0}} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{\boldsymbol{X}} \|q\|_M} \geq \inf_{\substack{q \in M^{\circ\circ} \\ q \neq 0}} \sup_{\substack{\boldsymbol{v} \in \boldsymbol{V} \cap \boldsymbol{W} \\ \boldsymbol{v} \neq 0}} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{\boldsymbol{W}} \|q\|_M} \geq \rho > 0.$$

with $\boldsymbol{W}$ and $\|\cdot\|_{\boldsymbol{W}}$ defined in (4.27) and (4.28), respectively.

Lemma 4.5, Remark 4.4, and the fact that $\big(\text{Ker } b \cap \boldsymbol{V}\big) \subset \big(\boldsymbol{X}_f \times \boldsymbol{H}(\text{div}^0, \Omega_p)\big)$ guarantee stability of the weak formulation (4.24); see [7, 15].

Recall that $\boldsymbol{Z} \subset \boldsymbol{H}(\text{div}^0, \Omega_f) \times \boldsymbol{H}(\text{div}^0, \Omega_p)$; see (4.25). To see that the weak formulation (4.26) is stable, the next lemma shows that the inf-sup condition between spaces $\boldsymbol{Z}$ and Λ° holds; see [7, 15]. The proof presented here follows the same ideas as [20], Lemma 3.4. The main difference is that we are working with the spaces Λ° and $\boldsymbol{Z}$.

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LEMMA 4.6. *There is a constant $\gamma > 0$, such that*

$$\inf_{\substack{\mu\in\Lambda^\circ\\ \mu\neq 0}} \sup_{\substack{\boldsymbol{v}\in\boldsymbol{Z}\\ \boldsymbol{v}\neq 0}} \frac{b_\Gamma(\boldsymbol{v},\mu)}{\|\boldsymbol{v}\|_{\boldsymbol{X}}|\mu|_{\Lambda^\circ}} \geq \gamma > 0.$$

Proof. Fix $\mu \in \Lambda^\circ$, then $\mu \in H^{1/2}(\Gamma)$ and $\int_\Gamma \mu = 0$, in particular if $\mu \neq 0$ then μ is not a constant. From Lemma 2.10 we have that there exists $f_\Gamma \in H^{-1/2}(\Gamma)$ such that $\langle f_\Gamma, 1\rangle_\Gamma = 0$ and

$$\frac{\langle f_\Gamma, \mu\rangle_\Gamma}{|f_\Gamma|_{H^{-1/2}(\Gamma)}} \geq \frac{1}{2}|\mu|_{H^{1/2}(\Gamma)} = \frac{1}{2}|\mu|_{\Lambda^\circ}. \tag{4.29}$$

From Remark 2.6, we introduce $f \in H^{-1/2}(\partial\Omega_p)$ given by

$$\langle f, \phi\rangle_{\partial\Omega_p} := \langle f_\Gamma, \phi|_\Gamma\rangle_\Gamma \quad \text{for all } \phi \in H^{1/2}(\partial\Omega_p) \tag{4.30}$$

with

$$|f|_{H^{-1/2}(\partial\Omega_p)} \leq C_1 |f_\Gamma|_{H^{-1/2}(\Gamma)} \tag{4.31}$$

and zero mean on $\partial\Omega_p$, i.e., $\langle f, 1\rangle_{\partial\Omega_p} = \langle f_\Gamma, 1\rangle_\Gamma = 0$. By using the normal trace theorem, and a continuous Stokes problem (f has zero mean on $\partial\Omega_p$) we can find $\boldsymbol{v}_p \in \boldsymbol{H}(\text{div}, \Omega_p)$ with $\nabla\cdot\boldsymbol{v}_p = 0$ in Ω_p such that

$$\|\boldsymbol{v}_p\|_{\boldsymbol{H}(\text{div},\Omega_p)} \leq C|f|_{H^{-1/2}(\partial\Omega_p)}, \tag{4.32}$$

$$\boldsymbol{v}_p\cdot\boldsymbol{\eta}_p = f \text{ on } \partial\Omega_p. \tag{4.33}$$

Observe that $\boldsymbol{v}_p \in \boldsymbol{X}_p^\circ$. Indeed, if $\phi \in H_{00}^{1/2}(\Gamma_p)$, then

$$\langle \boldsymbol{v}_p\cdot\boldsymbol{\eta}_p, \phi\rangle_{\partial\Omega_p} = \langle f, \phi\rangle_{\partial\Omega_p} = \langle f_\Gamma, \phi|_\Gamma\rangle_\Gamma = \langle f_\Gamma, 0\rangle_\Gamma = 0$$

and $\langle \boldsymbol{v}_p\cdot\boldsymbol{\eta}_p, 1\rangle_{\partial\Omega_p} = \langle f_\Gamma, 1\rangle_\Gamma = 0$.

Choosing $\boldsymbol{v}_f = 0$, we have $\boldsymbol{v} := (\boldsymbol{v}_f, \boldsymbol{v}_p) \in \boldsymbol{Z}$ and:

$$\begin{aligned}
\frac{b_\Gamma(\boldsymbol{v},\mu)}{\|\boldsymbol{v}\|_{\boldsymbol{X}}} &= \frac{0 + \langle \boldsymbol{v}_p\cdot\boldsymbol{\eta}_p, E_\Gamma^{1/2}\mu\rangle_{\partial\Omega_p}}{\|\boldsymbol{v}_p\|_{\boldsymbol{H}(\text{div},\Omega_p)}} && \text{by (4.14)}\\
&\geq \frac{1}{C}\frac{\langle f, E_\Gamma^{1/2}\mu\rangle_{\partial\Omega_p}}{|f|_{H^{-1/2}(\partial\Omega_p)}} && \text{by (4.32) and (4.33)}\\
&= \frac{1}{CC_1}\frac{\langle f_\Gamma, \mu\rangle_\Gamma}{|f_\Gamma|_{H^{-1/2}(\Gamma)}} && \text{by (4.30) and (4.31)}\\
&\geq \frac{1}{CC_1}\frac{1}{2}|\mu|_{H^{1/2}(\Gamma)} && \text{by (4.29).} \quad \square
\end{aligned}$$

For $(q,\mu) \in M^{\circ\circ} \times \Lambda^{\circ}$ define $|(p,\mu)|^2_{M\times\Lambda^{\circ}} := \|p\|^2_M + |\mu|^2_{\Lambda^{\circ}}$. From Lemmas 4.5 and 4.6 we can show

PROPOSITION 4.7. *There is a constant $\beta > 0$ such that:*

$$\inf_{\substack{(q,\mu)\in M^{\circ\circ}\times\Lambda^{\circ}\\(q,\mu)\neq(0,0)}} \sup_{\substack{\boldsymbol{v}\in\boldsymbol{X}^{\circ\circ}\\\boldsymbol{v}\neq 0}} \frac{b(\boldsymbol{v},q)+b_\Gamma(\boldsymbol{v},\mu)}{\|\boldsymbol{v}\|_{\boldsymbol{X}}|(q,\mu)|_{M\times\Lambda^{\circ}}} \geq \beta > 0. \tag{4.34}$$

Proof. Given $(q,\mu) \in M^{\circ\circ} \times \Lambda^{\circ}$, if $q \neq 0$, from Lemma 4.5 there exists $\hat{\boldsymbol{v}} \in \boldsymbol{V}$ such that

$$\frac{b(\hat{\boldsymbol{v}},q)}{\|\hat{\boldsymbol{v}}\|_{\boldsymbol{X}}} \geq \rho\|q\|_M > 0,$$

where ρ independent of q. If $\mu \neq 0$, from Lemma 4.6 there exists $\boldsymbol{z} \in \boldsymbol{Z}$ such that

$$\frac{b_\Gamma(\boldsymbol{z},\mu)}{\|\boldsymbol{z}\|_{\boldsymbol{X}}} \geq \gamma|\mu|_{\Lambda^{\circ}} > 0,$$

where γ independent of μ.

Observe that, if $q \neq 0$

$$\sup_{\substack{\boldsymbol{v}\in\boldsymbol{X}^{\circ\circ}\\\boldsymbol{v}\neq 0}} \frac{b(\boldsymbol{v},q)+b_\Gamma(\boldsymbol{v},\mu)}{\|\boldsymbol{v}\|_{\boldsymbol{X}}} \geq \frac{b(\hat{\boldsymbol{v}},q)+b_\Gamma(\hat{\boldsymbol{v}},\mu)}{\|\hat{\boldsymbol{v}}\|_{\boldsymbol{X}}} = \frac{b(\hat{\boldsymbol{v}},q)+0}{\|\hat{\boldsymbol{v}}\|_{\boldsymbol{X}}} \geq \rho\|q\|_M.$$

Analogously, if $\mu \neq 0$,

$$\sup_{\substack{\boldsymbol{v}\in\boldsymbol{X}^{\circ\circ}\\\boldsymbol{v}\neq 0}} \frac{b(\boldsymbol{v},q)+b_\Gamma(\boldsymbol{v},\mu)}{\|\boldsymbol{v}\|_{\boldsymbol{X}}} \geq \frac{0+b_\Gamma(\boldsymbol{z},\mu)}{\|\boldsymbol{z}\|_{\boldsymbol{X}}} \geq \gamma|\mu|_{\Lambda^{\circ}},$$

then

$$\begin{aligned}\sup_{\substack{\boldsymbol{v}\in\boldsymbol{X}^{\circ\circ}\\\boldsymbol{v}\neq 0}} \frac{b(\boldsymbol{v},q)+b_\Gamma(\boldsymbol{v},\mu)}{\|\boldsymbol{v}\|_{\boldsymbol{X}}} &\geq \frac{\min\{\rho,\gamma\}}{2}\Big(\|q\|_M + |\mu|_{\Lambda^{\circ}}\Big)\\ &\geq \frac{\min\{\rho,\gamma\}}{2}|(q,\mu)|_{M\times\Lambda^{\circ}}. \qquad \square\end{aligned}$$

Proposition 4.7 permits us to formulate problem (4.23) as

$$\left\{\begin{array}{lll} a(\boldsymbol{u},\boldsymbol{v})+c(\boldsymbol{v},(p,\lambda)) & =\ell(\boldsymbol{v}) & \text{for all } \boldsymbol{v}\in\boldsymbol{X}^{\circ\circ}\\ c(\boldsymbol{u},(q,\mu)) & =0 & \text{for all } (q,\mu)\in M^{\circ\circ}\times\Lambda^{\circ}, \end{array}\right. \tag{4.35}$$

where $c(\boldsymbol{v},(q,\mu)) := b(\boldsymbol{v},q) + b_\Gamma(\boldsymbol{v},\mu)$. Then (4.34) in Proposition 4.7 can be written as: there exists $\beta = \frac{\min\{\rho,\gamma\}}{2} > 0$ such that

$$\inf_{\substack{(q,\mu)\in M^{\circ\circ}\times\Lambda^{\circ}\\(q,\mu)\neq 0}} \sup_{\substack{\boldsymbol{v}\in\boldsymbol{X}^{\circ\circ}\\\boldsymbol{v}\neq 0}} \frac{c(\boldsymbol{v},(q,\mu))}{\|\boldsymbol{v}\|_{\boldsymbol{X}}|(q,\mu)|_{M\times\Lambda^{\circ}}} \geq \beta > 0.$$

This inf-sup condition, together with the fact that a is $\boldsymbol{X}_f \times \boldsymbol{H}(\mathrm{div}^0, \Omega_p)$-elliptic and a and c are bounded, (according to the abstract saddle point theory) guarantees the well-posedness of the problem (4.35) or (4.23); see [7, 15].

REMARK 4.8. We now obtain the inf-sup condition for the weak formulation (4.20). Consider $\boldsymbol{z}$ introduced in Remark 4.3. Note that in Remark 4.3 we only have required $\boldsymbol{z} \in \boldsymbol{X}^\circ$ and

$$\int_\Gamma \boldsymbol{z}_f \cdot \boldsymbol{\eta}_f = -\int_\Gamma \boldsymbol{z}_p \cdot \boldsymbol{\eta}_p = \frac{|\Omega_p|}{|\Omega_f| + |\Omega_p|}.$$

Now we also require the divergence of $\boldsymbol{z}$ to be constant on each subdomain and also that $\boldsymbol{z}_f \cdot \boldsymbol{\eta}_f = -\boldsymbol{z}_p \cdot \boldsymbol{\eta}_p$. For instance, we can solve a Stokes problem with constant divergence on the fluid side and a Darcy problem with the corresponding boundary data and constant divergence on the porous side, with divergences values satisfying the subdomain compatibility conditions. Then we have

$$b(\boldsymbol{z}, q^3) = 0 \quad \text{for all } q^3 \in M^{\circ\circ}, \quad \text{and} \quad b_\Gamma(\boldsymbol{z}, \mu^2) = 0 \quad \text{for all } \mu^2 \in \Lambda^\circ. \tag{4.36}$$

We now show show that the inf-sup condition for the weak formulation (4.20) holds. The spaces involved are $\boldsymbol{X}^\circ$ for velocities, and M° and Λ° for pressures and Lagrange multipliers, respectively; see (4.18), (4.13) and (4.19). Take $q^2 \in M^\circ$ and $\mu^2 \in \Lambda^\circ$ and let $p^c = (1, -\frac{|\Omega_f|}{|\Omega_p|}) \in M^\circ$ as in Remark 4.3. We can write $q^2 = q^3 + \overline{q} p^c$ where $q^3 \in M^{\circ\circ}$. Note that

$$\|q^2\|_M \leq \|q^3\|_M + |\overline{q}| \|p^c\|_M.$$

From Proposition 4.7 and a Poincaré inequality, there exists $\boldsymbol{v}^3 \in \boldsymbol{X}^{\circ\circ}$ such that

$$b(\boldsymbol{v}^3, q^3) + b_\Gamma(\boldsymbol{v}^3, \mu^2) \geq \tilde{\beta} \|\boldsymbol{v}^3\|_{\boldsymbol{X}} \left\{ \|q^3\|_M + \|\mu^2\|_{\Lambda^\circ} \right\},$$

where $\tilde{\beta}$ is a positive constant independent of $\boldsymbol{v}^3$. If $\overline{q} \neq 0$, let

$$\boldsymbol{v}^2 = \boldsymbol{v}^3 + \tilde{\beta} \|\boldsymbol{v}^3\|_{\boldsymbol{X}} \|p^c\|_M \frac{\overline{q}}{|\overline{q}|} \boldsymbol{z} = \boldsymbol{v}^3 + r\boldsymbol{z}, \quad \text{with } r = \tilde{\beta} \|\boldsymbol{v}^3\|_{\boldsymbol{X}} \|p^c\|_M \frac{\overline{q}}{|\overline{q}|}.$$

