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THEORY AND NUMERICS FOR MULTI-TERM PERIODIC DELAY DIFFERENTIAL EQUATIONS: SMALL SOLUTIONS AND THEIR DETECTION*

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Abstract. In this paper we consider scalar linear periodic delay differential equations of the form

$$x'(t) = \sum_{j=0}^{m} b_j(t)x(t - jw), x(t) = \phi(t) \text{ for } t \in [0, mw),\ t \geq mw \quad (\ddagger)$$

where b_j, $j = 0, ..., m$ are continuous periodic functions with period w. We summarise a theoretical treatment that analyses whether the equation has small solutions. We consider discrete equations that arise when a numerical method with fixed step-size is applied to approximate the solution to (‡) and we develop a corresponding theory. Our results show that small solutions can be detected reliably by the numerical scheme. We conclude with some numerical examples.

Key words. delay differential equations, small solutions, super-exponential solutions, numerical methods

AMS subject classifications. 34K28, 65P99, 37N30

1. Introduction. The analysis of delay differential equations of the form

$$x'(t) = \sum_{j=0}^{m} b_j(t)x(t - jw), x(t) = \phi(t) \text{ for } t \in [0, mw),\ t \geq mw \tag{1.1}$$

has become increasingly important in recent years; see, for example, [1, 2, 10]. It is known that, whereas a first order ordinary differential equation represents a 1-dimensional problem, a delay differential equation of the form (1.1) represents an infinite dimensional system, even in the case $m = 1$. We can see this by considering the nature of the initial data required in each case to specify a unique solution: for a first order ordinary differential equation, it is sufficient to specify a single initial value (for example the value of $x(0)$) to define a unique solution. However in the case of the delay equation (1.1) one would need to specify the solution over an initial *interval* of length mw.

This infinite dimensionality of the delay differential equation brings with it a far richer range of possible dynamical behaviour than would be the case for an ordinary differential equation (even of higher order). The solutions of delay equations exhibit behaviour that extends beyond phenomena that we see in solutions of ordinary differential equations.

In this paper we focus on so-called small (or super-exponential) solutions. These are solutions x that satisfy $x(t)e^{st} \to 0$ as $t \to \infty$ for every $s \in \mathbb{R}$. They are important both from an analytical viewpoint and for mathematical modellers: if an equation has non-trivial (*not identically zero*) small solutions then the solution space is not spanned by the set of eigenvectors and generalised eigenvectors of the solution map; see, for example, [10, 14, 15, 18]. This means, for example, that possibly important features of the true solutions would be lost if one attempted a series expansion in terms of eigenfunctions and generalised eigenfunctions. This analytical property then implies that parameter estimation problems may be improperly posed, and so the detection of small solutions, when they are present, is a key objective; see [3, 10, 11, 14, 16, 17]. Unfortunately, the detection of small solutions by analytical methods is, in general, a difficult and incompletely solved problem and one would like to develop alternative methods if possible.

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The problem has been solved quite effectively for periodic single-term nonautonomous delay equations. In our previous work (see [5, 6, 8, 9]) we showed that the use of a simple numerical scheme with fixed step length yields a finite dimensional solution map operator for the numerical approximation of a single term delay equation. One can then show that, as the step length of the numerical scheme is reduced to zero, the eigenspectra of the discrete operators behave in characteristically different ways according to whether or not small solutions are present in the underlying infinite dimensional system; see [5, 6, 8, 9] for more details.

Here we extend our investigations to scalar linear periodic delay equations of the form (1.1) where $b_j, j = 0, ..., m$, are continuous periodic functions with period w. We will assume that the zeros of b_m are isolated. We begin this paper by developing the necessary theoretical results that will underpin our approach and we conclude with a series of illustrative numerical examples.

2. Mathematical preliminaries. In order to gain a proper understanding of the concept of small solutions, it is helpful to start by reviewing briefly the analysis for the single term autonomous linear equation

$$x'(t) = ax(t-1). \tag{2.1}$$

The range of dynamical behaviour exhibited by members of the solution set is determined by the solution set Λ of the characteristic quasi-polynomial

$$\lambda e^{\lambda} = a. \tag{2.2}$$

REMARK 2.1. The equation (2.2) is obtained from (2.1) either by solving using Laplace transforms, or by substitution of the trial solution $x(t) = \alpha e^{\lambda t}$.

