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FACTORIZATION OF THE HYPERGEOMETRIC-TYPE DIFFERENCE EQUATION ON THE UNIFORM LATTICE*

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Abstract. We discuss factorization of the hypergeometric-type difference equations on the uniform lattices and show how one can construct a dynamical algebra, which corresponds to each of these equations. Some examples are exhibited, in particular, we show that several models of discrete harmonic oscillators, previously considered in a number of publications, can be treated in a unified form.

Key words. discrete polynomials, factorization method, discrete oscillators

AMS subject classifications. 33C45, 33C90, 39A13

1. Introduction. The study of discrete system has attracted the attention of many authors in the last years. Of special interest are the discrete analogs of the quantum harmonic oscillators [2, 5, 6, 9, 11, 12, 16, 17, 18, 21, 25] among others.

There are several methods for studying such systems. One of them is the factorization method (FM), first introduced for solving differential equations [28, 19]. This classical FM is based on the existence of the so-called raising and lowering operators for the corresponding equation, which allow to find the explicit solutions in a simple way, see e.g. [7, 22]. Later on, Miller extended it to difference equations [23] and q-differences –in the Hahn sense– [24]. In the case of difference equations this method has been also extensively used during the last years (see e.g. [9, 11, 14, 22, 29] for difference analogs on the uniform lattice and [4, 5, 6, 9, 11, 12, 15] for the q-case).

Later on, references [7, 8, 13] indicated a way of constructing the so-called "dynamical symmetry algebra" by applying the FM to differential or difference equations [3, 11, 12] and then this technique has been used to consider some particular instances of q-hypergeometric difference equations. Of special interest is also the paper by Smirnov [29], in which the equivalence of the FM and the Nikiforov *et al* formulation of theory of q-orthogonal polynomials [26], was established. In [4], following the papers [15, 22] for the classical case, it has been shown that one can factorize the hypergeometric-type difference equation (2.1) in terms of the above-mentioned raising and lowering operators.

Our main purpose here is to show how to deal with all different cases of difference equations on the uniform lattice $x(s) = s$ in an unified form. One should consider this paper as an attempt to provide a background for the more general q-linear case (since in the limit as q goes to 1, the q-linear case reduces to the uniform one). Some results concerning this general case will be also given in the last section.

The structure of the paper is as follows. In Section 2 some necessary results on classical polynomials are collected. In section 3 the factorization of the hypergeometric-type difference equation is discussed, which is used in section 4 to construct a dynamical symmetry algebra in the case of the Charlier polynomials. In section 5 the Kravchuk and the Meixner cases are considered in detail. Finally, in section 6 we briefly discuss a possibility of applying this technique to the q-case.

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2. Preliminaries: the classical "discrete" polynomials. The discretization of the hypergeometric differential equation on the lattice $x(s)$ [26, 27] leads to the *second order difference equation of the hypergeometric type*

$$\sigma(s)\frac{\Delta}{\Delta x(s-1/2)}\frac{\nabla y(x(s))}{\nabla x(s)}+\tau(s)\frac{\Delta y(x(s))}{\Delta x(s)}+\lambda y(x(s))=0, \tag{2.1}$$

where $\Delta f(s) := f(s+1)-f(s)$, $\nabla f(s) := f(s)-f(s-1)$.

The most simple lattice is the *uniform* one $x(s)=s$ and it corresponds to the equation

$$\sigma(s)\Delta\nabla y(s)+\tau(s)\Delta y(s)+\lambda y(s)=0. \tag{2.2}$$

The above equation have polynomial solutions $P_n(s)$, usually called classical *discrete* orthogonal polynomials, if and only if $\lambda=\lambda_n=-n(\tau'+(n-1)\sigma''/2)$.

It is well known [26] that under certain conditions the polynomial solutions of (2.2) are orthogonal. For example, if $\sigma(s)\rho(s)s^k\Big|_{s=a,b}=0$, for all $k=0,1,2,\ldots$, then the polynomial solutions $P_n(s)$ of (2.2) satisfy

$$\langle P_n,P_m\rangle_d=\sum_{s=a}^{b-1}P_n(s)\,P_m(s)\,\rho(s)=\delta_{nm}d_n^2, \tag{2.3}$$

where the weight functions $\rho(s)$ are solutions of the Pearson-type equation

$$\Delta\left[\sigma(s)\rho(s)\right]=\tau(s)\rho(s)\quad\text{or}\quad\sigma(s+1)\rho(s+1)=[\sigma(s)+\tau(s)]\rho(s). \tag{2.4}$$

In the following we will consider the monic polynomials, i.e., $P_n(s)=s^n+b_ns^{n-1}+\cdots$.

The polynomial solutions of (2.2) are the classical *discrete* orthogonal polynomials of Hahn, Meixner, Kravchuk and Charlier and their principal data are given in Table 2.1.

TABLE 2.1
The classical discrete orthogonal monic polynomials.

$P_n(s)$	Hahn $h_n^{\alpha,\beta}(s;N)$	Meixner $M_n^{\gamma,\mu}(s)$	Kravchuk $K_n^p(s)$	Charlier $C_n^\mu(s)$
$[a,b]$	$[0,N]$	$[0,\infty)$	$[0,N+1]$	$[0,\infty)$
$\sigma(s)$	$s(N+\alpha-s)$	s	s	s
$\tau(s)$	$(\beta+1)(N-1)-(\alpha+\beta+2)s$	$(\mu-1)s+\mu\gamma$	$\frac{Np-s}{1-p}$	$\mu-s$
$\sigma+\tau$	$(s+\beta+1)(N-1-s)$	$\mu s+\gamma\mu$	$-\frac{p}{1-p}(s-N)$	μ
λ_n	$n(n+\alpha+\beta+1)$	$(1-\mu)n$	$\frac{n}{1-p}$	n
$\rho(s)$	$\frac{\Gamma(N+\alpha-s)\Gamma(\beta+s+1)}{\Gamma(N-s)\Gamma(s+1)}$ $\alpha,\beta\geq-1,\ n\leq N-1$	$\frac{\mu^s\Gamma(\gamma+s)}{\Gamma(\gamma)\Gamma(s+1)}$ $\gamma>0,\mu\in(0,1)$	$\binom{N}{s}p^s(1-p)^{N-s}$ $p\in(0,1), n\leq N-1$	$\frac{e^{-\mu}\mu^s}{\Gamma(s+1)}$ $\mu>0$
d_n^2	$\frac{n!\Gamma(\alpha+\beta+N+n+1)}{(N-n-1)!(\alpha+\beta+n+1)_n}\binom{\alpha+\beta+2n+2}{\alpha+n+1}^{-1}$	$\frac{n!(\gamma)_n\mu^n}{(1-\mu)^{\gamma+2n}}$	$\binom{N}{n}p^n(1-p)^n$	$n!\mu^n$

They can be expressed in terms of the generalized hypergeometric function ${}_p\mathrm{F}_q$,

$$ {}_p\mathrm{F}_q\left(\begin{array}{c} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{array} \middle| x \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{x^k}{k!}, $$

shifted factorial)

$$ (a)_0 = 1, \quad (a)_k = a(a+1)(a+2)\cdots(a+k-1), \; k = 1, 2, 3, \dots \; . $$

Using the above notations, we have for the monic polynomials of Hahn, Meixner, Kravchuk and Charlier, respectively

$$ h_n^{\alpha,\beta}(s,N) = \frac{(1-N)_n (\beta+1)_n}{(\alpha+\beta+n+1)_n} \, {}_3\mathrm{F}_2\left(\begin{array}{c} -s, \alpha+\beta+n+1, -n \\ 1-N, \beta+1 \end{array} \middle| 1 \right), $$

$$ M_n^{\gamma,\mu}(s) = \frac{(\gamma)_n \mu^n}{(\mu-1)^n} \quad {}_2\mathrm{F}_1\left(\begin{array}{c} -n, -s \\ \gamma \end{array} \middle| 1 - \frac{1}{\mu} \right), $$

$$ K_n^{p}(s) = \frac{(-p)^n N!}{(N-n)!} \quad {}_2\mathrm{F}_1\left(\begin{array}{c} -n, -s \\ -N \end{array} \middle| \frac{1}{p} \right), $$

$$ C_n^{\mu}(s) = (-\mu)^n \quad {}_2\mathrm{F}_0\left(\begin{array}{c} -n, -s \\ - \end{array} \middle| -\frac{1}{\mu} \right). $$

A further information on orthogonal polynomials on the uniform lattice can be found in [1, 20, 26, 27].

