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ORTHOGONAL POLYNOMIALS AND RECURRENCE EQUATIONS, OPERATOR EQUATIONS AND FACTORIZATION*

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Abstract. This article surveys the classical orthogonal polynomial systems of the Hahn class, which are solutions of second-order differential, difference or q-difference equations.

Orthogonal families satisfy three-term recurrence equations. Example applications of an algorithm to determine whether a three-term recurrence equation has solutions in the Hahn class—implemented in the computer algebra system *Maple*—are given.

Modifications of these families, in particular associated orthogonal systems, satisfy fourth-order operator equations. A factorization of these equations leads to a solution basis.

Key words. orthogonal polynomials, Hahn class, differential equations, difference equations, q-difference equations, hypergeometric functions, factorization of operator polynomials, computer algebra, Maple

AMS subject classifications. 33C45, 33C20, 33D45, 33D15, 39A70

1. Classical Orthogonal Polynomials. To define families of orthogonal polynomials on the real line, one uses a *scalar product*

$$\langle f, g\rangle := \int_a^b f(x)g(x)d\mu(x)$$

with nonnegative measure $\mu(x)$ and support in a real interval $[a, b]$ (which can be infinite in one or both directions).

As special cases one considers

- absolutely continuous measures $d\mu(x) = \rho(x)dx$,
- discrete measures $\rho(x)$ supported in $\mathbb{Z}$,
- and discrete measures $\rho(x)$ supported in $q^{\mathbb{Z}}$ for some base $q \in \mathbb{R}$.

A family $P_n(x)$ of polynomials

$$P_n(x) = k_n x^n + k_n' x^{n-1} + k_n'' x^{n-2} + k_n''' x^{n-3} + \cdots, \quad k_n \neq 0 \tag{1.1}$$

is called orthogonal w. r. t. the measure $\mu(x)$, if

$$\langle P_m, P_n\rangle = \begin{cases} 0 & \text{if } m \neq n \\ d_n^2 \neq 0 & \text{if } m = n \end{cases} .$$

The classical orthogonal polynomials can be defined as the common polynomial solutions (1.1) of a differential equation of the type

$$\sigma(x)P_n''(x) + \tau(x)P_n'(x) + \lambda_n P_n(x) = 0 . \tag{1.2}$$

The case $n = 1$ shows that $\tau(x)$ must be a first order polynomial: $\tau(x) = dx + e, d \neq 0$, whereas because of $n = 2$ the function $\sigma(x)$ turns out to be a polynomial of degree $\leqq 2$: $\sigma(x) = ax^2 + bx + c$. Considering the coefficient of x^n, one finally gets $\lambda_n = -n(a(n - 1) + d)$.

A complete characterization of the solution families of the differential equation (1.1) can be given and leads to the following scheme ([2], 1929):

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$\sigma(x) = 0$	powers x^n
$\sigma(x) = 1$	Hermite polynomials
$\sigma(x) = x$	Laguerre polynomials
$\sigma(x) = x^2$	powers, Bessel polynomials
$\sigma(x) = x^2 - 1$	Jacobi polynomials

All other solutions of (1.1) are translations of the above systems.[1] It turns out that—besides the powers—all these polynomial systems are orthogonal. However, the weight function of the Bessel polynomials is not defined in a real interval but in the complex plane. However, in both the Jacobi and Bessel case, for specific values of the parameters finite real orthogonal families arise [1, 13, 15, 20].

The weight function $\rho(x)$ corresponding to the system satisfies *Pearson's differential equation*

$$\frac{d}{dx}\Big(\sigma(x)\rho(x)\Big) = \tau(x)\rho(x)$$

from which it follows that

$$\rho(x) = \frac{C}{\sigma(x)} e^{\int \frac{\tau(x)}{\sigma(x)} dx} \ .$$

Further details can be found in [17].

2. Classical Discrete Families. The classical discrete orthogonal polynomials can be defined as the polynomial solutions of the difference equation

$$\sigma(x)\Delta\nabla P_n(x) + \tau(x)\Delta P_n(x) + \lambda_n P_n(x) = 0 \ , \tag{2.1}$$

where $\Delta f(x) = f(x+1) - f(x)$ and $\nabla f(x) = f(x) - f(x-1)$ denote the forward and backward difference operators, respectively.

Again, from (2.1) it follows that $\tau(x) = d\,x + e, d \neq 0$ (using $n = 1$) and $\sigma(x) = a\,x^2 + b\,x + c$ (using $n = 2$). The coefficient of x^n yields also $\lambda_n = -n(a(n-1) + d)$.