One can express any solution of (2.1) as a linear combination of the functions $e^{\lambda t} : \lambda \in \Lambda$. Thus $x(t) = \sum_{\lambda \in \Lambda} \alpha_\lambda e^{\lambda t}$. The coefficients α_λ in the linear combination are determined by the initial conditions (initial function) and the dominant dynamical behaviour, as $t \to \infty$, of a particular solution will be determined by the value of λ with $\alpha_\lambda \neq 0$ which lies furthest to the right in the complex plane. Thus, not all solutions to (2.1) will have the same dominant behaviour as $t \to \infty$.

The solutions (which we shall call eigenvalues) to (2.2) all lie on a single trajectory in $\mathbb{C}$ and one can also predict how they will be distributed along the trajectory:

LEMMA 2.2. *For the equation (2.1), all the characteristic roots $\lambda = x + iy$ lie on the curve*

$$x = -y\cot(y)$$

and also satisfy

$$a = \frac{-ye^x}{\sin(y)}. \tag{2.3}$$

This result can be established by substituting $\lambda = x + iy$ in (2.2). (2.3) implies that, for each fixed value of a there will be precisely 2 values of λ for each 2π variation in y. This means, in turn, that the eigenvalues are isolated and do not have a limit point.

REMARK 2.3. It follows from this discussion that every non-zero solution of (2.1) will have a dominant eigenfunction (of the form $e^{\lambda t}$ (or pair of eigenfunctions $e^{\lambda t}$ and $e^{\bar{\lambda} t}$) for some $\lambda \in \Lambda$) for which α_λ and/or $\alpha_{\bar{\lambda}}$ is non-zero. This determines the exponential growth or decay rate of the particular solution. Hence there can be no non-zero small solutions to (2.1).

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Small solutions cannot arise with autonomous equations of the form (2.1). In fact, one can go further. For non-autonomous periodic delay equations of the form

$$x'(t) = b(t)x(t-1), \text{ where } b(t+1) = b(t), \tag{2.4}$$

we have the following lemma (see [8]):

LEMMA 2.4. *The equation (2.4) has no small solutions if the function b is of constant sign. In this case (2.4) is equivalent to an equation of the form (2.1).*

REMARK 2.5. The equivalence of the two equations is discussed in more detail in [8]. One makes the substitution $x(t) = z(\sigma(t))$ where $\sigma(t) = \int_0^t b(\alpha)d\alpha / \int_0^1 b(\alpha)d\alpha$. It follows that $z'(\sigma) = \beta z(\sigma - \tau)$.

The above discussion provides the key to the characterisation of small solutions by numerical techniques. We now know that, if the equation (2.4) has no small solutions then the eigenvalues of its solution operator must lie on a single trajectory in the Argand diagram. Thus, if the eigenvalues *do not lie on a single trajectory* then it follows that the equation must have small solutions.

As we have seen, even for the autonomous linear equation, one needs to solve a quasi-polynomial in order to find the eigenvalues. For the non-autonomous equation, the problem is even harder. Therefore we employ a numerical scheme to approximate the eigenvalues of (2.4). In doing this, we are using a finite dimensional approximaion of the infinite dimensional solution operator and therefore one might question the validity of the approach.The justification is provided by the following result from [4]:

THEOREM 2.6. *Let the parameter value $\alpha = \alpha_0$ be fixed and let $z_0 = x_0 + iy_0$ be a characteristic root of equation (2.1). With $h > 0$ (chosen so that $h = \frac{1}{m}$ with m some positive integer, as before) we apply a strongly stable linear multistep method (ρ, σ) of order $p \geq 1$ to (2.1) to yield a discrete equation that has m characteristic values. Now let $z_h = x_h + iy_h$ be such that $z_h^* = e^{z_h/m}$ is a characteristic value of the discrete equation for which $|e^{z_0} - (z_h^*)^m|$ is minimised. Then $|e^{z_0} - (z_h^*)^m| = \mathcal{O}(h^p)$ as $h \to 0$.*

