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ELECTROSTATICS AND GHOST POLES IN NEAR BEST FIXED POLE RATIONAL INTERPOLATION*

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Abstract. We consider points that are near best for rational interpolation with prescribed poles in the same sense that Chebyshev points are near best for polynomial interpolation. It is shown that these interpolation points satisfy an electrostatic equilibrium problem involving the fixed poles and certain 'ghost' poles. This problem is closely related to Lamé equations with residues of mixed sign.

Key words. Rational interpolation, Chebyshev weight, zeros, potential theory.

AMS subject classifications. Primary 33C45, secondary 42C05.

1. Introduction.

Recent years have witnessed a renewed interest in the electrostatic interpretation of zeros of orthogonal polynomials and related functions, a subject which lay dormant for almost a century. According to [9], this renewed interest is probably due partly to its connection with logarithmic potential theory, which provides a very powerful framework to study asymptotic properties of orthogonal polynomials and their zeros.

In the present article we study the zeros of a rational generalisation of the Chebyshev polynomials of the first kind and present an electrostatic interpretation in terms of the poles. These zeros are near best for rational interpolation with prescribed poles, as explained in [16]. Section 2 provides the relevant background. Some of the physics underlying this electrostatic interpretation is given in Section 3 and the main result is proved in Section 4.

The survey articles [14, 15] contain Stieltjes' results about the zeros of Jacobi, Laguerre and Hermite polynomials, as well as several pointers to the literature where consequences of Stieltjes' work, newer results and applications can be found. A result for the zeros of more general orthogonal polynomials was proved in [8] and a special case of Stieltjes polynomials (solutions to Lamé differential equations) is discussed in [5, 6]. There are many more important research papers in this field, but we only cite the ones most relevant to the present article. The connection between the results obtained in these references and our own work is discussed in Section 5.

Section 4 shows that the electrostatic interpretation involves not only the real poles, but also certain 'ghost' charges which are related to these poles. Some of their properties are discussed in the last section.

2. Near best fixed pole rational interpolation.

When approximating an analytic function $f(x)$ on the interval $[-1, 1]$ using a polynomial interpolant, a good choice for the interpolation points is often given by the zeros of a Chebyshev polynomial of the first kind. For functions $f(x)$ which are analytic in a large neighbourhood of the interval, this gives rise to an interpolation error which is very uniform on the interval, as opposed to interpolation in equidistant points, which leads to the Runge phenomenon (extreme oscillations between interpolation points).

There are several more or less equivalent ways to explain this attractive behaviour of Chebyshev points. Let $p_n(f; x)$ denote the polynomial interpolant of degree n to f in the

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points $x_1, \ldots, x_{n+1}$, then formula (3.1.8) in [3] says that

$$|f(x) - p_n(f;x)| \le \max_{-1\le t\le 1} |f^{(n+1)}(t)| \frac{|x-x_1|\cdots|x-x_{n+1}|}{(n+1)!} \,. \tag{2.1}$$

Maximising the right hand side of this expression over x and then minimising over all points x_k in the interval $[-1,1]$ leads to the zeros of a Chebyshev polynomial $T_{n+1}(x)$ of the first kind and degree $n+1$, according to [3, Theorem 3.3.4]. In other words, the monic polynomial of degree n which best approximates the zero function on the interval $[-1,1]$ according to the Chebyshev (minimax) norm, is a scaled Chebyshev polynomial. The above argument is given in more detail in [3, Section 3.3].

Another argument in favour of the Chebyshev points is given by [12, Theorem 5.6] or [2, Theorem 2.1], which both state that the interpolation error is of order $O(K^{-n})$ for $n \to \infty$, where K is the sum of the semimajor and semiminor axis lengths of f's ellipse of analyticity (in both references the argument is given for the Chebyshev extrema and not the Chebyshev zeros, but asymptotically this does not make any difference). Reference [12] also gives an interpretation in terms of potential theory: the asymptotic distribution of the interpolation points is exactly the equilibrium measure of the interval $[-1,1]$ without an external field.

When f is not analytic but merely continuous, the Chebyshev points are still a good choice, as explained by [2, Theorem 2.1] and the remarks following it: the Chebyshev interpolant of a continuous function f is within a factor 10 of the best uniform approximation if $n < 10^5$, or in the words of the authors, Chebyshev interpolants are near best. Again this relates to the Chebyshev extrema, but we assume that a similar statement can be given in terms of the Chebyshev zeros. The reason why the extrema are often used, is that they include the endpoints -1 and 1, which is often useful, e.g. in spectral methods to solve boundary value problems.