Observe that $\|\boldsymbol{v}^2\|_{\boldsymbol{X}} \leq (1 + \tilde{\beta} \|\boldsymbol{z}\|_{\boldsymbol{X}} \|p^c\|_M) \|\boldsymbol{v}^3\|_{\boldsymbol{X}}$. We have

$$\begin{aligned} b(\boldsymbol{v}^2, q^2) &= \{ b(\boldsymbol{v}^3, q^3) + \overline{q} b(\boldsymbol{v}^3, p^c) \} + r \{ b(\boldsymbol{z}, q^3) + \overline{q} b(\boldsymbol{z}, p^c) \} \\ &= \{ b(\boldsymbol{v}^3, q^3) + 0 \} + r \{ 0 + \overline{q} \} \qquad\qquad \text{(see (4.36))} \\ &= b(\boldsymbol{v}^3, q^3) + \tilde{\beta} |\overline{q}| \|\boldsymbol{v}^3\|_{\boldsymbol{X}} \|p^c\|_M \end{aligned}$$

and

$$b_\Gamma(\boldsymbol{v}^2, \mu^2) = b_\Gamma(\boldsymbol{v}^3, \mu^2) + r b_\Gamma(\boldsymbol{z}, \mu^2) = b(\boldsymbol{v}^3, \mu^2) + 0.$$

Then

$$\begin{aligned} b(\boldsymbol{v}^2, q^2) + b_\Gamma(\boldsymbol{v}^2, \mu^2) &= b(\boldsymbol{v}^3, q^3) + b_\Gamma(\boldsymbol{v}^3, \mu^2) + \tilde{\beta} |\overline{q}| \|\boldsymbol{v}^3\|_{\boldsymbol{X}} \|p^c\|_M \\ &\geq \tilde{\beta} \|\boldsymbol{v}^3\|_{\boldsymbol{X}} \left\{ \|q^3\|_M + \|\mu^2\|_{\Lambda^\circ} \right\} + \tilde{\beta} |\overline{q}| \|\boldsymbol{v}^3\|_{\boldsymbol{X}} \|p^c\|_M \\ &= \tilde{\beta} \|\boldsymbol{v}^3\|_{\boldsymbol{X}} \left\{ \|q^3\|_M + |\overline{q}| \|p^c\|_M + \|\mu^2\|_\Lambda \right\} \\ &\geq \tilde{\beta} \|\boldsymbol{v}^3\|_{\boldsymbol{X}} \left\{ \|q^2\|_M + \|\mu^2\|_{\Lambda^\circ} \right\} \\ &\geq \frac{\tilde{\beta}}{1 + \tilde{\beta} \|\boldsymbol{z}\|_{\boldsymbol{X}} \|p^c\|_M} \|\boldsymbol{v}^2\|_{\boldsymbol{X}} \left\{ \|q^2\|_M + \|\mu^2\|_{\Lambda^\circ} \right\}. \end{aligned}$$

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This gives the inf-sup condition for weak formulation (4.20).

We now obtain the inf-sup condition for the weak formulation (4.17). The spaces are $\boldsymbol{X}$ for velocities, M° for pressures, and Λ defined in (4.15) for Lagrange multipliers. Consider $\boldsymbol{w}$ introduced in Remark 4.2. Note that in Remark 4.2 we required $\boldsymbol{w} = (\boldsymbol{0}, \boldsymbol{w}_p)$ with

$$\boldsymbol{w}_p \cdot \boldsymbol{\eta}_p = \frac{1}{|\Gamma|} \text{ on } \Gamma \text{ and } \boldsymbol{w}_p \cdot \boldsymbol{\eta}_p = 0 \text{ on } \Gamma_p.$$

Now we also require that the divergence of $\boldsymbol{w}$ be a constant on Ω_p. Given $\mu^1 \in \Lambda$ and $q^1 \in M^\circ$, we write $\mu^1 = \mu^2 + \bar{\mu}$ where $\int_\Gamma \mu^2 = 0$, i.e., $\mu^2 \in \Lambda^\circ$. From the inf-sup for weak formulation (4.20) deduced above, we can find $\boldsymbol{v}^2 \in \boldsymbol{X}^\circ$ such that

$$b(\boldsymbol{v}^2, q^1) + b_\Gamma(\boldsymbol{v}^2, \mu^2) \geq \hat{\beta} \|v^2\|_{\boldsymbol{X}} \left\{ \|q^1\|_M + \|\mu^2\|_{\Lambda^\circ} \right\}$$

If $\bar{\mu} \neq 0$, define $\boldsymbol{v}^1 = \boldsymbol{v}^2 + \hat{\beta} \|\boldsymbol{v}^2\|_{\boldsymbol{X}} |\Gamma|^{\frac{1}{2}} \frac{\bar{\mu}}{|\bar{\mu}|} \boldsymbol{w}$. Note that

$$\|\boldsymbol{v}^1\|_{\boldsymbol{X}} \leq (1 + \hat{\beta} \|\boldsymbol{w}\|_{\boldsymbol{X}} |\Gamma|^{\frac{1}{2}}) \|\boldsymbol{v}^2\|_{\boldsymbol{X}} \quad \text{and} \quad \|\mu^1\|_\Lambda \asymp \|\mu^2\|_{\Lambda^\circ} + |\bar{\mu}| |\Gamma|^{\frac{1}{2}}.$$

And we proceed as before to obtain the inf-sup condition for the weak formulation (4.17).

5. Finite element approximation.

In Section 3 the problem for the coupling fluid flow with porous media flow in its continuous form was presented, while in Section 4 it was presented its variational formulation and well-posedness. Now a two dimensional non-matching grid finite element approximation is discussed. We choose the $P2\backslash P1$ triangular Taylor Hood finite elements for approximating the free fluid side velocity and pressure, while we use the lowest order triangular Raviart-Thomas finite element to approximate the filtration velocity and the porous pressure; see Section 5.1 below. In Section 5.2 a discrete non-conforming Lagrange multiplier space to couple the Taylor-Hood and Raviart-Thomas spaces is introduced. It is important for the analysis to choose the Stokes side as the mortar side, i.e., to place the discrete Lagrange multiplier on the Darcy side. In this case the discrete map from mortar to non-mortar side is continuous in $L^2(\Gamma)$ norm. Extensions of the results to other than Stokes and Darcy finite element spaces are straightforward; just take the Lagrange multiplier spaces that are used to hybridize mixed finite elements of the Darcy equation; see [7]. We establish the discrete inf-sup conditions related to the weak formulation (4.24), (4.26) and (4.23). The extension of the results to the three dimensional case is also straightforward.

5.1. Discretization.

From now on we assume that Ω has polygonal boundary. Let $\mathcal{T}_{h_j}$ be a triangulation of Ω_j, $j = f, p$. We do not assume that they match at the polyhedral interface Γ. We choose $P2\backslash P1$ triangular Taylor-Hood finite elements; see [6, 7, 15]. Define

$$\boldsymbol{X}_{h_f} := \left\{ \boldsymbol{v}_f \in \boldsymbol{X}_f \ : \ \boldsymbol{v}_{fK} = \hat{\boldsymbol{v}}_{fK} \circ F_K^{-1} \text{ on } K \text{ and } \hat{\boldsymbol{v}}_K \in P_2(\hat{K})^2 \right\} \cap C^0(\overline{\Omega}_f)^2, \tag{5.1}$$

and

$$\boldsymbol{X}^\circ_{h_f} := \{ \boldsymbol{v}_{h_f} \in \boldsymbol{X}_{h_f} : \int_\Gamma \boldsymbol{v}_{h_f} \cdot \boldsymbol{\eta}_f = 0 \}, \tag{5.2}$$

where $\boldsymbol{v}_{fK} := \boldsymbol{v}_f|_K$. We also define

$$M_{h_f} := \left\{ q_f \in M_f \ : \ q_{fK} = \hat{q}_{fK} \circ F_K^{-1} \text{ on } K \text{ and } \hat{q}_{fK} \in P_1(\hat{K}) \right\} \cap C^0(\overline{\Omega}_f),$$

$$M^\circ_{h_f} := \{ q_f \in M_{h_f} : \int_{\Omega_f} q_f = 0 \}. \tag{5.3}$$

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We have the following result.

LEMMA 5.1 (Approximation of Taylor-Hood elements). *Suppose that $\mathcal{T}_{h_f}$ is non-degenerate and has no triangle with two edges on $\partial\Omega_f$. Then, there exists a bounded linear operator $\boldsymbol{I}_{h_f}^{TH} : \boldsymbol{X}_f \to \boldsymbol{X}_{h_f}$ such that*

$$b_f\left(\boldsymbol{v}_f - \boldsymbol{I}_{h_f}^{TH}\boldsymbol{v}_f, p_{h_f}\right) = 0 \quad \textit{for all } p_{h_f} \in M_{h_f}^{\circ}$$

and $\|\boldsymbol{I}_{h_f}^{TH}\boldsymbol{v}_f\|_{\boldsymbol{X}_f} \preceq \|\boldsymbol{v}_f\|_{\boldsymbol{X}_f}$, with constant independent of h_f. In addition we have:

$$\|\boldsymbol{v}_f - \boldsymbol{I}_{h_f}^{TH}\boldsymbol{v}_f\|_{L^2(\Omega_f)^2} \preceq h_f^s|\boldsymbol{v}_f|_{H^s(\Omega_f)^2} \quad s = 1, 2. \tag{5.4}$$

$$|\boldsymbol{v}_f - \boldsymbol{I}_{h_f}^{TH}\boldsymbol{v}_f|_{H^1(\Omega_f)^2} \preceq h_f|\boldsymbol{v}_f|_{H^2(\Omega_f)^2} \tag{5.5}$$

$$\int_\Gamma \boldsymbol{I}_{h_f}^{TH}\boldsymbol{v}_f \cdot \boldsymbol{\eta}_f = \int_\Gamma \boldsymbol{v}_f \cdot \boldsymbol{\eta}_f \quad (\textit{which implies } \boldsymbol{I}_{h_f}^{TH} : \boldsymbol{X}_f^{\circ} \to \boldsymbol{X}_{h_f}^{\circ})$$

$$|\boldsymbol{I}_{h_f}^{TH}\boldsymbol{v}_f|_{H^{1/2}(\Gamma)^2} \preceq |\boldsymbol{v}_f|_{H^{1/2}(\Gamma)^2}. \tag{5.6}$$

A constructive and apparently new proof using Fortin interpolation is given in Appendix B, or see [6, 7, 15].

A direct consequence of Fortin's criterion and the previous lemma is that, if $\mathcal{T}_{h_f}$ is non-degenerate and has no triangle with two edges on $\partial\Omega_f$, then $(\boldsymbol{X}_{h_f}^{\circ}, M_{h_f}^{\circ})$ satisfies the inf-sup condition; see (5.2) and (5.3).

For the porous region we are going to use the lowest order Raviart-Thomas finite elements based on triangles. In general the Raviart-Thomas elements in a cell are defined by (see [5, 7, 15])

$$RT_k(K) := (P_k(K))^n + P_k(K)\boldsymbol{x},$$

and if $\boldsymbol{v} \in RT_k(K)$ then $\nabla\cdot\boldsymbol{v} \in P_k(K)$ and $\boldsymbol{v}\cdot\boldsymbol{\eta}|_{e_i} \in P_k(e_i)$, for all edge e_i. Then we choose

$$\boldsymbol{X}_{h_p}^{\circ} := \left\{\boldsymbol{v}_p \in \boldsymbol{X}_p \ : \ \boldsymbol{v}_p|_K \in RT_0(K) \text{ and } \int_\Gamma \boldsymbol{v}_p\cdot\boldsymbol{\eta}_p = 0\right\}, \tag{5.7}$$

and

$$M_{h_p}^{\circ} := \left\{p_p \in M_p \ : \ p_p|_K \in P_0(K) \text{ with } \int_{\Omega_p} p_p = 0\right\}. \tag{5.8}$$

Velocities of lowest order Raviart-Thomas finite elements, $RT_0(K)$, $K \in \mathcal{T}_{h_p}$, are then of the form

$$\boldsymbol{v}_p(x_1, x_2) = \begin{pmatrix} a \\ b \end{pmatrix} + c\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

We obtain the following result; see also [5, 7]. Recall the definition of $\boldsymbol{W}_p$ in (4.27).

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LEMMA 5.2 (Approximation of Raviart-Thomas elements). *For $K \in \mathcal{T}_{h_p}$, define $\boldsymbol{I}^{RT}_{h_p,K} : \boldsymbol{H}(\mathrm{div}, K) \cap H^1(K)^2 \to RT_0(K)$ by*

$$\boldsymbol{I}^{RT}_{h_p,K}\boldsymbol{v}_p \cdot \boldsymbol{\eta}_p|_e = \frac{1}{|e|}\int_e \boldsymbol{v}_p \cdot \boldsymbol{\eta}_p \tag{5.9}$$

and define $\boldsymbol{I}^{RT}_{h_p} : \boldsymbol{W}_p \to RT_0$ locally by: $\boldsymbol{I}^{RT}_{h_p}\boldsymbol{v}_p|_K = \boldsymbol{I}^{RT}_{h_p,K}\boldsymbol{v}_p$. Then

$$\int_{\Omega_p} \nabla\cdot(\boldsymbol{v}_p - \boldsymbol{I}^{RT}_{h_p}\boldsymbol{v}_p)q_{h_p} = 0 \quad \textit{for all } q_{h_p} \in M^\circ_{h_p} \tag{5.10}$$

and $\|\boldsymbol{I}^{RT}_{h_p}\boldsymbol{v}_p\|_{\boldsymbol{H}(div,\,\Omega_p)} \preceq \|\boldsymbol{v}_p\|_{\boldsymbol{W}_p}$ with $\|\cdot\|_{\boldsymbol{W}_p}$ defined in (4.28). The property (5.9) implies that $\boldsymbol{I}^{RT}_{h_p} : \boldsymbol{X}^\circ_p \cap \boldsymbol{W}_p \to \boldsymbol{X}^\circ_{h_p}$. In addition, with the property (5.10) we have $\boldsymbol{I}^{RT}_{h_p} : \boldsymbol{Z}_p \cap \boldsymbol{W}_p \to \boldsymbol{Z}^\circ_{h_p}$. Moreover, if $\boldsymbol{v}_p \in H^1(\Omega_p)^2$ then

$$\|\boldsymbol{v}_p - \boldsymbol{I}^{RT}_{h_p}\boldsymbol{v}_p\|_{L^2(\Omega_p)^2} \preceq h_p|\boldsymbol{v}_p|_{H^1(\Omega_p)^2}, \tag{5.11}$$

and

$$\|\nabla\cdot(\boldsymbol{v}_p - \boldsymbol{I}^{RT}_{h_p}\boldsymbol{v}_p)\|_{L^2(\Omega_p)} \preceq h_p|\nabla\cdot\boldsymbol{v}_p|_{H^1(\Omega_p)}.$$

By using Fortin's idea we can establish the inf-sup condition for the spaces $(X^\circ_{h_p}, M^\circ_{h_p})$ defined in (5.7) and (5.8), respectively.

5.2. Discrete inf-sup condition. Let $\Gamma \cap \mathcal{T}_{h_p}$ be the trace on Γ of the porous side triangulation. We consider piecewise constant Lagrange multiplier space

$$\Lambda^\circ_{h_p} = \left\{\mu_{h_p} \in L^2(\Gamma) : \ \mu_{h_p}|_{e_p} \text{is constant on each edge } e_p \in \Gamma \cap \mathcal{T}^{h_p}_p \text{ and } \int_\Gamma \mu = 0\right\}.$$

We note that this choice leads to non-conforming finite elements associated to Λ° since piecewise constant functions do not belong to $H^{1/2}(\Gamma)$; see (4.19).