Thus, we have a characterisation of non-autonomous single-term periodic delay differential equations that can be summarised in the following corollary:

COROLLARY 2.7. *For a sequence of steplengths $h_i = \frac{1}{N_i} \to 0$ apply the trapezoidal rule (for example) to equation (2.4) and calculate the eigenvalues of the resulting operator. It is sufficient for the existence of small solutions to equation (2.4) that the eigenvalues lie on more than one trajectory.*

This result forms the mathematical basis for our paper [7] in which we developed an automated system for the detection of small solutions in single term periodic equations.

REMARK 2.8. Notice that the above discussion also gives a characterisation for an equation of the form

$$y'(t) = b(t)y(t-d), b(t+1) = b(t), d \in \mathbb{N}$$

to have small solutions.

3. Multi-term equations and the Floquet theory. In this section we develop similar results to those of the previous section for multi-term equations of the form (1.1). Our aim is to give a numerical characterisation theorem of a similar form to the one for single-term equations. We begin by developing some basic Floquet theory.

Non-zero solutions of equation (1.1) which are such that $x(t + w) \equiv \tilde{\lambda}x(t)$, $-\infty < t < \infty$, are known as Floquet solutions (see [13]). The $\tilde{\lambda}$ are known as the characteristic multipliers. These solutions can be represented in the form $x(t) = e^{\mu t}p(t)$ where $p(t + w) = p(t)$ and $\tilde{\lambda} = e^{\mu w}$.

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If $X(t)$ is a Floquet type solution of equation (1.1) then it satisfies

$$X'(t) = \sum_{j=0}^{m} b_j(t)X(t - jw) \tag{3.1}$$

with

$$X(t) = e^{\mu t}p(t) \text{ where } p(t + w) = p(t).$$

This expression for X satisfies

$$X(t - jw) = \tilde{\lambda}^{m-j}X(t - mw) \text{ with } \tilde{\lambda} = e^{\mu\omega},$$

and

$$X'(t) = \sum_{j=0}^{m} \tilde{\lambda}^{m-j}b_j(t)X(t - mw). \tag{3.2}$$

We can summarise this in the following Lemma:

LEMMA 3.1. *If X is a Floquet solution of the multi-term equation (1.1) then X satisfies a single-term delay equation (with delay mw, the maximum delay from the multi-term equation.)*

Next we need to show that it is sufficient for our purposes (the detection of small solutions) to concentrate on Floquet solutions. Once we have established this fact, we can rely on the previous analysis for single-term equations to provide a characterisation of equations with small solutions. Fortunately, there is an established result [10, Chapter 8, Theorem 3.5]:

THEOREM 3.2. *The Floquet solutions of a periodic delay differential equation span the solution space if and only if the equation has no small solutions.*

REMARK 3.3. Of course, it would be inconvenient to have to undertake a separate Floquet analysis for each multi-term equation to derive an appropriate single-term equation for analysis. However, when an equation has small solutions, these will generate Floquet solutions with $\tilde{\lambda} = 0$ for which the single term equation is simply

$$x'(t) = b_m(t)x(t - mw). \tag{3.3}$$

It follows from this observation (see also [10, Chapter 8, Theorem 3.3]) that one can prove the following:

THEOREM 3.4.

1. *Suppose that the zeros of b_m are isolated, then equation (1.1) has no small solutions if and only if b_m has no sign change.*
2. *Equation (1.1) has small solutions if and only if (3.3) has small solutions.*

This leads to our detection algorithm:

ALGORITHM 3.5. *For a multi-term equation of the form (1.1) construct the single term equation (3.3). For a sequence of steplengths $h_i = \frac{1}{N_i} \to 0$ apply the trapezoidal rule to (3.3) and calculate the eigenvalues of the resulting operator. It is sufficient for the existence of small solutions to equation (1.1) that the eigenvalues lie on more than one trajectory.*

4. Numerical experiments. We begin by giving brief details of the numerical approach. We write down the numerical scheme for solving (3.3) using the trapezoidal rule with fixed step length $h = \frac{mw}{N}$ in the form

$$X_{n+1} = X_n + \frac{h}{2}(b_{m,n+1-N}X_{n+1-N} + b_{m,n-N}X_{n-N}).$$

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Here $X_n \approx x(nh), b_{m,n} = b_m(nh)$. We note that, for each fixed value of m, as n varies, $\{b_{m,n}\}$ is a periodic sequence of period N.