When the singularities of the function $f(x)$ are close to the interval, polynomial interpolation converges slowly (because the constant K above is very small) and a rational interpolant with poles at or close to the singularities of $f(x)$ might do better. In this case, assume a real polynomial $\pi_m(x)$ of degree m is given. Then following the same reasoning as [3, Section 3.1], it is not difficult to find that

$$\left| f(x) - \frac{p_n(f;x)}{\pi_m(x)} \right| \le \max_{-1\le t\le 1} |[\pi_m(t)f(t)]^{(n+1)}| \frac{|x-x_1|\cdots|x-x_{n+1}|}{\pi_m(x)} \frac{1}{(n+1)!} \,,$$

analogous to (2.1). Maximising the right hand side of this expression over x and minimising over all x_k leads to the interpolation points which we study in this article. They are the zeros of a rational equivalent of a Chebyshev polynomial. This is a rational function with fixed denominator of degree m and whose numerator of degree $n \ge m$ is such that the rational function has minimal Chebyshev norm on $[-1,1]$ (and, as a consequence, equi-oscillates).

Explicit formulas for this function were already known to Markoff and Bernstein and are derived by Achieser in Appendix A, Section 5 of his book on approximation theory [1]. They are given below. These functions are closely related to Bernstein–Szegő polynomials [11, pp. 31–32]. It was recently shown [16] that the zeros (interpolation points) can be obtained efficiently as the eigenvalues of a tridiagonal generalised eigenvalue problem, using a connection with orthogonal rational functions.

So let there be given a real monic polynomial $\pi_m(x)$ of strict degree m, whose zeros $\{\alpha_1, \ldots, \alpha_m\}$ are all outside $[-1,1]$. The Joukowski transformation, which maps the complex unit circle to the interval is denoted by $x = J(z) = (z+z^{-1})/2$ with $|z| \le 1$ and

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in everything which follows, x and z are always related by this transformation. The inverse transformation is denoted by $z = J^{-1}(x)$. It is given by

$$z = J^{-1}(x) = x - \sqrt{x^2 - 1},$$

where the branch of the square root is such that $|z| \le 1$. Now define the numbers β_k by

$$\beta_k = J^{-1}(\alpha_k), \quad k = 1, \ldots, m.$$

By definition, they are all inside the complex unit circle. Next we introduce the finite Blaschke product $B_n(z)$ as

$$B_n(z) = z^{n-m} \frac{z - \beta_1}{1 - \beta_1 z} \cdots \frac{z - \beta_m}{1 - \beta_m z} \,. \tag{2.2}$$

The standard definition of a Blaschke product contains complex conjugate β's in the denominator, but since they are real or appear in complex conjugate pairs, and only real values of z will be considered, we can omit the conjugation. The product $B_n(z)$ is obviously real for real z. The Chebyshev rational function under consideration is given by

$$\mathcal{T}_n(x) = \frac{1}{2}\left(B_n(z) + \frac{1}{B_n(z)}\right).$$

It can be shown that $\mathcal{T}_n(x)$ is of the form

$$\mathcal{T}_n(x) = \frac{p_n(x)}{\pi_m(x)}$$

(where $p_n(x)$ is a polynomial of strict degree n) and satifies the minimax and equi-oscillation properties mentioned above. For more information we refer to [16]. Note that in the polynomial case $m = 0$, we obtain a well-known formula for the Chebyshev polynomial $T_n(x)$.

It is known that the n zeros of $\mathcal{T}_n(x)$ are all real, simple and on the interval $[-1, 1]$. These are the near best interpolation points for rational interpolation with fixed poles in $\{\alpha_1, \ldots, \alpha_m\}$ which we mentioned above. In Section 4 we derive an electrostatic interpretation for these zeros, analogous to the one for the Chebyshev polynomials discussed in [11, Section 6.7].

3. The electrostatic model. We give a brief survey of the electrostatics of line charges, which is needed in the sequel. Although the content of this section is usually taken for granted by most authors, we believe it might be instructive to people with less background in electrostatics. The basic definitions were taken from [10, Chapters 13 and 14].

According to Coulomb's law, the electric field (i.e. the force acting on a unit charge) at a point $\mathbf{x}$ in space, generated by a point charge q at position $\mathbf{y}$, is proportional to

$$\mathbf{E} \sim \frac{q}{|\mathbf{x} - \mathbf{y}|^2} \mathbf{e_{yx}}$$

where $\mathbf{e_{yx}}$ is the unit vector pointing in the direction from $\mathbf{y}$ to $\mathbf{x}$. The constant of proportionality is irrelevant for our discussion.

Now assume electric charge is placed on an infinite straight line perpendicular to the complex plane and such that it intersects the plane in the point z. The charge has a uniform linear density of q. We wish to calculate the electric field generated by this line charge in a point x of the plane. Because of symmetry, this field has no vertical component and points

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from z to x. According to Coulomb's law, the contribution to the magnitude of the field from a small segment dh at height h is proportional to

$$dE \sim \frac{q\ dh}{|z-x|^2+h^2}\frac{|z-x|}{\sqrt{|z-x|^2+h^2}}$$

where the last factor comes from taking the horizontal component and is the cosine of the angle between the line connecting x and the point at height h above z, and the complex plane. Integrating for h from $-\infty$ to ∞ gives

$$E \sim \frac{2q}{|z-x|} \sim \frac{q}{|z-x|}\ .$$

For the sake of simplicity, we take the constant of proportionality equal to 1 so we may write this in vector form as

$$\mathbf{E} = \frac{q}{|z-x|}\frac{x-z}{|x-z|} = \frac{q}{\overline{x}-\overline{z}}\ . \tag{3.1}$$

This means that, mathematically, the electric field generated by a line charge in three-dimensional space according to Coulomb's inverse square law is the same as that generated by a point charge in a plane according to an inverse first-power law. Only the first interpretation has physical meaning, of course. Hence, in the sequel, when we speak of point charges in the complex plane, we actually mean line charges perpendicular to the plane.