3. Factorization of the difference equation. Let us consider the following second order linear difference operator

$$ \mathfrak{h}_1(s) = -\nu(s-1)\, e^{-\partial_s} - \nu(s)\, e^{\partial_s} + [2\sigma(s) + \tau(s)]I, \tag{3.1} $$

where $e^{\alpha \partial_s} f(s) = f(s+\alpha)$ for all $\alpha \in \mathbb{C}$, $\nu(s) = \sqrt{\sigma(s+1)[\sigma(s)+\tau(s)]}$, and I is the identity operator, and let $(\Phi_n)_n$ be the set of functions

$$ \Phi_n(s) = \frac{\sqrt{\rho(s)}}{d_n} P_n(s), \tag{3.2} $$

where d_n is a norm of the polynomials $P_n(s)$, which satisfy equation (2.2), and $\rho(s)$ is the solution of the Pearson-type equation (2.4). If $P_n(s)$ possess the discrete orthogonality property (2.3), then the functions $\Phi_n(s)$ have the property

$$ \langle \Phi_n(s), \Phi_m(s) \rangle_d = \sum_{s=a}^{b-1} \Phi_n(s) \Phi_m(s) = \delta_{n,m}. $$

Using the identity $\nabla = \Delta - \nabla\Delta$ and the equation (2.2), one finds that

$$ \mathfrak{h}_1(s) \Phi_n(s) = \lambda_n \Phi_n(s), \tag{3.3} $$

i.e., the functions $\Phi_n(s)$, defined in (3.2), are the eigenfunctions of $\mathfrak{h}_1(s)$. In the following we will refer to $\mathfrak{h}_1(s)$ as the *hamiltonian*.

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Our first step is to find two operators $a(s)$ and $b(s)$ such that the Hamiltonian $\mathfrak{h}_1(s) = b(s)a(s)$, i.e., the operators $a(s)$ and $b(s)$ *factorize* the Hamiltonian $\mathfrak{h}_1(s)$.

DEFINITION 3.1. *Let α be a real number. We define a family of α-down and α-up operators by*

$$\begin{aligned} \mathfrak{a}_\alpha^{\downarrow}(s) &:= e^{-\alpha\partial_s}\left(e^{\partial_s}\sqrt{\sigma(s)} - \sqrt{\sigma(s)+\tau(s)}\,I\right), \\ \mathfrak{a}_\alpha^{\uparrow}(s) &:= \left(\sqrt{\sigma(s)}e^{-\partial_s} - \sqrt{\sigma(s)+\tau(s)}\,I\right)e^{\alpha\partial_s}, \end{aligned} \tag{3.4}$$

respectively.

A straightforward calculation (by using the simple identity $e^{\partial_s}\nabla = \Delta$) shows that for all $\alpha \in \mathbb{R}$

$$\mathfrak{h}_1(s) = \mathfrak{a}_\alpha^{\uparrow}(s)\mathfrak{a}_\alpha^{\downarrow}(s),$$

i.e., the operators $\mathfrak{a}_\alpha^{\downarrow}(s)$ and $\mathfrak{a}_\alpha^{\uparrow}(s)$ factorize the Hamiltonian, defined in (3.1). Thus, we have the following

THEOREM 3.2. *Given a Hamiltonian $\mathfrak{h}_1(s)$, defined by (3.1), the operators $\mathfrak{a}_\alpha^{\downarrow}(s)$ and $\mathfrak{a}_\alpha^{\uparrow}(s)$, defined in (3.4), are such that for all $\alpha \in \mathbb{C}$, the relation $\mathfrak{h}_1(s) = \mathfrak{a}_\alpha^{\uparrow}(s)\mathfrak{a}_\alpha^{\downarrow}(s)$ holds.*

4. The dynamical algebra: The Charlier case. Our next step is to find a dynamical symmetry algebra, associated with the operator $\mathfrak{h}_1(s)$, or, equivalently, with the corresponding family of polynomials, i.e., *To find two operators $a(s)$ and $b(s)$, that factorize the hamiltonian $\mathfrak{h}_1(s)$, i.e., $\mathfrak{h}_1(s) = b(s)a(s)$, and are such that its commutator $[a(s), b(s)] = a(s)b(s) - b(s)a(s) = I$, where I denotes the identity operator.*

THEOREM 4.1. *Let $\mathfrak{h}_1(s)$ be the hamiltonian, defined in (3.1). The operators $b(s) = \mathfrak{a}_\alpha^{\uparrow}(s)$ and $a(s) = \mathfrak{a}_\alpha^{\downarrow}(s)$, given in (3.4), factorize the Hamiltonian $\mathfrak{h}_1(s)$ (3.1) and satisfy the commutation relation $[a(s), b(s)] = \Lambda$ for a certain complex number Λ, if and only if the following two conditions hold:*

$$\frac{\sigma(s-\alpha)[\sigma(s-\alpha)+\tau(s-\alpha)]}{\sigma(s)[\sigma(s-1)+\tau(s-1)]} = 1$$

and

$$\sigma(s-\alpha+1) + \sigma(s-\alpha) + \tau(s-\alpha) - 2\sigma(s) - \tau(s) = \Lambda.$$

Proof. Taking the expression for the operators $a_\alpha^{\uparrow}(s)$ and $a_\alpha^{\downarrow}(s)$, a straightforward calculation shows that $a_\alpha^{\downarrow}(s)a_\alpha^{\uparrow}(s) = A_1(s)e^{\partial_s} + A_2(s)e^{-\partial_s} + A_3(s)I$, where

$$A_1(s) = -\sqrt{\sigma(s+1-\alpha)[\sigma(s-\alpha+1)+\tau(s-\alpha+1)},$$

$$A_2(s) = -\sqrt{\sigma(s-\alpha)[\sigma(s-\alpha)+\tau(s-\alpha)},$$

$$A_3(s) = \sigma(s+1-\alpha) + \sigma(s-\alpha) + \tau(s-\alpha).$$

In the same way, $a_\alpha^{\uparrow}(s)a_\alpha^{\downarrow}(s) = \mathfrak{h}_1(s) = B_1(s)e^{\partial_s} + B_2(s)e^{-\partial_s} + B_3(s)I$, where

$$B_1(s) = -\nu(s), \quad B_2(s) = -\nu(s-1), \quad B_3(s) = 2\sigma(s) + \tau(s).$$

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Consequently,

$$[a^{\downarrow}_{\alpha}(s), a^{\uparrow}_{\alpha}(s)] = \Big(A_1(s) - B_1(s)\Big)e^{\partial_s} + \Big(A_2(s) - B_2(s)\Big)e^{-\partial_s} + \Big(A_3(s) - B_3(s)\Big)I. \tag{4.1}$$

To eliminate the two terms in the right-hand side of (4.1), which are proportional to $\exp(\pm\partial_s)$, one have to require that $A_1(s) - B_1(s) = 0$ and $A_2(s) - B_2(s) = 0$. But $A_1(s)/B_1(s) = A_2(s+1)/B_2(s+1)$, hence, the requirement that $A_1(s) = B_1(s)$ entails the relation $A_2(s) = B_2(s)$, and vice versa. Thus, from (4.1) it follows that the commutator $[a^{\uparrow}_{\alpha}(s), a^{\downarrow}_{\alpha}(s)] = \Lambda$, iff $A_1(s) = B_1(s)$ and $A_3(s) - B_3(s) = \Lambda$. ☐

Using the main data for the discrete polynomials (see Table 2.1), we see that the only possible solution of the problem 1 corresponds to the case when $\sigma(s) + \tau(s) = const.$ and $\alpha = 0$, i.e., the Charlier polynomials. Moreover, in this case $\lambda_n = n$.