Now we construct the companion matrix A_n corresponding to the method and calculate $C = \prod_{i=1}^{N} A_{N-i}$ which represents a discrete analogue of the period map of the solution operator for (3.3). It is the eigenvalues of the matrix C that we shall calculate. These will be marked on the figures with $'+'$.

For reference purposes in our diagram, we also calculate the matrix $\tilde{C}$ which corresponds to the autonomous system formed by replacing $b(t)$ by the constant $\frac{\int_0^{mw} b(s)ds}{mw}$. This time, the eigenvalues will be marked on the figures with $'*'$.

4.1. Numerical results. We illustrate our approach based on some simple equations of the form (1.1). We display the eigenspectrum arising from the discretisation of equation $x'(t) = b_m(t)x(t - mw)$ using the trapezium rule.

EXAMPLE 4.1. We consider two examples of equation (1.1) with $b_0(t) \equiv 0, w = 1, m = 2$. In this case if $b_2(t)$ changes sign on $[0, 1]$ then there can be small solutions. Our theory tells us that, whereas the eigenvalues marked by $'*'$ will always follow a single trajectory, those marked by $'+'$ may follow more than one trajectory, and this will imply the existence of small solutions.

The left-hand eigenspectrum of Figure 4.1 arises from (1.1) with $b_1(t) = \sin 2\pi t + c,\ b_2(t) = \sin 2\pi t + 1.8$ and the right-hand eigenspectrum arises from (1.1) with $b_1(t) = \sin 2\pi t + c,\ b_2(t) = \sin 2\pi t + 0.3$. As expected we observe additional eigenspectra in the case when $b_2(t)$ changes sign.

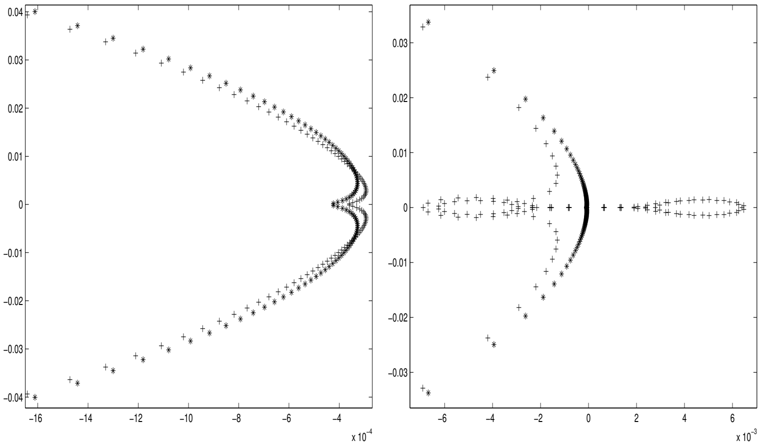

FIG. 4.1. *Left: $b_2(t)$ does not change sign Right: $b_2(t)$ changes sign*

EXAMPLE 4.2. We now give (in Figure 4.2) two eigenspectra resulting from equation (1.1) with $w = 1, m = 4$ and $b_0(t) \not\equiv 0$. (a) $b_0(t) = \sin 2\pi t + 0.6,\ b_1(t) = \sin 2\pi t + 0.3,\ b_2(t) = \sin 2\pi t + 0.2,\ b_3(t) = \sin 2\pi t + 0.7,\ b_4(t) = \sin 2\pi t + 1.4$. (b) $b_0(t) = \sin 2\pi t + 1.8,\ b_1(t) = \sin 2\pi t + 1.3,\ b_2(t) = \sin 2\pi t + 1.2,\ b_3(t) = \sin 2\pi t + 1.7,\ b_4(t) = \sin 2\pi t + 0.4$. As expected we observe additional trajectories in the case when $b_4(t)$ changes sign.

