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OSCILLATION OF FACTORED DYNAMIC EQUATIONS*

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Abstract. Results developed for the Euler–Cauchy dynamic equation are extended to a more general class of factored dynamic equations. The oscillation properties are studied in the case of isolated time scales, where a necessary and sufficient criterion for oscillation is developed.

Key words. time scales, factored dynamic equations

AMS subject classifications. 39A10

1. Introduction. We will assume that the reader is familiar with the time scale calculus (see Bohner and Peterson [2]). The factored form of the Euler–Cauchy dynamic equation

$$(tD - \lambda_2)(tD - \lambda_1)x = 0, \tag{1.1}$$

where D is the delta derivative operator with respect to t and λ_1, λ_2 are constants was introduced by Akin-Bohner and Bohner [1] and they used this to define and solve the nth order Euler–Cauchy dynamic equation. The oscillation of the second-order Euler–Cauchy dynamic equation (1.1) was studied by Huff et al [4]. We assume throughout that $\mathbb{T} \subset (0, \infty)$ and $f : \mathbb{T} \to (0, \infty)$. In this paper we solve and study the oscillation properties of the factored dynamic equation

$$(f(t)D - \lambda_2)(f(t)D - \lambda_1)x = 0, \tag{1.2}$$

where λ_1, λ_2 are constants, which we call the characteristic roots of (1.2). K. Messer studies nth order factored equations in [5].

We will assume that the regressivity condition

$$1 + \frac{(\lambda_1 + \lambda_2)\mu(t)}{f(t)} + \frac{\lambda_1 \lambda_2 \mu^2(t)}{f^2(t)} \neq 0 \tag{1.3}$$

holds throughout. This regressivity condition (1.3) is equivalent to the restriction that

$$\frac{\lambda_1}{f(t)}, \frac{\lambda_2}{f(t)} \in \mathcal{R},$$

where $\mathcal{R}$ is the regressive group defined in [2], page 58.

The next three results are motivated by results in [4].

THEOREM 1.1. *Let λ_1, λ_2 be the characteristic roots to (1.2). If $\lambda_1 \neq \lambda_2$, then*

$$x(t) = c_1 e_{\frac{\lambda_1}{f(t)}}(t, t_0) + c_2 e_{\frac{\lambda_2}{f(t)}}(t, t_0)$$

is a general solution of (1.2). If $\lambda_1 = \lambda_2$, then

$$x(t) = c_1 e_{\frac{\lambda_1}{f(t)}}(t, t_0) + c_2 e_{\frac{\lambda_1}{f(t)}}(t, t_0) \int_{t_0}^{t} \frac{1}{f(s) + \lambda_1 \mu(s)} \Delta s$$

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is a general solution of (1.2).

Proof. Assume that x solves (1.2) and take $y = (f(t)D - \lambda_1)x$, so that

$$(f(t)D - \lambda_2)y = 0. \tag{1.4}$$

This is equivalent to the dynamic equation

$$y^\Delta = \frac{\lambda_2}{f(t)} y$$

which is solved by

$$y(t) = c_2 e_{\frac{\lambda_2}{f(t)}}(t, t_0)$$

due to the regressivity condition (1.3). From (1.4) it follows that x satisfies

$$(f(t)D - \lambda_1)x = c_2 e_{\frac{\lambda_2}{f(t)}}(t, t_0),$$

or equivalently

$$\left(D - \frac{\lambda_1}{f(t)}\right) x = c_2 \frac{1}{f(t)} e_{\frac{\lambda_2}{f(t)}}(t, t_0).$$

