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PERIODIC POINTS OF SOME ALGEBRAIC MAPS*

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Abstract. We study the local dynamics of maps $f(z) = -z - \sum_{n=1}^{\infty} \alpha_n z^{n+1}$, where $f(z)$ is an irreducible branch of the algebraic curve

$$z + w + \sum_{i+j=n} a_{ij} z^i w^j = 0.$$

We show that the center and cyclicity problems have simple solutions when n is odd. For the case of even n some partial results are obtained.

Key words. discrete dynamical systems, polynomial maps, periodic points

AMS subject classifications. 37F10, 58F, 13P

1. Introduction.

Consider a map of the form

$$w = f(z) \equiv -z - \sum_{n=1}^{\infty} \alpha_n z^{n+1}, \;\; z \in \mathbb{R}. \tag{1.1}$$

Denote by f^p $(p \in \mathbb{N})$ the p-th iteration of the map (1.1).

DEFINITION 1.1. *A singular point $z = 0$ of the map (1.1) is called a center, if $\exists\, \epsilon > 0$ such that $\forall z : |z| < \epsilon$ the equality $f^2(z) = z$ holds, and a focus otherwise.*

Clearly, if the right hand side of (1.1) is a polynomial, then $z = 0$ is a center if and only if $f(z) \equiv -z$.

DEFINITION 1.2. *A point $z_0 > 0$ is called a limit cycle of the map (1.1) if z_0 is an isolated root of the equation*

$$f^2(z) - z = 0.$$

In the other words, a limit cycle is an isolated 2-periodic point of (1.1) [5].

Consider the equation

$$\Psi(z, w) = w + z + \sum_{i+j=2}^{n} a_{ij} z^i w^j = 0, \tag{1.2}$$

where $a_{ij}, w, z \in \mathbb{R}$. Obviously, (1.2) has an analytic solution of the form (1.1),

$$w = \tilde{f}(z) = -z + \dots. \tag{1.3}$$

DEFINITION 1.3. *We say that the polynomial (1.2) defines (or has) a center at the origin if the equation $\Psi(z, w) = 0$ has a solution (1.3) such that the map $\tilde{f}$ has a center at the origin, and we say that (1.2) defines a focus at the origin, if $\tilde{f}$ has a focus.*

Thus, the problem arises to find in the space of coefficients $\{a_{ij}\}$ the manifold on which the corresponding maps $\tilde{f}$ have a center at the origin and to investigate the limit cycles bifurcations of such maps. For the first time this problem has been stated in [8]. As should be

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remarked the center-focus problem and the problem of estimating the number of limit cycles near $z = 0$ (cyclicity) for the map $\tilde{f}$ are similar to the corresponding problems for the second order system of differential equations

$$\begin{aligned} \dot{x} &= -y + P(x, y), \\ \dot{y} &= x + Q(x, y), \end{aligned}$$

where P and Q are polynomials.

One possible way to investigate the behavior of trajectories of the map (1) near the origin is a transformation to the normal form [1, 6]

$$z \mapsto -z(1 + d_1 z^2 + d_2 z^4 + \cdots).$$

If the first coefficient which differs from zero, is d_k and if $d_k > 0$ then

$$f^2(z) = z + 2d_k z^{2k+1} + o(z^{2k+1}),$$

which implies an unstable focus at $z = 0$, otherwise, if $d_k < 0$ the focus is stable.

Another possible way, suggested by Żołądek [8], is based on making use of Lyapunov functions. Namely, for the map (1.1) it is possible to find a Lyapunov function of the form

$$\Phi(z) = z^2(1 + \sum_{k=1}^{\infty} b_k z^k)$$

with the property

$$\Phi(f(z)) - \Phi(z) = g_2 z^4 + g_4 z^6 + \cdots + g_{2m} z^{2m+2} + \cdots.$$

It is shown in [7] that if $g_{2k} = 0$ for all $k \in \mathbb{N}$ then the map (1.1) has a center in the origin (with $f^2(z) \equiv z$), and if $g_2 = \ldots = g_{2k-2} = 0,\ g_{2k} \neq 0$ then $z = 0$ is a stable focus, when $g_{2k} < 0$, and an unstable focus, when $g_{2k} > 0$.