The classical discrete systems can be classified according to the scheme ([11]–[12], 1962, [18], 1991):

$\sigma(x) = 0$	falling factorials $x^{\underline{n}} = x(x-1)\cdots(x-n+1)$
$\sigma(x) = 1$	translated Charlier polynomials
$\sigma(x) = x$	falling factorials, Charlier, Meixner, Krawtchouk polynomials
$\deg(\sigma(x), x) = 2$	Hahn polynomials

Again, all these families—besides the falling factorials—form orthogonal polynomial families.

The weight function $\rho(x)$ corresponding to the system satisfies Pearson's difference equation

$$\Delta\Big(\sigma(x)\rho(x)\Big) = \tau(x)\rho(x)$$

from which it follows that

$$\frac{\rho(x+1)}{\rho(x)} = \frac{\sigma(x)+\tau(x)}{\sigma(x+1)} \ .$$

[1] If $\sigma(x)$ is constant, one gets translated Hermite polynomials, if $\sigma(x)$ has degree 1, then translated Laguerre polynomials result, and if the degree of $\sigma(x)$ is 2, then Jacobi polynomials or Bessel polynomials follow depending on whether or not $\sigma(x)$ has two different zeros (giving the interval bounds a and b if they are real) or one double zero.

3. Hypergeometric Functions. The power series

$$ {}_pF_q\left(\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{array}\middle|\, z\right) = \sum_{k=0}^{\infty} A_k\, z^k \ , $$

whose coefficients $\alpha_k = A_k z^k$ have rational term ratio

$$ \frac{\alpha_{k+1}}{\alpha_k} = \frac{A_{k+1}\, z^{k+1}}{A_k\, z^k} = \frac{(k+a_1)\cdots(k+a_p)}{(k+b_1)\cdots(k+b_q)}\frac{z}{(k+1)} $$

is called the *generalized hypergeometric function*. The summand $\alpha_k = A_k z^k$ is called a *hypergeometric term* w. r. t. k.

Hence, because of

$$ \frac{\rho(x+1)}{\rho(x)} = \frac{\sigma(x)+\tau(x)}{\sigma(x+1)} \ , $$

and since $\sigma(x)$ and $\tau(x)$ are polynomials, the weight functions $\rho(x)$ of the classical discrete orthogonal polynomials form hypergeometric terms w. r. t. the variable x.

For the coefficients of the generalized hypergeometric function one obtains the formula

$$ {}_pF_q\left(\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{array}\middle|\, z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{z^k}{k!} \ , $$

where $(a)_k = a(a+1)\cdots(a+k-1)$ denotes the *Pochhammer symbol* or *shifted factorial*.

Simple examples of hypergeometric functions are the exponential function

$$ e^z = {}_0F_0(z) \ , $$

the sine function

$$ \sin z = z \cdot {}_0F_1\left(\begin{array}{c} - \\ 3/2 \end{array}\middle|\, -\frac{z^2}{4}\right) $$

as well as $\cos(z), \arcsin(z), \arctan(z), \ln(1+z), \mathrm{erf}(z), L_n^{(\alpha)}(z), \ldots$, but for example *not* $\tan(z)$.

From the difference equation, one can determine a hypergeometric representation; see [9, 18]. As an example, the Hahn polynomials are given by[2]

$$ Q_n(x;\alpha,\beta,N) = {}_3F_2\left(\begin{array}{c} -n,-x,n+1+\alpha+\beta \\ \alpha+1,-N \end{array}\middle|\, 1\right) . $$

4. *q*-Orthogonal Polynomials. To define orthogonal polynomials on the lattice $q^{\mathbb{Z}}$, we need some more notation.

The operator ([6], 1949)

$$ D_q\, f(x) = \frac{f(x)-f(qx)}{(1-q)x} $$

[2] In the Russian literature the parameters α and β are interchanged, N is replaced by $N-1$, and the standardization is different; see [18, p. 54]. The given definition is the one of the American school; see [7].

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is called *Hahn's q-difference operator*, and the q-brackets are defined by

$$[k]_q = \frac{1-q^k}{1-q} = 1 + q + \cdots + q^{k-1} .$$

Since

$$\lim_{q\to 1} D_q\, f(x) = f'(x),$$

by de l'Hospital's rule, the limit $q \to 1$ yields the continuous case. The q-brackets are the q-equivalent of the term k since

$$\lim_{q\to 1} [k]_q = k .$$

The q-orthogonal polynomials of the Hahn class can be defined as the polynomial solutions of the q-difference equation

$$\sigma(x) D_q\, D_{1/q}\, P_n(x) + \tau(x) D_q\, P_n(x) + \lambda_n P_n(x) = 0 .$$

Analogously to the classical case, one gets $\tau(x) = dx + e, d \neq 0$, $\sigma(x) = ax^2 + bx + c$ and $\lambda_n = -a[n]_{1/q}[n-1]_q - d[n]_q$.