We also introduce for later use

$$\Lambda_{h_p} = \left\{\mu_{h_p} \in L^2(\Gamma) : \ \mu_{h_p}|_{e_p} \text{is constant on each edge } e_p \in \Gamma \cap \mathcal{T}^{h_p}_p\right\}.$$

Define $h = (h_f, h_p)$,

$$\boldsymbol{X}^{\circ\circ}_h := \boldsymbol{X}^\circ_{h_f} \times \boldsymbol{X}^\circ_{h_p} \subset \boldsymbol{X}^{\circ\circ}, \qquad M^{\circ\circ}_h := M^\circ_{h_f} \times M^\circ_{h_p} \subset M^{\circ\circ} \tag{5.12}$$

and

$$\boldsymbol{V}_h = (\boldsymbol{V}_{h_f}, \boldsymbol{V}_{h_p}) := \left\{\boldsymbol{v}_h \in \boldsymbol{X}^{\circ\circ}_h : \ ([\![\boldsymbol{v}_h]\!]\cdot\boldsymbol{\eta}_f, \mu_{h_p})_\Gamma = 0 \text{ for all } \mu_{h_p} \in \Lambda^\circ_{h_p}\right\}, \tag{5.13}$$

where $[\![\boldsymbol{v}_h]\!] := \boldsymbol{v}_{h_f} - \boldsymbol{v}_{h_p}$ on Γ for all $\boldsymbol{v}_h \in \boldsymbol{X}^\circ_h$. Also define

$$\boldsymbol{Z}_h = (\boldsymbol{Z}_{h_f}, \boldsymbol{Z}_{h_p}) := \{\boldsymbol{v}_h \in \boldsymbol{X}^{\circ\circ}_h : \ b(\boldsymbol{v}_h, q_h) = 0 \text{ for all } q_h \in M^{\circ\circ}_h\}. \tag{5.14}$$

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For $z_{h_p} \in \boldsymbol{X}^{\circ}_{h_p} \cdot \boldsymbol{\eta}_p|_\Gamma = \Lambda^{\circ}_{h_p}$, i.e., z_{h_p} is piecewise constant on Γ relatively to $\mathcal{T}_{h_p}$ and with zero mean on Γ, define $\boldsymbol{E}_{h_p} z_{h_p} \in \boldsymbol{X}^{\circ}_{h_p}$ as the discrete velocity solution of the problem

$$\left\{\begin{array}{lll} a_p(\boldsymbol{E}_{h_p} z_{h_p}, \boldsymbol{v}_{h_p}) + b_p(\boldsymbol{v}_{h_p}, \hat{p}_{h_p}) & = 0 & \text{for all } \boldsymbol{v}_{h_p} \in \boldsymbol{X}^{\circ}_{h_p} \\ & & \text{such that } \boldsymbol{v}_{h_p} \cdot \boldsymbol{\eta}_p = 0 \text{ on } \Gamma_p \\ b_p(\boldsymbol{E}_{h_p} z_{h_p}, q_{h_p}) & = 0 & \text{for all } q_{h_p} \in M^{\circ}_{h_p} \\ \boldsymbol{E}_{h_p} z_{h_p} \cdot \boldsymbol{\eta}_p & = z_{h_p} & \text{on } \Gamma. \end{array}\right. \tag{5.15}$$

We note that a discrete divergence free Raviart-Thomas vector field is also a divergence free vector field. Therefore, using [22] we have

$$\|\boldsymbol{E}_{h_p} z_{h_p}\|^2_{\boldsymbol{L}^2(\Omega)} = \|\boldsymbol{E}_{h_p} z_{h_p}\|_{\boldsymbol{X}_p} \asymp |z_{h_p}|_{H^{-1/2}(\Gamma)}. \tag{5.16}$$

We have the following result.

PROPOSITION 5.3. *Suppose that $\mathcal{T}_{h_f}$ is non-degenerate and has no triangle with two edges on $\partial\Omega_f$ and consider $\boldsymbol{W}$ defined in (4.27). There exists a linear continuous operator*

$$\boldsymbol{\Pi}_h : (\boldsymbol{V} \cap \boldsymbol{W}) \to \boldsymbol{V}_h$$

such that

$$b(\boldsymbol{\Pi}_h \boldsymbol{v} - \boldsymbol{v}, q_h) = 0 \textit{ for all } q_h \in M^{\circ\circ}_h \tag{5.17}$$

and

$$\|\boldsymbol{\Pi}_h \boldsymbol{v}\|_{\boldsymbol{X}} \preceq \|\boldsymbol{v}_p\|_{\boldsymbol{W}_p} \leq \|\boldsymbol{v}\|_{\boldsymbol{W}}. \tag{5.18}$$

with $\|\cdot\|_{\boldsymbol{W}}$ defined in (4.28).

Proof. Write $\boldsymbol{\Pi}_h(\boldsymbol{v}) = (\boldsymbol{\Pi}_{h_f}\boldsymbol{v}, \boldsymbol{\Pi}_{h_p}\boldsymbol{v})$, where $\boldsymbol{\Pi}_{h_f}\boldsymbol{v} := \boldsymbol{I}^{TH}_{h_f}\boldsymbol{v}_f$ and

$$\boldsymbol{\Pi}_{h_p}\boldsymbol{v} := \boldsymbol{I}^{RT}_{h_p}\boldsymbol{v}_p + \boldsymbol{E}_{h_p}\left(Q_{h_p}(\boldsymbol{I}^{TH}_{h_f}\boldsymbol{v}_f \cdot \boldsymbol{\eta}_p) - \boldsymbol{I}^{RT}_{h_p}\boldsymbol{v}_p \cdot \boldsymbol{\eta}_p\right),$$

where Q_{h_p} denotes the L^2-projection on Λ^{h_p}, i.e., on the space of piecewise constant functions on Γ.

Let $\mu_{h_p} \in \Lambda^{\circ}_{h_p}$. We have

$$\begin{aligned} ([\![\boldsymbol{\Pi}_h \boldsymbol{v}]\!] \cdot \boldsymbol{\eta}, \mu_{h_p})_\Gamma &= (\boldsymbol{\Pi}_{h_p}\boldsymbol{v} \cdot \boldsymbol{\eta}_p, \mu_{h_p})_\Gamma - (\boldsymbol{\Pi}_{h_f}\boldsymbol{v} \cdot \boldsymbol{\eta}_p, \mu_{h_p})_\Gamma \\ &= (Q_{h_p}(\boldsymbol{I}^{TH}_{h_f}\boldsymbol{v}_f \cdot \boldsymbol{\eta}_p), \mu_{h_p})_\Gamma - (\boldsymbol{I}^{TH}_{h_f}\boldsymbol{v}_f \cdot \boldsymbol{\eta}_p, \mu_{h_p})_\Gamma \\ &= 0 \qquad \text{by definition of } Q_{h_p}, \end{aligned}$$

and then obtain $\boldsymbol{\Pi}_h \boldsymbol{v} \in \boldsymbol{V}_h$.

Now we show (5.18). Observe that

$$\begin{aligned} \|\boldsymbol{\Pi}_h \boldsymbol{v}\|_{\boldsymbol{X}} &\leq \|\boldsymbol{\Pi}_{h_f}\boldsymbol{v}\|_{\boldsymbol{X}_f} + \|\boldsymbol{\Pi}_{h_p}\boldsymbol{v}\|_{\boldsymbol{X}_p} \\ &\leq \|\boldsymbol{I}^{TH}_{h_f}\boldsymbol{v}_f\|_{\boldsymbol{X}_f} + \|\boldsymbol{I}^{RT}_{h_p}\boldsymbol{v}_p\|_{\boldsymbol{X}_p} \\ &\quad + \|\boldsymbol{E}_{h_p}\left(Q_{h_p}(\boldsymbol{I}^{TH}_{h_f}\boldsymbol{v}_f \cdot \boldsymbol{\eta}_p) - \boldsymbol{I}^{RT}_{h_p}\boldsymbol{v}_p \cdot \boldsymbol{\eta}_p\right)\|_{\boldsymbol{X}_p}. \end{aligned}$$

The bound (5.18) follows from the boundedness of $\boldsymbol{I}^{TH}_{h_f}$ (Lemma 5.1), $\boldsymbol{I}^{RT}_{h_p}$ (Lemma 5.2), $\boldsymbol{E}_{h_p}$ (Equation (5.16)), and from the following two bounds:

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1. From the boundedness of $\boldsymbol{I}_{h_f}^{TH}$ and Q_{h_p}, and from a trace theorem, we have

$$\begin{aligned} |Q_{h_p}(\boldsymbol{I}_{h_f}^{TH}\boldsymbol{v}_f{\cdot}\boldsymbol{\eta}_p)|_{H^{-1/2}(\Gamma)} &\preceq \|Q_{h_p}(\boldsymbol{I}_{h_f}^{TH}\boldsymbol{v}_f{\cdot}\boldsymbol{\eta}_p)\|_{L^2(\Gamma)} \\ &\leq \|\boldsymbol{I}_{h_f}^{TH}\boldsymbol{v}_f{\cdot}\boldsymbol{\eta}_p\|_{L^2(\Gamma)} \\ &\preceq |\boldsymbol{v}_f{\cdot}\boldsymbol{\eta}_p|_{H^{1/2}(\Gamma)} \\ &= |\boldsymbol{v}_p{\cdot}\boldsymbol{\eta}_p|_{H^{1/2}(\Gamma)} \\ &\preceq \|\boldsymbol{v}_p\|_{\boldsymbol{W}_p} \\ &\leq \|\boldsymbol{v}\|_{\boldsymbol{W}}. \end{aligned}$$

2. From the normal trace theorem and the boundedness of $\boldsymbol{I}_{h_p}^{RT}$, we have

$$|\boldsymbol{I}_{h_p}^{RT}\boldsymbol{v}_p{\cdot}\boldsymbol{\eta}_p|_{H^{-1/2}(\Gamma)} \preceq \|\boldsymbol{I}_{h_p}^{RT}\boldsymbol{v}_p\|_{\boldsymbol{X}_p} \preceq \|\boldsymbol{v}_p\|_{\boldsymbol{W}_p} \leq \|\boldsymbol{v}\|_{\boldsymbol{W}}. \qquad \square$$

REMARK 5.4. We note that when the mesh $\mathcal{T}_{h_f}(\Omega_f)$ restricted to Γ is a refinement of the mesh $\mathcal{T}_{h_p}(\Omega_p)$ restricted to Γ, then by using (B.8) in Appendix B we have $Q_{h_p}(\boldsymbol{I}_{h_f}^{TH}\boldsymbol{v}_f{\cdot}\boldsymbol{\eta}_p) = Q_{h_p}\boldsymbol{v}_f{\cdot}\boldsymbol{\eta}_p$. Also from (5.9) we have $\boldsymbol{I}_{h_p}^{RT}\boldsymbol{v}_p{\cdot}\boldsymbol{\eta}_p = Q_{h_p}\boldsymbol{v}_p{\cdot}\boldsymbol{\eta}_p$. Hence using that $\boldsymbol{v}_p{\cdot}\boldsymbol{\eta}_p = \boldsymbol{v}_f{\cdot}\boldsymbol{\eta}_f \in H_{00}^{1/2}(\Gamma)$ we obtain

$$\boldsymbol{E}_{h_p}\left(Q_{h_p}(\boldsymbol{v}_f{\cdot}\boldsymbol{\eta}_p) - \boldsymbol{I}_{h_p}^{RT}\boldsymbol{v}_p{\cdot}\boldsymbol{\eta}_p\right) = 0.$$

In the following result we establish the discrete inf-sup condition using Fortin's Lemma.

LEMMA 5.5. *Suppose that $\mathcal{T}_{h_f}$ is non-degenerate and has no triangle with two edges on $\partial\Omega_f$. Consider $\boldsymbol{V}$ and $M_h^{\circ\circ}$ defined in (5.13) and (5.12), respectively. Then $(\boldsymbol{V}_h, M_h^{\circ\circ})$ satisfies the discrete* inf-sup *condition, i.e., there is a constant $\tilde{\rho} > 0$ independent of h, such that*

$$\inf_{\substack{q_h\in M_h^{\circ\circ}\\ q_h\neq 0}} \sup_{\substack{\boldsymbol{v}_h\in\boldsymbol{V}_h\\ \boldsymbol{v}_h\neq 0}} \frac{b(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{\boldsymbol{X}}\|q_h\|_M} \geq \tilde{\rho} > 0.$$

Proof. Take $q_h \in M_h^{\circ\circ}$. From Lemma 4.5 we can find $\boldsymbol{v} \neq 0 \in \boldsymbol{V}\cap\boldsymbol{W}$ such that

$$\frac{b(\boldsymbol{v}, q_h)}{\|\boldsymbol{v}\|_{\boldsymbol{W}}} \geq \rho\|q_h\|_M.$$

Then from Proposition 5.3 we have

$$\rho\|q_h\|_M \leq \frac{b(\boldsymbol{v}, q_h)}{\|\boldsymbol{v}\|_{\boldsymbol{W}}} = \frac{b(\boldsymbol{\Pi}_h\boldsymbol{v}, q_h)}{\|\boldsymbol{v}\|_{\boldsymbol{W}}} \leq \frac{b(\boldsymbol{\Pi}_h\boldsymbol{v}, q_h)}{\frac{1}{C}\|\boldsymbol{\Pi}_h\boldsymbol{v}\|_{\boldsymbol{X}}},$$

where C is the constant in (5.18). $\square$

For $\mu_{h_p} \in \Lambda_{h_p}^{\circ}$, define $\widetilde{\boldsymbol{u}}_{h_p} = \widetilde{\boldsymbol{u}}_{h_p}(\mu_{h_p}) \in \boldsymbol{X}_{h_p}^{\circ}$ as velocity solution of the discrete problem,

$$(5.19)\quad \left\{\begin{array}{lll} a_p(\widetilde{\boldsymbol{u}}_{h_p}, \boldsymbol{v}_{h_p}) + b_p(\boldsymbol{v}_{h_p}, p_{h_p}) & = -(\boldsymbol{v}_{h_p}{\cdot}\boldsymbol{\eta}_p, \mu_{h_p})_\Gamma & \text{for all } \boldsymbol{v}_{h_p} \in \boldsymbol{X}_{h_p}^{\circ} \\ b_p(\widetilde{\boldsymbol{u}}_{h_p}, q_{h_p}) & = 0 & \text{for all } q_{h_p} \in M_{h_p}^{\circ} \end{array}\right.$$

and introduce

$$|\mu_{h_p}|^2_{\Lambda^\circ_{h_p}} := a_p(\widetilde{\boldsymbol{u}}_{h_p}(\mu_{h_p}), \widetilde{\boldsymbol{u}}_{h_p}(\mu_{h_p})). \tag{5.20}$$

In order to see that $|\cdot|^\circ_{\Lambda_{h_p}}$ is a norm on $\Lambda^\circ_{h_p}$, observe that if μ_{h_p} is such that $|\mu_{h_p}|_{\Lambda_{h_p}} = 0$, then $\widetilde{\boldsymbol{u}}_{h_p}(\mu_{h_p})$ vanishes. If we take $\boldsymbol{v}_{h_p}$ in (5.19) such that

$$\begin{cases} \boldsymbol{v}_{h_p} \cdot \boldsymbol{\eta}_p & = \mu_{h_p} \\ b_p(\boldsymbol{v}_{h_p}, q_{h_p}) & = 0 \qquad q_{h_p} \in M^\circ_{h_p}, \end{cases}$$

we see that $\|\mu_{h_p}\|_{L^2(\Gamma)} = 0$, that is $\mu_{h_p} = 0$. Then $|\cdot|_{\Lambda_{h_p}}$ is positive.

