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ETNA
Kent State University
etna@mcs.kent.edu

OSCILLATION CRITERIA FOR A CERTAIN CLASS OF SECOND ORDER EMDEN–FOWLER DYNAMIC EQUATIONS*

ELVAN AKIN–BOHNER†, MARTIN BOHNER†, AND SAMIR H. SAKER‡

Abstract. By means of Riccati transformation techniques we establish some oscillation criteria for the second order Emden–Fowler dynamic equation on a time scale. Such equations contain the classical Emden–Fowler equation as well as their discrete counterparts. The classical oscillation results of Atkinson (in the superlinear case) and Belohorec (in the sublinear case) are extended in this paper to Emden–Fowler dynamic equations on any time scale.

Key words. oscillation, dynamic equation, time scale, Riccati transformation technique

AMS subject classifications. 34K11, 39A10, 39A99, 34C10, 39A11

1. Introduction. In this paper we shall consider the second order Emden–Fowler dynamic equation

$$(px^{\Delta})^{\Delta}(t) + q(t)x^{\gamma}(\sigma(t)) = 0 \quad \text{for} \quad t \in [a, b] \tag{1.1}$$

on a time scale, where p and q are positive, real-valued rd-continuous functions, and γ is an odd positive integer. We shall also consider the two cases

$$\int_{t_0}^{\infty} \frac{\Delta t}{p(t)} = \infty \tag{1.2}$$

and

$$\int_{t_0}^{\infty} \frac{\Delta t}{p(t)} < \infty. \tag{1.3}$$

In the case of $\gamma > 1$, (1.1) is the prototype of a wide class of nonlinear dynamic equations called Emden–Fowler superlinear dynamic equations, and if $0 < \gamma < 1$, then (1.1) is the prototype of dynamic equations called Emden–Fowler sublinear dynamic equations. It is interesting to study (1.1) because the continuous version, i.e., when t is a continuous variable, has several physical applications, see e.g., [20] and when t is a discrete variable it is a difference equation of Emden–Fowler type and also is important in applications. By a solution of (1.1) we mean a nontrivial real-valued function x satisfying equation (1.1) for $t \geq t_0 \geq a$ for some $t_0 \geq a > 0$. A solution x of (1.1) is said to be *oscillatory* if it is neither eventually positive nor eventually negative; otherwise it is called *nonoscillatory*. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory. Our attention is restricted to those solutions of (1.1) which exist on some half line $[t_x, \infty)$ and satisfy $\sup\{|x(t)| : t > t_0\} > 0$ for any $t_0 \geq t_x$.

Much recent attention has been given to dynamic equations on time scales (or measure chains), and we refer the reader to the landmark paper of Hilger [20] for a comprehensive treatment of the subject. Since then several authors have expounded on various aspects of this new theory; see the survey paper by Agarwal, Bohner, O'Regan, and Peterson [1] and the references cited therein. Two books on the subject of time scales, by Bohner and Peterson [9, 10], summarize and organize much of the time scale calculus.

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†Department of Mathematics and Statistics, University of Missouri–Rolla, Rolla, MO 65409 (akine, bohner@umr.edu).
‡Mansoura University, Department of Mathematics, Mansoura 35516, Egypt (shsaker@mans.edu.eg).

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Kent State University
etna@mcs.kent.edu

In recent years there has been much research activity concerning the oscillation and nonoscillation of solutions of dynamic equations on time scales. We refer the reader to the papers [2, 3, 4, 7, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21, 23].

In [3], Akın-Bohner and Hoffacker considered the second order dynamic equation

$$x^{\Delta\Delta}(t) + q(t)x^{\gamma}(\sigma(t)) = 0 \tag{1.4}$$

and gave necessary and sufficient conditions for oscillation of all solutions when $\gamma > 1$ and $0 < \gamma < 1$. Their results cannot be applied in the case when $\gamma = 1$ and applied only to discrete time scales.

In this paper we use the Riccati transformation technique to obtain some oscillation criteria for (1.1) when (1.2) or (1.3) holds. Our results can be applied in the case $\gamma = 1$ and also for any time scale. So our results extend and improve the results established by Akın-Bohner and Hoffacker [3]. The paper is organized as follows. In the next section we present some basic definitions concerning the calculus on time scales. In Section 3 we develop a Riccati transformation technique and give some fundamental lemmas. These lemmas are used to give sufficient conditions for oscillation of all solutions of (1.1) in the superlinear case, i.e., when $\gamma \geq 1$ (in Section 4) and in the sublinear case, i.e., when $\gamma \in (0, 1)$ (see Section 5). Our results when (1.2) holds are sufficient for oscillation of all solutions of (1.1), and when (1.3) holds our results ensure that all solutions are either oscillatory or converge to zero. For the superlinear case we present an extension of the classical Atkinson [5] result, and for the sublinear case we present an extension of the classical Belohorec [6] result.