The classical q-systems can be classified according to the scheme ([19], 1993)[3]

$\sigma(x) = 0$	powers and q-Pochhammer symbols (5.1)
$\sigma(x) = 1$	discrete q-Hermite II polynomials
$\sigma(x) = x$	q-Charlier-, q-Laguerre-, q-Meixner polynomials
$\deg(\sigma(x), x) = 2$	q-Hahn polynomials, Big q-Jacobi polynomials

The weight function $\rho(x)$ corresponding to the system satisfies the q-Pearson difference equation

$$D_q\Big(\sigma(x)\rho(x)\Big) = \tau(x)\rho(x) .$$

Hence we have

$$\frac{\rho(qx)}{\rho(x)} = \frac{\sigma(x) + (q-1)x\tau(x)}{\sigma(qx)} .$$

5. Basic Hypergeometric Series. Instead of considering series whose coefficients A_k have rational term ratio $A_{k+1}/A_k \in \mathbb{C}(k)$, we can also consider such series whose coefficients A_k have term ratio $A_{k+1}/A_k \in \mathbb{C}(q^k)$ w. r. t. some base $q \in \mathbb{R}$.

This leads to the q-hypergeometric (or basic hypergeometric) series

$${}_r\varphi_s\left(\left.\begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array}\right| q\,;\, x\right) = \sum_{k=0}^{\infty} A_k\, x^k .$$

Now the coefficients—that are called q-hypergeometric terms—are given by the formula

$$A_k = \frac{(a_1;q)_k \cdots (a_r;q)_k}{(b_1;q)_k \cdots (b_s;q)_k} \frac{x^k}{(q;q)_k} \left((-1)^k\, q^{\binom{k}{2}}\right)^{1+s-r} ,$$

[3]Historically many more systems were introduced that fit in this list. A complete classification that boils down to essentially seven different types can be found in [14], see also [16].

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where

$$(a;q)_k = \prod_{j=0}^{k-1}(1-aq^j) \tag{5.1}$$

denotes the q-Pochhammer symbol. Since

$$\frac{\rho(qx)}{\rho(x)} = \frac{\sigma(x)+(q-1)x\tau(x)}{\sigma(qx)}$$

is rational, the weight $\rho(x)$ of a q-orthogonal system of the Hahn tableau is a q-hypergeometric term w. r. t. $x = q^k$.

Classical orthogonal systems have (generally several) q-hypergeometric equivalents. For example, the Big q-Jacobi polynomials are q-equivalents of the Jacobi polynomials and are given by

$$P_n(x;a,b,c;q) = {}_3\varphi_2\left(\begin{array}{c} q^{-n}, a, b, q^{n+1}, x \\ aq, cq \end{array}\middle| q\,;\,q\right).$$

The families of the Hahn class that we considered in this article—with absolutely continuous, arithmetically discrete and geometrically discrete weights—can be generated by suitable limit procedures from the Big q-Jacobi polynomials.

6. Computing the Difference Equation from a Recurrence Equation. From the differential or (q)-difference equation one can determine the three-term recurrence equation for $P_n(x)$ in terms of the coefficients of $\sigma(x)$ and $\tau(x)$. Using a computer algebra system like *Maple* one can easily compute how the three-term recurrence equation corresponding to a given system can be expressed by the five parameters a, b, c, d and e.

As an example case, we consider the discrete situation.
This defines the forward and backward difference operators

```
> Delta:=(f,x)->subs(x=x+1,f)-f:
> nabla:=(f,x)->f-subs(x=x-1,f):
```

We consider the three highest coefficients of the orthogonal polynomial

```
> p:=k[n]*x^n+kprime[n]*x^(n-1)+kprimeprime[n]*x^(n-2);
```

$$p := k_n\, x^n + \mathit{kprime}_n\, x^{(n-1)} + \mathit{kprimeprime}_n\, x^{(n-2)}$$

and we define the polynomials $\sigma(x)$ and $\tau(x)$ with symbolically given coefficients
a, b, c, d, e:

```
> sigma:=a*x^2+b*x+c:
> tau:=d*x+e:
```