The norm $\Lambda^\circ_{h_p}$ is the natural discrete version of the norm $H^{1/2}(\Gamma)$ scaled by the factor $\sqrt{\frac{\kappa}{\nu}}$ for the space $\Lambda^\circ_{h_p}$. Indeed, by using (5.15) and (5.19), we have

$$\sup_{z_{h_p} \in \boldsymbol{X}^\circ_{h_p} \cdot \boldsymbol{\eta}_p|_\Gamma = \Lambda^\circ_{h_p}} \frac{(z_{h_p}, \mu_{h_p})}{\sqrt{\frac{\nu}{\kappa}} |z_{h_p}|_{H^{-1/2}(\Gamma)}} \asymp \sup_{z_{h_p} \in \Lambda^\circ_{h_p}} \frac{(\boldsymbol{E}_{h_p} z_{h_p} \cdot \boldsymbol{\eta}_p, \mu_{h_p})}{\sqrt{\frac{\nu}{\kappa}} \|\boldsymbol{E}_{h_p} z_{h_p}\|_{\boldsymbol{L}^2(\Omega)}} = |\mu_{h_p}|_{\Lambda^\circ_{h_p}}. \tag{5.21}$$

We obtain the following result.

LEMMA 5.6. *The spaces* $(\boldsymbol{Z}_h, \Lambda^\circ_{h_p})$ *satisfy the discrete* inf-sup *condition, i.e., there is a constant* $\tilde{\gamma} > 0$ *such that*

$$\inf_{\substack{\mu_{h_p} \in \Lambda^\circ_{h_p} \\ \lambda_{h_p} \neq 0}} \sup_{\substack{\boldsymbol{v}_h \in \boldsymbol{Z}_h \\ \boldsymbol{v}_h \neq 0}} \frac{([\![\boldsymbol{v}_h]\!] \cdot \boldsymbol{\eta}_f, \mu_{h_p})_\Gamma}{\|\boldsymbol{v}_h\|_{\boldsymbol{X}} |\mu_{h_p}|_{\Lambda^\circ_{h_p}}} \geq \tilde{\gamma} > 0.$$

Proof. Take $\mu_{h_p} \in \Lambda^\circ_{h_p}$ and let $\widetilde{\boldsymbol{u}}_{h_p}(\mu_{h_p})$ be the velocity solution of (5.19). Since $\widetilde{\boldsymbol{u}}_{h_p}(\mu_{h_p}) \in \boldsymbol{Z}_{h_p}$ then $\nabla \cdot \widetilde{\boldsymbol{u}}_{h_p} = 0$. Take $\boldsymbol{v}_h = (\boldsymbol{0}, \widetilde{\boldsymbol{u}}_{h_p}(\mu_{h_p})) \in \boldsymbol{Z}_{h_f} \times \boldsymbol{Z}_{h_p}$, then from (5.19) we obtain

$$\frac{([\![\boldsymbol{v}_h]\!] \cdot \boldsymbol{\eta}_f, \mu_{h_p})_\Gamma}{\|\boldsymbol{v}_h\|_{\boldsymbol{X}} |\mu_{h_p}|_{\Lambda^\circ_{h_p}}} = \frac{a_p(\widetilde{\boldsymbol{u}}_{h_p}(\mu_{h_p}), \widetilde{\boldsymbol{u}}_{h_p}(\mu_{h_p}))}{\|\widetilde{\boldsymbol{u}}_{h_p}(\mu_{h_p})\|_{L^2(\Omega_p)} |\mu_{h_p}|_{\Lambda^\circ_{h_p}}} = \sqrt{\frac{\nu}{\kappa}} > 0. \qquad \square$$

For $(q_h, \mu_{h_p}) \in M^{\circ\circ}_h \times \Lambda^\circ_{h_p}$ define $|(q_h, \mu_{h_p})|^2_{M \times \Lambda^\circ_{h_p}} := \|q_h\|^2_M + |\mu_{h_p}|^2_{\Lambda^\circ_{h_p}}$. Then using the same argument of Proposition 4.7 we have

PROPOSITION 5.7. *Under assumptions of Lemmas 5.5 and 5.6 we have that there exists* $\tilde{\beta} > 0$ *such that*

$$\inf_{\substack{(q_h, \mu_{h_p}) \in M^{\circ\circ}_h \times \Lambda^\circ_{h_p} \\ (q_h, \mu_{h_p}) \neq (0,0)}} \sup_{\substack{\boldsymbol{v}_h \in \boldsymbol{X}^{\circ\circ}_h \\ \boldsymbol{v}_h \neq 0}} \frac{b(\boldsymbol{v}_h, q_h) + ([\![\boldsymbol{v}_h]\!] \cdot \boldsymbol{\eta}_f, \mu_{h_p})_\Gamma}{\|\boldsymbol{v}\|_{\boldsymbol{X}} \|(q_h, \mu_{h_p})\|_{M \times \Lambda^\circ_{h_p}}} \geq \tilde{\beta} > 0. \tag{5.22}$$

REMARK 5.8. With the inf-sup condition (5.22) of Proposition 5.7 we can establish the inf-sup conditions corresponding to the discrete versions of the first and the second weak formulations in (4.17) and (4.20), respectively. Here we consider $\|\mu_{h_p}\|^2_{\Lambda_{h_p}} := \|\mu_{h_p}\|^2_{L^2(\Gamma)} + |\mu_{h_p}|^2_{\Lambda^\circ_{h_p}}$. This is done using similar arguments to those given in Section 4.2; see Remark 4.8.

6. Error analysis. We remark that the constants involved in the notation $\preceq$ are all independent, not only of the mesh size but also independent of the parameters ν and κ. In addition, using scaling arguments, it is easy to see that $\frac{1}{\sqrt{\nu}}p_f, \sqrt{\nu}\boldsymbol{u}_f, \sqrt{\frac{\kappa}{\nu}}p_p$ and $\sqrt{\frac{\nu}{\kappa}}\boldsymbol{u}_p$ are all $O(1)$, therefore, we keep those factors on the a priori error estimates.

We introduce the following energy norms,

$$|\boldsymbol{v}_f|_{a_f}^2 := a_f(\boldsymbol{v}_f, \boldsymbol{v}_f), \tag{6.1}$$

$$\|\boldsymbol{v}_p\|_{a_p}^2 := a_p(\boldsymbol{v}_p, \boldsymbol{v}_p), \tag{6.2}$$

and

$$\|\boldsymbol{v}\|_a^2 := a(\boldsymbol{v}, \boldsymbol{v}). \tag{6.3}$$

We next establish a priori error estimates for the Stokes and Darcy velocities.

PROPOSITION 6.1. *Suppose that $\mathcal{T}_{h_f}$ is non-degenerate and has no triangle with two edges on $\partial\Omega_f$. Let $h := \max\{h_f, h_p\}$. Then we have the following estimate*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_a \preceq h\left(\sqrt{\nu}|\boldsymbol{u}_f|_{H^2(\Omega_f)^2} + \sqrt{\frac{\nu}{\kappa}}|\boldsymbol{u}_p|_{H^1(\Omega_p)^2}\right) + h_p\frac{1}{\sqrt{\nu}}|p_f|_{H^1(\Omega_f)}.$$

Moreover, if the refinement condition of Remark 5.4 is satisfied then

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_a \preceq h_f\sqrt{\nu}|\boldsymbol{u}_f|_{H^2(\Omega_f)^2} + h_p\sqrt{\frac{\nu}{\kappa}}|\boldsymbol{u}_p|_{H^1(\Omega_p)^2}.$$

Proof. From Proposition 5.7 we have that $\boldsymbol{Z}_h \cap \boldsymbol{V}_h$ is not empty, where $\boldsymbol{Z}_h$ and $\boldsymbol{V}_h$ are defined in (5.14) and (5.13), respectively. Then, the discrete problem associated with (4.24) can also be described as: find $\boldsymbol{u}_h \in \boldsymbol{Z}_h \cap \boldsymbol{V}_h$ such that

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) = \ell(\boldsymbol{v}_h) \quad \boldsymbol{v}_h \in \boldsymbol{Z}_h \cap \boldsymbol{V}_h,$$

where a is *elliptic* in $\boldsymbol{Z}_h \cap \boldsymbol{V}_h$. Furthermore, $\boldsymbol{u}_h$ is also the only velocity solution of

$$\left\{\begin{array}{lll} a(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h) + (\llbracket \boldsymbol{v}_h \rrbracket \cdot \boldsymbol{\eta}_f, \lambda_{h_p}) & = \ell(\boldsymbol{v}_h) & \text{for all } \boldsymbol{v}_h \in \boldsymbol{X}_h^{\circ\circ} \\ b(\boldsymbol{u}_h, q_h) & = 0 & \text{for all } q_h \in M_h^{\circ\circ} \\ (\llbracket \boldsymbol{u}_h \rrbracket \cdot \boldsymbol{\eta}_f, \mu_{h_p}) & = 0 & \text{for all } \mu_{h_p} \in \Lambda_{h_p}^{\circ}. \end{array}\right. \tag{6.4}$$

For any $\boldsymbol{w}_h \in \boldsymbol{Z}_h \cap \boldsymbol{V}_h$ we have that $\boldsymbol{v}_h := \boldsymbol{u}_h - \boldsymbol{w}_h \in \boldsymbol{Z}_h \cap \boldsymbol{V}_h$ and

$$a(\boldsymbol{v}_h, \boldsymbol{v}_h) = a(\boldsymbol{u}_h, \boldsymbol{v}_h) - a(\boldsymbol{w}_h, \boldsymbol{v}_h) = \ell(\boldsymbol{v}_h) - a(\boldsymbol{w}_h, \boldsymbol{v}_h). \tag{6.5}$$

Let $(\boldsymbol{u}, p, \lambda)$ be the solution of the continuous problem (4.9). Then

$$\ell(\boldsymbol{v}_h) = a(\boldsymbol{u}, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p) + b_\Gamma(\boldsymbol{v}_h, \lambda)$$

and using (6.5) it follows that

$$a(\boldsymbol{v}_h, \boldsymbol{v}_h) = a(\boldsymbol{u} - \boldsymbol{w}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p) + b_\Gamma(\boldsymbol{v}_h, \lambda),$$

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and

$$\|\boldsymbol{u}_h - \boldsymbol{w}_h\|_a = \|\boldsymbol{v}_h\|_a \leq \|\boldsymbol{u} - \boldsymbol{w}_h\|_a + \sup_{\boldsymbol{z}_h \in \boldsymbol{Z}_h \cap \boldsymbol{V}_h} \frac{b(\boldsymbol{z}_h, p)}{\|\boldsymbol{z}_h\|_a} + \sup_{\boldsymbol{z}_h \in \boldsymbol{Z}_h \cap \boldsymbol{V}_h} \frac{b_\Gamma(\boldsymbol{z}_h, \lambda)}{\|\boldsymbol{z}_h\|_a}.$$

Hence, using

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_a \leq \|\boldsymbol{u} - \boldsymbol{w}_h\|_a + \|\boldsymbol{u}_h - \boldsymbol{w}_h\|_a,$$

we obtain

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_a \leq 2 \inf_{\boldsymbol{w}_h \in \boldsymbol{Z}_h \cap \boldsymbol{V}_h} \|\boldsymbol{u} - \boldsymbol{w}_h\|_a + \sup_{\boldsymbol{z}_h \in \boldsymbol{Z}_h \cap \boldsymbol{V}_h} \frac{b(\boldsymbol{z}_h, p)}{\|\boldsymbol{z}_h\|_a} + \sup_{\boldsymbol{z}_h \in \boldsymbol{Z}_h \cap \boldsymbol{V}_h} \frac{b_\Gamma(\boldsymbol{z}_h, \lambda)}{\|\boldsymbol{z}_h\|_a}. \tag{6.6}$$

To bound the first term on the right-hand size of (6.6) we let $\boldsymbol{w}_h = \boldsymbol{\Pi}_h \boldsymbol{u}$, where $\boldsymbol{\Pi}_h$ is defined in Proposition 5.3. Proposition 5.3 guarantees that $\boldsymbol{w}_h \in \boldsymbol{V}_h$. In addition, since $b(\boldsymbol{u}, q_h) = 0$ for all $q_h \in M_h^{\circ\circ}$, (5.17) guarantees that $\boldsymbol{w}_h = \boldsymbol{\Pi}_h \boldsymbol{u} \in \boldsymbol{Z}_h$ and we have

$$\begin{aligned}
\|\boldsymbol{u} - \boldsymbol{\Pi}_h \boldsymbol{u}\|_a &\leq \|\boldsymbol{u}_f - \boldsymbol{\Pi}_{h_f} \boldsymbol{u}\|_{a_f} + \|\boldsymbol{u}_p - \boldsymbol{\Pi}_{h_p} \boldsymbol{u}\|_{a_p} \\
&\leq \|\boldsymbol{u}_f - \boldsymbol{I}_{h_f}^{TH} \boldsymbol{u}_f\|_{a_f} + \|\boldsymbol{u}_p - \boldsymbol{I}_{h_p}^{RT} \boldsymbol{u}_p\|_{a_p} \\
&\quad + \|\boldsymbol{E}_{h_p} \left(Q_{h_p}(\boldsymbol{I}_{h_f}^{TH} \boldsymbol{u}_f \cdot \boldsymbol{\eta}_p) - \boldsymbol{I}_{h_p}^{RT} \boldsymbol{u}_p \cdot \boldsymbol{\eta}_p \right) \|_{a_p}.
\end{aligned}$$

From (5.5) in Lemma 5.1 we obtain

$$\|\boldsymbol{u}_f - \boldsymbol{I}_{h_f}^{TH} \boldsymbol{u}_f\|_{a_f} \preceq h_f \sqrt{\nu} |\boldsymbol{u}_f|_{H^2(\Omega_f)^2}$$

and from (5.11) in Lemma 5.2 we obtain

$$\|\boldsymbol{u}_p - \boldsymbol{I}_{h_p}^{RT} \boldsymbol{u}_p\|_{a_p} \preceq h_p \sqrt{\frac{\nu}{\kappa}} |\boldsymbol{u}_p|_{H^1(\Omega_p)^2}$$

since $\nabla \cdot \boldsymbol{u}_p = 0$.