Using the variation of constants formula [2], page 77, we get that

$$\begin{aligned} x(t) &= c_1 e_{\frac{\lambda_1}{f(t)}}(t, t_0) + c_2 \int_{t_0}^t e_{\frac{\lambda_1}{f(t)}}(t, \sigma(s)) \left(\frac{1}{f(s)} e_{\frac{\lambda_2}{f(t)}}(s, t_0)\right) \Delta s \\ &= c_1 e_{\frac{\lambda_1}{f(t)}}(t, t_0) + c_2 e_{\frac{\lambda_1}{f(t)}}(t, t_0) \int_{t_0}^t \frac{1}{f(s)} e_{\frac{\lambda_1}{f(t)}}(t_0, \sigma(s)) e_{\frac{\lambda_2}{f(t)}}(s, t_0) \Delta s \\ &= c_1 e_{\frac{\lambda_1}{f(t)}}(t, t_0) + c_2 e_{\frac{\lambda_1}{f(t)}}(t, t_0) \int_{t_0}^t \frac{1}{f(s)} e_{\ominus\frac{\lambda_1}{f(t)}}(\sigma(s), t_0) e_{\frac{\lambda_2}{f(t)}}(s, t_0) \Delta s \\ &= c_1 e_{\frac{\lambda_1}{f(t)}}(t, t_0) + c_2 e_{\frac{\lambda_1}{f(t)}}(t, t_0) \int_{t_0}^t \frac{1}{f(s) + \lambda_1 \mu(s)} e_{\ominus\frac{\lambda_1}{f(t)}}(s, t_0) e_{\frac{\lambda_2}{f(t)}}(s, t_0) \Delta s \\ &= c_1 e_{\frac{\lambda_1}{f(t)}}(t, t_0) + c_2 e_{\frac{\lambda_1}{f(t)}}(t, t_0) \int_{t_0}^t \frac{1}{f(s) + \lambda_1 \mu(s)} e_{\frac{\lambda_2}{f(t)} \ominus \frac{\lambda_1}{f(t)}}(s, t_0) \Delta s. \end{aligned}$$

If $\lambda_1 = \lambda_2$, we have the desired result that

$$x(t) = c_1 e_{\frac{\lambda_1}{f(t)}}(t, t_0) + c_2 e_{\frac{\lambda_1}{f(t)}}(t, t_0) \int_{t_0}^t \frac{1}{f(s) + \lambda_1 \mu(s)} \Delta s.$$

If $\lambda_1 \neq \lambda_2$,, then the formula

$$\int_{t_0}^t \frac{1}{f(s) + \lambda_1 \mu(s)} e_{\frac{\lambda_2}{f(t)} \ominus \frac{\lambda_1}{f(t)}}(s, t_0) \Delta s = \frac{1}{\lambda_2 - \lambda_1} \left[e_{\frac{\lambda_2}{f(t)} \ominus \frac{\lambda_1}{f(t)}}(t, t_0) - 1 \right]$$

completes the proof. ∎

If the characteristic roots are complex conjugates of each each other, we can write the general solution in terms of the generalized expontential and trigonometric functions (see [2] for the definitions of these functions).

THEOREM 1.2. *If the characteristic roots of* (1.2) *are* $\lambda_{1,2} = \alpha \pm i\beta$, *where* $\beta > 0$, *and the regressivity condition* (1.3) *holds, then*

$$x(t) = c_1 e_{\frac{\alpha}{f(t)}}(t, t_0) \cos_{\frac{\beta}{f(t) + \alpha\mu(t)}}(t, t_0) + c_2 e_{\frac{\alpha}{f(t)}}(t, t_0) \sin_{\frac{\beta}{f(t) + \alpha\mu(t)}}(t, t_0)$$

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is a general solution of equation (1.2).

Proof. By Theorem 1.1,

$$e_{\frac{\alpha+i\beta}{f(t)}}(t,t_0), \quad e_{\frac{\alpha-i\beta}{f(t)}}(t,t_0)$$

are solutions. Define $\widetilde{\beta}$ by

$$\frac{\widetilde{\beta}(t)}{f(t)} = \frac{\beta}{f(t)+\alpha\mu(t)}, \quad t \in \mathbb{T}$$

Then the following two conditions hold:

$$\begin{aligned} \frac{\alpha}{f(t)} + i\frac{\beta}{f(t)} &= \frac{\alpha}{f(t)} \oplus i\frac{\widetilde{\beta}(t)}{f(t)} \\ \frac{\alpha}{f(t)} - i\frac{\beta}{f(t)} &= \frac{\alpha}{f(t)} \oplus (-i)\frac{\widetilde{\beta}(t)}{f(t)}. \end{aligned}$$