5. Direct discretisation of (1.1). Our long-term goal is to be able to use the numerical approach for the determination of whether or not a delay equation has small solutions without detailed analytical theory being known. This will be important because of the slow progress

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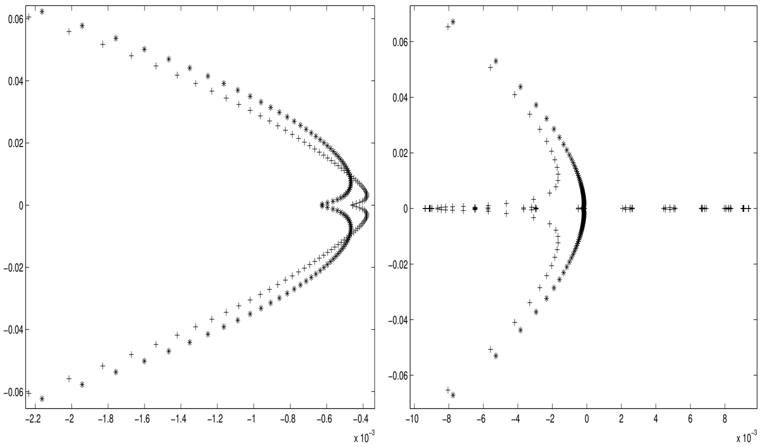

FIG. 4.2. *Left:* $b_4(t)$ *does not change sign* *Right:* $b_4(t)$ *changes sign*

towards obtaining analytical results for more complicated delay equations. Therefore, although the results of the previous section are interesting, it would be more useful to be able to detect small solutions by direct application of our numerical scheme to the delay equation. In this section, we consider the question of whether such a direct application of the numerical techniques will provide a reliable result.

5.1. The numerical approach. We need to introduce some notation: we let $x_n = x(nh)$ and $b_{i,j} = b_i(jh)$ as before. We continue to use numerical methods with constant step size $h = \frac{1}{N} = \frac{m\omega}{N^*}$. We introduce $D_1 \in \mathbb{R}^{1\times(N+1)}$, $D_j \in \mathbb{R}^{1\times N}$ for $j = 2,3,...,m-1$, $D_m \in \mathbb{R}^{1\times(N-1)}$, $D(n) \in \mathbb{R}^{1\times mN}$ and $A(n) \in \mathbb{R}^{(mN+1)\times(mN+1)}$ defined by

1. $D_1 = \begin{pmatrix} \frac{(2+hb_{0,n})}{(2-hb_{0,n+1})} & 0 & \dots & 0 & \frac{h}{(2-hb_{0,n+1})}b_{1,n+1} & \frac{h}{(2-hb_{0,n+1})}b_{1,n} \end{pmatrix}$
2. $D_j = \begin{pmatrix} 0 & \dots & \dots & 0 & \frac{h}{(2-hb_{0,n+1})}b_{j,n+1} & \frac{h}{(2-hb_{0,n+1})}b_{j,n} \end{pmatrix}$
   for $j = 2,3,...,m-1$.
3. $D_m = \begin{pmatrix} 0 & \dots & \dots & 0 & \frac{h}{(2-hb_{0,n+1})}b_{m,n+1} \end{pmatrix}$
4. $D(n) = \begin{pmatrix} D_1 & D_2 & D_3 & \dots & \dots & D_m \end{pmatrix}$
5. $A(n) = \begin{pmatrix} D(n) & \frac{h}{(2-hb_{0,n+1})}b_{m,n} \\ I & 0 \end{pmatrix}$
6. $y_n = \begin{pmatrix} x_n & x_{n-1} & \dots & x_{n-N} & x_{n-1-N} & \dots & x_{n-2N} & x_{n-1-2N} & \dots & x_{n-mN} \end{pmatrix}^T$