From the boundedness of $\boldsymbol{E}_{h_p}$ in (5.16), we have

$$\|\boldsymbol{E}_{h_p} \left(Q_{h_p}(\boldsymbol{I}_{h_f}^{TH} \boldsymbol{u}_f \cdot \boldsymbol{\eta}_p) - \boldsymbol{I}_{h_p}^{RT} \boldsymbol{u}_p \cdot \boldsymbol{\eta}_p \right) \|_{a_p} \preceq \sqrt{\frac{\nu}{\kappa}} |Q_{h_p}(\boldsymbol{I}_{h_f}^{TH} \boldsymbol{u}_f \cdot \boldsymbol{\eta}_p) - \boldsymbol{I}_{h_p}^{RT} \boldsymbol{u}_p \cdot \boldsymbol{\eta}_p|_{H^{-1/2}(\Gamma)}.$$

Therefore, we need to estimate the following three terms:

$$\begin{aligned}
|Q_{h_p}(\boldsymbol{I}_{h_f}^{TH} \boldsymbol{u}_f \cdot \boldsymbol{\eta}_p) - \boldsymbol{I}_{h_p}^{RT} \boldsymbol{u}_p \cdot \boldsymbol{\eta}_p|_{H^{-1/2}(\Gamma)} &\leq |Q_{h_p}(\boldsymbol{I}_{h_f}^{TH} \boldsymbol{v}_f \cdot \boldsymbol{\eta}_p) - \boldsymbol{I}_{h_f}^{TH} \boldsymbol{u}_f \cdot \boldsymbol{\eta}_p|_{H^{-1/2}(\Gamma)} \\
&\quad + |\boldsymbol{I}_{h_f}^{TH} \boldsymbol{u}_f \cdot \boldsymbol{\eta}_p - \boldsymbol{u}_f \cdot \boldsymbol{\eta}_p|_{H^{-1/2}(\Gamma)} + \|\boldsymbol{u}_f \cdot \boldsymbol{\eta}_p - \boldsymbol{I}_{h_p}^{RT} \boldsymbol{u}_p \cdot \boldsymbol{\eta}_p\|_{H^{-1/2}(\Gamma)}
\end{aligned}$$

1. Approximation property (6.7), boundedness of $\boldsymbol{I}_{h_f}^{TH}$ in (5.6) and the trace theorem give

$$\begin{aligned}
|Q_{h_p}(\boldsymbol{I}_{h_f}^{TH} \boldsymbol{u}_f \cdot \boldsymbol{\eta}_p) - \boldsymbol{I}_{h_f}^{TH} \boldsymbol{u}_f \cdot \boldsymbol{\eta}_p|_{H^{-1/2}(\Gamma)} &\preceq h_p |\boldsymbol{I}_{h_f}^{TH} \boldsymbol{u}_f \cdot \boldsymbol{\eta}_p|_{H^{1/2}(\Gamma)} \\
&\preceq h_p |\boldsymbol{u}_f \cdot \boldsymbol{\eta}_p|_{H^{1/2}(\Gamma)} \\
&= h_p |\boldsymbol{u}_p \cdot \boldsymbol{\eta}_p|_{H^{1/2}(\Gamma)} \\
&\leq h_p |\boldsymbol{u}_p|_{H^{1/2}(\Gamma)^2} \\
&\preceq h_p |\boldsymbol{u}_p|_{H^1(\Omega_p)^2}.
\end{aligned}$$

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2. The trace theorem and approximation properties of $\boldsymbol{I}_{h_f}^{TH}$ (Lemma 5.1) give

$$\begin{aligned}|\boldsymbol{I}_{h_f}^{TH}\boldsymbol{u}_f\cdot\boldsymbol{\eta}_p - \boldsymbol{u}_f\cdot\boldsymbol{\eta}_p|_{H^{-1/2}(\Gamma)} &\preceq h_f\|\boldsymbol{u}_f\cdot\boldsymbol{\eta}_p\|_{H^{1/2}(\Gamma)}\\ &= h_f\|\boldsymbol{u}_p\cdot\boldsymbol{\eta}_p\|_{H^{1/2}(\Gamma)}\\ &\preceq h_f|\boldsymbol{u}_p|_{H^1(\Omega_p)^2}.\end{aligned}$$

3. The normal trace theorem and the approximation property (5.11) of $\boldsymbol{I}_{h_p}^{RT}$ imply

$$\begin{aligned}|\boldsymbol{u}_p\cdot\boldsymbol{\eta}_p - \boldsymbol{I}_{h_p}^{RT}\boldsymbol{u}_p\cdot\boldsymbol{\eta}_p|_{H^{-1/2}(\Gamma)} &\preceq h_p|\boldsymbol{u}_p\cdot\boldsymbol{\eta}_p|_{H^{1/2}(\Gamma)}\\ &\preceq h_p|\boldsymbol{u}_p|_{H^1(\Omega_p)^2}.\end{aligned}$$

We note that we have used

$$|Q_{h_p}\mu - \mu|_{H^{-1/2}(\Gamma)} \preceq h_p|\mu|_{H^{1/2}(\Gamma)}, \tag{6.7}$$

since by using local arguments we have $\|Q_{h_p}\mu-\mu\|_{L^2(\Gamma)} \preceq h_p^{1/2}|\mu|_{H^{1/2}(\Gamma)}$ and then

$$\begin{aligned}|Q_{h_p}\mu - \mu|_{H^{-1/2}(\Gamma)} &= \sup_{\phi\in H^{1/2}(\Gamma)}\frac{\langle Q_{h_p}\mu-\mu,\phi\rangle_\Gamma}{|\phi|_{H^{1/2}(\Gamma)}}\\ &\le \sup_{\phi\in H^{1/2}(\Gamma)}\frac{\|Q_{h_p}\mu-\mu\|_{L^2(\Gamma)}\|Q_{h_p}\phi-\phi\|_{L^2(\Gamma)}}{|\phi|_{H^{1/2}(\Gamma)}}\\ &\preceq h_p|\mu|_{H^{1/2}(\Gamma)}.\end{aligned}$$

We now bound the second term on the right-hand size of (6.6). Note that since we are using lowest order Raviart-Thomas elements, the porous side components of $\boldsymbol{Z}_h$ defined in (5.14) are divergence free, i.e., $\boldsymbol{Z}_{h_p}\subset\boldsymbol{Z}_p$, where $\boldsymbol{Z}_p$ is defined in (4.25). Therefore, $b_p(\boldsymbol{z_h},q)=0$ for all $q=(q_f,q_p)\in M^{\circ\circ}$. In addition, we have $b(\boldsymbol{z_h},p-q_h)=0$ for $\boldsymbol{z}_h\in\boldsymbol{Z}_h\cap\boldsymbol{V}_h$. In summary, we have

$$|b(\boldsymbol{z}_h,p)| = |b_f(\boldsymbol{z}_{h_f},p_f)| = |b_f(\boldsymbol{z}_{h_f},p_f-Q_fp_f)| \preceq h_f\frac{1}{\sqrt{\nu}}|p_f|_{H^1(\Omega_f)}\|\boldsymbol{z}_h\|_a,$$

where we have used the first order approximation of the L_2-projection operator Q_f on the fluid pressure space $M_{h_f}^{\circ}$.

To bound the third term on the right-hand size of (6.6) we have

$$\begin{aligned}b_\Gamma(\boldsymbol{z}_h,\lambda) &= \langle\lambda,\boldsymbol{z}_{h_f}\cdot\boldsymbol{\eta}_f\rangle_\Gamma + \langle\lambda,\boldsymbol{z}_{h_p}\cdot\boldsymbol{\eta}_p\rangle_\Gamma\\ &= \langle\lambda,\boldsymbol{z}_{h_f}\cdot\boldsymbol{\eta}_f\rangle_\Gamma + \langle Q_{h_p}\lambda,\boldsymbol{z}_{h_p}\cdot\boldsymbol{\eta}_p\rangle_\Gamma \qquad \boldsymbol{z}_{h_p}\cdot\boldsymbol{\eta}_p \text{ is constant in } e\\ &= \langle\lambda-Q_{h_p}\lambda,\boldsymbol{z}_{h_f}\cdot\boldsymbol{\eta}_f\rangle_\Gamma \qquad \boldsymbol{z}_{h_p}\in\boldsymbol{Z}_h,\end{aligned}$$

hence,

$$|b_\Gamma(\boldsymbol{z}_h,\lambda)| \preceq h_p\frac{1}{\sqrt{\nu}}|\lambda|_{H^{1/2}(\Gamma)}\sqrt{\nu}|\boldsymbol{z}_{h_f}\cdot\boldsymbol{\eta}_f|_{H^{1/2}(\Gamma)}. \tag{6.8}$$

By using (4.8) on Γ (on the Ω_f side) and trace theorems, we obtain

$$|b_\Gamma(\boldsymbol{z}_h,\lambda)| \preceq h_p\left(\frac{1}{\sqrt{\nu}}|p_f|_{H^1(\Omega_f)}+\sqrt{\nu}|\boldsymbol{u}_f|_{H^2(\Omega_f)^2}\right)\|\boldsymbol{z}_h\|_a, \tag{6.9}$$

and the proposition follows. □

REMARK 6.2. We note that we could have used the porous media side in (4.8) to bound $|\lambda|_{H^{1/2}(\Gamma)}$ in (6.8). In this case, we would have obtained

$$|b_\Gamma(\boldsymbol{z}_h,\lambda)| \preceq \frac{h_p}{\sqrt{\nu}}|p_p|_{H^1(\Omega_p)}\|\boldsymbol{z}_h\|_a. \tag{6.10}$$

Even thought we obtain the term h_p multiplying p_p in (6.10), the bound (6.9) is qualitatively better than the bound (6.10). Note that by using scaling arguments we have $\sqrt{\frac{\kappa}{\nu}}p_p = O(1)$. Therefore, the factor $\frac{h_p}{\sqrt{\nu}}|p_p|_{H^1(\Omega_p)}$ is very pessimistic due to the fact that in practice the value of κ is very small.

We next establish a priori error estimates for the Stokes and Darcy pressures.

PROPOSITION 6.3. *Suppose that $\mathcal{T}_{h_f}$ is non-degenerate and has no triangle with two edges on $\partial\Omega_f$. Let $h := \max\{h_f, h_p\}$. Then we have the following estimate,*

$$\begin{aligned}
&\frac{1}{\sqrt{\nu}}\|p_f - p_{h_f}\|_{L^2(\Omega_f)} + \sqrt{\frac{\kappa}{\nu}}\|p_p - p_{h_p}\|_{L^2(\Omega_p)} \\
&\qquad \preceq h\left(\sqrt{\nu}|\boldsymbol{u}_f|_{H^2(\Omega_f)^2} + \sqrt{\frac{\nu}{\kappa}}|\boldsymbol{u}_p|_{H^1(\Omega_p)^2} + \frac{1}{\sqrt{\nu}}|p_f|_{H^1(\Omega_f)}\right) \\
&\qquad + h_p\sqrt{\frac{\kappa}{\mu}}|p_p|_{H^1(\Omega_p)}.
\end{aligned}$$

Moreover, if the refinement condition of Remark 5.4 is satisfied then

$$\begin{aligned}
&\frac{1}{\sqrt{\nu}}\|p_f - p_{h_f}\|_{L^2(\Omega_f)} + \sqrt{\frac{\kappa}{\nu}}\|p_p - p_{h_p}\|_{L^2(\Omega_p)} \\
&\qquad \preceq h_f\left(\sqrt{\nu}|\boldsymbol{u}_f|_{H^2(\Omega_f)^2} + \frac{1}{\sqrt{\nu}}|p_f|_{H^1(\Omega_f)}\right) \\
&\qquad + h_p\left(\sqrt{\frac{\kappa}{\mu}}|p_p|_{H^1(\Omega_p)} + \sqrt{\frac{\nu}{\kappa}}|\boldsymbol{u}_p|_{H^1(\Omega_p)^2}\right).
\end{aligned}$$

Proof. To obtain an expression for the pressure error, observe that for all $\boldsymbol{v}_h \in \boldsymbol{V}_h \cap (H_0^1(\Omega_f) \times \boldsymbol{H}_0(\mathrm{div},\Omega_p))$ (i.e., $\boldsymbol{v}_{h_f} = 0$ on $\partial\Omega_f$ and $\boldsymbol{v}_{h_p}\cdot\boldsymbol{\eta}_p = 0$ on $\partial\Omega_p$) and all $q_h \in M_h^{\circ\circ}$

$$b(\boldsymbol{v}_h, p_h - q_h) = a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p - q_h). \tag{6.11}$$

This holds true in particular for $\boldsymbol{v}_h = (\boldsymbol{v}_{h_f}, 0)$ and $q_h = (q_{h_f}, 0)$. If we take $q_{h_f} = Q_f p_f$, i.e., the L_2-projection on the discrete fluid pressure space, we obtain

$$b_f(\boldsymbol{v}_{h_f}, p_{h_f} - Q_f p_f) = a_f(\boldsymbol{u}_f - \boldsymbol{u}_{h_f}, \boldsymbol{v}_{h_f}) + b_f(\boldsymbol{v}_{h_f}, p_f - Q_f p_f).$$

Then, using the standard discrete inf-sup condition for the fluid problem, we have

$$\begin{aligned}
\frac{1}{\sqrt{\nu}}\|p_{h_f} - Q_f p_f\|_{L^2(\Omega_f)} &\preceq \sup_{\boldsymbol{v}_{h_f}\in\boldsymbol{V}_{h_f}\cap H_0^1(\Omega_f)} \frac{a_f(\boldsymbol{u}_f - \boldsymbol{u}_{h_f}, \boldsymbol{v}_{h_f}) + b_f(\boldsymbol{v}_{h_f}, p_f - Q_f p_f)}{\|\boldsymbol{v}_{h_f}\|_{a_f}} \\
&\preceq \|\boldsymbol{u}_f - \boldsymbol{u}_{h_f}\|_{a_f} + \frac{1}{\sqrt{\nu}}\|p_f - Q_f p_f\|_{L^2(\Omega_f)} \\
&\preceq \|\boldsymbol{u}_f - \boldsymbol{u}_{h_f}\|_{a_f} + h_f\frac{1}{\sqrt{\nu}}|p_f|_{H^1(\Omega_f)},
\end{aligned}$$

and from a triangle inequality we obtain

$$\frac{1}{\sqrt{\nu}}\|p_f - p_{h_f}\|_{L^2(\Omega_f)} \preceq \|\boldsymbol{u}_f - \boldsymbol{u}_{h_f}\|_{a_f} + h_f \frac{2}{\sqrt{\nu}}|p_f|_{H^1(\Omega_f)}.$$

Analogously we obtain

$$\sqrt{\frac{\kappa}{\mu}}\|p_p - p_{h_p}\|_{L^2(\Omega_p)} \preceq \|\boldsymbol{u_p} - \boldsymbol{u}_{h_p}\|_{p_f} + 2h_p\sqrt{\frac{\kappa}{\mu}}|p_p|_{H^1(\Omega_p)}.$$

The proposition follows from the bound on velocity error given on Proposition 6.1. □

Now we analyze a priori error estimate for λ in the discrete norm $|\cdot|^{\circ}_{\Lambda_{h_p}}$ defined in (5.20); see also [1]. Note that the norm $\Lambda^{\circ}_{h_p}$ was defined for piecewise constant functions on the Γ_{h_p} triangulation. For functions $\mu \in L^2(\Gamma)$, we define

$$|\mu|_{\Lambda^{\circ}_{h_p}} := |Q_{h_p}\mu|_{\Lambda^{\circ}_{h_p}}, \tag{6.12}$$

where Q_{h_p} is the L^2-projection onto $\Lambda^{\circ}_{h_p}$. We have the following result.