By definition, the potential drop from a point a to a point b in the electric field $\mathbf{E}$ is given by

$$V_{ab} = \int_{ab} E_t dl$$

where E_t is the component of the field tangential to the path from a to b. This potential drop is independent of the path taken. Since the field (3.1) is radial, it is also independent of the angle between $z-a$ and $z-b$. It is then not difficult to show that

$$V_{ab} = V(a) - V(b) = q\log|z-b| - q\log|z-a|$$

so we may define the electrostatic potential $V(x)$ in a point x by

$$V(x) = -q\log|z-x|$$

(recall that the potential is only defined up to an additive constant).

Next assume we are given n free positive unit charges at points x_i of the real line and m fixed charges q_i at points z_i such that the z_i are real or appear in complex conjugate pairs. The total potential energy of this system, which is the work done against the electric field to assemble the n free charges one by one in the presence of the m fixed charges, is given by

$$L(\mathbf{x}) = L(x_1,\ldots,x_n) = -\sum_{1\le j<k\le n}\log|x_j-x_k| - \sum_{j=1}^{n}\sum_{k=1}^{m} q_k\log|x_j-z_k|.$$

Since the x_i are real and the z_i are real or complex conjugate, the total electric field of this system is everywhere parallel to the real axis, so the free charges cannot move into the complex plane. The system is in electrostatic equilibrium if

$$\nabla L = (\frac{\partial L}{\partial x_1},\cdots,\frac{\partial L}{\partial x_n})^T = 0. \tag{3.2}$$

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If $\nabla L(\mathbf{x}) = 0$ and $L(\mathbf{x})$ is minimal, the equilibrium is stable. It is possible that $\mathbf{x}$ is only a local minimum, but not a global one. In mechanics, this situation is usually referred to as a "metastable equilibrium". If $L(\mathbf{x})$ is maximal, the equilibrium is obviously unstable. It may also happen that $\mathbf{x}$ is a saddle point for L. It is not difficult to show that

$$\frac{\partial L}{\partial x_j} = -\sum_{\substack{k=1\\k\neq j}}^{n} \frac{1}{x_j - x_k} - \sum_{k=1}^{m} \frac{q_k}{x_j - z_k} \tag{3.3}$$

(this would not be the case if the z_i were arbitrary instead of real or complex conjugate). So in view of (3.1) and the principle of superposition, the condition $\nabla L(\mathbf{x}) = 0$ only states that the sum of the forces on each particle of the system is equal to zero.

4. Equilibrium. We finally arive at the main purpose of this article, which is to show that the zeros x_i of $\mathcal{T}_n(x)$ satisfy an electrostatic equilibrium condition of the form (3.2), where the points z_i are related to the poles α_k. However, it is not certain whether this equilibrium is stable, as is discussed below. First we need a simple lemma.

LEMMA 4.1. *Let $B_n(z)$ be defined by (2.2). Then it follows that*

$$B_n'(z) = \frac{1}{z} B_n(z) g(z)$$

where the prime means differentiation with respect to z and the function $g(z)$ is given by

$$g(z) = n - m - \sum_{k=1}^{m} \frac{1-\beta_k^2}{2\beta_k} \cdot \frac{1}{x - \alpha_k} \ .$$

Recall that x and z are related by $x = J(z)$.

Proof. This follows from the fact that

$$(z^{n-m})' = \frac{1}{z}(n-m)z^{n-m}$$

and

$$\begin{aligned}\left(\frac{z-\beta_k}{1-\beta_k z}\right)' &= \frac{1}{z} \cdot \frac{z-\beta_k}{1-\beta_k z} \cdot \frac{(1-\beta_k^2)z}{(z-\beta_k)(1-\beta_k z)} \ , \\ &= -\frac{1}{z} \cdot \frac{z-\beta_k}{1-\beta_k z} \cdot \frac{1-\beta_k^2}{2\beta_k} \cdot \frac{1}{x-\alpha_k} \ ,\end{aligned}$$

and the definition of $B_n(z)$. □

Before we proceed, it is convenient to make explicit the possible repetition of poles. So let us assume that there are only s distinct poles among the m poles $\{\alpha_1, \dots, \alpha_m\}$. Since the order of the poles does not matter, we may assume that these s different poles are $\{\alpha_1, \dots, \alpha_s\}$, with multiplicities $\{m_1, \dots, m_s\}$ such that

$$\sum_{k=1}^{s} m_k = m.$$

The function $g(z)$ of the previous lemma may then be written

$$g(z) = n - m - \sum_{k=1}^{s} \frac{1-\beta_k^2}{2\beta_k} \cdot \frac{m_k}{x - \alpha_k} \ . \tag{4.1}$$

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The zeros of $g(z)$ are important in what follows and we denote them by $\tilde{\alpha}_k$. In Section 6 it is shown that they are outside $[-1, 1]$, and that when $n > m$ and all α_k are real, so are the $\tilde{\alpha}_k$. Furthermore, for fixed m they converge to the α_k as n goes to infinity.