COROLLARY 4.2. *For the hamiltonian, associated with the Charlier polynomials,*

$$\mathfrak{h}^C_1(s) = -\sqrt{s\mu}\, e^{-\partial_s} - \sqrt{(s+1)\mu}\, e^{\partial_s} + (s+\mu)I,$$

$$\mathfrak{h}^C_1(s)\Phi^C_n(s) = n\Phi^C_n(s), \qquad \Phi^C_n(s) = \sqrt{\frac{e^{-\mu}\mu^{s-n}}{s!\,n!}}\, C^{\mu}_n(s), \quad \mu > 0, \quad n = 0, 1, 2, \dots.$$

Furthermore, the operators

$$\mathfrak{a}^{\downarrow}_0(s) = \sqrt{s+1}\, e^{\partial_s} - \sqrt{\mu}\, I, \qquad \mathfrak{a}^{\uparrow}_0(s) = \sqrt{s}e^{-\partial_s} - \sqrt{\mu}\, I,$$

are such that $\mathfrak{h}^C_1 = \mathfrak{a}^{\uparrow}_0(s)\mathfrak{a}^{\downarrow}_0(s)$ *and* $[a^{\downarrow}_0(s), a^{\uparrow}_0(s)] = 1$.

Notice that, since $\mathfrak{h}_1(s)\Phi(s) = \lambda\Phi(s)$,

$$\begin{aligned}
\mathfrak{h}_1(s)\{\mathfrak{a}^{\downarrow}_0(s)\Phi(s)\} &= \mathfrak{a}^{\uparrow}_0(s)\mathfrak{a}^{\downarrow}_0(s)\{\mathfrak{a}^{\downarrow}_0(s)\Phi(s)\} = (\mathfrak{a}^{\downarrow}_0(s)\mathfrak{a}^{\uparrow}_0(s) - 1)\{\mathfrak{a}^{\downarrow}_0(s)\Phi(s)\} \\
&= (\lambda - 1)\{\mathfrak{a}^{\downarrow}_0(s)\Phi(s)\}, \\
\mathfrak{h}_1(s)\{\mathfrak{a}^{\uparrow}_0(s)\Phi(s)\} &= \mathfrak{a}^{\uparrow}_0(s)\mathfrak{a}^{\downarrow}_0(s)\mathfrak{a}^{\uparrow}_0(s)\Phi(s) = \mathfrak{a}^{\uparrow}_0(s)(\lambda+1)\Phi(s) \\
&= (\lambda + 1)\{\mathfrak{a}^{\uparrow}_0(s)\Phi(s)\}.
\end{aligned}$$

In other words, if $\Phi(s)$ is an eigenvector of the hamiltonian $\mathfrak{h}_1(s)$, then $\mathfrak{a}^{\downarrow}_0(s)\Phi(s)$ is the eigenvector of $\mathfrak{h}_1(s)$, associated with the eigenvalue $\lambda - 1$, and $\mathfrak{a}^{\uparrow}_0(s)\Phi(s)$ is the eigenvector of $\mathfrak{h}_1(s)$, associated with the eigenvalue $\lambda+1$. In general then $[\mathfrak{a}^{\downarrow}_0]^k(s)\Phi(s)$ and $[\mathfrak{a}^{\uparrow}_0]^k(s)\Phi(s)$ are also eigenvectors corresponding to the eigenvalues $\lambda - k$ and $\lambda + k$, respectively.

Using the preceding formulas for the Charlier polynomials, one finds

$$\mathfrak{a}^{\uparrow}_0(s)\Phi^C_n(s) = U_n\Phi^C_{n+1}(s), \qquad \mathfrak{a}^{\downarrow}_0(s)\Phi^C_n(s) = D_n\Phi^C_{n-1}(s), \tag{4.2}$$

where U_n and D_n are some constants.

If we now apply $\mathfrak{a}^{\uparrow}_0(s)$ to the first equation of (4.2) and then use the second one and (3.3), we find that $\lambda_n = D_nU_{n-1}$. On the other hand, applying $\mathfrak{a}^{\downarrow}_0(s)$ to the second equation in (4.2) and using the first one, as well as the fact that $\mathfrak{a}^{\downarrow}_0(s)\mathfrak{a}^{\uparrow}_0(s)\Phi^C_n(s) = (\lambda_n + 1)\Phi^C_n(s)$, one obtains that $1 + \lambda_n = U_nD_{n+1} = \lambda_{n+1}$, from which it follows that λ_n should be a linear function of n (that is also obvious from Table 2.1).

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If we use the boundary conditions $\sigma(s)\rho(s)\big|_{s=a,b} = 0$, as well as the formula of summation by parts, we obtain

$$\langle \mathfrak{a}_0^{\downarrow}(s)\Phi_m(s), \Phi_n(s)\rangle_d = \langle \Phi_m(s), \mathfrak{a}_0^{\uparrow}(s)\Phi_n(s)\rangle_d,$$

i.e., the operators $\mathfrak{a}_0^{\downarrow}(s)$ and $\mathfrak{a}_0^{\uparrow}(s)$ are mutually adjoint.

From the above equality (the adjointness property) and (4.2) it follows that $D_{n+1} = U_n$, thus $U_n^2 = \lambda_{n+1}$, therefore $U_n = \sqrt{\lambda_{n+1}}$ and $D_n = \sqrt{\lambda_n}$, i.e., we have the following

COROLLARY 4.3. *The operators $\mathfrak{a}_0^{\uparrow}(s)$ and $\mathfrak{a}_0^{\downarrow}(s)$ are mutually adjoint with respect to the inner product $\langle \cdot, \cdot \rangle_d$ and*

$$\mathfrak{a}_0^{\uparrow}(s)\Phi_n^C(s) = \left(\sqrt{s}\, e^{\partial_s} - \sqrt{\mu}\, I\right) \Phi_n^C(s) = \sqrt{n+1}\, \Phi_{n+1}^C(s),$$
$$\mathfrak{a}_0^{\downarrow}(s)\Phi_n^C(s) = \left(\sqrt{s+1}\, e^{\partial_s} - \sqrt{\mu}\, I\right) \Phi_n^C(s) = \sqrt{n}\, \Phi_{n-1}^C(s).$$

From the above corollary one can deduce that

$$\sqrt{s+1}\, \Phi_0^C(s+1) - \sqrt{\mu}\, \Phi_0^C(s) = 0 \quad \Rightarrow \quad \Phi_0^C(s) = N_0 \sqrt{\frac{\mu^s}{s!}}.$$

Using the orthonormality of $\Phi_0^C(s)$, one obtains that $N_0 = e^{-\mu/2}$. Thus

$$\Phi_n^C(s) = \frac{1}{\sqrt{n!}} [\mathfrak{a}_0^{\uparrow}(s)]^n \Phi_0^C(s) = \frac{1}{\sqrt{n!}} \left[\sqrt{s}\, e^{\partial_s} - \sqrt{\mu}\, I\right]^n \left(\sqrt{\frac{e^{-\mu}\mu^s}{s!}}\right).$$

Notice that

$$[\mathfrak{h}_1(s), \mathfrak{a}_0^{\uparrow}(s)] = \sqrt{\mu}(\mu-1) + \mu\mathfrak{a}_0^{\uparrow}(s), \qquad [\mathfrak{h}_1(s), \mathfrak{a}_0^{\downarrow}(s)] = -\sqrt{\mu}(\mu-1) - \mu\mathfrak{a}_0^{\downarrow}(s).$$

This example constitute a discrete analog of the quantum harmonic oscillator [9].