PROPOSITION 6.4. *Suppose that $\mathcal{T}_{h_f}$ is non-degenerate and has no triangle with two edges on $\partial\Omega_f$. Let $h := \max\{h_f, h_p\}$. Then we have the following estimates:*

$$|\lambda - \lambda_{h_p}|_{\Lambda^{\circ}_{h_p}} \preceq h\left(\sqrt{\nu}|\boldsymbol{u}_f|_{H^2(\Omega_f)^2} + \sqrt{\frac{\nu}{\kappa}}|\boldsymbol{u}_p|_{H^1(\Omega_p)^2}\right) + h_p\frac{1}{\sqrt{\nu}}|p_f|_{H^1(\Omega_f)}, \tag{6.13}$$

and

$$\sqrt{\frac{\kappa}{\nu}}|\lambda - \lambda_{h_p}|_{H^{-1/2}(\Gamma)} \preceq h_p\sqrt{\frac{\kappa}{\nu}}|p_p|_{H^1(\Omega_p)} + |\lambda - \lambda_{h_p}|_{\Lambda^{\circ}_{h_p}}. \tag{6.14}$$

Moreover, if the refinement condition of Remark 5.4 is satisfied then

$$|\lambda - \lambda_{h_p}|_{\Lambda^{\circ}_{h_p}} \preceq h_f\sqrt{\nu}|\boldsymbol{u}_f|_{H^2(\Omega_f)^2} + h_p\sqrt{\frac{\nu}{\kappa}}|\boldsymbol{u}_p|_{H^1(\Omega_p)^2}.$$

Proof. Let $\widetilde{\boldsymbol{u}}_{h_p}(Q_{h_p}\lambda)$ and $\widetilde{p}_{h_p}(Q_{h_p}\lambda)$ be the solution of (5.19). Note that the solution of (6.4) satisfies $\boldsymbol{u}_{h_p} = \widetilde{\boldsymbol{u}}_{h_p}(\lambda_{h_p})$ and $p_{h_p} = \widetilde{p}_{h_p}(\lambda_{h_p})$. Then, using the definition of the discrete norm $\Lambda^{\circ}_{h_p}$ we have

$$|\lambda - \lambda_{h_p}|_{\Lambda^{\circ}_{h_p}} = \|\widetilde{\boldsymbol{u}}_{h_p}(Q_{h_p}\lambda) - \boldsymbol{u}_{h_p}\|_{a_p},$$

which can be bounded by

$$\|\widetilde{\boldsymbol{u}}_{h_p}(Q_{h_p}\lambda) - \boldsymbol{u}_{h_p}\|_{a_p} \le \|\widetilde{\boldsymbol{u}}_{h_p}(Q_{h_p}\lambda) - \boldsymbol{u}_p\|_{a_p} + \|\boldsymbol{u}_p - \boldsymbol{u}_{h_p}\|_{a_p}. \tag{6.15}$$

We use Proposition 6.1 to estimate the second term on the right-hand side of (6.15). We next estimate the first term of the right-hand side of (6.15). Note that

$$a_p(\widetilde{\boldsymbol{u}}_{h_p}(Q_{h_p}\lambda) - \boldsymbol{u}_p, \boldsymbol{v}_{h_p}) + b_p(\boldsymbol{v}_{h_p}, \widetilde{p}_{h_p}(Q_{h_p}\lambda) - p_p) = 0. \tag{6.16}$$

Inserting $\boldsymbol{v}_{h_p} = \widetilde{\boldsymbol{u}}_{h_p}(Q_{h_p}\lambda) - \boldsymbol{u}_{h_p} \in \boldsymbol{Z}_{h_p}$ into (6.16) and recalling that $\boldsymbol{Z}_{h_p} \subset \boldsymbol{Z}_p$ where $\boldsymbol{Z}_{h_p}$ and $\boldsymbol{Z}_p$ are defined in (4.25) and (5.14), respectively, we have

$$a_p(\widetilde{\boldsymbol{u}}_{h_p}(Q_{h_p}\lambda) - \boldsymbol{u}_p, \widetilde{\boldsymbol{u}}_{h_p}(Q_{h_p}\lambda) - \boldsymbol{u}_{h_p}) = 0.$$

Hence,

$$a_p(\widetilde{\boldsymbol{u}}_{h_p}(Q_{h_p}\lambda) - \boldsymbol{u}_p, \widetilde{\boldsymbol{u}}_{h_p}(Q_{h_p}\lambda) - \boldsymbol{u}_p) + a_p(\widetilde{\boldsymbol{u}}_{h_p}(Q_{h_p}\lambda) - \boldsymbol{u}_p, \boldsymbol{u}_p - \boldsymbol{u}_{h_p}) = 0,$$

and by using a Cauchy-Schwarz inequality we obtain

$$\|\widetilde{\boldsymbol{u}}_{h_p}(Q_{h_p}\lambda) - \boldsymbol{u}_p\|_{a_p} \leq \|\boldsymbol{u}_{h_p} - \boldsymbol{u}_p\|_{a_p}$$

and (6.13) follows. To obtain the estimate (6.14), we note that from (5.21) we have

$$\sqrt{\frac{\kappa}{\mu}}\|Q_{h_p}\lambda - \lambda_{h_p}\|_{L^2(\Gamma)} = \sup_{z_{h_p}\in\Lambda^{\circ}_{h_p}} \frac{(z_{h_p}, Q_{h_p}\lambda - \lambda_{h_p})}{\sqrt{\frac{\nu}{\kappa}}|z_{h_p}|_{L^2(\Gamma)}} \preceq |\lambda - \lambda_{h_p}|_{\Lambda^{\circ}_{h_p}},$$

therefore,

$$|\lambda - \lambda_{h_p}|_{H^{-1/2}(\Gamma)} \preceq |\lambda - Q_{h_p}\lambda|_{H^{-1/2}(\Gamma)} + \|Q_{h_p}\lambda - \lambda_{h_p}\|_{L^2(\Gamma)}, \tag{6.17}$$

and (6.14) follows from (6.17) and (6.7). □

REMARK 6.5. Note that we are discretizing the third weak formulation (4.23). We have to recover the piecewise constant pressure in each subdomain. Recall the function $\boldsymbol{z}$ of Remark 4.3. Note that we can compute $\boldsymbol{z}_h := \boldsymbol{\Pi}_h(\boldsymbol{z}) = (\boldsymbol{\Pi}_{h_f}\boldsymbol{z}, \boldsymbol{\Pi}_{h_p}\boldsymbol{z})$; see Proposition 5.3. Then

$$\gamma_h := \ell(\boldsymbol{z}_h) - a(\boldsymbol{u}_h, \boldsymbol{z}_h) - b(\boldsymbol{z}_h, p_h) - ([\![\boldsymbol{z}_h]\!]\cdot\boldsymbol{\eta}_f, \lambda_{h_p})_\Gamma,$$

and $\gamma_h\, p^c = \gamma_h(p_f^c, p_p^c)$ is the approximation for piecewise constant pressure in each subdomain Ω_j, $j = f, p$. Observe that

$$|\gamma - \gamma_h| \preceq |a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{z}_h)| + |b(\boldsymbol{z}_h, \boldsymbol{p} - \boldsymbol{p}_h)| + |([\![\boldsymbol{z}_h]\!]\cdot\boldsymbol{\eta}_f, \gamma - \gamma_{h_p})_\Gamma|.$$

These last terms can be estimated using the results of this section. Analogously we can recover the mean value $\bar{\lambda}$ of the Lagrange multiplier. Indeed, we can find

$$\boldsymbol{w}_h = (0, \boldsymbol{w}_{h_p}) \in \boldsymbol{X}_f \times \boldsymbol{X}_p$$

such that

$$\boldsymbol{w}_{h_p}\cdot\boldsymbol{\eta}_p = \frac{1}{|\Gamma|} \text{ on } \Gamma \text{ and } \boldsymbol{w}_{h_p}\cdot\boldsymbol{\eta}_p = 0 \text{ on } \Gamma_p,$$

and so we can define (see Remark 4.2)

$$\bar{\lambda}_h := \ell(\boldsymbol{w}) - a(\boldsymbol{u}_h, \boldsymbol{w}) - b(\boldsymbol{w}, p_h).$$

In this case

$$|\bar{\lambda} - \bar{\lambda}_h| \preceq |a(\boldsymbol{u} - \boldsymbol{u}_h, w)| + |b(\boldsymbol{w}, p - p_h)|.$$

The last two terms can be estimated using the results of this section.

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7. Numerical results. In this section we present numerical experiments in order to verify the estimates established in the paper. We consider $\Omega_f = (1,2) \times (0,1)$ and $\Omega_p = (0,1) \times (0,1)$. We consider $\alpha_f = 0$. The velocity solution for Stokes is given by $\boldsymbol{u}_f(x,y) = (y(1-y), -x+2+2(x-1)y)$ with pressure $p_f(x,y) = -2x - \frac{\nu}{\kappa} y + 5/2 + \frac{5\nu}{12\kappa}$. Note that $\boldsymbol{u}_f$ is not divergence free. The velocity solution for Darcy is $\boldsymbol{u}_p(x,y) = (1-2x+x^2+y-y^2, -1+x+2y-2xy)$ with pressure $p_p(x,y) = \frac{\nu}{\kappa}((1-x)y(1-y)-x+x^2-\frac{x^3}{3}+\frac{3}{4}-y)+\frac{1}{2}$. Note that the normal component of $\boldsymbol{u}_p$ has a parabolic profile on the interface $\Gamma = 1 \times (0,1)$ while its tangential component is zero. Note also that $\boldsymbol{D}\boldsymbol{u}_f = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}$ on Γ, and $p_f = p_p$ on Γ. The exact solution is compatible with (3.5) with (3.6) when $\alpha_f = 0$. A similar example is presented in [8], where the term $\nabla \cdot \boldsymbol{D}\boldsymbol{u}_f$ is replaced by $\Delta \boldsymbol{u}_f$ in the Stokes equations.

In Figure 7.1 we show the computed solution of the coupled problem. On the porous side we have plotted the velocity in the center of each triangle. In Figure 7.2 we zoom part of the interface and plot the y component of the velocities. In Figure 7.3 we show the behavior of the error (in the scaled norms defined in (6.1), (6.2) and (6.3)) with respect to the discretization parameters. Here we also show $\|\lambda - \lambda_{h_p}\|_{\Lambda^{h_p}}$, i.e., the Lagrange multiplier approximation error in the discrete norm defined in (6.12). We observe according to Figure 7.3, the error in the norm $\|\cdot\|_a^2$ defined in (6.3), which is the sum of the fluid velocity and porous velocity errors in the scaled norms, is of linear order. This agree with Proposition 6.1. Analogously, the pressure error is of linear order. This also agree with the result about the pressure error, see Proposition 6.3. We finally observe that the Lagrange multiplier error in the discrete norm defined in (6.12) is also of linear order.

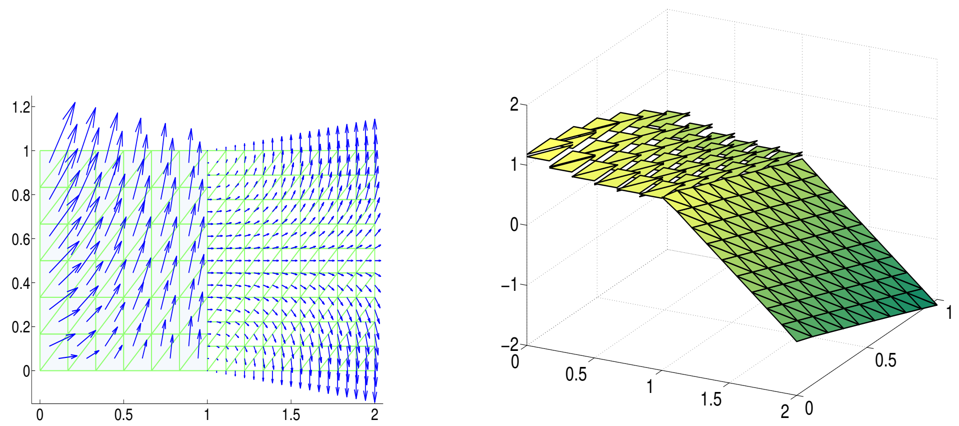

FIG. 7.1. *Computed velocities (left figure) and pressures (right figure). On the porous side (left subdomain) we have plotted the value of the velocity at the centroid of each triangle.*

8. Conclusion. We studied the coupling across an interface of fluid and porous media flows, consisting of *Stokes equations* in the fluid region Ω_f and *Darcy law* for the filtration velocity in the porous medium region Ω_p. After discussing the adequate choice of $H^{1/2}(\Gamma)$, rather than $H^{1/2}_{00}(\Gamma)$, as the Lagrange multiplier space, we presented a complete analysis for the inf-sup and approximation results associated with the continuous and discrete formulations of this Stokes-Darcy system. We chose the triangular $P2\backslash P1$ Taylor Hood finite elements and the lower order Raviart-Thomas elements as discrete spaces for the free and porous medium subdomains, respectively. Optimal a priori discrete error estimates do not

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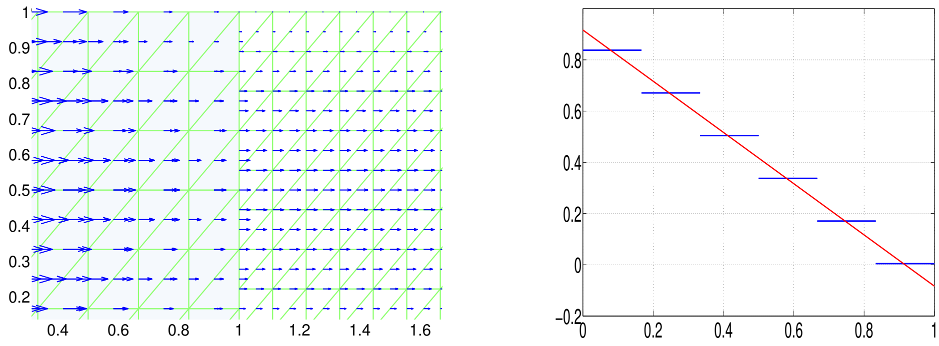

FIG. 7.2. *The x-component of the discrete velocity (left figure), where on the porous side (left subdomain) we plot the two values of the x-component of the velocities at the midpoint of each edge; recall that Raviart-Thomas elements allow discontinuous tangential velocities on interior edges. The discrete (in blue) and the exact (in red) Lagrange multipliers on the interface (right figure).*

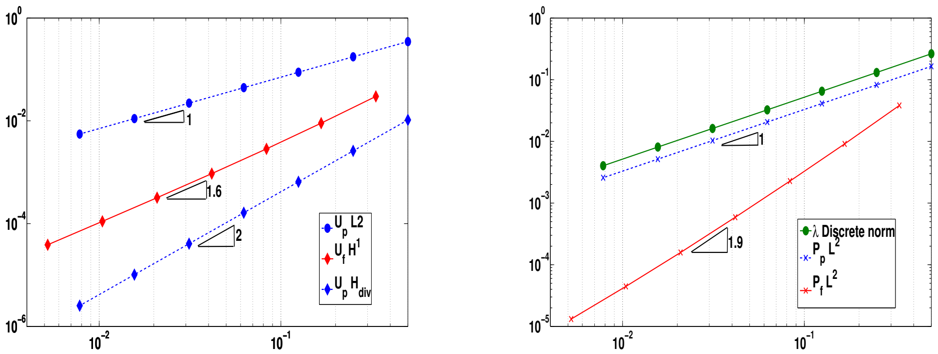

FIG. 7.3. *Velocities errors (left) and pressures errors (right).*

depend on the coefficients ν and κ and ratio of mesh parameters. Sharper local estimates can also be obtained for the case where the fluid mesh on the interface Γ is a refinement of the porous mesh on Γ. The numerical experiments show good agreements with our theoretical results.