To investigate bifurcations of limit cycles of the map (1.1) one can also find the return (Poincaré) map

$$\mathcal{P}(z) = f^2(z) = z + p_2 z^3 + p_3 z^4 + \cdots. \tag{1.4}$$

We call the coefficient p_m of the return map (1.4) the *mth focus quantity*. All focus quantities are polynomials in coefficients of (1.1).

The case of the cubic polynomial

$$\Psi(z, w) = z + w + Az^2 + Bzw + Cw^2 + Dz^3 + Ez^2 w + Fzw^2 + Gw^3,$$

where $A, B, \ldots, G \in \mathbb{C}$, was considered in [8, 7].

In this paper we consider maps defined by (1.2) in the form of the sum of the homogeneous linear polynomial $w + z$ and a homogeneous polynomial of the degree n, that is,

$$\Psi^{(n)}(z, w) = w + z + \sum_{j=0}^{n} a_{n-j,j} z^{n-j} w^j = 0. \tag{1.5}$$

Here and below the superscript (n) denotes the degree of the polynomial in (1.5) and indicates the focus quantities relevant to (1.5), so

$$\Psi^{(2)}(z, w) = w + z + a_{20} z^2 + a_{11} zw + a_{02} w^2, \tag{1.6}$$

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$$\Psi^{(3)}(z,w) = w + z + a_{30}z^3 + a_{21}z^2w + a_{12}zw^2 + a_{03}w^3. \tag{1.7}$$

Correspondingly, $p_2^{(2)}, p_3^{(2)}, \ldots$, are the focus quantities of (1.6), $p_2^{(3)}, p_3^{(3)}, \ldots$, are the focus quantities of (1.7), etc. To compute focus quantities of (1.5) we first look for the branch of the algebraic curve $\Psi^{(n)}(z,w)$ passing through the origin (that is, for a function $w = f(z)$ of the form (1.1) such that $\Psi^{(n)}(z, f(z)) \equiv 0$). Then, the coefficients of the Taylor expansion of the second iteration of $f(z)$ are the focus quantities $p_2^{(n)}, p_3^{(n)}, \ldots$. For example, (1.6) implicitly defines the function

$$w = f(z) = -z - (a_{02} - a_{11} + a_{20})z^2 - (2a_{02} - a_{11})(a_{02} - a_{11} + a_{20})z^3 + \ldots.$$

The second iteration of f is

$$f^2 = z - 2(a_{20} - a_{02})(a_{02} - a_{11} + a_{20})z^3 - (a_{20} - a_{02})(a_{02} - a_{11} + a_{20})^2 z^4 + \ldots,$$

so the focus quantities of $\Psi^{(2)}$ are

$$p_2^{(2)} = -2(a_{20} - a_{02})(a_{20} - a_{11} + a_{02}), \quad p_3^{(2)} = -(a_{20} - a_{02})(a_{02} - a_{11} + a_{20})^2,$$

and so on.

2. The cyclicity of maps defined by (1.5). Denote the real space of coefficients of polynomial (1.5) by $\mathcal{E}$, the δ-ball centered at $\sigma^* = (a^*_{n,0}, a^*_{n-1,1}, \ldots, a^*_{0,n}) \in \mathcal{E}$ by $U_\delta(\sigma^*)$, and let $\tilde{f}_\sigma$ be the map (1.3) corresponding to a given point $\sigma = (a_{n,0}, a_{n-1,1}, \ldots, a_{0,n})$ of the parameter space, that is,

$$f_\sigma = -z(1 + \sum_{k=1}^{\infty} \alpha_k(a_{n,0}, a_{n-1,1}, \ldots, a_{0,n})z^k). \tag{2.1}$$