5. The dynamical algebra: The Meixner and Kravchuk cases. From the previous results we see that only the Charlier polynomials (functions) have a closed simple oscillator algebra. What to do in the other cases? To answer to this question, we can use the following operators:

$$a(s) = \sqrt{\sigma(s+1)}\, e^{\frac{1}{2}\partial_s} - \sqrt{\sigma(s-1)+\tau(s-1)}\, e^{-\frac{1}{2}\partial_s},$$

$$a^+(s) = e^{-\frac{1}{2}\partial_s} \sqrt{\sigma(s+1)} - e^{\frac{1}{2}\partial_s} \sqrt{\sigma(s-1)+\tau(s-1)}.$$

For this operators

$$\mathfrak{h}_1(s) = a(s)a^+(s) + \tau' - \sigma''.$$

We will define a new hamiltonian $\mathfrak{h}_2(s)$ and operators $b(s)$ and $b^+(s)$

$$\mathfrak{h}_2(s) = C_a^2 \mathfrak{h}_1(s) + E, \qquad b(s) = C_a a(s) \qquad \text{and} \qquad b^+(s) = C_a a^+(s),$$

where C_a and E are some constants (to be fixed later on). Notice that from (3.3) it follows that the eigenfunctions of $\mathfrak{h}_2(s)$ are the same functions (3.2), but the eigenvalues are $C_a^2\lambda_n + E$, i.e.,

$$\mathfrak{h}_2(s)\Phi_n(s) = (C_a^2\lambda_n + E)\Phi_n(s).$$

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A straightforward computation yields

$$\mathfrak{h}_2(s) = b(s)b^+(s) + (\tau' - \sigma'')C_a^2 + E, \tag{5.1}$$

and

$$\begin{aligned}[b(s), b^+(s)] =&C_a^2\sqrt{\sigma(s+\tfrac{1}{2})(\sigma(s-\tfrac{3}{2}) + \tau(s-\tfrac{3}{2}))}\, e^{-\partial_s} \\ &+ C_a^2\sqrt{\sigma(s+\tfrac{3}{2})(\sigma(s-\tfrac{1}{2}) + \tau(s-\tfrac{1}{2}))}\, e^{\partial_s} \\ &+ \mathfrak{h}_2(s) - C_a^2(2\sigma(s) + \tau(s)) + \tfrac{1}{2}(\tfrac{3}{2}\,\sigma'' - \tau')C_a^2,\end{aligned} \tag{5.2}$$

or, equivalently,

$$\begin{aligned}[a(s), a^+(s)] =&\sqrt{\sigma(s+\tfrac{1}{2})(\sigma(s-\tfrac{3}{2}) + \tau(s-\tfrac{3}{2}))}\, e^{-\partial_s} \\ &+ \sqrt{\sigma(s+\tfrac{3}{2})(\sigma(s-\tfrac{1}{2}) + \tau(s-\tfrac{1}{2}))}\, e^{\partial_s} \\ &+ \mathfrak{h}_1(s) - (2\sigma(s) + \tau(s))I + \tfrac{1}{2}(\tfrac{3}{2}\,\sigma'' - \tau').\end{aligned}$$

The right-hand side of (5.2) suggests us to use the following new operators

$$\begin{aligned} c(s) &= C_b b(s)\, e^{-\frac{1}{2}\partial_s}\sqrt{\sigma(s+1)} = C_bC_a(\sigma(s+1) - e^{-\partial_s}\nu(s)), \\ c^+(s) &= C_b\sqrt{\sigma(s+1)}\, e^{\frac{1}{2}\partial_s} b^+(s) = C_bC_a(\sigma(s+1) - \nu(s)\, e^{\partial_s}), \end{aligned} \tag{5.3}$$

where, as before, $\nu(s) = \sqrt{\sigma(s+1)(\sigma(s) + \tau(s))}$. So,

$$[\mathfrak{h}_2(s), c(s)] = -C_a^2(\sigma''-\tau')c(s) + C_aC_b\left[\mathfrak{h}_2(s) + \left((\sigma''-\tau')C_a^2 - E\right) I\right]\sigma'(s+\tfrac{1}{2}),$$

$$[\mathfrak{h}_2(s), c^+(s)] = C_a^2(\sigma'' - \tau')c^+(s) - C_aC_b\sigma'(s+\tfrac{1}{2})[\mathfrak{h}_2(s) + \left((\sigma'' - \tau')C_a^2 - E\right) I],$$

$$[c(s), c^+(s)] = C_a^2C_b^2\left(\sigma'(s+\tfrac{1}{2})e^{-\partial_s}\nu(s) + \nu(s)e^{\partial_s}\sigma'(s+\tfrac{1}{2}) - [\nu^2(s) - \nu^2(s-1)]I\right).$$

The above expression leads to the following

THEOREM 5.1. *If $\sigma'' = 0$, then the operators $\mathfrak{h}_2(s)$, $c(s)$ and $c^+(s)$, defined by (5.1) and (5.3), respectively, form a closed algebra such that*

$$[\mathfrak{h}_2(s), c(s)] = \tau' C_a^2 c(s) + C_bC_a\sigma'(0)\left(\mathfrak{h}_2(s) - \tau' C_a^2 - E\right),$$

$$[\mathfrak{h}_2(s), c^+(s)] = -\tau' C_a^2 c^+(s) - C_bC_a\sigma'(0)\left(\mathfrak{h}_2(s) - \tau' C_a^2 - E\right),$$

$$[c(s), c^+(s)] = C_b^2\Big[\sigma(s) - \sigma'(0)(\mathfrak{h}_2(s) - E)\Big].$$

Observe also that with this particular choice $\sigma'(s+\frac{1}{2}) = \sigma'(0)$ and

$$c(s) + c^+(s) = C_bC_a(\mathfrak{h}_1(s) + 2\sigma'(0) + \tau(s)),$$

$$\nu^2(s) - \nu^2(s-1) = \sigma'(0)(2\sigma(s) + \tau(s)) + \tau'\sigma(s).$$

Furthermore, using the boundary conditions $\sigma(s)\rho(s)\big|_{s=a,b} = 0$, one finds

$$\begin{aligned}\langle c\,\Phi_n, \Phi_m\rangle_d &= C_a C_b \sum_{s=a}^{b-1} \sigma(s+1)\Phi_n(s)\Phi_m(s) - \sum_{s=a}^{b-1} \nu(s-1)\Phi_n(s-1)\Phi_m(s) \\ &= C_a C_b \sum_{s=a}^{b-1} \sigma(s+1)\Phi_n(s)\Phi_m(s) - C_a C_b \sum_{s=a}^{b-1} \nu(s)\Phi_n(s)\Phi_m(s+1) \\ &= \langle \Phi_n, c^+\Phi_m\rangle_d,\end{aligned}$$

i.e., the following theorem follows.

THEOREM 5.2. *The operators* $c(s)$ *and* $c^+(s)$ *are mutually adjoint.*

Notice also that the operators $\mathfrak{h}_1(s)$ and $\mathfrak{h}_2(s)$ are selfadjoint operators.

REMARK 5.3. *Since* $\lambda = \lambda_n = -n(\tau' + (n-1)\sigma''/2)$*, the identity* $\sigma'' = 0$ *is equivalent to the statement that* λ_n *is a linear function of* n*. In this case* $\lambda_n = -n\tau'$.

In the following we will consider only the case when $\sigma'' = 0$, i.e., the case of the Meixner, the Kravchuk and the Charlier polynomials.

If we define the operators

$$\begin{aligned} K_0(s) &= \mathfrak{h}_2(s)(-\tau' C_a^2)^{-1} \\ K_-(s) &= -\tau' C_a^2\, c(s) - C_b C_a \sigma'(0)\left(\mathfrak{h}_2(s) - \tau' C_a^2 - E\right), \\ K_+(s) &= -\tau' C_a^2\, c^+(s) - C_b C_a \sigma'(0)\left(\mathfrak{h}_2(s) - \tau' C_a^2 - E\right), \end{aligned} \tag{5.4}$$

then

$$[K_0(s), K_\pm(s)] = \pm K_\pm(s) \ \text{ y } \ [K_-(s), K_+(s)] = A_0 K_0(s) + A_1,$$

where

$$\begin{aligned} A_0 &= -2\tau'\sigma'(0) C_b^2 C_a^4 (-\tau' C_a^2)(\sigma'(0) + \tau') \quad \text{and} \\ A_1 &= -E A_0 (-\tau' C_a^2)^{-1} + C_b^2 C_a^6 \tau'^2 [\sigma'(0)\tau(0) - \sigma(0)\tau']. \end{aligned}$$

The case $A_0 = 0$ corresponds to the Charlier case (see the previous section). If $A_0 \neq 0$, we have two possibilities: $A_0 > 0$ and $A_0 < 0$. In the following we will choose $C_a^2 = -1/\tau'$, i.e., $-\tau' C_a^2 = 1$.