Since we are interested in oscillatory behavior, we suppose that the time scale under consideration is not bounded above, i.e., it is a time scale interval of the form $[a, \infty)$.

2. Some Preliminaries on Time Scales. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. On any time scale $\mathbb{T}$ we define the *forward* and *backward jump operators* by

$$\sigma(t) := \inf\{s \in \mathbb{T} : \ s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} : \ s < t\}.$$

A point $t \in \mathbb{T}$ with $t > \inf \mathbb{T}$ is said to be left-dense if $\rho(t) = t$ and right-dense if $\sigma(t) = t$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. The *graininess* function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$. For a function $f : \mathbb{T} \to \mathbb{R}$ (the range $\mathbb{R}$ of f may actually be replaced by any Banach space) the (delta) *derivative* is defined by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} \tag{2.1}$$

if f is continuous at t and t is right-scattered. If t is not right-scattered, then the derivative is defined by

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{t - s} = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} \tag{2.2}$$

provided this limit exists. A function $f : [a, b] \to \mathbb{R}$ is said to be rd-continuous if it is continuous at each right-dense point and there exists a finite left limit at all left-dense points, and f is said to be differentiable if its derivative exists. A useful formula is

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t). \tag{2.3}$$

We will make use of the product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^{\sigma} \neq 0$) of two differentiable functions f and g

$$(fg)^{\Delta} = f^{\Delta}g + f^{\sigma}g^{\Delta} = fg^{\Delta} + f^{\Delta}g^{\sigma} \quad \text{and} \quad \left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta}g - fg^{\Delta}}{gg^{\sigma}}. \tag{2.4}$$

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Kent State University
etna@mcs.kent.edu

For $a, b \in \mathbb{T}$ and a differentiable function f, the Cauchy integral of f^Δ is defined by

$$\int_a^b f^\Delta(t)\Delta t = f(b) - f(a).$$

An integration by parts formula reads

$$\int_a^b f(t)g^\Delta(t)\Delta t = [f(t)g(t)]_a^b - \int_a^b f^\Delta(t)g(\sigma(t))\Delta t,$$

and improper integrals are defined as

$$\int_a^\infty f(t)\Delta t = \lim_{b\to\infty}\int_a^b f(t)\Delta t.$$

Note that in the case $\mathbb{T} = \mathbb{R}$ we have

$$\sigma(t) = \rho(t) = t, \quad f^\Delta(t) = f'(t), \quad \int_a^b f(t)\Delta t = \int_a^b f(t)dt,$$

and in the case $\mathbb{T} = \mathbb{Z}$ we have

$$\sigma(t) = t+1, \quad \rho(t) = t-1, \quad f^\Delta(t) = \Delta f(t) = f(t+1)-f(t), \quad \int_a^b f(t)\Delta t = \sum_{i=a}^{b-1} f(i).$$

3. A Riccati Transformation. Crucial for our calculations is the following lemma.

LEMMA 3.1. *If z and x are differentiable on a time scale $\mathbb{T}$ with $x(t) \neq 0$ for all $t \in \mathbb{T}$, then we have*

$$x^\Delta\left(\frac{z^2}{x}\right)^\Delta = \left(z^\Delta\right)^2 - xx^\sigma\left[\left(\frac{z}{x}\right)^\Delta\right]^2. \tag{3.1}$$

Proof. Using (2.3), (2.4), and $(z^2)^\Delta = z^\Delta(z+z^\sigma) = z^\Delta(2z+\mu z^\Delta)$, we obtain

$$\begin{aligned}
&x^\Delta\left(\frac{z^2}{x}\right)^\Delta + xx^\sigma\left[\left(\frac{z}{x}\right)^\Delta\right]^2\\
&= x^\Delta\frac{z^\Delta(2z+\mu z^\Delta)x - z^2x^\Delta}{xx^\sigma} + xx^\sigma\left[\frac{z^\Delta x - zx^\Delta}{xx^\sigma}\right]^2\\
&= \frac{2xx^\Delta zz^\Delta - \left(zx^\Delta\right)^2 + \left(z^\Delta x - zx^\Delta\right)^2 + \mu x^\Delta x\left(z^\Delta\right)^2}{xx^\sigma}\\
&= \frac{\left(z^\Delta x\right)^2 + \left(x^\sigma - x\right)x\left(z^\Delta\right)^2}{xx^\sigma}\\
&= \left(z^\Delta\right)^2.
\end{aligned}$$

This shows that (3.1) holds. ∎

Using Lemma 3.1, we now derive the following result.