Discretisation of (1.1) using the trapezium rule yields

$$x_{n+1} = x_n + \frac{h}{2}\sum_{j=0}^{m}\left(b_{j,n}x_{n-jN} + b_{j,n+1}x_{n+1-jN}\right).$$

which can be written in the form

$$y_{n+1} = A(n)y_n$$

It follows that $y(t+m\omega) \approx y_{n+N^*} = Cy_n$ where $C = \prod_{i=0}^{N^*-1} A(n+i)$. For the single term delay equation, we considered the autonomous problem arising from the replacement of $b_1(t)$, in the non-autonomous problem, by $\int_0^1 b_1(t)dt$. We then compared the eigenspectrum arising from the autonomous problem with that from the non-autonomous problem.

Here we consider the autonomous problem in which we replace each $b_i(t)$ with $\frac{1}{\omega}\int_0^\omega b_i(t)dt$ and we use this to create a constant matrix A.

REMARK 5.1. Our motivation for this approach arises from the fact that the characteristic equation for the Floquet exponents is $det\left(e^{\mu\omega} - e^{w\sum_{j=0}^m \hat{b}_j e^{-j\mu\omega}}\right) = 0$ where $\hat{b}_j = \frac{1}{\omega}\int_0^\omega b_j(s)ds$, for $j = 0, 1, ..., m$. The characteristic matrix for the exponents may be taken to be $\mu = \sum_{j=0}^m \hat{b}_j e^{-j\omega\mu}$, which is the characteristic matrix for the autonomous equation $x'(t) = \sum_{j=0}^m \hat{b}_j x(t - j\omega)$; see [10, p. 249].

By analogy with the previous approach, we are then able to compare the eigenvalues of C with the eigenvalues of A^{N^*}. We want to show that, as before, additional eigenvalue trajectories imply the existence of small solutions.

5.2. A revised characterisation of small solutions. To understand the results of the direct application of the numerical scheme to the multi-term equation (1.1) we shall find it helpful to undertake a discrete Floquet analysis.

The discrete scheme corresponding to (3.1) (using the trapezium rule, as usual) is

$$X_{n+1} = X_n + \frac{h}{2}\sum_{j=0}^{m}\left(b_{j,n}X_{n-jN} + b_{j,n+1}X_{n+1-jN}\right) \tag{5.1}$$

and, for Floquet solutions we let

$$X_n = e^{\mu n h}p_n = \Lambda^n p_n \text{ where } \Lambda = e^{\mu h} \text{ and } \Lambda^N = \lambda.$$

so that

$$X_n = \lambda X_{n-N} = \lambda^m X_{n-mN} \tag{5.2}$$

and we put

$$p_n = p_{n-N}.$$

We can use (5.2) to write (5.1) as

$$X_{n+1} = X_n + \frac{h}{2}\sum_{j=0}^{m}\lambda^{m-j}\left(b_{j,n}X_{n-mN} + b_{j,n+1}X_{n+1-mN}\right).$$

The conclusion of this analysis is more significant than is at first evident. Clearly, for each of the Floquet exponents λ, we have reduced the discrete problem to one with a single delay of mw just as we did in the continuous case. More significantly, (5.2) is the discretisation of (3.2) using the trapezium rule. Therefore the analysis shows that one can undertake the discretisation either before or after the Floquet analysis without affecting the outcome. This is summarised in Figure 5.1.

The Floquet theory now leads to a simple, and possibly more intuitive, characterisation of small solutions to multi-term delay equations.

Assume that equation (1.1) does not possess small solutions. By Theorem 3.1 we know that the Floquet solutions span the solution space of (1.1). In other words, the characteristic roots of (1.1) will be approximated by roots of equations of the form (5.2) for various values of λ. The absence of small solutions will lead, for each fixed λ, to a trajectory of the general form given in Lemma 2.1.