The polynomial $P_n(x)$ satisfies the difference equation $DE = 0$ with:

```
> DE:=sigma*Delta(nabla(p,x),x)+tau*Delta(p,x)+lambda[n]*p;
```

$$\begin{aligned} DE := {} & (a\,x^2+b\,x+c)(k_n\,(x+1)^n + \mathit{kprime}_n\,(x+1)^{(n-1)} + \mathit{kprimeprime}_n\,(x+1)^{(n-2)} \\ & - 2\,k_n\,x^n - 2\,\mathit{kprime}_n\,x^{(n-1)} - 2\,\mathit{kprimeprime}_n\,x^{(n-2)} + k_n\,(x-1)^n \\ & + \mathit{kprime}_n\,(x-1)^{(n-1)} + \mathit{kprimeprime}_n\,(x-1)^{(n-2)}) + (d\,x+e)(k_n\,(x+1)^n \\ & + \mathit{kprime}_n\,(x+1)^{(n-1)} + \mathit{kprimeprime}_n\,(x+1)^{(n-2)} - k_n\,x^n - \mathit{kprime}_n\,x^{(n-1)} \\ & - \mathit{kprimeprime}_n\,x^{(n-2)}) + \lambda_n\,(k_n\,x^n + \mathit{kprime}_n\,x^{(n-1)} + \mathit{kprimeprime}_n\,x^{(n-2)}) \end{aligned}$$

We replace the powers $(x+1)^n$ and $(x-1)^n$ by the binomial theorem

```
>   DE:=subs(
>   {(x+1)^n=x^n+n*x^(n-1)+n*(n-1)/2*x^(n-2),
>   (x+1)^(n-1)=subs(n=n-1,x^n+n*x^(n-1)+n*(n-1)/2*x^(n-2)),
>   (x+1)^(n-2)=subs(n=n-2,x^n+n*x^(n-1)+n*(n-1)/2*x^(n-2)),
>   (x-1)^n=x^n-n*x^(n-1)+n*(n-1)/2*x^(n-2),
>   (x-1)^(n-1)=subs(n=n-1,x^n-n*x^(n-1)+n*(n-1)/2*x^(n-2)),
>   (x-1)^(n-2)=subs(n=n-2,x^n-n*x^(n-1)+n*(n-1)/2*x^(n-2))},
>   DE):
```

and collect coefficients:

```
>   de:=collect(simplify(DE/x^(n-4)),x);
```

$$
\begin{aligned}
de &:= (a\,k_n\,n^2 + d\,k_n\,n - a\,k_n\,n + \lambda_n\,k_n)\,x^4 + (a\,kprime_n\,n^2 + b\,k_n\,n^2 + \lambda_n\,kprime_n \\
&+ 2\,a\,kprime_n - b\,k_n\,n + e\,k_n\,n + \frac{1}{2}\,d\,k_n\,n^2 - d\,kprime_n - 3\,a\,kprime_n\,n - \frac{1}{2}\,d\,k_n\,n \\
&+ d\,kprime_n\,n)x^3 + (d\,kprimeprime_n\,n + d\,kprime_n + a\,kprimeprime_n\,n^2 - \frac{1}{2}\,e\,k_n\,n \\
&- e\,kprime_n + 6\,a\,kprimeprime_n + b\,kprime_n\,n^2 - 3\,b\,kprime_n\,n + \frac{1}{2}\,e\,k_n\,n^2 \\
&+ c\,k_n\,n^2 - c\,k_n\,n - 2\,d\,kprimeprime_n - 5\,a\,kprimeprime_n\,n + e\,kprime_n\,n \\
&+ \frac{1}{2}\,d\,kprime_n\,n^2 - \frac{3}{2}\,d\,kprime_n\,n + \lambda_n\,kprimeprime_n + 2\,b\,kprime_n)x^2 + (\\
&6\,b\,kprimeprime_n + 2\,c\,kprime_n - 3\,c\,kprime_n\,n + 3\,d\,kprimeprime_n \\
&+ e\,kprimeprime_n\,n - \frac{5}{2}\,d\,kprimeprime_n\,n - 2\,e\,kprimeprime_n + \frac{1}{2}\,e\,kprime_n\,n^2 \\
&+ c\,kprime_n\,n^2 + b\,kprimeprime_n\,n^2 - 5\,b\,kprimeprime_n\,n + e\,kprime_n \\
&- \frac{3}{2}\,e\,kprime_n\,n + \frac{1}{2}\,d\,kprimeprime_n\,n^2)x + 6\,c\,kprimeprime_n + 3\,e\,kprimeprime_n \\
&+ c\,kprimeprime_n\,n^2 - 5\,c\,kprimeprime_n\,n + \frac{1}{2}\,e\,kprimeprime_n\,n^2 \\
&- \frac{5}{2}\,e\,kprimeprime_n\,n
\end{aligned}
$$