In the first case $A_0 > 0$ one can choose C_b and E in such a way that $A_0 = 2$ and $A_1 = 0$. Thus

$$C_b^2 = \frac{-\tau'}{\sigma'(0)[\tau' + \sigma'(0)]}, \qquad E = -\frac{C_b^2[\sigma'(0)\tau(0) - \sigma(0)\tau']}{2\tau'}. \tag{5.5}$$

Consequently, the operators $K_\pm$ and K_0 are such that

$$[K_0(s), K_\pm(s)] = \pm K_\pm(s) \ \text{ and } \ [K_-(s), K_+(s)] = 2K_0(s). \tag{5.6}$$

This case corresponds to the Lie algebra Sp$(2, \mathfrak{R})$.

In the second case one can choose C_b and E in such a way that $A_0 = -2$ and $A_1 = 0$. Thus

$$C_b^2 = \frac{\tau'}{\sigma'(0)[\tau' + \sigma'(0)]}, \qquad E = \frac{C_b^2[\sigma'(0)\tau(0) - \sigma(0)\tau']}{2\tau'}. \tag{5.7}$$

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Consequently, the operators $K_\pm$ and K_0 are such that

$$[K_0(s), K_\pm(s)] = \pm K_\pm(s) \ \text{ and } \ [K_+(s), K_-(s)] = 2K_0(s). \tag{5.8}$$

This case corresponds to the Lie algebra $\mathrm{so}(3)$.

Notice that since the operator $\mathfrak{h}_2(s)$ is selfadjoint, the operators $K_\pm(s)$ are mutually adjoint in both cases, i.e.

$$\langle K_+\Phi_m, \Phi_n\rangle_d = \langle \Phi_m, K_-\Phi_n\rangle_d.$$

5.1. Dynamical symmetry algebra Sp$(2,\mathfrak{R})$. Let us consider the first case. We start with the operator

$$K^2(s) = K_0^2(s) - K_0(s) - K_+(s)K_-(s),$$

where $K_0(s)$, $K_+(s)$, and $K_-(s)$ are the operators given in (5.4). A straightforward calculation gives

$$K^2(s) = E(E-1)I, \qquad E = \frac{\tau(0)\sigma'(0) - \tau'\sigma(0)}{2\sigma'(0)(\sigma'(0)+\tau')},$$

where E is given by (5.5), i.e., the $K^2(s)$ is the invariant Casimir operator.

Furthermore, if we define the normalized functions

$$\Phi_n(s) = \sqrt{\frac{\rho(s)}{d_n^2}}P_n(s),$$

we have

$$K^2(s)\Phi_n(s) = E(E-1)\Phi_n(s), \qquad K_0(s)\Phi_n(s) = (n+E)\Phi_n(s). \tag{5.9}$$

Now using the commutation relation (5.6), it is easy to show that

$$K_0(s)[K_\pm(s)\Phi_n(s)] = (n+E\pm 1)K_\pm(s)\Phi_n(s).$$

Consequently, from (5.9) and the above equation we deduce that

$$K_+(s)\Phi_n(s) = \widetilde{\kappa}_n\Phi_{n+1}(s), \qquad K_-(s)\Phi_n(s) = \kappa_n\Phi_{n-1}(s).$$

Employing the mutual adjointness of the operators $K_\pm$, one obtains

$$\widetilde{\kappa}_n = \langle K_+\Phi_n(s), \Phi_{n+1}(s)\rangle_d = \langle \Phi_n(s), K_-\Phi_{n+1}(s)\rangle_d = \kappa_{n+1},$$

thus

$$K_+(s)\Phi_n(s) = \kappa_{n+1}\Phi_{n+1}(s), \qquad K_-(s)\Phi_n(s) = \kappa_n\Phi_{n-1}(s). \tag{5.10}$$

In order to compute κ_n, use (5.9) and (5.10); this yields

$$E(E-1) = (n+E)^2 - (n+E) - \kappa_n^2 \quad \Rightarrow \quad \kappa_n = \sqrt{n(n+2E-1)}.$$

In this case the functions $(\Phi_n)_n$ define a basis for the irreducible unitary representation $D^+(-E)$ of the Lie group (algebra) Sp$(2,\mathfrak{R})$.

From the above formula it follows that the functions $\Phi_n(s)$ can be obtained recursively via the application of the operator $K_+(s)$, i.e.,

$$\Phi_n(s) = \frac{1}{\kappa_1\cdots\kappa_n}K_+^n(s)\Phi_0(s), \quad \Phi_0(s) = \frac{\sqrt{\rho(s)}}{d_0}.$$

where $\rho(s)$ is the weight function of the corresponding orthogonal polynomial family and d_0 is the norm of the $P_0(s)$.

5.1.1. Example: The Meixner functions. Let consider the Meixner functions

$$\Phi_n^M(s) = \mu^{(s-n)/2}(1-\mu)^{\gamma/2+n}\sqrt{\frac{(\gamma)_s}{s!n!(\gamma)_n}}\,M_n^{\gamma,\mu}(s), \quad n \geq 0,$$

and the hamiltonian $\mathfrak{h}_1(s)$

$$\mathfrak{h}_1^M(s) = -\sqrt{\mu s(s+\gamma-1)}e^{-\partial_s} - \sqrt{\mu(s+1)(s+\gamma)}e^{\partial_s} + (s+\mu\,(s+\gamma))\,I,$$

thus $\mathfrak{h}_1^M(s)\Phi_n^M(s) = n\Phi_n^M(s)$. In this case we have $C_b = \sqrt{\frac{1-\mu}{\mu}}$, $E = \frac{\gamma}{2}$, $C_a = \sqrt{\frac{1}{1-\mu}}$. Therefore

$$b(s) = -\sqrt{\frac{(s-1+\gamma)\mu}{1-\mu}}e^{-\frac{1}{2}\partial_s} + \sqrt{\frac{s+1}{1-\mu}}e^{\frac{1}{2}\partial_s},$$

$$b^+(s) = -\sqrt{\frac{(s-\frac{1}{2}+\gamma)\mu}{1-\mu}}e^{\frac{1}{2}\partial_s} + \sqrt{\frac{s+\frac{1}{2}}{1-\mu}}e^{-\frac{1}{2}\partial_s}.$$

Consequently,

$$\mathfrak{h}_2(s) = \frac{1}{1-\mu}\,\mathfrak{h}_1(s) + \frac{\gamma}{2} = b(s)b^+(s) + \frac{\gamma}{2} - 1.$$

Moreover,

$$\mathfrak{h}_2(s)\Phi_n^M(s) = \left(n + \frac{\gamma}{2}\right)\Phi_n^M(s),$$

$$K_0(s) = -\sqrt{s(s-1+\gamma)}\frac{\sqrt{\mu}}{1-\mu}\,e^{-\partial_s} - \sqrt{(s+1)(s+\gamma)}\frac{\sqrt{\mu}}{1-\mu}\,e^{\partial_s} + \left(s+\frac{\gamma}{2}\right)\frac{1+\mu}{1-\mu}\,I,$$

$$K_+(s) = -\frac{\sqrt{s(s-1+\gamma)}}{1-\mu}\,e^{-\partial_s} - \frac{\mu\sqrt{(s+1)(s+\gamma)}}{1-\mu}\,e^{\partial_s} + \frac{\sqrt{\mu}}{1-\mu}\,(2s+\gamma)\;I,$$

$$K_-(s) = -\frac{\mu\sqrt{s(s-1+\gamma)}}{1-\mu}\,e^{-\partial_s} - \frac{\sqrt{(s+1)(s+\gamma)}}{1-\mu}\,e^{\partial_s} + \frac{\sqrt{\mu}}{1-\mu}\,(2s+\gamma)\;I,$$

and

$$K_0(s)\Phi_n^M(s) = \left(n+\frac{\gamma}{2}\right)\Phi_n^M(s), \qquad K^2(s)\Phi_n^M(s) = \frac{\gamma}{2}\left(\frac{\gamma}{2}-1\right)\Phi_n^M(s),$$

$$\begin{aligned} K_+(s)\Phi_n^M(s) &= \sqrt{(n+1)(n+\gamma)}\Phi_{n+1}^M(s), \\ K_-(s)\Phi_n^M(s) &= \sqrt{n(n+\gamma-1)}\Phi_{n-1}^M(s). \end{aligned} \tag{5.11}$$