DEFINITION 2.1. *Let $n_{\sigma,\epsilon}$ be the number of limit cycles of the map f_σ in the interval $0 < z < \epsilon$. We say that a singular point $z = 0$ of the map f_{σ^*} has the cyclicity m with respect to the space $\mathcal{E}$ if there exist $\delta_0 > 0$ and $\epsilon_0 > 0$, such that for every $0 < \epsilon < \epsilon_0$ and $0 < \delta < \delta_0$*

$$\max_{\sigma \in U_\delta(\sigma^*)} n_{\sigma,\epsilon} = m.$$

In order to simplify notations we denote by (a) the n-tuple $(a_{n,0}, a_{n-1,1}, \ldots, a_{0,n})$ and by $k[a]$ the ring of polynomials in $a_{n,0}, a_{n-1,1}, \ldots, a_{0,n}$ over the field k. Also we denote by $F^{(n)}(a)$ the map (1.1) defined by the polynomial (1.5) (more precisely, $F^{(n)}(a)$ is a family of maps depending upon the parameter (a)).

Given polynomials $f_1, \ldots, f_s \in k[a]$ (k is a field) we denote by $\langle f_1, \ldots, f_s\rangle$ the ideal of $k[a]$ generated by $f_1, \ldots, f_s$ and by $\mathbf{V}(I)$ the (affine) variety of the ideal I,

$$\mathbf{V}(I) = \{(a) \in k^{n+1} : f(a) = 0, \ \forall\, f \in I\}.$$

DEFINITION 2.2. *The ideal of $\mathbb{C}[a]$ generated by all focus quantities, $\mathcal{B}^{(n)} = \langle p_2^{(n)}, p_3^{(n)}, \ldots\rangle$, is called the Bautin ideal of the map $F^{(n)}(a)$. The set $V_{\mathcal{B}^{(n)}} = \mathbf{V}(\mathcal{B}^{(n)}) \subset \mathbb{C}^{n+1}$ is called the center variety of $F^{(n)}(a)$.*

Speaking about the center varieties we will assume that the coefficients of (1.5) are complex and speaking on the cyclicity we restrict ourself to maps (1.5) with real coefficients.

Let $\mathcal{D} = \langle p^{(n)}_{k_1}, p^{(n)}_{k_2}, \ldots, p^{(n)}_{k_m}\rangle$ be a basis of $\mathcal{B}^{(n)}$ such that for any $p^{(n)}_{k_i}, p^{(n)}_{k_j}$ from $\mathcal{D}$ $k_i < k_j$ if $i < j$, and for any k_s such that $k_i < k_s < k_{i+1}$ the polynomial $p^{(n)}_{k_s}$ belongs to

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$\langle p_{k_1}^{(n)}, p_{k_2}^{(n)}, \ldots, p_{k_i}^{(n)} \rangle$. Using the results of [2, 7] it is easy to conclude that the following statement holds.

THEOREM 2.3. *If $\mathcal{D}$ is the basis of $\mathcal{B}^{(n)}$ defined above then the cyclicity of the origin for any map (2.1) is at most $m-1$.*

We have seen at the end of the previous section that for the map defined by (1.5) with $n=2$

$$p_2^{(2)} = -2(a_{20} - a_{11} + a_{02})(a_{20} - a_{02}),$$

and, as it was shown in [8], $\Psi^{(2)}(z,w)$ defines a center in the origin if and only if one of conditions (i) $a_{20} - a_{11} + a_{02} = 0$ or (ii) $a_{20} - a_{02} = 0$ holds. According to [7] for (1.5) with $n = 3$

$$p_2^{(3)} = 2(a_{30} - a_{21} + a_{12} - a_{03})$$

and the map defines a center if and only if $p_2^{(3)} = 0$. Because $p_2^{(2)}$, $p_2^{(3)}$ are linear polynomials they generate the corresponding Bautin ideals, $\mathcal{B}^{(2)} = \langle p_2^{(2)} \rangle$, $\mathcal{B}^{(3)} = \langle p_2^{(3)} \rangle$. Thus the cyclicity of the origin for every map $\tilde{f}$ defined by (1.5) with $n=2$ and $n=3$ is equal to zero.

THEOREM 2.4. *1) The polynomial (1.5) with n odd defines a center in the origin if and only if*

$$\sum_{l+j=n} (-1)^j a_{lj} = 0. \tag{2.2}$$

2) The cyclicity of the map defined by (1.5) with n odd is equal to zero.