Equating the highest coefficient gives the already mentioned identity for λ_n:

```
>   rule1:=lambda[n]=solve(coeff(de,x,4),lambda[n]);
```

$$rule1 := \lambda_n = -n\,(d - a + a\,n)$$

This result can be substituted into the differential equation:

```
>   de:=expand(subs(rule1,de)):
```

Equating the second highest coefficient gives k'_n as rational multiple of k_n

```
>   rule2:=kprime[n]=solve(coeff(de,x,3),kprime[n]);
```

$$rule2 := kprime_n = \frac{1}{2}\,\frac{k_n\,n\,(d\,n - 2\,b + 2\,b\,n - d + 2\,e)}{d - 2\,a + 2\,a\,n}$$

and equating the third highest coefficient gives k''_n as rational multiple of k_n:[4]

```
>   rule3:=kprimeprime[n]=solve(coeff(subs(rule2,de),x,2),
>   kprimeprime[n]);
```

[4]Of course, taking into consideration the fourth highest coefficients yields k'''_n as rational multiple of k_n, and so forth.

$$\mathit{rule3} := \mathit{kprimeprime}_n = \frac{1}{8} k_n\, n(-8\,d\,n\,e - 16\,d\,n^2\,b + 4\,b\,n^3\,d + 20\,d\,n\,b + 4\,e\,n^2\,d + 8\,c\,n^2\,a + 8\,e\,n^2\,b - 20\,e\,n\,b + 4\,c\,n\,d - 16\,c\,n\,a + 4\,e\,n^2\,a - 8\,e\,n\,a - 4\,d^2\,n^2 + 5\,d^2\,n + 4\,e^2\,n + 20\,b^2\,n - 8\,b^2 + 12\,b\,e - 8\,b\,d + 8\,c\,a - 4\,c\,d + 4\,e\,a + 4\,e\,d - 16\,b^2\,n^2 - 2\,d^2 - 4\,e^2 + 4\,b^2\,n^3 + d^2\,n^3)/((-2\,a + 2\,a\,n + d)\,(2\,a\,n - 3\,a + d))$$

We consider the monic case, hence

```
> k[n]:=1;
```

$$k_n := 1$$

and therefore

```
> rule2;
```

$$\mathit{kprime}_n = \frac{n\,(d\,n - 2\,b + 2\,b\,n - d + 2\,e)}{2\,(d - 2\,a + 2\,a\,n)}$$

```
> rule3;
```

$$\mathit{kprimeprime}_n = n(d^2\,n^3 - 16\,b^2\,n^2 + 5\,d^2\,n + 8\,c\,n^2\,a + 4\,c\,n\,d - 16\,c\,n\,a + 4\,b\,n^3\,d - 16\,b\,n^2\,d + 8\,b\,n^2\,e + 20\,b\,d\,n + 4\,e\,n^2\,a - 8\,e\,n\,d - 8\,e\,n\,a + 4\,e\,n^2\,d - 20\,e\,n\,b - 4\,c\,d + 8\,c\,a + 4\,b^2\,n^3 - 8\,b^2 + 20\,b^2\,n - 8\,b\,d + 12\,b\,e + 4\,e\,d + 4\,e\,a + 4\,e^2\,n - 2\,d^2 - 4\,e^2 - 4\,d^2\,n^2)/(8\,(d - 2\,a + 2\,a\,n)\,(-3\,a + d + 2\,a\,n))$$

Now we would like to find the coefficients β_n and γ_n in the recurrence equation $RE = 0$ (see (7.1)):

```
> RE:=P(n+1)-(x-beta[n])*P(n)+gamma[n]*P(n-1);
```

$$RE := \mathrm{P}(n+1) - (x - \beta_n)\,\mathrm{P}(n) + \gamma_n\,\mathrm{P}(n-1)$$

```
> RE:=subs(
> {P(n)=p,P(n+1)=subs(n=n+1,p),P(n-1)=subs(n=n-1,p)},RE);
```

$$RE := x^{(n+1)} + \mathit{kprime}_{n+1}\,x^n + \mathit{kprimeprime}_{n+1}\,x^{(n-1)} - (x - \beta_n)\,(x^n + \mathit{kprime}_n\,x^{(n-1)} + \mathit{kprimeprime}_n\,x^{(n-2)}) + \gamma_n\,(x^{(n-1)} + \mathit{kprime}_{n-1}\,x^{(n-2)} + \mathit{kprimeprime}_{n-1}\,x^{(n-3)})$$