THEOREM 3.2. *Suppose x solves (1.1) such that $x(t) \neq 0$ for all $t \in \mathbb{T}$. Let z be any differentiable function and define w by the* Riccati substitution

$$w = \frac{z^2px^\Delta}{x^\gamma}. \tag{3.2}$$

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Kent State University
etna@mcs.kent.edu

Then we have

$$-w^{\Delta} = q\left(z^{\sigma}\right)^2 - \frac{px^{\Delta}}{\left(x^{\gamma}\right)^{\Delta}}\left(z^{\Delta}\right)^2 + \frac{px^{\Delta}\left(xx^{\sigma}\right)^{\gamma}}{\left(x^{\gamma}\right)^{\Delta}}\left[\left(\frac{z}{x^{\gamma}}\right)^{\Delta}\right]^2. \tag{3.3}$$

Proof. We use again (2.3) and (2.4) to find

$$\begin{aligned}
-w^{\Delta} &= -\left[\frac{z^2}{x^{\gamma}}px^{\Delta}\right]^{\Delta} \\
&= -\left\{\left(\frac{z^2}{x^{\gamma}}\right)^{\sigma}\left(px^{\Delta}\right)^{\Delta} + \left(\frac{z^2}{x^{\gamma}}\right)^{\Delta} px^{\Delta}\right\} \\
&= q\left(z^{\sigma}\right)^2 - px^{\Delta}\left(\frac{z^2}{x^{\gamma}}\right)^{\Delta} \\
&= q\left(z^{\sigma}\right)^2 - \frac{px^{\Delta}}{\left(x^{\gamma}\right)^{\Delta}}\left(x^{\gamma}\right)^{\Delta}\left(\frac{z^2}{x^{\gamma}}\right)^{\Delta}.
\end{aligned}$$

Now applying formula (3.1) from Lemma 3.1 (with x replaced by x^{γ}), we arrive at (3.3). ☐

We will use the above Theorem 3.2 several times in the remainder of this paper, in conjunction with the formula

$$\frac{\left(x^{\gamma}\right)^{\Delta}}{x^{\Delta}} = \gamma\int_0^1 \left[hx^{\sigma} + (1-h)x\right]^{\gamma-1} dh, \tag{3.4}$$

which is a simple consequence of Keller's chain rule [9, Theorem 1.90].

The following result is used frequently in the remainder of this paper.

LEMMA 3.3. *Assume that* (1.2) *holds. If x is a solution of* (1.1) *such that $x(t) > 0$ for all $t \geq t_0$, then*

$$x^{\Delta}(t) \geq 0 \quad \text{for} \quad t \geq t_0. \tag{3.5}$$

Proof. In view of (1.1) we have

$$\left(px^{\Delta}\right)^{\Delta}(t) = -q(t)x^{\gamma}(\sigma(t)) < 0 \tag{3.6}$$

for all $t \geq t_0$, and so $y := px^{\Delta}$ is an eventually decreasing function. We first show that y is eventually nonnegative. Suppose there exists $t_1 \geq t_0$ such that $y(t_1) < 0$. Then from (3.6) we have $y(t) < y(t_1)$ for $t \geq t_1$. Hence

$$x^{\Delta}(t) \leq \frac{y(t_1)}{p(t)},$$

which implies by (1.2) that

$$x(t) \leq x(t_1) + y(t_1)\int_{t_1}^{t}\frac{\Delta s}{p(s)} \to -\infty \quad \text{as} \quad t \to \infty, \tag{3.7}$$

contradicting the fact that $x(t) > 0$ for all $t \geq t_0$. Hence $y(t) = p(t)x^{\Delta}(t)$ is eventually nonnegative. Therefore, we see that there exists some t_0 such that

$$x(t) > 0, \quad x^{\Delta}(t) \geq 0, \quad y(t) \geq 0, \quad y^{\Delta}(t) < 0 \quad \text{for} \quad t \geq t_0.$$

Hence (3.5) holds. ☐

ETNA
Kent State University
etna@mcs.kent.edu

4. Oscillation Criteria in the Superlinear Case. In this section we give some oscillation criteria when $\gamma \in \mathbb{N}$ is odd.

First we consider the case when (1.2) holds.

THEOREM 4.1. *Assume that* (1.2) *holds. Furthermore, assume that there exists a differentiable function* z *such that*

$$\limsup_{t\to\infty} \int_a^t \left[q(s) \left(z^\sigma(s) \right)^2 - K^{\gamma-1} p(s) \left(z^\Delta(s) \right)^2 \right] \Delta s = \infty \tag{4.1}$$

holds for all constants $K > 0$. *Then every solution of* (1.1) *is oscillatory on* $[a, \infty)$.