The following theorem gives an electrostatic interpretation to the zeros of $\mathcal{T}_n(x)$.

THEOREM 4.2. *Assume $n > m$ and let $\{x_k\}_{k=1}^n$ denote the n zeros of $\mathcal{T}_n(x)$ and $\{\tilde{\alpha}_k\}_{k=1}^s$ the s zeros of $g(z)$. Then*

$$-\sum_{\substack{k=1\\k\neq j}}^{n}\frac{1}{x_j-x_k}-\frac{1/4}{x_j-1}-\frac{1/4}{x_j+1}+\sum_{k=1}^{s}\left(\frac{m_k-1/2}{x_j-\alpha_k}+\frac{1/2}{x_j-\tilde{\alpha}_k}\right)=0 \tag{4.2}$$

for $j = 1, \ldots, n$.

The electrostatic interpretation is as follows. Fix negative charges $-m_k + 1/2$ at the points α_k and $-1/2$ at $\tilde{\alpha}_k$, for $k = 1, \ldots, s$, and positive charges $1/4$ at -1 and 1. Assume n positive unit charges can move freely on the interval $[-1, 1]$. Then the zeros of $\mathcal{T}_n(x)$ correspond to an electrostatic equilibrium for this system (in the weakest sense of the vanishing gradient of the total energy).

If $n = m$, then there are only $s - 1$ zeros $\tilde{\alpha}_k$ of $g(z)$ and then the last sum in (4.2), and hence also the electrostatic interpretation, should be modified appropriately.

Proof. We only give the proof for the case $n > m$ and note that the case $n = m$ requires only minor changes. In what follows, a prime means differentiation with respect to the variable between brackets and we write $z(x)$ to mean $J^{-1}(x)$. According to the chain rule it then follows that

$$\begin{aligned}\mathcal{T}_n'(x) &= \frac{1}{2}\left(B_n(z)+\frac{1}{B_n(z)}\right)' z'(x),\\ &= \frac{1}{2z}B_n(z)g(z)\left(1-\frac{1}{B_n^2(z)}\right)z'(x),\end{aligned}$$

where the last equality follows from Lemma 4.1. Similarly, some computations yield

$$\begin{aligned}\mathcal{T}_n''(x) = -\frac{1}{z}\mathcal{T}_n'(x)z'(x)+\frac{1}{z}g(z)\mathcal{T}_n'(x)z'(x)+&\\ \frac{g'(z)}{g(z)}\mathcal{T}_n'(x)z'(x)+\frac{1}{z^2}\frac{g^2(z)}{B_n(z)}(z'(x))^2+\mathcal{T}_n'(x)\frac{z''(x)}{z'(x)}&.\end{aligned}$$

Then write $z_j = J^{-1}(x_j)$ for $j = 1, \ldots, n$. It follows from the definition of $\mathcal{T}_n(x)$ and the fact that x_j is a zero of $\mathcal{T}_n(x)$ that $B_n^2(z_j) = -1$. Substituting this in the previous two expressions gives

$$\mathcal{T}_n'(x_j)=\frac{1}{z}B_n(z_j)g(z_j)z'(x_j)$$

and

$$\mathcal{T}_n''(x_j)=\mathcal{T}_n'(x_j)\left(-\frac{1}{z_j}z'(x_j)+\frac{z''(x_j)}{z'(x_j)}+\frac{g'(z_j)z'(x_j)}{g(z_j)}\right).$$

From the definition of $J(z)$ it follows that

$$z'(x)=\frac{2z^2}{z^2-1}\quad\text{and}\quad z''(x)=-\frac{4z}{(z^2-1)^2}z'(x)$$

and also that

$$x^2 - 1 = \frac{(z^2-1)^2}{4z^2}\,.$$

We then obtain

$$\mathcal{T}_n''(x_j) = \mathcal{T}_n'(x_j)\left(\frac{g'(z_j)z'(x_j)}{g(z_j)} - \frac{x_j}{x_j^2-1}\right). \tag{4.3}$$

It follows from formula (4.1) that we may write

$$g(z) = \frac{h_s(x)}{\pi_s(x)}$$

where

$$\begin{aligned} h_s(x) &= c_s(x-\tilde{\alpha}_1)\cdots(x-\tilde{\alpha}_s) \\ \pi_s(x) &= (x-\alpha_1)\cdots(x-\alpha_s) \end{aligned}$$

and the constant c_s is further irrelevant. Combining this with equation (4.3) yields

$$\frac{\mathcal{T}_n''(x_j)}{\mathcal{T}_n'(x_j)} = \frac{h_s'(x_j)}{h_s(x_j)} - \frac{\pi_s'(x_j)}{\pi_s(x_j)} - \frac{x_j}{x_j^2-1}\,.$$