We substitute the already known formulas:

```
> RE:=subs(
> {rule2,subs(n=n+1,rule2),subs(n=n-1,rule2),
> rule3,subs(n=n+1,rule3),subs(n=n-1,rule3)},RE):
```

and get a highly complicated expansion

```
> re:=simplify(numer(normal(RE))/x^(n-3)):
```

Equating the highest coefficient gives β_n as rational function of a, b, c, d, e and n:

```
> rule4:=beta[n]=
> factor(solve(coeff(re,x,3),beta[n]));
```

$$\mathit{rule4} := \beta_n = -\frac{d\,n^2\,a + 2\,b\,n^2\,a - d\,a\,n - 2\,b\,a\,n - 2\,e\,a + d^2\,n + 2\,b\,d\,n + e\,d}{(d - 2\,a + 2\,a\,n)\,(d + 2\,a\,n)}$$

and equating the second highest coefficient yields finally γ_n as rational function, too:

```
> rule5:=gamma[n]=factor(subs(rule4,
> solve(coeff(re,x,2),gamma[n])));
```

$$\mathit{rule5} := \gamma_n = (d + a\,n - 2\,a)(4\,b\,e\,d - 8\,a^2\,e - 16\,a^2\,c + 4\,a\,b^2 - d^3 + 4\,a\,b\,d + 8\,a\,e\,d + 16\,a\,c\,d + 4\,n\,b^2\,d - 4\,a\,e^2 + n^2\,a\,d^2 - 2\,n\,a\,d^2 + a\,d^2 + 4\,n^2\,a\,b\,d - 16\,n^2\,a^2\,c - 8\,e\,n^2\,a^2 + 4\,n^2\,a\,b^2 + 16\,n\,a^2\,e + 32\,n\,a^2\,c - 8\,n\,a\,b^2 - 8\,n\,a\,b\,d - 8\,n\,a\,e\,d - 16\,n\,a\,c\,d + 4\,n\,b\,d^2 - 4\,c\,d^2 - 4\,b^2\,d - 4\,b\,d^2 + d^3\,n)n/(4\,(2\,a\,n - 3\,a + d)(-2\,a + 2\,a\,n + d)^2\,(d - a + 2\,a\,n))$$

Starting from a given three-term recurrence equation, one can use these identities in the opposite direction to find the corresponding differential or (q)-difference equation by solving a quadratic system of equations; see [10].

Example 1: Let the recurrence equation

$$P_{n+2}(x) - (x - n - 1)\,P_{n+1}(x) + \alpha(n+1)^2 P_n(x) = 0 \tag{6.1}$$

be given. The computations

```
> read "hsum9.mpl";
```

Package "Hypergeometric Summation", Maple V − Maple 9

Copyright 1998 − 2004, Wolfram Koepf, University of Kassel

```
> read "retode.mpl";
```

Package "REtoDE", Maple V − Maple 8

Copyright 2002, Wolfram Koepf, University of Kassel

```
> RE:=P(n+2)-(x-n-1)*P(n+1)+alpha*(n+1)^2*P(n)=0;
```

$$\mathit{RE} := \mathrm{P}(n+2) - (x - n - 1)\,\mathrm{P}(n+1) + \alpha\,(n+1)^2\,\mathrm{P}(n) = 0$$

```
> REtoDE(RE,P(n),x);
```

$$\mathit{Warning:\ parameters\ have\ the\ values},\ \{a = 0,\ e = 0,\ b = 2\,c,\ c = c,\ \alpha = \frac{1}{4},\ d = -4\,c\}$$

$$[\frac{1}{2}\,(2\,x+1)\,(\tfrac{\partial^2}{\partial x^2}\,\mathrm{P}(n,\,x)) - 2\,x\,(\tfrac{\partial}{\partial x}\,\mathrm{P}(n,\,x)) + 2\,n\,\mathrm{P}(n,\,x) = 0,$$
$$[I = [\frac{-1}{2},\,\infty],\ \rho(x) = 2\,e^{(-2\,x)}],\ \frac{k_{n+1}}{k_n} = 1]$$

```
> REtodiscreteDE(RE,P(n),x);
```

$$\mathit{Warning:\ parameters\ have\ the\ values},\{a = 0,\ e = -g\,d,$$
$$c = \frac{1}{2}\,g\,d\,f + \frac{1}{2}\,g\,d + \frac{1}{4}\,d - \frac{1}{4}\,f^2\,d, b = -\frac{1}{2}\,f\,d - \frac{1}{2}\,d,\ f = f,\ d = d,\ \alpha = \frac{f^2-1}{4\,f^2},\ g = g\}$$