Proof. Suppose to the contrary that x is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that x is an eventually positive solution of (1.1) such that $x(t) > 0$ for all $t \geq t_0 > a$. We shall consider only this case, since the substitution $\tilde{x} = -x$ transforms equation (1.1) into an equation of the same form. By Lemma 3.3 we obtain that (3.5) holds. Now note that $\gamma \geq 1$ and (3.4) imply

$$\begin{aligned} \frac{(x^\gamma)^\Delta(t)}{x^\Delta(t)} &\overset{(3.4)}{=} \gamma \int_0^1 \left[h x^\sigma(t) + (1-h) x(t) \right]^{\gamma-1} dh \\ &\geq \gamma \int_0^1 \left[h x(t) + (1-h) x(t) \right]^{\gamma-1} dh \\ &= \gamma (x(t))^{\gamma-1} \\ &\overset{(3.5)}{\geq} \gamma (x(t_0))^{\gamma-1} \\ &= \frac{1}{M^{\gamma-1}}, \end{aligned}$$

where we put $M := \left(\gamma^{1/(\gamma-1)} x(t_0) \right)^{-1}$. Note $M > 0$. Now define the function w on $[t_0, \infty)$ by (3.2). Then (3.5) implies $w(t) \geq 0$ for all $t \geq t_0$. Therefore, using (3.3) from Theorem 3.2, we obtain

$$\begin{aligned} w(t_0) &\geq w(t_0) - w(t) \\ &= -\int_{t_0}^t w^\Delta(s) \Delta s \\ &= \int_{t_0}^t \left\{ q(s) \left(z^\sigma(s) \right)^2 - p(s) \frac{x^\Delta(s)}{(x^\gamma)^\Delta(s)} \left(z^\Delta(s) \right)^2 \right\} \Delta s \\ &\quad + \int_{t_0}^t p(s) \frac{x^\Delta(s)}{(x^\gamma)^\Delta(s)} \left(x(s) x^\sigma(s) \right)^\gamma \left[\left(\frac{z}{x^\gamma} \right)^\Delta (s) \right]^2 \Delta s \\ &\geq \int_{t_0}^t \left\{ q(s) \left(z^\sigma(s) \right)^2 - p(s) \frac{x^\Delta(s)}{(x^\gamma)^\Delta(s)} \left(z^\Delta(s) \right)^2 \right\} \Delta s \\ &\geq \int_{t_0}^t \left\{ q(s) \left(z^\sigma(s) \right)^2 - M^{\gamma-1} p(s) \left(z^\Delta(s) \right)^2 \right\} \Delta s \\ &\overset{(4.1)}{\to} \infty \quad \text{as} \quad t \to \infty, \end{aligned}$$

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Kent State University
etna@mcs.kent.edu

which is impossible. The proof is complete. ∎

COROLLARY 4.2. *Assume that (1.2) holds. Furthermore, assume that there exists a positive differentiable function δ such that*

$$\limsup_{t\to\infty}\int_a^t\left[q(s)\delta^\sigma(s)-K^{\gamma-1}p(s)\left(\frac{\delta^\Delta(s)}{\sqrt{\delta(s)}+\sqrt{\delta^\sigma(s)}}\right)^2\right]\Delta s=\infty \tag{4.2}$$

holds for all constants $K>0$. Then every solution of (1.1) is oscillatory on $[a,\infty)$.

Proof. Define $z=\sqrt{\delta}$ and note that [9]

$$z^\Delta=\frac{\delta^\Delta}{\sqrt{\delta}+\sqrt{\delta^\sigma}}.$$

Since (4.2) holds for δ, (4.1) holds for z. So the claim follows by Theorem 4.1. ∎

From Theorem 4.1 and Corollary 4.2 we can obtain different conditions for oscillation of all solutions of (1.1) by different choices of δ. For instance, we obtain the following two corollaries if we choose $\delta(t)\equiv 1$ and $\delta(t)=t$, respectively. The first choice confirms that the Leighton–Wintner theorem is valid for Emden–Fowler dynamic equations.

COROLLARY 4.3 (Leighton–Wintner). *Assume*

$$\int_a^\infty\frac{\Delta t}{p(t)}=\infty\quad\textit{and}\quad\int_a^\infty q(t)\Delta t=\infty. \tag{4.3}$$

Then every solution of (1.1) is oscillatory on $[a,\infty)$.