Writing $\mathcal{T}_n(x) = p_n(x)/\pi_m(x)$ and using the fact that $p_n(x_j) = 0$ then finally gives

$$-\frac{p_n''(x_j)}{p_n'(x_j)} - \frac{x_j}{x_j^2-1} + \frac{h_s'(x_j)}{h_s(x_j)} - \frac{\pi_s'(x_j)}{\pi_s(x_j)} + 2\frac{\pi_m'(x_j)}{\pi_m(x_j)} = 0. \tag{4.4}$$

Formula (4.2) now easily follows.

Since $g(z)$ is real for real x, the zeros $\tilde{\alpha}_k$ are real or appear in complex conjugate pairs. We may thus use the results from Section 3 and the electrostatic interpretation of the zeros follows from formulas (3.2) and (3.3). This concludes the proof. □

Note that the previous theorem gives a very precise meaning to the statement "poles attract zeros" which we made in [17]. The asymptotic zero distribution was already studied in terms of the asymptotic distribution of the poles using logarithmic potential theory in [4], but the case of finite n had hitherto been ignored.

Following Marcellán et. al. in [9, p. 10], we choose to call the charges at $\tilde{\alpha}_k$ 'ghost' charges, or in this case more appropriately, ghost poles. More information about these ghost poles is given in the last section and their relation to the more general framework of Ismail [8] is discussed in the next section.

But before we move on, two important questions remain: Do the x_k correspond to a stable equilibrium? And is this equilibrium position unique? Of course, since any permutation of the equilibrium points again yields an equilibrium position, we impose the ordering $-1 < x_1 < \ldots < x_n < 1$ when studying uniqueness. Unfortunately, at present no conclusive answer to these questions is available. To study the stability of the system, one can investigate the Hessian matrix

$$H = \left[\frac{\partial^2 L}{\partial x_i \partial x_j}\right]_{1\le i,j\le n}\,.$$

If this matrix is positive definite, the equilibrium corresponds to a minimum of L and is therefore stable. In the polynomial cases discussed in [7, 8, 14], it turns out that the Hessian

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is not just positive definite in the equilibrium position, but in the entire region $x_1 < \ldots < x_n$ of the hypercube $[-1,1]^n$, although this is not always mentioned explicitly. This implies that the equilibrium is stable and unique, since L is then a convex function. The positive definiteness in these cases is deduced from the strict diagonal dominance of H. However, as will be shown momentarily, in our case H may not be positive definite in this entire region, and it need not be strictly diagonally dominant, although several numerical experiments seem to indicate that the equilibrium is indeed stable and unique. It appears likely that a possible proof of this property will be considerably more complicated than in the polynomial case, but in the next section we indicate how stability and uniqueness may be studied in the context of Lamé differential equations, which might provide an alternative solution.

It can easily be shown that the Hessian may fail to be positive definite in some regions of $[-1,1]^n$. To see this, consider the n-th diagonal element

$$h_{nn} = \sum_{k=1}^{n-1} \frac{1}{(x_k - x_n)^2} + \frac{1/4}{(x_n-1)^2} + \frac{1/4}{(x_n+1)^2} - \sum_{k=1}^{s} \left(\frac{m_k - 1/2}{(x_n - \alpha_k)^2} + \frac{1/2}{(x_n - \tilde{\alpha}_k)^2} \right).$$

It is possible to choose the poles α_k and the points x_j such that this element is negative, e.g. when $s = 1$, $\alpha_1 > 1$ is close to 1 and m_1 is very large, x_n is close to 1 and all the other x_k are close to -1. A matrix which has one or more negative diagonal elements cannot be positive definite.

In [9] the authors mention the possibility of a *Nash-type* equilibrium, which means that the total energy, as a function of a single variable x_k, also attains its minimum at the zeros of $\mathcal{T}_n(x)$, and this for each k. This would mean that if the Hessian is evaluated in the equilibrium position, its diagonal elements are all positive. This is obviously a weaker condition than requiring the Hessian to be positive definite, but we have not been able to prove that the equilibrium is a Nash-type equilibrium.