$$[\frac{1}{2}\,\frac{(f - 1 + 2\,x\,f)\,(\mathrm{Nabla}(\mathrm{P}(n,\,x\,f + f + g),\,x+1) - \mathrm{Nabla}(\mathrm{P}(n,\,x\,f + g),\,x))}{f}$$
$$- \frac{2\,x\,(\mathrm{P}(n,\,x\,f + f + g) - \mathrm{P}(n,\,x\,f + g))}{f+1} + \frac{2\,n\,\mathrm{P}(n,\,x\,f + g)}{(f+1)\,f} = 0,$$
$$[\sigma(x) = \frac{f}{2} - \frac{1}{2} + x - g,\ \sigma(x) + \tau(x) = \frac{(f-1)\,(-2\,g + 1 + 2\,x + f)}{2\,(f+1)}],\ \rho(x) = (\frac{f-1}{f+1})^x,$$
$$\frac{k_{n+1}}{k_n} = \frac{1}{f}]$$

show that for $\alpha = 1/4$ translated Laguerre polynomials and for $\alpha < 1/4$ Meixner and Krawtchouk polynomials are solutions of (6.1).[5]

[5]The parameters f and g that appear in Maple's output correspond to the translation $x \mapsto fx + g$.

Example 2: Let the recurrence equation

$$P_{n+2}(x) - xP_{n+1}(x) + \alpha\, q^n (q^{n+1} - 1) P_n(x) = 0 \tag{6.2}$$

be given. The computations

```
> RE:=P(n+2)-x*P(n+1)+alpha*q^n*(q^(n+1)-1)*P(n)=0;
```

$$RE := \mathrm{P}(n+2) - \mathrm{P}(n+1)\,x + \alpha\, q^n\,(q^{(n+1)} - 1)\,\mathrm{P}(n) = 0$$

```
> REtoqDE(RE,P(n),q,x);
```

$$Warning:\ parameters\ have\ the\ values,$$
$$\{e = 0,\, d = d,\, c = -\alpha\, d\, q + \alpha\, d,\, a = -d\, q + d,\, b = 0\}$$

$$[(x^2 + \alpha)\,\mathrm{Dq}(\mathrm{Dq}(\mathrm{P}(n,\, x),\, \frac{1}{q},\, x),\, q,\, x) - \frac{x\,\mathrm{Dq}(\mathrm{P}(n,\, x),\, q,\, x)}{q-1} + \frac{q\,(-1+q^n)\,\mathrm{P}(n,\, x)}{(q-1)^2\, q^n} = 0,$$
$$\frac{\rho(q\,x)}{\rho(x)} = \frac{\alpha}{q^2\, x^2 + \alpha},\ \frac{k_{n+1}}{k_n} = 1]$$

show that for every $\alpha \in \mathbb{R}$ there are q-orthogonal polynomial solutions of (6.2).

7. Associated Orthogonal Polynomials. A monic orthogonal system

$$P_n(x) = x^n + k_n' x^{n-1} + k_n'' x^{n-2} + \cdots$$

satisfies a recurrence equation of the form (see, e. g., [3])

$$P_{n+1}(x) = (x - \beta_n)\, P_n(x) - \gamma_n\, P_{n-1}(x)\ . \tag{7.1}$$

The polynomials defined by

$$P_{n+1}^{(r)}(x) = (x - \beta_{n+r})\, P_n^{(r)}(x) - \gamma_{n+r}\, P_{n-1}^{(r)}(x)\ ,$$

called the rth *associated orthogonal polynomials*, are also orthogonal by Favards Theorem; see [3].

It turns out that the associated polynomials can be represented as linear combinations

$$P_n^{(r)}(x) = \frac{P_{r-1}(x)}{\Gamma_{r-1}}\, P_{n+r-1}^{(1)}(x) - \frac{P_{r-2}^{(1)}(x)}{\Gamma_{r-1}}\, P_{n+r}(x)$$

where $\Gamma_n = \prod_{k=1}^n \gamma_k$; see [4].

As examples, we consider the classical discrete polynomials. Then it turns out that the associated polynomials $y(x) = P_n^{(r)}(x)$ satisfy a fourth order recurrence equation of the form

$$R_n^{(r)} y(x) = \sum_{k=0}^{4} J_k(x,n)\, \mathcal{S}^k\, y(x) = \sum_{k=0}^{4} J_k(x,n)\, y(x+k) = 0$$

with polynomials $J_k(x,n) \in \mathbb{R}[x,n]$, where $\mathcal{S}$ denotes the shift operator.