So

$$\begin{aligned} x_1(t) &= \frac{1}{2}e_{\frac{\alpha+i\beta}{f(t)}}(t,t_0) + \frac{1}{2}e_{\frac{\alpha-i\beta}{f(t)}}(t,t_0) \\ &= \frac{1}{2}e_{\frac{\alpha}{f(t)}\oplus\frac{i\widetilde{\beta}(t)}{f(t)}}(t,t_0) + \frac{1}{2}e_{\frac{\alpha}{f(t)}\oplus(-\frac{i\widetilde{\beta}(t)}{f(t)})}(t,t_0) \\ &= e_{\frac{\alpha}{f(t)}}(t,t_0)\left(\frac{e_{i\frac{\widetilde{\beta}(t)}{f(t)}}(t,t_0) + e_{-i\frac{\widetilde{\beta}(t)}{f(t)}}(t,t_0)}{2}\right) \\ &= e_{\frac{\alpha}{f(t)}}(t,t_0)\cos_{\frac{\widetilde{\beta}(t)}{f(t)}}(t,t_0) \\ &= e_{\frac{\alpha}{f(t)}}(t,t_0)\cos_{\frac{\beta}{f(t)+\alpha\mu(t)}}(t,t_0) \end{aligned}$$

is a solution. Likewise

$$x_2(t) = e_{\frac{\alpha}{f(t)}}(t,t_0)\sin_{\frac{\beta}{f(t)+\alpha\mu(t)}}(t,t_0)$$

is a solution. Since x_1, x_2 are linearly independent solutions, we have the desired result. ∎

The properties of the generalized trigonometric functions are not fully known, so we write the solution in terms of the classical trigonometric functions. This leads to a useful forumla on **isolated time scales**, that is time scales where every point is isolated.

LEMMA 1.3. *If the characteristic roots are* $\lambda_{1,2} = \alpha \pm i\beta$, *where* $\beta > 0$, *then*

$$x(t) = A(t)\left(c_1 \cos B(t) + c_2 \sin B(t)\right),$$

where

$$A(t) = e^{\int_{t_0}^{t} \Re\left(\xi_{\mu(\tau)}\left(\frac{\alpha+i\beta}{f(\tau)}\right)\right)\Delta\tau} > 0, \quad B(t) = \int_{t_0}^{t} \Im\left(\xi_{\mu(\tau)}\left(\frac{\alpha+i\beta}{f(\tau)}\right)\right)\Delta\tau, \tag{1.5}$$

where ξ_h *is the cylinder transformation (see page 57 in [2]), is a general solution of the dynamic equation* (1.2). *If, in addition,* $\mathbb{T}$ *is a isolated time scale, then for* $t \in \mathbb{T}$,

$$A(t) = \prod_{\tau=t_0}^{\rho(t)} \frac{1}{f(\tau)}\sqrt{(f(\tau)+\mu(\tau)\alpha)^2 + \beta^2\mu^2(t)}, \quad B(t) = \sum_{\tau=t_0}^{\rho(t)} \arctan\left(\frac{\beta\mu(\tau)}{f(\tau)+\alpha\mu(\tau)}\right).$$

Proof. From [2], page 59, we have

$$\begin{aligned} e_{\frac{\alpha+i\beta}{f(t)}}(t,t_0) &= e^{\int_{t_0}^t \xi_{\mu(\tau)}\left(\frac{\alpha+i\beta}{f(\tau)}\right)\Delta\tau} \\ &= e^{\int_{t_0}^t \Re\left(\xi_{\mu(\tau)}\left(\frac{\alpha+i\beta}{f(\tau)}\right)\right)+i\Im\left(\xi_{\mu(\tau)}\left(\frac{\alpha+i\beta}{f(\tau)}\right)\right)\Delta\tau} \\ &= A(t)e^{iB(t)} \\ &= A(t)\left(\cos B(t) + i\sin B(t)\right). \end{aligned}$$

The real and imaginary parts

$$x_1(t) := A(t)\cos B(t), \quad x_2(t) := A(t)\sin B(t)$$

are linearly independent solutions of (1.2), and the result follows.
Suppose that every point in $\mathbb{T}$ is isolated, then

$$\begin{aligned} &\xi_{\mu(\tau)}\left(\frac{\alpha+i\beta}{f(\tau)}\right) \\ &= \frac{1}{\mu(\tau)} Log\left(1+\mu(\tau)\frac{\alpha+i\beta}{f(\tau)}\right) \\ &= \frac{1}{\mu(\tau)}\log\left|\frac{f(\tau)+\alpha\mu(\tau)}{f(\tau)} + i\frac{\beta\mu(\tau)}{f(\tau)}\right| + \frac{i}{\mu(\tau)} Arg\left(\frac{f(\tau)+\alpha\mu(\tau)}{f(\tau)} + i\frac{\beta\mu(\tau)}{f(\tau)}\right) \\ &= \frac{1}{\mu(\tau)}\log\left(\frac{1}{f(\tau)}\sqrt{(f(\tau)+\alpha\mu(\tau))^2+\beta^2\mu^2(\tau)}\right) + \frac{i}{\mu(\tau)}\arctan\left(\frac{\beta\mu(\tau)}{f(\tau)+\alpha\mu(\tau)}\right). \end{aligned}$$