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Appendix A. Non-homogeneous boundary conditions. The non-homogeneous boundary condition can be reduced to the homogeneous case when $\boldsymbol{h}_f \in H^{1/2}(\Gamma_f)^2$ and

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$h_p \in H^{-1/2}(\Gamma_p)$. First construct $\boldsymbol{\omega}_f \in H^1(\Omega_f)^2$ such that

$$\text{(A.1)} \qquad \begin{cases} -\nabla\cdot T(\boldsymbol{w}_f, \tilde{p}_f) & = 0 \quad \text{in } \Omega_f \\ \nabla\cdot \boldsymbol{w}_f & = g_f \quad \text{in } \Omega_f \\ \boldsymbol{w}_f & = \boldsymbol{h}_f \quad \text{on } \Gamma_f \\ T(\boldsymbol{w}_f, \tilde{p}_f)\cdot \boldsymbol{\eta}_f & = 0 \quad \text{on } \Gamma. \end{cases}$$

From the divergence theorem

$$\text{(A.2)} \qquad \int_{\Gamma_f} \boldsymbol{w}_f \cdot \boldsymbol{\eta}_f = \int_{\Omega_f} g_f - \int_{\Gamma_f} \boldsymbol{h}_f \cdot \boldsymbol{\eta}_f.$$

Now put $\boldsymbol{u}_f = \boldsymbol{\omega}_f + \boldsymbol{\zeta}_f$ where $\boldsymbol{u}_f$ satisfies the non-homogeneous system (3.1). So we are looking for $\boldsymbol{\zeta}_f$ that satisfy

$$\begin{cases} -\nabla\cdot T(\boldsymbol{\zeta}_f, p_f) & = \boldsymbol{f}_f + \nabla\cdot 2\nu \boldsymbol{D}(\boldsymbol{\omega}_f) \quad \text{in } \Omega_f \\ \nabla\cdot \boldsymbol{\zeta}_f & = 0 \quad \text{in } \Omega_f \\ \boldsymbol{\zeta}_f & = 0 \quad \text{on } \Gamma_f. \end{cases}$$

Analogously, on the porous region, the non-homogeneous case can be reduced to the homogeneous one. In this case $h_p \in H^{-1/2}(\Gamma_p)$. Construct $\boldsymbol{w}_p \in \boldsymbol{H}(\mathrm{div}, \Omega_p)$ such that

$$\text{(A.3)} \qquad \begin{cases} \frac{\nu}{\kappa}\boldsymbol{w}_p + \nabla \tilde{p}_p & = 0 \quad \text{in } \Omega_p \\ \nabla\cdot \boldsymbol{w}_p & = g_p \quad \text{in } \Omega_p \\ \boldsymbol{w}_p\cdot \boldsymbol{\eta}_p & = h_p \quad \text{on } \Gamma_p, \\ \boldsymbol{w}_p\cdot \boldsymbol{\eta}_p & = \boldsymbol{w}_f\cdot \boldsymbol{\eta}_p \quad \text{on } \Gamma, \end{cases}$$

with $\boldsymbol{w}_f$ defined in (A.1). This construction is possible since the compatibility condition (3.3) and (A.2) imply that the system (A.3) is compatible. Put $\boldsymbol{u}_p = \boldsymbol{\omega}_p + \boldsymbol{\zeta}_p$. Then we look for $\boldsymbol{\zeta}_p$ such that

$$\begin{cases} \frac{\nu}{\kappa}\boldsymbol{\zeta}_p + \nabla p_p & = -\frac{\nu}{\kappa}\boldsymbol{\omega}_p \quad \text{in } \Omega_p \\ \nabla\cdot \boldsymbol{\zeta}_p & = 0 \quad \text{in } \Omega_p \\ \boldsymbol{\zeta}_p\cdot \boldsymbol{\eta}_p & = 0 \quad \Gamma_p. \end{cases}$$

In terms of weak formulation, with $\boldsymbol{\omega} := (\boldsymbol{\omega}_f, \boldsymbol{\omega}_p)$, we have:
find $(\boldsymbol{\zeta}, p, \lambda) \in \boldsymbol{X} \times M^{\circ} \times \Lambda$ *satisfying*

$$\begin{cases} a(\boldsymbol{\zeta}, \boldsymbol{v}) + b(\boldsymbol{v}, p) + b_\Gamma(\boldsymbol{v}, \lambda) & = \ell(\boldsymbol{v}) - a(\boldsymbol{\omega}, \boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in \boldsymbol{X} \\ b(\boldsymbol{\zeta}, q) & = 0 \quad \text{for all } q \in M^{\circ} \\ b_\Gamma(\boldsymbol{\zeta}, \mu) & = 0 \quad \text{for all } \mu \in \Lambda, \end{cases}$$

which is the same problem (4.17) with a different right hand side.

Appendix B. Approximation properties of Taylor-Hood finite elements. In this appendix, the domain of reference is Ω_f. Recall the definitions of $\boldsymbol{X}_f$ and $M^{\circ}_{h_f}$ on (5.1) and (5.3), respectively. In order to simplify the notation in some cases we omit the subscript that refers to the domain. In particular, all the operators defined in this section act on velocities defined on Ω_f.

Let $\boldsymbol{\mathcal{Q}} : \boldsymbol{X}_f \to \boldsymbol{X}_{h_f}$ be Clement interpolation; see [5, 9, 28]. It is know that $\boldsymbol{\mathcal{Q}}$ is bounded, i.e.,

$$\text{(B.1)} \qquad |\boldsymbol{\mathcal{Q}}\boldsymbol{v}_f|_{H^1(\Omega_f)^2} \preceq |\boldsymbol{v}|_{H^1(\Omega_f)^2},$$

and we have

(B.2) $$\|\boldsymbol{v}_f - \boldsymbol{\mathcal{Q}}\boldsymbol{v}_f\|_{L^2(\Omega_f)^2} \preceq h^s |\boldsymbol{v}_f|_{H^s(\Omega_f)^2}, \quad s = 1, 2.$$

(B.3) $$|\boldsymbol{v}_f - \boldsymbol{\mathcal{Q}}\boldsymbol{v}_f|_{H^1(\Omega_f)^2} \preceq h |\boldsymbol{v}_f|_{H^2(\Omega_f)^2},$$

(B.4) $$|\boldsymbol{\mathcal{Q}}\boldsymbol{v}_f|_{H^{1/2}(\Gamma)^2} \preceq |\boldsymbol{v}_f|_{H^{1/2}(\Gamma)^2},$$

(B.5) $$\|\boldsymbol{v}_f - \boldsymbol{\mathcal{Q}}\boldsymbol{v}_f\|_{L(\Gamma)^2} \preceq h^{\frac{1}{2}} |\boldsymbol{v}_f|_{H^{1/2}(\Gamma)^2}.$$

This interpolation is basically a Clement interpolation on Γ, i.e., values zero at the interface relative boundary points and a Clement interpolation at the interior nodes.

Given $K \in \mathcal{T}_{h_f}$ and e edge of K, let $\boldsymbol{\eta}_e^{(K)} = (\eta_e^1, \eta_e^2)$ denote the normal to e exterior to K, $\boldsymbol{\tau}_e^{(K)} = (\tau_e^1, \tau_e^2)$ the tangential vector to e (with ∂K anticlockwise oriented), and x_e the midpoint of the edge e. Each interior edge belongs to two triangles K_1 and K_2. Let $\boldsymbol{\eta}_e$ denote one of the directions $\boldsymbol{\eta}_e^{(K_1)}$ or $\boldsymbol{\eta}_e^{(K_2)}$. For boundary edges $\boldsymbol{\eta}_e$ denotes $\boldsymbol{\eta}_e^{(K)}$. Analogously, for interior edges let $\boldsymbol{\tau}_e$ denote one of the directions $\boldsymbol{\tau}_e^{(K_1)}$ or $\boldsymbol{\tau}_e^{(K_2)}$, and for boundary edges $\boldsymbol{\tau}_e = \boldsymbol{\tau}_e^{(K)}$.

Let $\phi_i^{(K)}$, $i = 1, 2, 3$, be the edge bubble Taylor-Hood basis functions based on the midpoints of the edges of K. Let $\boldsymbol{\psi}_i^{(K)} := \phi_i^{(K)} \boldsymbol{\eta}_{e_i}$, $i = 1, 2, 3$, and $\boldsymbol{\vartheta}_i^{(K)} := \phi_i^{(K)} \boldsymbol{\tau}_{e_i}$, $i = 1, 2, 3$. Observe that

$$\int_K \boldsymbol{\psi}_i^{(K)} \cdot \boldsymbol{\eta}_{e_i} \neq 0, \qquad \boldsymbol{\psi}_i^{(K)} \cdot \boldsymbol{\tau}_{e_i} = 0 \quad i = 1, 2, 3.$$

$$\boldsymbol{\vartheta}_i^{(K)}(x_{e_i}) \cdot \boldsymbol{\tau}_{e_i} \neq 0, \quad \boldsymbol{\vartheta}_i^{(K)} \cdot \boldsymbol{\eta}_{e_i} = 0 \quad i = 1, 2, 3.$$

Now consider the following subspaces of $\boldsymbol{X}_{h_f}$:

$$\boldsymbol{W}_{h_f}^{\boldsymbol{\eta}} := \left\{ \boldsymbol{v}_{h_f} \in \boldsymbol{X}_{h_f} : v|_K \in \mathrm{Span}\{\boldsymbol{\psi}_1^{(K)}, \boldsymbol{\psi}_2^{(K)}, \boldsymbol{\psi}_3^{(K)}\} \right\} \cap \boldsymbol{X}_{h_f}$$

and

$$\boldsymbol{W}_{h_f}^{\boldsymbol{\tau}} := \{ \boldsymbol{v}_{h_f} \in \boldsymbol{X}_{h_f} : v|_K \in \mathrm{Span}\{\boldsymbol{\vartheta}_1^{(K)}, \boldsymbol{\vartheta}_2^{(K)}, \boldsymbol{\vartheta}_3^{(K)}\} \} \cap \boldsymbol{X}_{h_f}.$$

Note that if $\boldsymbol{v}_{h_f} \in \boldsymbol{W}_{h_f}^{\boldsymbol{\eta}}$ then $\boldsymbol{v}_{h_f} \cdot \boldsymbol{\eta}_f|_\Gamma \in H_{00}^{1/2}(\Gamma)$ and $\boldsymbol{v}_{h_f} \cdot \boldsymbol{\tau}_f|_{\partial\Omega_f} = 0$. Also note that if $\boldsymbol{v}_{h_f} \in \boldsymbol{W}_{h_f}^{\boldsymbol{\tau}}$ then $\boldsymbol{v}_{h_f} \cdot \boldsymbol{\tau}_f|_\Gamma \in H_{00}^{1/2}(\Gamma)$ and $\boldsymbol{v}_{h_f} \cdot \boldsymbol{\eta}_f|_{\partial\Omega_f} = 0$.

Let $\boldsymbol{\Pi}_\eta : \boldsymbol{X}_f \to \boldsymbol{W}_{h_f}^{\boldsymbol{\eta}}$ be (locally) defined by

$$\boldsymbol{\Pi}_\eta \boldsymbol{v}_f \in \mathrm{Span}\{\boldsymbol{\psi}_1^{(K)}, \boldsymbol{\psi}_2^{(K)}, \boldsymbol{\psi}_3^{(K)}\}, \quad \text{s.t.} \int_{e_i} \boldsymbol{\Pi}_\eta \boldsymbol{v}_f \cdot \boldsymbol{\eta} = \frac{1}{|e_i|} \int_{e_i} \boldsymbol{v}_f \cdot \boldsymbol{\eta}_{e_i}, \;\; i = 1, 2, 3,$$

for all $K \in \mathcal{T}_h$. In other words, $\boldsymbol{\Pi}_\eta \boldsymbol{v}_f = \alpha_1 \boldsymbol{\psi}_1 + \alpha_2 \boldsymbol{\psi}_2 + \alpha_3 \boldsymbol{\psi}_3$, where

$$\alpha_i := \frac{\int_{e_i} \boldsymbol{v}_f \cdot \boldsymbol{\eta}_{e_i}}{\int_{e_i} \boldsymbol{\psi}_i \cdot \boldsymbol{\eta}_{e_i}} = \frac{\int_{e_i} \boldsymbol{v}_f \cdot \boldsymbol{\eta}_{e_i}}{\int_{e_i} \phi_i^{(K)}}.$$

From a trace theorem and a scaling argument we have that

$$|\alpha_i|^2 \preceq \frac{1}{h_f^2} \|\boldsymbol{v}_f\|_{L^2(K)^2}^2 + |\boldsymbol{v}_f|_{H^1(K)^2}^2.$$

Then

$$|\boldsymbol{\Pi}_\eta \boldsymbol{v}_f|_{H^1(\Omega_f)^2} \preceq \max_{1 \le i \le 3} |\alpha_i|^2 \preceq \frac{1}{h_f^2} \|\boldsymbol{v}_f\|_{L^2(\Omega_f)^2}^2 + |\boldsymbol{v}_f|_{H^1(\Omega_f)^2}^2$$

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and

$$\|\mathbf{\Pi}_\eta \boldsymbol{v}_f\|^2_{L^2(\Omega_f)^2} \preceq h_f^2 \max_{1\le i\le 3} |\alpha_i|^2 \preceq \|\boldsymbol{v}_f\|^2_{L^2(\Omega_f)^2} + h_f^2 |\boldsymbol{v}_f|^2_{H^1(\Omega_f)^2}. \tag{B.6}$$

Observe that

$$\int_K \nabla\cdot \mathbf{\Pi}_\eta \boldsymbol{v}_f = \int_{\partial K} \mathbf{\Pi}_\eta(\boldsymbol{v}_f)\cdot\boldsymbol{\eta} = \int_{\partial K} \boldsymbol{v}_f\cdot\boldsymbol{\eta} = \int_K \nabla\cdot\boldsymbol{v}_f.$$

We also have

$$\|\mathbf{\Pi}_\eta \boldsymbol{v}_f\|_{L^2(\Gamma)^2} \preceq \|\boldsymbol{v}_f\|_{L^2(\Gamma)^2}. \tag{B.7}$$

Define $\mathbf{\Upsilon}_\eta : \boldsymbol{X}_f \to \boldsymbol{X}_{h_f}$ by

$$\mathbf{\Upsilon}_\eta \boldsymbol{v}_f := \boldsymbol{\mathcal{Q}} \boldsymbol{v}_f + \mathbf{\Pi}_\eta(\boldsymbol{v}_f - \boldsymbol{\mathcal{Q}}\boldsymbol{v}_f), \tag{B.8}$$

then we have the following result.