COROLLARY 4.4. *Assume that (1.2) holds. Furthermore, assume that*

$$\limsup_{t\to\infty}\int_a^t\left[q(s)\sigma(s)-K^{\gamma-1}p(s)\frac{1}{\left(\sqrt{s}+\sqrt{\sigma(s)}\right)^2}\right]\Delta s=\infty \tag{4.4}$$

holds for all constants $K>0$. Then every solution of (1.1) is oscillatory on $[a,\infty)$.

The next result is the same as [3, Theorem 5] when $p(t)\equiv 1$. But we note that [3, Theorem 5] cannot be applied in the case when $\gamma=1$ and also not for the second order Emden–Fowler differential equation, i.e., when $\mathbb{T}=\mathbb{R}$. So the following result extends and improves in various ways the results established in [3]. Its classical version was given in 1955 by Atkinson [5].

THEOREM 4.5. *Assume that (1.2) holds. Define*

$$P(t)=\int_a^t\frac{\Delta s}{p(s)}.$$

If

$$\limsup_{t\to\infty}\int_a^t P(\sigma(s))q(s)\Delta s=\infty, \tag{4.5}$$

then every solution of (1.1) is oscillatory on $[a,\infty)$.

Proof. Again we suppose x is a solution of (1.1) such that $x(t)>0$ for all $t\ge t_0$. By Lemma 3.3 we obtain (3.5). Now we let $z=\sqrt{P}$ and define the Riccati substitution w by

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Kent State University
etna@mcs.kent.edu

(3.2). Using the product rule from (2.4), we calculate

$$\begin{aligned} w^{\Delta} &= \left\{ \frac{1}{p} p x^{\Delta} + P^{\sigma} (p x^{\Delta})^{\Delta} \right\} (x^{-\gamma})^{\sigma} + P p x^{\Delta} (x^{-\gamma})^{\Delta} \\ &= x^{\Delta} (x^{-\gamma})^{\sigma} - P^{\sigma} q + P p x^{\Delta} (x^{-\gamma})^{\Delta} \\ &\leq \frac{(x^{1-\gamma})^{\Delta}}{1-\gamma} - P^{\sigma} q, \end{aligned}$$

where the last inequality is true because $(x^{-\gamma})^{\Delta} \leq 0$ due to (3.4) and because

$$\begin{aligned} \frac{\left(x^{1-\gamma}\right)^{\Delta}(t)}{x^{\Delta}(t)} &\overset{(3.4)}{=} (1-\gamma) \int_0^1 \left[h x^{\sigma}(t) + (1-h) x(t) \right]^{-\gamma} dh \\ &\leq (1-\gamma) \int_0^1 \left[h x^{\sigma}(t) + (1-h) x^{\sigma}(t) \right]^{-\gamma} dh \\ &= (1-\gamma)(x^{\sigma}(t))^{-\gamma}. \end{aligned}$$

Upon integration we arrive at

$$\begin{aligned} \int_{t_0}^{t} P(\sigma(s)) q(s) \Delta s &\leq \int_{t_0}^{t} \left\{ \frac{x^{1-\gamma}}{1-\gamma} - w \right\}^{\Delta} (s) \Delta s \\ &= \frac{x^{1-\gamma}(t)}{1-\gamma} - w(t) - \frac{x^{1-\gamma}(t_0)}{1-\gamma} + w(t_0) \\ &\leq \frac{x^{1-\gamma}(t_0)}{\gamma - 1} + w(t_0). \end{aligned}$$

This contradicts (4.5) and finishes the proof. ◻

Putting $p(t) \equiv 1$, i.e., $P(t) = t$ in Theorem 4.5, we obtain the following corollary.

COROLLARY 4.6. *Assume* $p(t) \equiv 1$. *If*

$$\limsup_{t \to \infty} \int_a^t \sigma(s) q(s) \Delta s = \infty, \tag{4.6}$$

then every solution of (1.1) *is oscillatory on* $[a, \infty)$.

EXAMPLE 4.7. *Consider the dynamic equation*

$$x^{\Delta\Delta} + \frac{1}{t\sigma(t)} (x^{\sigma})^2 = 0 \quad \textit{for} \quad t \geq 1. \tag{4.7}$$

Here $p(t) \equiv 1$ *and* $q(t) = \frac{1}{t\sigma(t)}$. *Using [8, Theorem 5.11], we find*

$$\int_{t_0}^{\infty} \sigma(s) q(s) \Delta s = \int_{t_0}^{\infty} \frac{\Delta s}{s} = \infty.$$

Hence, by Corollary 4.6, equation (4.7) *is oscillatory on* $[1, \infty)$.