Using the fact that $\mathfrak{h}_2\Phi_0^M(s) = \frac{\gamma}{2}\Phi_0^M(s)$, together with the formulas (5.4) and (5.11), one finds

$$0 = K_-(s)\Phi_0^M(s) = \sqrt{s(s-1+\gamma)}\,\Phi_0^M(s-1) - s\mu^{-\frac{1}{2}}\Phi_0^M(s),$$

therefore the normalized function $\Phi_0^M(s)$ is

$$\Phi_0^M(s) = \sqrt{\frac{(1-\mu)^\gamma}{\Gamma(\gamma)}}\sqrt{\frac{\mu^s\Gamma(\gamma+s)}{\Gamma(\gamma)\Gamma(s+1)}},$$

and

$$\Phi_n^M(s) = \sqrt{\frac{(1-\mu)^\gamma}{n!\Gamma(\gamma+n)}}\,(K_+(s))^n\left[\sqrt{\frac{\mu^s\Gamma(\gamma+s)}{\Gamma(s+1)}}\right].$$

A similar result have been obtained before in [9].

5.2. Dynamical symmetry algebra so(3). Let us consider the second case and define the following operator

$$K^2(s) = K_0^2(s) + K_0(s) + K_-(s)K_+(s).$$

where $K_0(s)$, $K_+(s)$, and $K_-(s)$ are the operators given in (5.4). Substituting the value of E, given by (5.7), and doing some straightforward computations yield

$$K^2(s) = E(E-1)I, \qquad E = \frac{\tau(0)\sigma'(0) - \tau'\sigma(0)}{2\sigma'(0)(\sigma'(0)+\tau')},$$

i.e., the $K^2(s)$ is the invariant Casimir operator.

Moreover, if we define the normalized functions as

$$\Phi_n(s) = \sqrt{\frac{\rho(s)}{d_n^2}}P_n(s),$$

we have

$$K^2(s)\Phi_n(s) = E(E-1)\Phi_n(s), \qquad K_0(s)\Phi_n(s) = (n+E)\Phi_n(s).$$

Now using the commutation relation (5.8), we have

$$K_0(s)[K_\pm(s)\Phi_n(s)] = (n+E\pm 1)K_\pm(s)\Phi_n(s).$$

Consequently, from (5.9) and the above equation, we conclude that

$$K_+(s)\Phi_n(s) = \widetilde{\kappa}_n\Phi_{n+1}(s), \qquad K_-(s)\Phi_n(s) = \kappa_n\Phi_{n-1}(s).$$

Using the mutual adjointness of the operators $K_\pm$, one obtains

$$\widetilde{\kappa}_n = \langle K_+\Phi_n(s), \Phi_{n+1}(s)\rangle_d = \langle \Phi_n(s), K_-\Phi_{n+1}(s)\rangle_d = \kappa_{n+1},$$

thus

$$K_+(s)\Phi_n(s) = \kappa_{n+1}\Phi_{n+1}(s), \qquad K_-(s)\Phi_n(s) = \kappa_n\Phi_{n-1}(s). \tag{5.12}$$

To compute κ_n, use (5.9) and (5.12); this leads to

$$E(E-1) = (n+E)^2 + (n+E) + \kappa_{n+1}^2 \quad\Rightarrow\quad \kappa_n = \sqrt{-(n)(n+2E-1)}.$$

In this case the functions $(\Phi_n)_n$ define a basis for the irreducible unitary representation $D^+(-E)$ of the Lie algebra so(3).

As in the previous case, from the above formula it follows that the functions $\Phi_n(s)$ can be obtained recursively via the application of the operator $K_+(s)$, i.e.,

$$\Phi_n(s) = \frac{1}{\kappa_1\cdots\kappa_n}K_+^n(s)\Phi_0(s), \quad \Phi_0(s) = \frac{\sqrt{\rho(s)}}{d_0}.$$

where $\rho(s)$ is the weight function for the associated orthogonal polynomial family and d_0 is the norm of the $P_0(s)$.

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5.2.1. Example: The Kravchuk functions. Let us consider now the Kravchuk functions

$$\Phi_n^K(s) = p^{(s-n)/2}(1-p)^{(N-n-s)/2}\sqrt{\frac{n!(N-n)!}{s!(N-s)!}}K_n^p(s,N), \quad 0 \le n \le N,$$

and the corresponding hamiltonian $\mathfrak{h}_1(s)$

$$\mathfrak{h}_1^K(s) = -\frac{\sqrt{ps(N-s+1)}}{\sqrt{1-p}}e^{-\partial_s} + \frac{Np+s-2ps}{1-p}I - \frac{\sqrt{p(s+1)(N-s)}}{\sqrt{1-p}}e^{\partial_s},$$

thus $\mathfrak{h}_1^K(s)\Phi_n^K(s) = n\Phi_n^K(s)$. In this case $C_a = \sqrt{1-p}$, $C_b = \sqrt{p^{-1}}$, $E = -\frac{N}{2}$, therefore

$$b(s) = -\sqrt{p(N-s+1)}e^{-\frac{1}{2}\partial_s} + \sqrt{(1-p)(s+1)}e^{\frac{1}{2}\partial_s},$$

$$b^+(s) = -\sqrt{p(N-s+\tfrac{1}{2})}e^{\frac{1}{2}\partial_s} + \sqrt{(1-p)(s+\tfrac{1}{2})}e^{\frac{1}{2}\partial_s}.$$

Consequently,

$$\mathfrak{h}_2(s) = (1-p)\,\mathfrak{h}_1(s) - \frac{N}{2} = b(s)b^+(s) - \frac{N}{2} - 1.$$

Moreover,

$$\mathfrak{h}_2(s)\Phi_n^K(s) = \left(n - \frac{N}{2}\right)\Phi_n^K(s),$$

$$\begin{aligned} K_0(s) = &-\sqrt{p(1-p)s(N-s+1)}e^{-\partial_s} - \sqrt{p(1-p)(s+1)(N-s)}e^{\partial_s} \\ &+[N(p-\tfrac{1}{2}) - s(2p-1)]I, \end{aligned}$$

$$K_+(s) = (1-p)\sqrt{s(N-s+1)}e^{-\partial_s} + p\sqrt{(s+1)(N-s)}e^{\partial_s} - \sqrt{p(1-p)}(2s-N)I,$$

$$K_-(s) = p\sqrt{s(N-s+1)}e^{-\partial_s} + (1-p)\sqrt{(s+1)(N-s)}e^{\partial_s} - \sqrt{p(1-p)}(2s-N)I,$$

and

$$K_0(s)\Phi_n^K(s) = \left(n - \frac{N}{2}\right)\Phi_n^K(s), \qquad K^2(s)\Phi_n^K(s) = \frac{N}{4}(N+2)\,\Phi_n^K(s),$$

$$\begin{aligned} K_+(s)\Phi_n^K(s) &= \sqrt{(n+1)(N-n)}\Phi_{n+1}^K(s), \\ K_-(s)\Phi_n^K(s) &= \sqrt{n(N-n+1))}\Phi_{n-1}^K(s). \end{aligned} \tag{5.13}$$