8. Factorization of Fourth Order Difference Equations. By linear algebra, one can prove that a certain multiple of the difference operator $R_n^{(r)}$ can be factorized as product of two difference operators of second order [5]

$$X(\sigma, \tau, P_{r-1})\, R_n^{(r)} = S_n^{(r)}\, T_n^{(r)}$$

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for some function $X(\sigma, \tau, P_{r-1})$. Using computer algebra, in each specific case this factorization can be computed explicitly.

For example, let's consider the Charlier polynomials and their associated. The monic Charlier polynomials are given by

$$P_n(x) = (-a)^n\, c_n^{(a)}(x) = (-a)^n\, {}_2F_0\left(\begin{array}{c} -n, -x \\ - \end{array} \middle| -\frac{1}{a}\right) .$$

The fourth order difference operator of the rth associated Charlier polynomials is given by

$$\begin{aligned} R_n^{(r)} &= a\,(n + 2\,\zeta)\,(x + 4)\,\mathcal{S}^4 \\ &\quad + (-2\,a\,x - 4\,\zeta - 2\,\zeta^3 + 2\,n^2 - 6\,a + 6\,\zeta^2 - 3\,n\,\zeta^2 - n^2\,\zeta + 7\,n\,\zeta - 2\,n)\,\mathcal{S}^3 \\ &\quad + (2\,a\,x - 5\,a\,n + 2\,\zeta + 4\,\zeta^3 - n^2 - 4\,\zeta\,a\,x - 10\,\zeta\,a + n^3 + 4\,a - 6\,\zeta^2 + 6\,n\,\zeta^2 \\ &\qquad + 4\,n^2\,\zeta - 4\,n\,\zeta - 2\,a\,x\,n)\mathcal{S}^2 \\ &\qquad + (2\,a\,x + 2\,\zeta - 2\,\zeta^3 + 4\,a - 3\,n\,\zeta^2 - n^2\,\zeta + n\,\zeta)\,\mathcal{S} \\ &\qquad + a\,(n - 2 + 2\,\zeta)\,(x + 1), \end{aligned}$$

where $\zeta = r - x - a - 2$. The factorization yields the second order right factor[6]

$$\begin{aligned} T_n^{(r)} &= P_{r-1}(x+1)\,P_{r-1}(x)\,(x+2)^2\,a\,\mathcal{S}^2 \\ &\quad + (-(x+1)(n+\zeta+1)(x+2)P_{r-1}(x)^2 - \zeta(n+\zeta+1)(x+2)P_{r-1}(x+1)P_{r-1}(x))\mathcal{S} \\ &\quad + (-a\,(x+1)\,(x+2)\,P_{r-1}(x+1)\,P_{r-1}(x) - \zeta\,a\,(x+2)\,P_{r-1}(x+1)^2), \end{aligned}$$

$P_n(x)$ still denoting the monic Charlier polynomial. The function X as well as the left factor $S_n(r)$ turn out to be rather complicated.

One main advantage of the factorization is the following: In the general case, using the right factor $T_n^{(r)}$, one can find a solution basis for the fourth order difference equation of the rth associated polynomials (for arbitrary r) consisting of the four linearly independent functions

$$\begin{aligned} A_n^{(r)}(x) &= \rho(x)\,P_{r-1}(x)\,P_{n+r}(x), \\ B_n^{(r)}(x) &= \rho(x)\,P_{r-1}(x)\,Q_{n+r}(x), \\ C_n^{(r)}(x) &= \rho(x)\,Q_{r-1}(x)\,P_{n+r}(x), \\ D_n^{(r)}(x) &= \rho(x)\,Q_{r-1}(x)\,Q_{n+r}(x). \end{aligned}$$

In a similar manner, the fourth order difference equations and their factorizations of differently modified polynomials like the generalized co-recursive and the generalized co-dilated polynomials can be detected [5].

9. Conclusion. The software used was written in connection with my book [8] and is available from my home page.[7]

I hope to have shown that new and interesting research results in the classical topic of orthogonal polynomials can be obtained using computer algebra algorithms.

The most important computer algebra algorithms utilized are the algorithms of linear algebra, polynomial factorization and the solution of polynomial systems, e. g. by Gröbner bases.

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[6] which doesn't necessarily look simpler but is of second instead of fourth order

[7] http://www.mathematik.uni-kassel.de/~koepf

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Software development is a time consuming activity! Software developers love when their software is used. But they need your support. Hence my suggestion: If you use a computer algebra package for your research, please cite its use!

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