THEOREM 4.8. *Assume that* (1.2) *holds. If there exists a positive differentiable function* δ *and an odd integer* $m \in \mathbb{N}$ *such that*

$$\limsup_{t \to \infty} \frac{1}{t^m} \int_a^t (t-s)^m \left[\delta(\sigma(s)) q(s) - \frac{K^{\gamma-1} (\delta^{\Delta}(s))^2 p(s)}{\left(\sqrt{\delta(s)} + \sqrt{\delta(\sigma(s))} \right)^2} \right] \Delta s = \infty \tag{4.8}$$

holds for all $K > 0$*, then every solution of* (1.1) *oscillates on* $[a, \infty)$.

Proof. The proof is similar to [23, Theorem 3.2] and hence is omitted. ∎

Note that when $\delta(t) \equiv 1$, then (4.8) reduces to

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_a^t (t-s)^m q(s)\Delta s = \infty,$$

which can be considered as an extension of Kamenev type oscillation criteria for second order differential equations (see [22]).

Next we consider the case when (1.3) **holds.** Now we give some sufficient conditions when (1.3) holds, which guarantee that every solution of (1.1) oscillates or converges to zero on $[a, \infty)$.

THEOREM 4.9. *Assume that* (1.3) *holds. Furthermore, assume that there exists a positive function* z *such that* (4.1) *holds, and*

$$\int_a^\infty \left[\frac{1}{p(t)} \int_a^t q(s)\Delta s\right] \Delta t = \infty. \tag{4.9}$$

Then every solution of equation (1.1) *is oscillatory or converges to zero on* $[a, \infty)$.

THEOREM 4.10. *Assume that* (1.3) *holds. Furthermore, assume that there exists a positive function* δ *such that* (4.8) *and* (4.9) *hold. Then every solution of equation* (1.1) *is oscillatory or converges to zero on* $[a, \infty)$.

5. Oscillation Criteria in the Sublinear Case. In this section we give some new oscillation criteria for (1.1) when $\gamma \in (0, 1)$ is a quotient of odd positive integers.

First we consider the case when (1.2) **holds.**

THEOREM 5.1. *Assume that* (1.2) *holds and suppose that* p *is differentiable and nondecreasing. Furthermore, assume that there exists a differentiable function* z *such that*

$$\limsup_{t\to\infty} \int_a^t \left[q(s)\left(z^\sigma(s)\right)^2 - K^{\gamma-1}(\sigma(s))^{1-\gamma} p(s)\left(z^\Delta(s)\right)^2\right] \Delta s = \infty \tag{5.1}$$

holds for all constants $K > 0$*. Then every solution of* (1.1) *is oscillatory on* $[a, \infty)$.

Proof. Suppose to the contrary that x is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that x is an eventually positive solution of (1.1) such that $x(t) > 0$ for all $t \ge t_0 > a$. Hence, by Lemma 3.3, we obtain (3.5), which implies

$$0 \ge (px^\Delta)^\Delta(t) = p^\Delta(t)x^\Delta(t) + p^\sigma(t)x^{\Delta\Delta}(t)$$

so that, again by using (3.5), $x^{\Delta\Delta}(t) \le 0$ for all $t \ge t_0$. Hence x^Δ is nonincreasing on $[t_0, \infty)$, and therefore we obtain

$$x(t) = x(t_0) + \int_{t_0}^t x^\Delta(s)\Delta s \le \alpha + \beta t,$$

where $\alpha = x(t_0) - t_0 x^\Delta(t_0)$ and $\beta = x^\Delta(t_0)$. By putting $L = |\alpha| + \beta$ and $t_1 \ge \max\{t_0, 1\}$, we find that

$$x(t) \le Lt \quad \text{for all} \quad t \ge t_1. \tag{5.2}$$

ETNA
Kent State University
etna@mcs.kent.edu

Now note that $\gamma \in (0,1)$ and (3.4) imply

$$\begin{aligned} \frac{(x^\gamma)^\Delta(t)}{x^\Delta(t)} &\overset{(3.4)}{=} \gamma \int_0^1 \left[hx^\sigma(t) + (1-h)x(t)\right]^{\gamma-1} dh \\ &\geq \gamma \int_0^1 \left[hx^\sigma(t) + (1-h)x^\sigma(t)\right]^{\gamma-1} dh \\ &= \gamma (x^\sigma(t))^{\gamma-1} \\ &\overset{(5.2)}{\geq} \gamma (L\sigma(t))^{\gamma-1} \\ &= \frac{(\sigma(t))^{\gamma-1}}{M^{\gamma-1}}, \end{aligned}$$

where we put $M := \left(\gamma^{1/(\gamma-1)}L\right)^{-1}$. Note $M > 0$. Now define the function w on $[t_0, \infty)$ by (3.2). Then (3.5) implies $w(t) \geq 0$ for all $t \geq t_0$. Therefore, using (3.3) from Theorem 3.2, we obtain