Then

$$\Re\left(\xi_{\mu(\tau)}\left(\frac{\alpha+i\beta}{f(\tau)}\right)\right) = \frac{1}{\mu(\tau)}\log\left(\frac{1}{f(\tau)}\sqrt{(f(\tau)+\alpha\mu(\tau))^2+\beta^2\mu^2(\tau)}\right) \tag{1.6}$$

and

$$\Im\left(\xi_{\mu(\tau)}\left(\frac{\alpha+i\beta}{f(\tau)}\right)\right) = \frac{1}{\mu(\tau)}\arctan\left(\frac{\beta\mu(\tau)}{f(\tau)+\alpha\mu(\tau)}\right). \tag{1.7}$$

From (1.5) and (1.6) we have

$$\begin{aligned} A(t) &= e^{\int_{t_0}^t \frac{1}{\mu(\tau)}\log\left(\frac{1}{f(\tau)}\sqrt{(f(\tau)+\alpha\mu(\tau))^2+\beta^2\mu^2(\tau)}\right)\Delta\tau} \\ &= e^{\sum_{\tau=t_0}^{\rho(t)}\log\left(\frac{1}{f(\tau)}\sqrt{(f(\tau)+\alpha\mu(\tau))^2+\beta^2\mu^2(\tau)}\right)} \\ &= \prod_{\tau=t_0}^{\rho(t)}\left(\frac{1}{f(\tau)}\sqrt{(f(\tau)+\alpha\mu(\tau))^2+\beta^2\mu^2(\tau)}\right). \end{aligned}$$

Furthermore, from (1.5) and (1.7) we have

$$\begin{aligned} B(t) &= \int_{t_0}^t \frac{1}{\mu(\tau)}\arctan\left(\frac{\beta\mu(\tau)}{f(\tau)+\alpha\mu(\tau)}\right)\Delta\tau \\ &= \sum_{\tau=t_0}^{\rho(t)}\arctan\left(\frac{\beta\mu(\tau)}{f(\tau)+\alpha\mu(\tau)}\right), \end{aligned}$$

which is the desired result. ∎

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2. Oscillation Results. For the remainder of the paper we assume that $\mathbb{T}$ is unbounded above and that the characteristic roots of the dynamic equation (1.2) are $\lambda_{1,2} = \alpha \pm i\beta$, where $\beta > 0$. Recall from [4] the definition of oscillatory:

DEFINITION 2.1. *If the characteristic roots of* (1.2) *are* $\lambda_{1,2} = \alpha \pm i\beta$, $\beta > 0$, *then we say the dynamic equation* (1.2) *is* **oscillatory** *iff* $B(t)$ *is unbounded.*

For example, let $\mathbb{T}$ be the real interval $[1, \infty)$ and let $f(t) = t^k$. Then

$$B(t) = \beta \int_1^t \frac{1}{f(\tau)}\, d\tau = \beta \int_1^t \frac{1}{\tau^k}\, d\tau.$$

So $B(t)$ is unbounded if and only if $k \leq 1$. Thus we have oscillation only in the case where $k \leq 1$.

We now restrict ourselves to isolated time scales, for which we have the following criterion for oscillation.

THEOREM 2.2. *Let* $\mathbb{T}$ *be an isolated time scale. The dynamic equation* (1.2) *is oscillatory on* $\mathbb{T}$ *if and only if* $\sum_{\tau=t_0}^{\infty} \frac{\mu(\tau)}{f(\tau)}$ *diverges.*

Proof. Suppose that $\sum_{\tau=t_0}^{\infty} \frac{\mu(\tau)}{f(\tau)}$ diverges. We break the proof into two cases. If $\lim_{\tau\to\infty} \frac{f(\tau)}{\mu(\tau)} \neq \infty$, then clearly

$$\lim_{\tau\to\infty} \arctan\left(\frac{\beta}{\frac{f(\tau)}{\mu(\tau)} + \alpha}\right) \neq 0.$$

So

$$\begin{aligned}
\lim_{t\to\infty} B(t) &= \sum_{\tau=t_0}^{\infty} \arctan\left(\frac{\beta\mu(\tau)}{f(\tau) + \mu(\tau)\alpha}\right) \\
&= \sum_{\tau=t_0}^{\infty} \arctan\left(\frac{\beta}{\frac{f(\tau)}{\mu(\tau)} + \alpha}\right) \\
&= \infty.
\end{aligned}$$

Thus (1.2) is oscillatory.