To end this section, we analyse a specific example for which it can be proved that the equilibrium position is stable, even though the Hessian is not strictly diagonally dominant at this position. Take $n = 3$ and $m = 2$ real poles α and $-\alpha$, symmetric with respect to the origin. The equation $\mathcal{T}_n(x) = 0$ leads to

$$z^2(z^2 - \beta^2)^2 + (1 - \beta^2 z^2)^2 = 0.$$

Dividing by z^3 and using the formula $T_n(x) = (z^n + z^{-n})/2$ for the Chebyshev polynomial of the first kind and degree n, some algebra yields

$$T_3(x) + (\beta^4 - 2\beta^2) T_1(x) = 0$$

or, collecting like powers of x,

$$x(4x^2 - 3 + \beta^4 - 2\beta^2) = 0$$

from which it readily follows that

$$x_1 = -\frac{\sqrt{(1+\beta^2)(3-\beta^2)}}{2}, \quad x_2 = 0, \quad x_3 = \frac{\sqrt{(1+\beta^2)(3-\beta^2)}}{2}. \tag{4.5}$$

To find the ghost poles, we need to solve the equation

$$1 - \frac{1-\beta^2}{2\beta} \left(\frac{1}{x-\alpha} - \frac{1}{x+\alpha} \right) = 0$$

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which shows that they are given by $-\tilde{\alpha}$ and $\tilde{\alpha}$ with

$$\tilde{\alpha} = \frac{\sqrt{(1+\beta^2)(3-\beta^2)}}{2\beta}\,. \tag{4.6}$$

The computation of the Hessian matrix is now straightforward using formulas (3.3), (4.5) and (4.6), but the computations become very unelegant and we only give the final result (a computer algebra package such as Maple comes in handy at this point). It follows that $H = (1+\beta^2)^{-1}(3-\beta^2)^{-1}M$ where the matrix M is equal to

$$M = \begin{bmatrix} u & -4 & -1 \\ -4 & v & -4 \\ -1 & -4 & u \end{bmatrix},$$

and the values u and v are given by

$$u = \frac{16(3-\beta^4)}{(1-\beta^2)^4} - 1\,,$$
$$v = \frac{1}{2}\frac{19 - 11\beta^2 + \beta^4 - \beta^6}{1+\beta^2}\,.$$

It is easily checked that u is increasing and v is decreasing with β for $0 < \beta < 1$. We find that $\min u = 47$ and $\min v = 2$, so the matrix is certainly not strictly diagonally dominant for all $0 < \beta < 1$. However, a standard criterion for positive definiteness is to verify that the principal minors of the matrix M are all strictly positive. Those are

$$u, \quad uv - 16, \quad \text{and} \quad (u+1)(v(u-1) - 32),$$

and since the minimal value of u is 47 and that of v is 2, we can conclude that H is positive definite and hence the equilibrium is stable.

Of course, this direct approach is unfeasible to study stability in the general case, but at least it shows that the equilibrium may be stable even though H is not strictly diagonally dominant.

5. Orthogonal polynomials and Lamé equations. The equilibrium problem of the previous section is closely related to a more general theory for the electrostatic interpretation of zeros of orthogonal polynomials as described in [8], as we show next.

In the abovementioned article, the author considers polynomials $p_n(x)$ orthonormal with respect to a weight function $w(x)$ on $[a, b]$. Upon writing

$$w(x) = e^{-v(x)}, \quad x \in (a, b),$$

he then defines a function $A_n(x)$ which is related in a rather complicated way to p_n, w and v. The zeros of $p_n(x)$ are shown to be the points of electrostatic equilibrium of n movable unit charges in $[a, b]$ in the presence of the external potential[1] $V(x) = v(x) + \log(|A_n(x)|)$. The uniqueness and stability of the equilibrium can be guaranteed if $V(x)$ is convex and $A_n(x)$ does not change sign on (a, b). To prove this result, he derives the formula

$$-v'(x_j) - \frac{A_n'(x_j)}{A_n(x_j)} + \frac{p_n''(x_j)}{p_n'(x_j)} = 0 \tag{5.1}$$

[1] We have omitted a multiplicative constant in A_n found in [8] which is irrelevant for our discussion.

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which can be interpreted in the same way as formula (4.4).

It is well-known [1, p. 250] that the numerator polynomial $p_n(x)$ of $\mathcal{T}_n(x)$ is orthogonal to x^k for $k = 1, \ldots, n-1$ with respect to the weight function

$$w(x) = [\pi_m^2(x)\sqrt{1-x^2}]^{-1}. \tag{5.2}$$

It follows from (5.2) that

$$v'(x) = 2\frac{\pi'_m(x)}{\pi_m(x)} + \frac{x}{x^2-1}$$

and comparing (5.1) to (4.4) then gives

$$\frac{A'_n(x)}{A_n(x)} = -\frac{2x}{x^2-1} + \frac{h'_s(x)}{h_s(x)} - \frac{\pi'_s(x)}{\pi_s(x)}$$

from which it follows that

$$A_n(x) = \frac{c}{1-x^2} \cdot \frac{h_s(x)}{\pi_s(x)}$$

where c is an arbitrary (irrelevant) constant. Computing A_n from the general formulas in [8] seems considerably more complicated than the derivation given in the previous section.

The electric field created by $v(x)$ is referred to as the long range field, while the one created by $\log(|A_n(x)|)$ is called the short range field. Note how the charges at -1 and 1 and at the poles are distributed between these fields, and that the ghost poles are related only to the short range field. Unfortunately, the convexity of $V(x)$ cannot be proved in our case, since this would show that the Hessian is strictly diagonally dominant [8, p. 360] and we know that this is not necessarily true.