Using the fact that $\mathfrak{h}_2\Phi_0^K(s) = -\frac{N}{2}\Phi_0^K(s)$, together with the formulas (5.4) and (5.13), we find

$$0 = K_-(s)\Phi_0^K(s) = \sqrt{\frac{p}{1-p}}\left(s\Phi_0^K(s) - \sqrt{\frac{ps(N-s+1)}{1-p}}\Phi_0^K(s-1)\right),$$

therefore the normalized function $\Phi_0^K(s)$ is equal to

$$\Phi_0^K(s) = p^{(s-n)/2}(1-p)^{(N-n-s)/2}\sqrt{\frac{n!(N-n)!}{s!(N-s)!}},$$

and

$$\Phi_n^K(s) = \sqrt{\frac{(N-n)!(1-p)^{N-n}}{N!p^n}}\,(K_+(s))^n\left[\binom{N}{s}\left(\frac{p}{1-p}\right)^s\right].$$

6. The q-case. To conclude this paper we will discuss here briefly what happens in the q-case. The preliminary results, related with this case, have been presented during the Bexbach Conference 2002 [3]. A more detailed exposition of these results is under preparation.

$$\Phi_n(s) = \frac{A(s)\sqrt{\rho(s)}}{d_n}P_n(s;q),$$

where d_n is the norm of the q-polynomials $P_n(s;q)$, $\rho(s)$ is the solution of the Pearson-type equation

$$\frac{\Delta}{\Delta x(s-\frac{1}{2})}[\sigma(s)\rho(s)] = \tau(s)\rho(s) \quad \text{or} \quad \sigma(s+1)\rho(s+1) = \sigma(-s-\mu)\rho(s),$$

and $A(s)$ is an arbitrary continuous function, not vanishing in the interval (a,b) of orthogonality of P_n. If $P_n(s;q)$ possess the discrete orthogonality property (2.3), then the functions $\Phi_n(s)$ satisfy

$$\langle\Phi_n(s),\Phi_m(s)\rangle = \sum_{s=a}^{b-1}\Phi_n(s)\Phi_m(s)\frac{\nabla x_1(s)}{A^2(s)} = \delta_{n,m}. \tag{6.1}$$

Notice that if $A(s) = \sqrt{\nabla x_1(s)}$, then the set $(\Phi_n)_n$ is an orthonormal set. Obviously, in the case of a continuous orthogonality (as for the Askey-Wilson polynomials) one needs to change the sum in (6.1) by a Riemann integral [10, 26].

Next, we define the q-Hamiltonian $\mathfrak{H}_q(s)$ of the form

$$\mathfrak{H}_q(s) := \frac{1}{\nabla x_1(s)}A(s)H_q(s)\frac{1}{A(s)}, \tag{6.2}$$

where

$$\begin{aligned} H_q(s) := &-\frac{\sqrt{\sigma(-s-\mu+1)\sigma(s)}}{\nabla x(s)}e^{-\partial_s} - \frac{\sqrt{\sigma(-s-\mu)\sigma(s+1)}}{\Delta x(s)}e^{\partial_s} \\ &+\left(\frac{\sigma(-s-\mu)}{\Delta x(s)} + \frac{\sigma(s)}{\nabla x(s)}\right)I. \end{aligned}$$

As in the previous case, one can easily check that

$$\mathfrak{H}_q(s)\Phi_n(s) = \lambda_n\Phi_n(s).$$

Now we define the α operators:

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DEFINITION 6.1. *Let α be a real number and $A(s)$ and $B(s)$ are two arbitrary continuous non-vanishing functions. We define a family of α-down and α-up operators by*

$$(6.3)\qquad \begin{aligned} \mathfrak{a}_\alpha^{\downarrow}(s) &:= \frac{B(s)}{\sqrt{\nabla x_1(s)}} e^{-\alpha\partial_s}\left(e^{\partial_s}\sqrt{\frac{\sigma(s)}{\nabla x(s)}} - \sqrt{\frac{\sigma(-s-\mu)}{\Delta x(s)}}\right)\frac{1}{A(s)},\\ \mathfrak{a}_\alpha^{\uparrow}(s) &:= \frac{1}{\nabla x_1(s)} A(s)\left(\sqrt{\frac{\sigma(s)}{\nabla x(s)}} e^{-\partial_s} - \sqrt{\frac{\sigma(-s-\mu)}{\Delta x(s)}}\right) e^{\alpha\partial_s}\frac{\sqrt{\nabla x_1(s)}}{B(s)}, \end{aligned}$$

respectively. The first result in this case is [3]:

THEOREM 6.2. *Given a q-Hamiltonian (6.2) $\mathfrak{H}_q(s)$, then the operators $\mathfrak{a}_\alpha^{\downarrow}(s)$ and $\mathfrak{a}_\alpha^{\uparrow}(s)$, defined in (3.4), are such that for all $\alpha\in\mathbb{C}$, $\mathfrak{H}_q(s)=\mathfrak{a}_\alpha^{\uparrow}(s)\mathfrak{a}_\alpha^{\downarrow}(s)$.*

Our next step is again to find a dynamical symmetry algebra, associated with the operator $\mathfrak{H}_q(s)$, or equivalently, with the corresponding family of q-polynomials.

DEFINITION 6.3. *Let ς be a complex number, and let $a(s)$ and $b(s)$ be two operators. We define the ς-*commutator of a and b *as*

$$[a(s), b(s)]_\varsigma = a(s)b(s) - \varsigma b(s)a(s).$$

We want to know whether the following problem: *To find two operators $a(s)$ and $b(s)$ and a constant ς such that the Hamiltonian $\mathfrak{H}_q(s) = b(s)a(s)$ and $[a(s), b(s)]_\varsigma = I$,* has a non-trivial solution.

Obviously, we already know the answer to the first part: these are the operators $b(s) = \mathfrak{a}_\alpha^{\uparrow}(s)$ and $a(s) = \mathfrak{a}_\alpha^{\downarrow}(s)$, given in (6.3). The answer to the second part of this problem is summarized in the following two theorems (in what follows we assume that $A(s) = B(s)$).

THEOREM 6.4. *[3]Let $(\Phi_n)_n$ be the eigenfunctions of $\mathfrak{H}_q(s)$, corresponding to the eigenvalues $(\lambda_n)_n$, and suppose that the problem 1 has a solution for $\Lambda\neq 0$. Then the eigenvalues λ_n of the difference equation (3.3) are q-linear or q^{-1}-linear functions of n, i.e., $\lambda_n = C_1q^n + C_3$ or $\lambda_n = C_2q^{-n} + C_3$, respectively.*

THEOREM 6.5. *[3] Let $\mathfrak{H}_q(s)$ be the q-Hamiltonian (6.2). The operators $b(s) = \mathfrak{a}_\alpha^{\uparrow}(s)$ and $a(s) = \mathfrak{a}_\alpha^{\downarrow}(s)$, given in (6.3) with $B(s) = A(s)$, factorize the Hamiltonian $\mathfrak{H}_q(s)$ (6.2) and satisfy the commutation relation $[a(s), b(s)]_\varsigma = \Lambda$ for a certain complex number ς, if and only if the following two conditions hold*

$$\frac{\nabla x(s)}{\nabla x_1(s-\alpha)}\sqrt{\frac{\nabla x_1(s-1)\nabla x_1(s)}{\nabla x(s-\alpha)\Delta x(s-\alpha)}}\sqrt{\frac{\sigma(s-\alpha)\sigma(-s-\mu+\alpha)}{\sigma(s)\sigma(-s-\mu+1)}} = \varsigma,$$

$$\frac{1}{\Delta x(s-\alpha)}\left(\frac{\sigma(s-\alpha+1)}{\nabla x_1(s-\alpha+1)} + \frac{\sigma(-s-\mu+\alpha)}{\nabla x_1(s-\alpha)}\right) - \varsigma\frac{1}{\nabla x_1(s)}\left(\frac{\sigma(s)}{\nabla x(s)} + \frac{\sigma(-s-\mu)}{\Delta x(s)}\right) = \Lambda.$$

The proof of the theorem 6.5 is similar to the proof of the theorem 4.1, presented here for the case of the uniform lattice $x(s) = s$.