Thus, in the absence of small solutions, we would expect to find that the characteristic roots of (1.1) will lie along multiple trajectories of the same general shape as those we met

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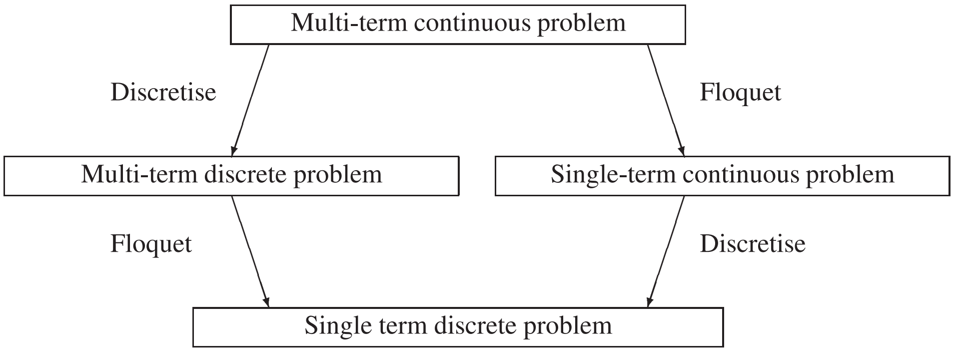

FIG. 5.1. *Rival approaches to the problem*

for single term equations without small solutions. Therefore the appearance of additional trajectories close to the real-axis may be taken to imply the existence of small solutions. We give this formally:

ALGORITHM 5.2. *For a multiterm equation of the form (1.1) apply the trapezoidal rule with a sequence of step lengths $h_i = \frac{1}{N_i} \to 0$ and calculate the eigenvalues of the resulting operator. It is sufficient for the existence of small solutions to (1.1) that either*

1. *the eigenvalues lie on one or more additional trajectories compared to those for the appropriate autonomous equation*
2. *the eigenvalue trajectories include some which loop close to the x-axis.*

5.3. Numerical examples. We present some examples illustrating the results of this aproach. In our diagrams we illustrate the eigenspectrum arising from the non-autonomous problem by '+' and that from the autonomous problem formed by replacing each $b_i(t)$ by $\frac{1}{w}\int_0^w b_i(s)ds$ by '*'.

EXAMPLE 5.3. In our first example we consider four cases of equation (1.1) with $b_0(t) \equiv 0, w = 1, m = 2$. In this case the theory tells us that $b_2(t)$ changes sign on $[0, 1]$ if and only if the equation has small solutions. In Figure 5.2 $b_2(t)$ does not change sign and we observe the proximity of the two trajectories. In Figure 5.3 $b_2(t)$ does change sign and we observe the presence of two additional trajectories in the non-autonomous eigenspectrum which indicates the presence of small solutions.

Our numerical experiments included (especially) cases when $b_2(t) = \sin 2\pi t + c$ and $|c|$ was close to 1. We found that it was still possible to detect the presence of small solutions precisely when $|c| \leq 1$, that is, when $b_2(t)$ changes sign.

EXAMPLE 5.4. We conclude with two eigenspectra resulting from equation (1.1) with $w = 1, m = 4$ and $b_0(t) \not\equiv 0$. (a) $b_0(t) = \sin 2\pi t + 0.6,\ b_1(t) = \sin 2\pi t + 0.3,\ b_2(t) = \sin 2\pi t + 0.2,\ b_3(t) = \sin 2\pi t + 0.7,\ b_4(t) = \sin 2\pi t + 1.4$. (b) $b_0(t) = \sin 2\pi t + 1.8,\ b_1(t) = \sin 2\pi t + 1.3,\ b_2(t) = \sin 2\pi t + 1.2,\ b_3(t) = \sin 2\pi t + 1.7,\ b_4(t) = \sin 2\pi t + 0.4$. In Figure 5.4 we observe the presence of additional trajectories in the right hand eigenspectrum, that is when $b_4(t)$ changes sign, which is in accordance with the theory.

6. Conclusions. In our previous work we successfully used a numerical method to identify whether or not equations of the form (1.1) with $m = 1$ admit small solutions. The discussion above shows that we can adapt our method to identify whether equations of the form (1.1) with any number of terms admit small solutions.