Another difference between our case and that of [8] is that our derivation only holds for $n \geq m$ so for fixed m the $p_n(x)$ do not form a complete set of orthogonal polynomials. However, as explained in [16, Section 3], it is possible to recursively define $p_n(x)$ for $n = m-1, m-2, \ldots, 1$ and although there is no mention of orthogonality in that article, it is easy to show that the polynomials so defined are also orthogonal with respect to $w(x)$. Explicit formulas, however, are unavailable.

To study the stability and uniqueness of the electrostatic equilibrium there is another, perhaps more important, connection worth pointing out, i.e. that to the theory of Lamé differential equations. The article [5] is particularly relevant to our discussion and it contains many more references concerning this theory. A Lamé equation is a differential equation of the form

$$A(x)y'' + 2B(x)y' + C(x)y = 0,$$

where $A(x) = (x-a_0)\cdots(x-a_p)$ and $B(x)$ and $C(x)$ are polynomials of degree p and $p-1$, and

$$\frac{B(x)}{A(x)} = \sum_{j=0}^{p} \frac{r_j}{x-a_j} \ .$$

Of particular interest is the question of characterising the polynomial solutions of this equation. It is known that, given $A(x)$ and $B(x)$, there exist at most $(n+p-1)!/(n!(p-1)!)$ polynomials $C(x)$ such that this equation has a polynomial solution $y(x)$ of degree n. The

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polynomial $C(x)$ is called a Van Vleck polynomial and the corresponding solution $y(x)$ is called a Stieltjes polynomial.

The case where all the residues r_j are positive has been thoroughly studied and much is known about the location of the zeros of the Stieltjes polynomials in this case. However, the case of both positive and negative residues appears to be more complicated and only some specific cases have been studied.

The relation to electrostatic equilibrium problems is that the residues r_j can be interpreted as charges located at the points a_j and then the zeros of the Stieltjes polynomials correspond to equilibrium positions for n free unit charges. More specifically, if the residues (charges) are positive and the points a_j are real, then for each $j = 1, \ldots, p$ there exists exactly one pair (C, y) of a Van Vleck and Stieltjes polynomial such that the n zeros of y are in the interval (a_{j-1}, a_j). This corresponds to a unique and stable equilibrium of n movable unit charges on that interval.

In [5] a configuration of two positive and two negative charges is studied and the unicity of the (C, y) pair is established under certain conditions, which then leads to an electrostatic interpretation of the zeros of Gegenbauer-Laurent polynomials.

To see how all of this is related to our problem, consider equation (4.4). It follows from this equation that the expression

$$(x^2-1)h_s(x)\pi_s(x)p_n''(x) + \Bigg[xh_s(x)\pi_s(x) - h_s'(x)(x^2-1)\pi_s(x) + \left(\pi_s'(x) - 2\frac{\pi_m'(x)\pi_s(x)}{\pi_m(x)}\right)(x^2-1)h_s(x)\Bigg]p_n'(x)$$

is a polynomial of degree $2s+n$ which vanishes in the points $x_1, \ldots, x_n$ and thus

$$A(x)p_n''(x) + 2B(x)p_n'(x) + C(x)p_n(x) = 0$$

for some polynomial $C(x)$ of degree $2s$, where

$$\frac{B(x)}{A(x)} = \frac{1/4}{x-1} + \frac{1/4}{x+1} - \sum_{k=1}^{s}\left(\frac{m_k - 1/2}{x-\alpha_k} + \frac{1/2}{x-\tilde{\alpha}_k}\right).$$

This is a Lamé equation with both positive and negative residues, which clearly has the polynomial solution $p_n(x)$. Following the same reasoning as in [5], it follows that establishing the unicity of the (C, y) pair where y has all its n zeros in $[-1, 1]$, is equivalent to establishing the unicity and stability of the electrostatic equilibrium problem of the previous section.

6. Ghost poles. A great deal can be said about the location of the ghost poles $\tilde{\alpha}_j$, as shown in the next theorem.

THEOREM 6.1. *The ghost poles $\tilde{\alpha}_j$ are outside the interval $[-1, 1]$ and if m is fixed they converge to the poles α_k as $n \to \infty$. Furthermore, if $n > m$ and if all α_k are real, then so are all $\tilde{\alpha}_j$.*

Proof. It follows from the proof of Lemma 4.1 that we may write

$$-\frac{1-\beta^2}{2\beta} \cdot \frac{1}{x-\alpha} = \frac{(1-\beta^2)z}{(z-\beta)(1-\beta z)},$$

where $\alpha = J(\beta)$ and as usual $x = J(z)$. If $x \in [-1, 1]$ then $|z| = 1$ and thus $\overline{z} = 1/z$. If α is real, then so is β and it follows that

$$-\frac{1-\beta^2}{2\beta} \cdot \frac{1}{x-\alpha} = \frac{1-\beta^2}{|z-\beta|^2} > 0, \quad x \in [-1, 1].$$