Let us point out that the q(respectively, q^{-1})-linearity of the eigenvalues is a necessary condition in order to provide that the solution of problem 1 exists. But this condition is not sufficient. For example, if we take the discrete q-Laguerre polynomials $L_n^\alpha(x;q)$ (for more details see [3]) with $a\neq q^{-1/2}$, the problem has not a solution, but λ_n is a q-linear function of n.

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6.1. Examples. We present here only two examples, others can be found in [3].

6.1.1. The Al-Salam & Carlitz I q-polynomials $U_n^{(a)}(x;q)$. We start with the very well known case: the Al-Salam & Carlitz I polynomials [20]. The corresponding normalized functions (3.2) are

$$\Phi_n(x) = \sqrt{\frac{(qx, a^{-1}qx;q)_\infty(-a)^n q^{\binom{n}{2}}}{(1-q)(q;q)_n(q,a,q/a;q)_\infty} A^2(s)}\, {}_2\phi_1\left(\begin{array}{c} q^{-n}, x^{-1} \\ 0 \end{array} \middle| \, q; \frac{qx}{a}\right), \quad x := q^s.$$

Putting $A(s) = \sqrt{\nabla x_1(s)} = \sqrt{xk_q}$, where $k_q = q^{\frac12} - q^{-\frac12}$, we have that the functions $(\Phi_n)_n$ satisfy the orthogonality condition

$$\int_a^1 \Phi_n(x)\Phi_m(x)\frac{d_qx}{k_qx} = \delta_{n,m},$$

where the integral $\int_a^1 f(x)d_qx$ denotes the classical Jackson q-integral.

For these polynomials $\sigma(s) + \tau(s)\nabla x_1(s) = a$, therefore $\alpha = 0$ and $\varsigma = q^{-1}$. The q-Hamiltonian has the form

$$\begin{aligned} \mathfrak{H}_q(s) = \quad & -\frac{q^2\sqrt{a(x-1)(x-a)}}{(q-1)^2x^2}e^{-\partial_s} - \frac{\sqrt{a(1-qx)(a-qx)}}{x^2}e^{\partial_s} \\ & + \left(\frac{\sqrt{q}\,(q\,(x-1)\,x + a\,(1+q-qx))}{(q-1)^2x^2}\right) I, \quad x = q^s. \end{aligned}$$

Consequently, $\mathfrak{H}_q(s)\Phi_n(s) = q^{\frac32}\frac{1-q^{-n}}{(1-q)^2}\Phi_n(s)$ and the operators ($x = q^s$)

$$\mathfrak{a}^{\downarrow}(s) \equiv \mathfrak{a}_0^{\downarrow}(s) = \frac{q^{\frac14}}{k_qx}\left(\sqrt{(x-1/q)(x-a/q)}\, e^{\partial_s} - \sqrt{a}\, I\right),$$

$$\mathfrak{a}^{\uparrow}(s) \equiv \mathfrak{a}_0^{\uparrow}(s) = \frac{q^{\frac14}}{k_qx}\left(\sqrt{(x-1)(x-a)}\, e^{-\partial_s} - \sqrt{a/q}\, I\right),$$

are such that

$$\mathfrak{a}^{\uparrow}(s)\mathfrak{a}^{\downarrow}(s) = \mathfrak{H}_q(s), \quad \text{and} \quad [\mathfrak{a}^{\downarrow}(s), \mathfrak{a}^{\uparrow}(s)]_{q^{-1}} = \frac{1}{k_q}.$$

A straightforward calculation shows that the operators $\mathfrak{a}^{\uparrow}(s)$ and $\mathfrak{a}^{\downarrow}(s)$ are mutually adjoint. A similar factorization was obtained earlier in [6] and more recently in [4] with the aid of a different technique.

The special case of the Al-Salam & Carlitz I polynomials are the discrete q-Hermite I $h_n(x;q)$, $x = q^s$, polynomials, which correspond to the parameter $a = -1$ [20].

6.2. Continuous q-Hermite polynomials. Let us now consider the particular case of the Askey-Wilson polynomials when all their parameters are equal to zero, i.e., the continuous q-Hermite polynomials [20]. In this case $\sigma(s) = C_\sigma q^{2s}$.

Let us choose $A(s) = B(s) = \sqrt{\nabla x_1(s)}$. In this case $\alpha = 1/2$ and $\varsigma = 1/q$. The corresponding Hamiltonian is given by

$$\mathfrak{H}_q(s) = \frac{-1}{k_q^2\sqrt{\sin\theta}}\left(\frac{C_\sigma q}{\sin(\theta+\frac{i}{2}\log q)\sqrt{\sin(\theta+i\log q)}}\,e^{-\partial_s} + \frac{C_\sigma q}{\sin(\theta-\frac{i}{2}\log q)\sqrt{\sin(\theta-i\log q)}}\,e^{\partial_s}\right)$$
$$+\frac{1}{k_q^2\sin\theta}\left(\frac{C_\sigma q^{2s}}{\sin(\theta+\frac{i}{2}\log q)} + \frac{C_\sigma q^{-2s}}{\sin(\theta-\frac{i}{2}\log q)}\right) I,$$

and the α-operators

$$a^{\downarrow}_{1/2}(s) = e^{\frac{1}{2}\partial_s}\sqrt{\frac{C_\sigma q^{2s}}{-k_q^2\sin\theta\sin(\theta+\frac{i}{2}\log q)}} - e^{-\frac{1}{2}\partial_s}\sqrt{\frac{C_\sigma q^{-2s}}{-k_q^2\sin\theta\sin(\theta-\frac{i}{2}\log q)}},$$
$$a^{\uparrow}_{1/2}(s) = \sqrt{\frac{C_\sigma q^{2s}}{-k_q^2\sin\theta\sin(\theta+\frac{i}{2}\log q)}}\,e^{-\frac{1}{2}\partial_s} - \sqrt{\frac{C_\sigma q^{-2s}}{-k_q^2\sin\theta\sin(\theta-\frac{i}{2}\log q)}}\,e^{\frac{1}{2}\partial_s},$$

are such that

$$a^{\uparrow}(s)a^{\downarrow}(s) = \mathfrak{H}_q(s) \quad \text{and} \quad [a^{\downarrow}(s), a^{\uparrow}(s)]_{1/q} = \frac{4C_\sigma}{k_q}.$$

Another possible choice is $A(s) = B(s) = 1$ [12], hence a straightforward calculation shows that the two conditions in Theorem 6.5 hold if $\varsigma = q^{-1}$, thus $\Lambda = 4C_\sigma k_q^{-1}$. With this choice the orthogonality of the functions Φ_n is $\int_{-1}^{1}\Phi_n(s)\Phi_m(s)dx = \delta_{n,m}$. In this case, the Hamiltonian is

$$\mathfrak{H}_q(s) = \frac{C_\sigma q}{k_q^2}\left\{\frac{e^{-\partial_s}}{\sin\theta\sin(\theta+\frac{i}{2}\ln q)} + \frac{e^{\partial_s}}{\sin(\theta-\frac{i}{2}\ln q)\sin\theta} - \frac{4}{\sqrt{q}}\left(1-\frac{1+q}{q+q^{-1}-2\cos 2\theta}\right) I\right\},$$

and

$$\mathfrak{a}^{\downarrow}(s) \equiv \mathfrak{a}^{\downarrow}_{1/2}(s) = \frac{\sqrt{-C_\sigma}}{k_q\sin\theta}\left(e^{\frac{1}{2}\partial_s}q^s - e^{-\frac{1}{2}\partial_s}q^{-s}\right),$$
$$\mathfrak{a}^{\uparrow}(s) \equiv \mathfrak{a}^{\uparrow}_{1/2}(s) = \frac{\sqrt{-C_\sigma}}{k_q\sin\theta}\left(q^s e^{-\frac{1}{2}\partial_s} - q^{-s}e^{\frac{1}{2}\partial_s}\right).$$

With these operators

$$\mathfrak{H}_q(s) = a^{\uparrow}(s)a^{\downarrow}(s) \quad \text{and} \quad [a^{\downarrow}(s), a^{\uparrow}(s)]_{q^{-1}} = \frac{4C_\sigma}{k_q}.$$

This case was first considered in [12].

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