Proof. Assume that the first different from zero coefficient of the map (1.1) is α_k (with $k>1$), then the first different from zero coefficient in the Poincaré map is

$$\begin{array}{rcll} p_k & = & 2\alpha_k, & \text{if } k \text{ is even,} \\ p_{k+1} & = & 2\alpha_{k+1}, & \text{if } k \text{ is odd.} \end{array} \tag{2.3}$$

Note that when $k=1$ the series expansion of the Poincaré map starts from $p_2 = 2\alpha_2 - 2\alpha_1^2$, however below we will deal only with the cases $k > 2$.

The map (1.1) defined by the polynomial (1.5) has the expansion

$$w = -z - \sum_{l+j=n} (-1)^l a_{lj} z^n - \ldots.$$

Therefore if n is odd, $n = 2m+1$, then the first different from zero coefficient of the Poincaré map is

$$p_{2m} = 2 \sum_{l+j=2m+1} (-1)^j a_{lj}.$$

We have to prove that if $p_{2m} = 0$ then $p_{2m+k} = 0$ for all positive integer k. To do so, it is sufficiently to show that if $p_{2m} = 0$ then the polynomial (1.5) has a branch symmetric with respect to the line $w = z$. We show that under the condition $p_{2m} = 0$ the line $w + z = 0$ is an irreducible branch of (1.5). Indeed, consider the equality

$$(w+z)(1 + \sum_{j=0}^{2m} b_j w^j z^{2m-j}) = w + z + \sum_{l+j=2m+1} a_{lj} w^l z^j.$$

Equaling the coefficient of the same terms in the both sides we get the system

$$\begin{aligned} a_{2m+1,0} &= b_{2m} \\ a_{2m,1} &= b_{2m} + b_{2m-1} \\ &\cdots \\ a_{1,2m} &= b_1 + b_0 \\ a_{0,2m+1} &= b_0, \end{aligned}$$

which has a solution if and only if the coefficients of (1.5) satisfy (2.2).

2) As we have shown if $p_{2m} = 0$ then $p_{2m+k} = 0$ for all positive integer k. Therefore $p_{2m+k} = p_{2m} h$ (with some polynomial h) for all such k. Hence,

$$\mathcal{I} = \langle p_{2m+1} \rangle$$

and, due to Theorem 2.3, the cyclicity of the map defined by (1.5) is equal to zero. $\square$

Consider now the case of (1.5) with n even. Then in the map (1.1) defined by (1.5) the two first different from zero coefficients are

$$\alpha_{n-1} = \sum_{l+j=n} (-1)^j a_{lj}, \quad \alpha_{2(n-1)} = \left(\sum_{l+j=n} (-1)^j a_{lj} \right) \left(\sum_{l+j=n} (-1)^j \, l \, a_{lj} \right).$$

Thus according to (2.3), $p_n = 2\alpha_n = 0$. It is easily seen that in this case the first different from zero coefficient of the Poincaré map is

$$p_{2(n-1)} = 2\alpha_{2(n-1)} - n\alpha_{n-1}^2.$$

PROPOSITION 2.5. *The the center varieties of the maps $F^{(4)}(a)$ and $F^{(6)}(a)$ are, correspondingly,*

$$V^{(4)} = \mathbf{V}(J_1^{(4)}) \cup \mathbf{V}(J_2^{(4)}),$$

where $J_1^{(4)} = \langle a_{04} - a_{13} + a_{22} - a_{31} + a_{40} \rangle$, $J_2^{(4)} = \langle a_{13} - a_{31}, a_{04} - a_{40} \rangle$, *and*

$$V^{(6)} = \mathbf{V}(J_1^{(6)}) \cup \mathbf{V}(J_2^{(6)}),$$

where $J_1^{(6)} = \langle a_{06} - a_{15} + a_{24} - a_{33} + a_{42} - a_{51} + a_{60} \rangle$, $J_2^{(6)} = \langle a_{24} - a_{42}, a_{15} - a_{51}, a_{06} - a_{60} \rangle$.