LEMMA B.1. *The operator $\mathbf{\Upsilon}_\eta$ defined in (B.8) is bounded*

$$|\mathbf{\Upsilon}_\eta \boldsymbol{v}_f|_{H^1(\Omega_f)^2} \preceq |\boldsymbol{v}_f|_{H^1(\Omega_f)^2}, \tag{B.9}$$

moreover,

$$\|\boldsymbol{v}_f - \mathbf{\Upsilon}_\eta \boldsymbol{v}_f\|_{L^2(\Omega_f)^2} \preceq h^s \|\boldsymbol{v}_f\|_{H^s(\Omega_f)^2} \quad s = 1,2. \tag{B.10}$$

and

$$|\boldsymbol{v}_f - \mathbf{\Upsilon}_\eta \boldsymbol{v}_f|_{H^1(\Omega_f)^2} \preceq h |\boldsymbol{v}_f|_{H^2(\Omega_f)^2}. \tag{B.11}$$

We also have

$$|\mathbf{\Upsilon}_\eta \boldsymbol{v}_f|_{H^{1/2}(\Gamma)^2} \preceq \|\boldsymbol{v}_f\|_{H^{1/2}(\Gamma)^2}, \tag{B.12}$$

and

$$\int_e \mathbf{\Upsilon}_\eta \boldsymbol{v}_f\cdot\boldsymbol{\eta}_e = \int_e \boldsymbol{v}_f\cdot\boldsymbol{\eta}_e \quad \textit{for all edge } e. \tag{B.13}$$

Proof. From (B.6) we have, for $s = 1, 2$,

$$\begin{aligned}
\sum_{K\in\mathcal{T}_h} \|\mathbf{\Pi}_\eta(\boldsymbol{v}_f - \boldsymbol{\mathcal{Q}}\boldsymbol{v}_f)\|^2_{L^2(K)^2} &\preceq \sum_{K\in\mathcal{T}_h} \left(\|\boldsymbol{v}_f - \boldsymbol{\mathcal{Q}}\boldsymbol{v}_f\|^2_{L^2(K)^2} + h_f^2 |\boldsymbol{v}_f - \boldsymbol{\mathcal{Q}}\boldsymbol{v}_f|^2_{H^1(K)^2} \right) \\
&\preceq h_f^{2s} |\boldsymbol{v}_f|^2_{H^s(\Omega_f)^2} + h_f^{2(s-1)+2} |\boldsymbol{v}_f|^2_{H^s(\Omega_f)^2} \quad \text{by (B.2), (B.3) and (B.1).} \\
&\preceq h_f^{2s} |\boldsymbol{v}_f|^2_{H^s(\Omega_f)^2}.
\end{aligned} \tag{B.14}$$

Then, using an inverse estimate (see [5]) and (B.14) we get

$$|\mathbf{\Pi}_\eta(\boldsymbol{v}_f - \boldsymbol{\mathcal{Q}}\boldsymbol{v}_f)|_{H^1(K)^2} \preceq \frac{1}{h_f} \|\mathbf{\Pi}_\eta(\boldsymbol{v}_f - \boldsymbol{\mathcal{Q}}\boldsymbol{v}_f)\|_{L^2(K)^2} \preceq |\boldsymbol{v}_f|_{H^1(\Omega_f)^2},$$

and hence

$$\begin{aligned}
|\mathbf{\Upsilon}_\eta \boldsymbol{v}_f|_{H^1(\Omega_f)^2} &\le |\boldsymbol{\mathcal{Q}}\boldsymbol{v}_f|_{H^1(\Omega_f)^2} + |\mathbf{\Pi}_\eta(\boldsymbol{v}_f - \boldsymbol{\mathcal{Q}}\boldsymbol{v}_f)|_{H^1(\Omega_f)^2} \text{ by definition of } \mathbf{\Upsilon}_\eta \\
&\preceq |\boldsymbol{v}_f|_{H^1(\Omega_f)^2} + |\boldsymbol{v}_f|_{H^1(\Omega_f)^2} \preceq |\boldsymbol{v}_f|_{H^1(\Omega_f)^2}.
\end{aligned}$$

To show (B.10) we have that

$$\begin{aligned}
\|\boldsymbol{v}_f - \boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f\|_{L^2(\Omega_f)^2} &= \|\boldsymbol{v}_f - \mathcal{Q}\boldsymbol{v}_f - \boldsymbol{\Pi}_\eta(\boldsymbol{v}_f - \mathcal{Q}\boldsymbol{v}_f)\|_{L^2(\Omega_f)^2} \quad \text{by definition of } \boldsymbol{\Upsilon}_\eta.\\
&\le \|\boldsymbol{v}_f - \mathcal{Q}\boldsymbol{v}_f\|_{L^2(\Omega_f)^2} + \|\boldsymbol{\Pi}_\eta(\boldsymbol{v}_f - \mathcal{Q}\boldsymbol{v}_f)\|_{L^2(\Omega_f)^2}\\
&\preceq h_f^s|\boldsymbol{v}_f|_{H^s(\Omega_f)^2} + h^s|\boldsymbol{v}_f|_{H^s(\Omega_f)^2} \quad \text{by (B.2) and (B.14)}\\
&\preceq h_f^s|\boldsymbol{v}_f|_{H^s(\Omega_f)^2}, \quad s = 1, 2.
\end{aligned}$$

Analogously we get (B.11). To prove (B.12), observe that

$$\begin{aligned}
|\boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f|_{H^{1/2}(\Gamma)^2} &\le |\mathcal{Q}\boldsymbol{v}_f|_{H^{1/2}(\Gamma)^2} + |\boldsymbol{\Pi}_\eta(\boldsymbol{v}_f - \mathcal{Q}\boldsymbol{v}_f)|_{H^{1/2}(\Gamma)^2}\\
&\preceq \|\mathcal{Q}\boldsymbol{v}_f\|_{H^{1/2}(\Gamma)^2} + h_f^{-\frac{1}{2}}|\boldsymbol{\Pi}_\eta(\boldsymbol{v}_f - \mathcal{Q}\boldsymbol{v}_f)|_{L^2(\Gamma)^2}\\
&\preceq |\boldsymbol{v}_f|_{H^{1/2}(\Gamma)^2} + h_f^{-\frac{1}{2}}\|\boldsymbol{v}_f - \mathcal{Q}\boldsymbol{v}_f\|_{L^2(\Gamma)^2} \quad \text{by (B.4) and (B.7)}\\
&\preceq |\boldsymbol{v}_f|_{H^{1/2}(\Gamma)^2} \quad \text{by (B.5)}.
\end{aligned}$$

The last assertion, (B.13), is straightforward. □

Given $q_{h_f} \in M_{h_f}$, define (locally) $\widehat{\boldsymbol{\Pi}}_\tau q_{h_f} \in \boldsymbol{W}_{h_f}^{\boldsymbol{\tau}}$ by

$$\widehat{\boldsymbol{\Pi}}_\tau q_{h_f}|_K \in \text{Span}\{\boldsymbol{\vartheta}_1^{(K)}, \boldsymbol{\vartheta}_2^{(K)}, \boldsymbol{\vartheta}_3^{(K)}\}$$

with

$$\widehat{\boldsymbol{\Pi}}_\tau q_{h_f}(x_e)\cdot\boldsymbol{\eta} = 0 \text{ and } \widehat{\boldsymbol{\Pi}}_\tau q_{h_f}(x_e)\cdot\boldsymbol{\tau} = \nabla q_{h_f}(x_e)\cdot\boldsymbol{\tau} \tag{B.15}$$

at midpoints x_e of all interior edges e. For edges on Γ_f we define $\widehat{\boldsymbol{\Pi}}_\tau q|_e = 0$. Note that $\widehat{\boldsymbol{\Pi}}_\tau q_{h_f}$ is zero at the vertices of all elements of $\mathcal{T}_{h_f}$ and observe that $\widehat{\boldsymbol{\Pi}}_\tau q_{h_f} \in H^1(\Omega_f)^2$ because the above equation are consistent in neighbor triangles which gives $\widehat{\boldsymbol{\Pi}}_\tau q_{h_f}$ continuous; see [5], Chapter II, Theorem 5.2.

LEMMA B.2. *Suppose that $\mathcal{T}_{h_f}$ is non-degenerate and has no triangle with two edges on $\partial\Omega_f$ and consider the operator $\widehat{\boldsymbol{\Pi}}_\tau$ defined in (B.15). Then*

$$\|\widehat{\boldsymbol{\Pi}}_\tau q_{h_f}\|_{\boldsymbol{L}^2(\Omega_f)^2} \preceq |q_{h_f}|_{H^1(\Omega_f)} \quad \textit{for all } q_{h_f} \in M_{h_f}^\circ,$$

and there exists a positive constant such that:

$$\int_{\Omega_f} \widehat{\boldsymbol{\Pi}}_\tau q_{h_f}\cdot\nabla q_{h_f} \succeq |q_{h_f}|^2_{H^1(\Omega_f)} \succeq \|q_{h_f}\|^2_{L^2(\Omega_f)} \quad \textit{for all } q_{h_f} \in M_{h_f}^\circ.$$

From Lemma B.2 and the boundedness of $\widehat{\boldsymbol{\Pi}}_\tau$, the spaces $\boldsymbol{W}_{h_f}^{\boldsymbol{\tau}}$ (with the $L^2(\Omega_f)$-norm) and $M_{h_f}^\circ$ (with the $H^1(\Omega_f)$-norm) satisfy the inf-sup condition independent of h_f with respect to the bilinear form defined in (4.4) by

$$b_f(\boldsymbol{v}_f, q_f) := -(q_f, \nabla\cdot\boldsymbol{v}_f)_{\Omega_f} \quad \text{for all } \boldsymbol{v}_f \in \boldsymbol{X}_f \text{ and } q_f \in M_f^\circ.$$

Also observe that if $\boldsymbol{v}_f \in \boldsymbol{W}_{h_f}^{\boldsymbol{\tau}}$ then $\boldsymbol{v}_f\cdot\boldsymbol{\eta} = 0$ on $\partial\Omega_f$ and then $b_f(\boldsymbol{v}_f, q_f) = \int_{\Omega_f} \boldsymbol{v}_f\cdot\nabla q_f$ by the Green formula. Then, according to the Brezzi's splitting theorem (see [5, 15]), we can always obtain a stable solution $\boldsymbol{w} \in \boldsymbol{W}_{h_f}^{\boldsymbol{\tau}}$ of

$$\left\{\begin{array}{lll} (\boldsymbol{w}, \boldsymbol{v}_f)_{\Omega_f} + b_f(\boldsymbol{v}_f, p_f) & = (\boldsymbol{z}, \boldsymbol{v}_f)_{\Omega_f} & \text{for all } \boldsymbol{v}_f \in \boldsymbol{W}_{h_f}^{\boldsymbol{\tau}}\\ b_f(\boldsymbol{w}, q_f) & = b_f(\boldsymbol{z}, q_f)_{\Omega_f} & \text{for all } q_f \in M_{h_f}^\circ, \end{array}\right. \tag{B.16}$$

where $\boldsymbol{z} \in L^2(\Omega_f)^2$.

Given $\boldsymbol{z}$, denote by $\boldsymbol{\Upsilon}_\tau \boldsymbol{z}$ the solution of (B.16). Then

$$\|\boldsymbol{\Upsilon}_\tau \boldsymbol{z}\|_{L^2(\Omega_f)^2} \preceq \|\boldsymbol{z}\|_{L^2(\Omega_f)^2}, \tag{B.17}$$

and $b_f(\boldsymbol{\Upsilon}_\tau \boldsymbol{z}, q_{h_f}) = b_f(\boldsymbol{z}, q_{h_f})$ for $q_{h_f} \in M_{h_f}^\circ$.

In order to prove Lemma 5.1 define

$$\boldsymbol{I}_{h_f}^{TH} \boldsymbol{v}_f := \boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f + \boldsymbol{\Upsilon}_\tau(\boldsymbol{v}_f - \boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f).$$

Observe that

$$\begin{aligned}
|\boldsymbol{I}_{h_f}^{TH} \boldsymbol{v}_f|_{H^1(\Omega_f)^2} &\leq |\boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f|_{H^1(\Omega_f)^2} + |\boldsymbol{\Upsilon}_\tau(\boldsymbol{v}_f - \boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f)|_{H^1(\Omega_f)^2} \\
&\preceq |\boldsymbol{v}_f|_{H^1(\Omega_f)^2} + \frac{1}{h_f}\|\boldsymbol{\Upsilon}_\tau(\boldsymbol{v}_f - \boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f)\|_{L^2(\Omega_f)^2} \quad \text{by (B.9) and inverse estimate.} \\
&\preceq |\boldsymbol{v}_f|_{H^1(\Omega_f)^2} + \frac{1}{h_f}\|\boldsymbol{v}_f - \boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f\|_{L^2(\Omega_f)^2} \quad \text{by (B.17)} \\
&\preceq |\boldsymbol{v}_f|_{H^1(\Omega_f)^2} + |\boldsymbol{v}_f|_{H^1(\Omega_f)^2} \quad \text{by (B.10).}
\end{aligned}$$

Then the operator $\boldsymbol{I}_{h_f}^{TH}$ is bounded (with constant independent of h_f). In addition for $p_{h_f} \in M_{h_f}^\circ$ we get

$$\begin{aligned}
b_f(\boldsymbol{I}_{h_f}^{TH} \boldsymbol{v}_f, p_{h_f}) &= b_f(\boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f, p_{h_f}) + b_f(\boldsymbol{\Upsilon}_\tau(\boldsymbol{v}_f - \boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f), p_{h_f}) \\
&= b_f(\boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f, p_{h_f}) + b_f(\boldsymbol{v}_f - \boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f, p_{h_f}) \quad \text{by definition of } \boldsymbol{\Upsilon}_\tau. \\
&= b_f(\boldsymbol{v}_f, p_{h_f}).
\end{aligned}$$

To obtain (5.4) observe that from definition of $\boldsymbol{I}_{h_f}^{TH}$ we have

$$\begin{aligned}
\|\boldsymbol{v}_f - \boldsymbol{I}_{h_f}^{TH} \boldsymbol{v}_f\|_{L^2(\Omega_f)^2} &\leq \|\boldsymbol{v}_f - \boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f\|_{L^2(\Omega_f)^2} + \|\boldsymbol{\Upsilon}_\tau(\boldsymbol{v}_f - \boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f)\|_{L^2(\Omega_f)^2} \\
&\preceq \|\boldsymbol{v}_f - \boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f\|_{L^2(\Omega_f)^2} + \|\boldsymbol{v}_f - \boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f\|_{L^2(\Omega_f)^2} \quad \text{by (B.17)} \\
&\preceq h_f^s |\boldsymbol{v}_f|_{H^s(\Omega_f)^2} \quad s = 1, 2. \quad \text{by (B.10)}
\end{aligned}$$

The proof of (5.5) is similar. Inequality (5.6) is obtained from (B.12).