If $\lim_{\tau\to\infty} \frac{f(\tau)}{\mu(\tau)} = \infty$, then there is a $t_1 \geq t_0$ such that

$$\begin{aligned}
\sum_{\tau=t_1}^{\infty} \arctan\left(\frac{\beta}{\frac{f(\tau)}{\mu(\tau)} + \alpha}\right) &\geq \sum_{\tau=t_1}^{\infty} \arctan\left(\frac{\beta}{2 \cdot \frac{f(\tau)}{\mu(\tau)}}\right) \\
&= \sum_{\tau=t_1}^{\infty} \arctan\left(\frac{\beta}{2} \cdot \frac{\mu(\tau)}{f(\tau)}\right) \\
&\geq \sum_{\tau=t_1}^{\infty} \left(\frac{\beta}{2} \cdot \frac{\mu(\tau)}{f(\tau)} - \frac{\beta^3}{24} \cdot \frac{\mu^3(\tau)}{f^3(\tau)}\right).
\end{aligned}$$

Since $\lim_{\tau\to\infty} \frac{f(\tau)}{\mu(\tau)} = \infty$, we have $\lim_{\tau\to\infty} \frac{\mu(\tau)}{f(\tau)} = 0$. We apply the limit comparison test,

$$\lim_{\tau\to\infty} \frac{\frac{\beta}{2} \cdot \frac{\mu(\tau)}{f(\tau)} - \frac{\beta^3}{24} \cdot \frac{\mu^3(\tau)}{f^3(\tau)}}{\frac{\mu(\tau)}{f(\tau)}} = \lim_{\tau\to\infty} \left(\frac{\beta}{2} - \frac{\beta^3}{24} \cdot \frac{\mu^2(\tau)}{f^2(\tau)}\right) = \frac{\beta}{2}.$$

We have $0 < \frac{\beta}{2} < \infty$, and $\sum_{\tau=t_0}^{\infty} \frac{\mu(\tau)}{f(\tau)} = \infty$, so $\lim_{t\to\infty} B(t) = \infty$ and therefore we have oscillation.

To prove the converse, we deal with the contrapositive. Suppose that $\sum_{\tau=t_0}^{\infty} \frac{\mu(\tau)}{f(\tau)}$ converges. Then $\lim_{\tau\to\infty} \frac{\mu(\tau)}{f(\tau)} = 0$, so $\lim_{\tau\to\infty} \frac{f(\tau)}{\mu(\tau)} = \infty$. Thus for t_1 sufficiently large

$$\begin{aligned}\sum_{\tau=t_1}^{\infty} \arctan\left(\frac{\beta}{\frac{f(\tau)}{\mu(\tau)} + \alpha}\right) &\leq \sum_{\tau=t_1}^{\infty} \arctan\left(\frac{\beta}{\frac{1}{2}\cdot\frac{f(\tau)}{\mu(\tau)}}\right)\\ &= \sum_{\tau=t_1}^{\infty} \arctan\left(2\beta\cdot\frac{\mu(\tau)}{f(\tau)}\right)\\ &\leq \sum_{\tau=t_1}^{\infty} 2\beta\cdot\frac{\mu(\tau)}{f(\tau)}\\ &= 2\beta\sum_{\tau=t_1}^{\infty}\frac{\mu(\tau)}{f(\tau)}.\end{aligned}$$

So $B(t)$ is bounded, and therefore the solutions are nonoscillatory. ∎

To show the utility of this result, consider the Euler–Cauchy equation (1.1) on the time scale $\mathbb{N}$. In this case,

$$\sum_{\tau=t_0}^{\infty}\frac{\mu(\tau)}{f(\tau)} = \sum_{n=1}^{\infty}\frac{1}{n} = \infty.$$

So (1.1) is oscillatory on $\mathbb{N}$.

We can also use standard series comparisons between time scales in this manner. On the time scale $\mathbb{N}^2$ we have

$$\frac{\mu(t)}{f(t)} = \frac{2n+1}{n^2} \geq \frac{2}{n} \geq \frac{1}{n}.$$

So $\sum_{\tau=t_0}^{\infty} \frac{\mu(\tau)}{f(\tau)} = \infty$, and we have oscillation of the Euler–Cauchy equation (1.1) on $\mathbb{N}^2$.