Proof. In the case $n = 4$ the calculation of the return map yields

$$p_6^{(4)} = 2(2a_{04} - a_{13} + a_{31} - 2a_{40})(a_{04} - a_{13} + a_{22} - a_{31} + a_{40}), \; p_9^{(4)} \equiv 0 \bmod \langle p_6 \rangle,$$

$$p_{12}^{(4)} \equiv \frac{1}{4}(a_{13} - a_{31}) \times$$

$$(a_{13} - 2a_{22} + 3a_{31} - 4a_{40})^2 (a_{04} - a_{13} + a_{22} - a_{31} + a_{40}) \bmod \langle p_6 \rangle.$$

With Singular [4] by making use of the routine $minAssGTZ$ we found that the minimal associate primes of $\sqrt{\langle p_6, p_{12} \rangle}$ are the ideals $J_1^{(4)}, J_2^{(4)}$ given above. This yields that

$$V^{(4)} \subseteq \mathbf{V}(J_1^{(4)}) \cup \mathbf{V}(J_2^{(4)}).$$

To see that the opposite inclusion holds,

$$V^{(4)} \supseteq \mathbf{V}(J_1^{(4)}) \cup \mathbf{V}(J_2^{(4)}),$$

one can check that for the points from $\mathbf{V}(J_1^{(4)})$ the curve $\Psi^{(4)}(z,w) = 0$ has the branch $w+z=0$ and the curves $\Psi^{(4)}(z,w)=0$ corresponding to the points from $\mathbf{V}(J_2^{(4)})$ are symmetric with respect to the line $w=z$.

Similar reasoning applies also to the case $n=6$, however in this case the variety $V^{(6)}$ is defined by three focus quantities, $V^{(6)} = \mathbf{V}(\langle p_{10}^{(6)}, p_{20}^{(6)}, p_{30}^{(6)}\rangle)$, where $p_{10}^{(6)} = 2(3a_{06} - 2a_{15} + a_{24} - a_{42} + 2a_{51} - 3a_{60})(a_{06} - a_{15} + a_{24} - a_{33} + a_{42} - a_{51} + a_{60})$, $p_{20}^{(6)} \equiv \frac{2}{27}(5\,a_{15} - 4\,a_{24} + 4\,a_{42} - 5\,a_{51})\,(a_{15} - 2\,a_{24} + 3\,a_{33} - 4\,a_{42} + 5\,a_{51} - 6\,a_{60})^2\,(a_{06} - a_{15} + a_{24} - a_{33} + a_{42} - a_{51} + a_{60}) \bmod \langle p_{10}\rangle$, $p_{30}^{(6)} = (2(a_{24} - a_{42})(a_{15} - 2a_{24} + 3a_{33} - 4a_{42} + 5a_{51} - 6a_{60})^2(a_{06} - a_{15} + a_{24} - a_{33} + a_{42} - a_{51} + a_{60})(2a_{24} - 5a_{33} + 8a_{42} - 10a_{51} + 10a_{60})^2)/1125 \bmod \langle p_{10}^{(6)}, p_{20}^{(6)}\rangle$ and for the polynomial reduction we used the lexicographic order with $a_{06} > a_{15} > a_{24} > a_{33} > a_{42} > a_{51} > a_{60}$. ∎

PROPOSITION 2.6. *The cyclicities of the maps $F^{(4)}(a)$ and $F^{(6)}(a)$ with a focus in the origin are equal, correspondingly, to 1 and 2.*

Proof. Consider the case $n=4$ (the case $n=6$ is similar). The variety $\mathbf{V}(\mathcal{B}^{(4)})$ is defined by $p_6^{(4)}, p_{12}^{(4)}$. Therefore the return map of $F^{(4)}(a^*)$ with a focus in the origin has the expansion

$$\mathcal{P}(a^*;z) = z + p_6^{(4)}(a^*)z^7 + p_9^{(4)}(a^*)z^{10} + \dots$$

or

$$\mathcal{P}(a^*;z) = z + p_{12}^{(4)}(a^*)z^{13} + p_{15}^{(4)}(a^*)z^{16} + \dots.$$

Obviously, in the first case the equation $\mathcal{P}(a;z) - z = 0$ has no roots if $\|a - a^*\|$ is sufficiently small, and in the second case this equation has at most one root for such a. Therefore, the cyclicity of $F^{(4)}(a)$ with a focus at the origin is at most one.