$$\begin{aligned} w(t_1) &\geq w(t_1) - w(t) \\ &= -\int_{t_1}^t w^\Delta(s)\Delta s \\ &= \int_{t_1}^t \left\{ q(s)\left(z^\sigma(s)\right)^2 - p(s)\frac{x^\Delta(s)}{(x^\gamma)^\Delta(s)}\left(z^\Delta(s)\right)^2 \right\}\Delta s \\ &\quad + \int_{t_1}^t p(s)\frac{x^\Delta(s)}{(x^\gamma)^\Delta(s)}\left(x(s)x^\sigma(s)\right)^\gamma \left[\left(\frac{z}{x^\gamma}\right)^\Delta(s)\right]^2 \Delta s \\ &\geq \int_{t_1}^t \left\{ q(s)\left(z^\sigma(s)\right)^2 - p(s)\frac{x^\Delta(s)}{(x^\gamma)^\Delta(s)}\left(z^\Delta(s)\right)^2 \right\}\Delta s \\ &\geq \int_{t_1}^t \left\{ q(s)\left(z^\sigma(s)\right)^2 - M^{\gamma-1}(\sigma(s))^{1-\gamma}p(s)\left(z^\Delta(s)\right)^2 \right\}\Delta s \\ &\overset{(5.1)}{\to} \infty \quad \text{as} \quad t \to \infty, \end{aligned}$$

which is impossible. The proof is complete. $\square$

The next result follows as in the proof of Corollary 4.2.

COROLLARY 5.2. *Assume that* (1.2) *holds and suppose that p is differentiable and nondecreasing. Furthermore, assume that there exists a positive differentiable function δ such that*

$$\limsup_{t\to\infty} \int_a^t \left[q(s)\delta^\sigma(s) - K^{\gamma-1}p(s)(\sigma(s))^{1-\gamma}\left(\frac{\delta^\Delta(s)}{\sqrt{\delta(s)}+\sqrt{\delta^\sigma(s)}}\right)^2 \right]\Delta s = \infty \tag{5.3}$$

holds for all constants $K > 0$. Then every solution of (1.1) *is oscillatory on* $[a, \infty)$.

From Theorem 5.1 and Corollary 5.2 we can obtain different conditions for oscillation of all solutions of (1.1) by different choices of δ. For instance, we obtain the following two corollaries if we choose $\delta(t) \equiv 1$ and $\delta(t) = t$, respectively.

COROLLARY 5.3 (Leighton–Wintner). *Suppose p is differentiable and nondecreasing and assume*

$$\int_a^\infty \frac{\Delta t}{p(t)} = \infty \quad \textit{and} \quad \int_a^\infty q(t)\Delta t = \infty. \tag{5.4}$$

ETNA
Kent State University
etna@mcs.kent.edu

Then every solution of (1.1) *is oscillatory on* $[a, \infty)$.

COROLLARY 5.4. *Assume that* (1.2) *holds and suppose that* p *is differentiable and nondecreasing. Furthermore, assume that*

$$\limsup_{t\to\infty} \int_a^t \left[q(s)\sigma(s) - K^{\gamma-1}(\sigma(s))^{1-\gamma} p(s) \frac{1}{\left(\sqrt{s} + \sqrt{\sigma(s)}\right)^2} \right] \Delta s = \infty \tag{5.5}$$

holds for all constants $K > 0$. *Then every solution of* (1.1) *is oscillatory on* $[a, \infty)$.

The next result gives a condition for oscillation in the sublinear case. Its classical version was given in 1961 by Belohorec [6].

THEOREM 5.5. *Assume that* (1.2) *holds. If*

$$\int_a^\infty \left(\frac{t}{p(t)} \right)^\gamma q(t) \Delta t = \infty, \tag{5.6}$$

then every solution of (1.1) *is oscillatory on* $[a, \infty)$.

Proof. We suppose that x is a solution of (1.1) satisfying $x(t) > 0$ for all $t \geq t_0$ and let $y = px^\Delta$. Then by Lemma 3.3 we obtain that $y(t) > 0$ and $y^\Delta(t) < 0$ for all $t \geq t_0$. First observe that

$$x(t) = x(t_0) + \int_{t_0}^t x^\Delta(s)\Delta s \geq x^\Delta(t)(t - t_0) \geq \frac{t}{2} x^\Delta(t)$$

for all $t \geq t_2$ if $t_2 > 2t_0$. Next note that

$$\begin{aligned} \frac{\left(y^{1-\gamma}\right)^\Delta(t)}{y^\Delta(t)} &\overset{(3.4)}{=} (1-\gamma) \int_0^1 \left[hy^\sigma(t) + (1-h)y(t) \right]^{-\gamma} dh \\ &\geq (1-\gamma) \int_0^1 \left[hy(t) + (1-h)y(t) \right]^{-\gamma} dh \\ &= (1-\gamma)(y(t))^{-\gamma}. \end{aligned}$$