Still considering the Euler–Cauchy equation, oscillation on the time scale $\mathbb{T}_p = \{t_n \mid t_0 = 1, t_{n+1} = t_n + \frac{1}{t_n^p}, n \in \mathbb{N}_0\}$ is determined under the condition that $p \geq 0$ after some effort in [4]. By using Theorem 2.2 we can establish the same result quickly. Note that $t_n \leq n+1$ for all $n \in \mathbb{N}_0$. So

$$\begin{aligned}\sum_{\tau=t_0}^{\infty}\frac{\mu(\tau)}{f(\tau)} &= \sum_{n=0}^{\infty}\frac{\frac{1}{t_n^p}}{t_n}\\ &= \sum_{n=0}^{\infty}\frac{1}{t_n^{p+1}}\\ &\geq \sum_{n=0}^{\infty}\frac{1}{(n+1)^{p+1}}\\ &= \infty.\end{aligned}$$

THEOREM 2.3. *Let $f(t) = t^k$, then the dynamic equation (1.2) is oscillatory on $\mathbb{N}^p$ for $p > 0$ if and only if $k \leq 1$.*

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Proof. For $t_n \in \mathbb{N}^p$ we have $\mu(t_n) = (n+1)^p - n^p$. If $p = 1$, we have

$$\sum_{\tau=t_0}^{\infty} \frac{\mu(\tau)}{f(\tau)} = \sum_{n=1}^{\infty} \frac{1}{n^k}$$

which is divergent if and only if $k \leq 1$.

If $p > 1$ we have

$$p(n+1)^{p-1} \geq (n+1)^p - n^p \geq pn^{p-1} \tag{2.1}$$

by the mean value theorem, and

$$(n+1)^{p-1} \leq 2n^{p-1} \tag{2.2}$$

for sufficiently large n. So we have for an integer n_0 sufficiently large that

$$\begin{aligned} \sum_{n=n_0}^{\infty} \frac{\mu(t_n)}{f(t_n)} &= \sum_{n=n_0}^{\infty} \frac{(n+1)^p - n^p}{n^{pk}} \\ &\geq \sum_{n=n_0}^{\infty} \frac{pn^{p-1}}{n^{pk}} \\ &= \sum_{n=n_0}^{\infty} \frac{p}{n^{p(k-1)+1}}. \end{aligned}$$

When $k \leq 1$, we have oscillation since $p(k-1)+1 \leq 1$.

The other half of (2.1) gives

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(n+1)^p - n^p}{n^{pk}} &\leq \sum_{n=1}^{\infty} \frac{p(n+1)^{p-1}}{n^{pk}} \\ &\leq \sum_{n=1}^{\infty} \frac{2pn^{p-1}}{n^{pk}} \\ &= \sum_{n=1}^{\infty} \frac{2p}{n^{p(k-1)+1}}. \end{aligned}$$

When $k > 1$, the solutions are nonoscillatory since $p(k-1)+1 > 1$.

If $p < 1$, inequalities (2.1) and (2.2) become

$$pn^{p-1} \geq (n+1)^p - n^p \geq p(n+1)^{p-1}$$

and

$$(n+1)^{p-1} \geq \frac{1}{2} n^{p-1}.$$

In this case,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(n+1)^{p-1}-n^{p}}{n^{pk}} &\geq \sum_{n=1}^{\infty}\frac{p(n+1)^{p-1}}{n^{pk}}\\ &\geq \sum_{n=1}^{\infty}\frac{pn^{p-1}}{2n^{pk}}\\ &= \sum_{n=1}^{\infty}\frac{p}{2n^{p(k-1)+1}}.\end{aligned}$$

So for $k \leq 1$, we have $p(k-1)+1 \leq 1$ and thus the solutions are oscillatory.

We can also form an upper bound,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(n+1)^{p}-n^{p}}{n^{pk}} &\leq \sum_{n=1}^{\infty}\frac{pn^{p-1}}{n^{pk}}\\ &= \sum_{n=1}^{\infty}\frac{p}{n^{p(k-1)+1}}.\end{aligned}$$

As before, the solutions are nonoscillatory if $k > 1$.

Therefore, for each case we have oscillation if and only if $k \leq 1$ which is the desired result. $\square$

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