It is easy to see that it is equal to one, because there are maps such that the equation $\mathcal{P}(a;z) - z = 0$ has a small positive real root. Indeed, let $a^* = (a_{40}, a_{31}, a_{22}, 2a_{04} + a_{31} - 2a_{40} + \delta, a_{04})$. Then $p_6^{(4)} = 2\delta(a_{04} - a_{22} + 2a_{31} - 3a_{40} + \delta)$ and when $\delta = 0$, $p_{12}^{(4)} = -2(a_{04} - a_{22} + 2a_{31} - 3a_{40})^3(a_{04} - a_{40})$ (and $p_{12}^{(4)} \neq 0$ due to our assumption that $F^{(4)}(a^*)$ has a focus at the origin). Obviously, we can choose δ such that $|p_6^{(4)}| \ll |p_{12}^{(4)}|$ and the sign of $p_6^{(4)}$ is opposite from that of $p_{12}^{(4)}$. That yields a small positive root of the function $\mathcal{P}(a;z) - z$. ∎

Recall that an ideal I is called radical if $f^l \in I$ for any integer $l \geq 1$ implies that $f \in I$. The radical of an ideal I is denoted by $\sqrt{I}$.

PROPOSITION 2.7. *The ideals $\langle p_6^{(4)}, p_{12}^{(4)}\rangle$ and $\langle p_{10}^{(6)}, p_{20}^{(6)}, p_{30}^{(6)}\rangle$ are not radical ideals in $\mathbb{C}[a]$.*

Proof. It is easy to check the statement of the lemma using any computer algebra system with an implemented routine for computing the radical of a polynomial ideal (Singular, Macaulay, CALI etc.). Computing with Singular [4] we found that $\sqrt{\langle p_6^{(4)}, p_{12}^{(4)}\rangle}$ and $\langle p_6^{(4)}, p_{12}^{(4)}\rangle$ have different reduced Gröbner bases. That means, that $\langle p_6^{(4)}, p_{12}^{(4)}\rangle$ is not a radical ideal. Similarly one can check that the second ideal is not radical as well. ∎

To conclude, we have shown that the center and cyclicity problems for the map defined by the polynomial (1.5) with odd n has a simple solution. The case of even n is more difficult. Basing on Proposition 2.5 we conjecture that the center variety of the maps $F^{(2m)}(a)$ consists of two components:

$$\sum_{l+j=2m} (-1)^j a_{lj} = 0$$

and

$$a_{2m-i,i} - a_{i,2m-i} = 0, \quad i = 0, 1, \ldots, m-1,$$

where the first component corresponds to (1.5) of the form $(w+z)(1+\sum_{i+j=n-1} a_{ij} z^i w^j)$ and the second one to those invariant under the involution $w \mapsto z,\ z \mapsto w$.

We have checked that $p_{18}^{(4)}, p_{24}^{(4)} \in \langle p_6^{(4)}, p_{12}^{(4)} \rangle$ and $p_{40}^{(6)} \in \langle p_{10}^{(6)}, p_{20}^{(6)}, p_{30}^{(6)} \rangle$. Therefore, probably,

$$\mathcal{B}^{(4)} = \langle p_6^{(4)}, p_{12}^{(4)} \rangle \ \text{ and } \ \mathcal{B}^{(6)} = \langle p_{10}^{(6)}, p_{20}^{(6)}, p_{30}^{(6)} \rangle, \tag{2.4}$$

yielding that the cyclicity of maps $F^{(4)}(a)$ and $F^{(6)}(a)$ with a center in the origin are, respectively, 1 and 2 as well. If the ideals $\langle p_6^{(4)}, p_{12}^{(4)} \rangle$ and $\langle p_{10}^{(6)}, p_{20}^{(6)}, p_{30}^{(6)} \rangle$ were radical ideals then (2.4) would be true. However Proposition 2.7 shows that these ideals are not radical ones. Thus there remains an open problem to find the center variety of map defined by (1.5) with n even and to investigate the cyclicity of this map.

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