Using these two inequalities, we obtain after dividing (1.1) by $(y(t))^\gamma$ for all $t \geq t_2$,

$$\begin{aligned} 0 &= \frac{y^\Delta(t) + q(t)(x^\sigma)^\gamma(t)}{(y(t))^\gamma} \\ &\geq y^\Delta(t)(y(t))^{-\gamma} + q(t) \left(\frac{x(t)}{p(t)x^\Delta(t)} \right)^\gamma \\ &\geq \left(\frac{y^{1-\gamma}}{1-\gamma} \right)^\Delta (t) + q(t) \left(\frac{t}{2p(t)} \right)^\gamma. \end{aligned}$$

Upon integration we arrive at

$$\begin{aligned} \int_{t_2}^t q(s) \left(\frac{s}{p(s)} \right)^\gamma \Delta s &\leq \int_{t_2}^t \frac{2^\gamma}{\gamma - 1} (y^{1-\gamma})^\Delta(s) \Delta s \\ &= \frac{2^\gamma}{1-\gamma} y^{1-\gamma}(t_2) - \frac{2^\gamma}{1-\gamma} y^{1-\gamma}(t) \\ &\leq \frac{2^\gamma}{1-\gamma} y^{1-\gamma}(t_2). \end{aligned}$$

This contradicts (5.6) and finishes the proof. ☐

Putting $p(t) \equiv 1$ in Theorem 5.5, we obtain the following corollary.

COROLLARY 5.6. *Assume that* $p(t) \equiv 1$. *If*

$$\int_a^\infty t^\gamma q(t)\Delta t = \infty, \tag{5.7}$$

then every solution of (1.1) *is oscillatory on* $[a, \infty)$.

EXAMPLE 5.7. *Consider the dynamic equation*

$$x^{\Delta\Delta} + \frac{1}{t^{4/3}}\left(x^\sigma\right)^{1/3} = 0 \quad \textit{for} \quad t \geq 1. \tag{5.8}$$

Here $p(t) \equiv 1$ *and* $q(t) = 1/t^{4/3}$. *As in Example 4.7 it follows from Corollary 5.6 that* (5.8) *is oscillatory on* $[1, \infty)$.

THEOREM 5.8. *Assume that* p *is differentiable and nondecreasing and suppose that* (1.2) *holds. If there exists a positive differentiable function* δ *and an odd integer* $m \in \mathbb{N}$ *such that*

$$\limsup_{t\to\infty} \frac{1}{t^m}\int_a^t (t-s)^m \left[\delta(\sigma(s))q(s) - \frac{K^{\gamma-1}(\sigma(s))^{1-\gamma}(\delta^\Delta(s))^2 p(s)}{\left(\sqrt{\delta(s)} + \sqrt{\delta(\sigma(s))}\right)^2}\right]\Delta s = \infty \tag{5.9}$$

for all constants $K > 0$, *then every solution of* (1.1) *oscillates on* $[a, \infty)$.

Proof. The proof is similar to [23, Theorem 3.2] and hence is omitted. ☐

Next we consider the case when (1.3) holds. Now we give some sufficient conditions when (1.3) holds, which guarantee that every solution of (1.1) oscillates or converges to zero on $[a, \infty)$.

THEOREM 5.9. *Assume that* $p^\Delta(t) \geq 0$ *and that* (1.3) *holds. Furthermore, assume that there exists a positive function* z *such that* (5.1) *holds, and*

$$\int_a^\infty \frac{1}{p(t)}\int_a^t q(s)\Delta s\Delta t = \infty. \tag{5.10}$$

Then every solution of equation (1.1) *is oscillatory or converges to zero on* $[a, \infty)$.

THEOREM 5.10. *Assume that* $p^\Delta(t) \geq 0$ *and that* (1.3) *holds. Furthermore, assume that there exists a positive function* δ *such that* (5.9) *and* (5.10) *hold. Then every solution of equation* (1.1) *is oscillatory or converges to zero on* $[a, \infty)$.

REMARK 5.11. *Note that our results also can be extended to the more general equation*

$$(p(t)x^\Delta(t))^\Delta + q(t)\left|x(\sigma(t))\right|^\gamma \operatorname{sgn}\left(x(\sigma(t))\right) = 0 \quad \textit{for} \quad t \in [a, b],$$

where $\gamma > 0$ *to cover the case when* γ *is even.*

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ETNA
Kent State University
etna@mcs.kent.edu

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