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If α is complex, then there must be a term in the sum (4.1) which is complex conjugate to the one above. Adding these two terms gives

$$\begin{aligned}\frac{(1-\beta^2)z}{(z-\beta)(1-\beta z)}+\frac{(1-\overline{\beta}^2)z}{(z-\overline{\beta})(1-\overline{\beta}z)} &= \frac{\beta}{z-\beta}+\frac{1}{1-\beta z}+\frac{\overline{\beta}}{z-\overline{\beta}}+\frac{1}{1-\overline{\beta}z}\\ &= \frac{\beta}{z-\beta}+\frac{1}{1-\overline{\beta}z}+\frac{\overline{\beta}}{z-\overline{\beta}}+\frac{1}{1-\beta z}\\ &= \frac{(1-|\beta|^2)z}{(z-\beta)(1-\overline{\beta}z)}+\frac{(1-|\beta|^2)z}{(z-\overline{\beta})(1-\beta z)}\\ &= \frac{1-|\beta|^2}{|z-\beta|^2}+\frac{1-|\beta|^2}{|z-\overline{\beta}|^2}>0,\end{aligned}$$

for $x\in[-1,1]$, where the last equality follows from the fact that $\overline{z}=1/z$. From the above formulas and the fact that $n\geq m$, we conclude that

$$g(z)>0,\quad x\in[-1,1].$$

Since $\tilde{\alpha}_j$ are the zeros of $g(z)$, they must be outside the interval $[-1,1]$.

Concerning the convergence of the ghost poles to the actual poles for $n\to\infty$ and m fixed, assume that $n>m$ and write the equation for $\tilde{\alpha}_j$ as

$$1+\frac{1}{n-m}\sum_{k=1}^{s}\frac{w_k}{\alpha_k-\tilde{\alpha}_j}=0,$$

where $w_k=m_k(1-\beta_k^2)/(2\beta_k)$. In the case where $w_k>0$, this is called a secular equation and it arises in the context of divide-and-conquer algorithms to solve eigenvalue problems [13, Lect. 30]. In our more general case, define the vector $\mathbf{w}=[\sqrt{w_1},\cdots,\sqrt{w_s}]^T$ where the branch of the square root does not matter. Then following the same reasoning as in [13] it can be shown that the values $\tilde{\alpha}_j$ are given by the eigenvalues of the matrix

$$\begin{bmatrix}\alpha_1 & & \\ & \ddots & \\ & & \alpha_s\end{bmatrix}+\frac{1}{n-m}\mathbf{w}\mathbf{w}^T.$$

It is clear that for $n\to\infty$ these eigenvalues converge to the poles $\alpha_1,\ldots,\alpha_s$.

Finally, if all poles α_k are real, then the function $\tilde{g}(x)=g(z)$ has simple poles at the points α_k and it is easily checked that

$$\begin{aligned}&\tilde{g}(\alpha_k-)=+\infty,\quad \tilde{g}(\alpha_k+)=-\infty\quad\text{if}\quad \alpha_k>0,\\ &\tilde{g}(\alpha_k-)=-\infty,\quad \tilde{g}(\alpha_k+)=+\infty\quad\text{if}\quad \alpha_k<0.\end{aligned}$$

Because of continuity, this means that there must be one ghost pole between any two consecutive positive poles and similarly between any two consecutive negative poles. This yields a total of $s-2$ ghost poles ($s-1$ if all poles are either all positive or all negative). Then since $\tilde{g}(\pm\infty)=n-m$, if $n>m$ there must be one more positive ghost pole which is greater than the largest positive pole and one more negative ghost pole which is larger (in absolute value) than the largest negative pole (or either one of these cases if all poles are either all positive or all negative). This gives a total of s real ghost poles. $\square$

The previous theorem shows that for large n and fixed m, the electrostatic equilibrium problem approximates the situation where each pole α_k is given a negative charge equal in magnitude to the multiplicity of this pole, since the ghost poles will then almost coincide with the real poles. Asymptotically, however, for n tending to infinity and m fixed, the equilibrium problem corresponds to the case without poles and the equilibrium distribution is the equilibrium measure of $[-1, 1]$. More interesting asymptotic behaviour occurs when n and m both tend to infinity such that the limit m/n exists. The study of the asymptotic behaviour and zero distribution of the $\mathcal{T}_n$, however, is outside the scope of this article.

Furthermore, the fact that the ghost poles are always outside the interval $[-1, 1]$ is quite fortunate since that way we do not have to give an interpretation to the case where (negative) ghost charges collide with (positive) unit charges x_k. This case cannot occur.

7. Conclusion. The electrostatic equilibrium problem discussed in this article nicely compliments some of the existing theory for polynomial problems. As we have shown, it can be regarded as a special case of a more general theory discussed in [8], and more importantly, it is closely connected to the theory of Lamé equations with residues of mixed sign. Establishing the uniqueness and stability of the equilibrium would mean a considerable step forward in the study of this kind of Lamé equations. However, at present this remains an open problem and, as the discussion at the end of Section 4 shows, the method of proof used in the polynomial case cannot simply be adapted to our case.

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