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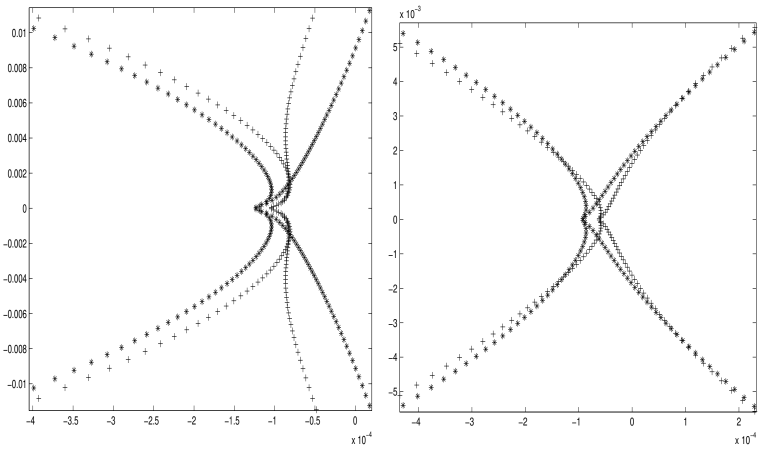

FIG. 5.2.

Left: $b_1(t) = \sin 2\pi t + 0.5$, $b_2(t) = \sin 2\pi t + 1.8$ Right: $b_1(t) = \sin 2\pi t + 1.5$, $b_2(t) = \sin 2\pi t + 1.3$

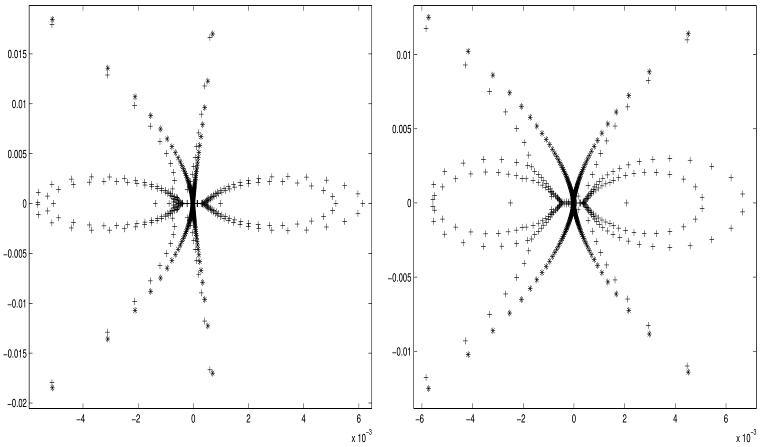

FIG. 5.3.

Left: $b_1(t) = \sin 2\pi t + 0.5$, $b_2(t) = \sin 2\pi t + 0.3$ $Right:$ $b_1(t) = \sin 2\pi t + 1.5$, $b_2(t) = \sin 2\pi t + 0.3$

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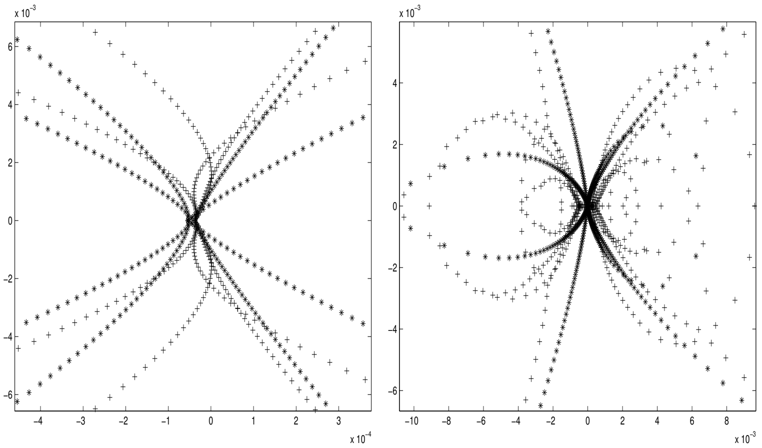

FIG. 5.4. *Left: $b_4(t)$ does not change sign* *Right: $b_4(t)$ changes sign*

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