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LANGENHOP'S INEQUALITY AND APPLICATIONS FOR DYNAMIC EQUATIONS*

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Abstract. A Langenhop-type inequality is given for dynamic equations on time scales. This result is further employed to obtain lower bounds for solutions of certain dynamic equations. As an application, usage of the derived Langenhop's inequality in determining the oscillatory behavior of a damped second order delay dynamic equation is illustrated. The results obtained are important in the qualitative sense.

Key words. Langenhop inequality, time scale, lower bounds, oscillation

AMS subject classifications. 26D20, 34A40, 34K11

1. Introduction. There is no doubt that the Gronwall inequality and its generalization, the Bihari inequality in continuous and discrete cases, have been very powerful tools in studying the qualitative behavior of differential and difference equations. These inequalities have been applied very successfully to investigate the global existence, uniqueness, stability, boundedness and other properties of solutions of various nonlinear differential and difference equations. Langenhop-type inequalities have also been used quite successfully in studying the qualitative behavior of differential equations [9, 10] and difference equations [11, 12].

2. Preliminaries. Time scale calculus which unifies continuous and discrete analysis was first introduced by Stefan Hilger [3]. Later, the theory developed very rapidly, see the monographs [2, 6] and the references cited therein. Here we shall only provide some basic facts on time scales extracted from [2].

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The most well-known examples are $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$. An interval $[a, b]$ in $\mathbb{T}$ is defined to be the set $\{t \in \mathbb{T} : a \le t \le b\}$. Other types of intervals are defined similarly. To define a continuity on a time scale we need the concept of forward and backward jump operators, $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ given by

$$\sigma(t) = \min\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) = \max\{s \in \mathbb{T} : s < t\}$$

together with the convention that $\min \emptyset = \max \mathbb{T}$ and $\max \emptyset = \min \mathbb{T}$. A point $t \in \mathbb{T}$ is called *right-scattered, right-dense, left-scattered, left-dense,* if $\sigma(t) > t$, $\sigma(t) = t$, $\rho(t) > t$, $\rho(t) = t$ is satisfied, respectively. The graininess at t is then defined by $\mu(t) = \sigma(t) - t$. The set $\mathbb{T}^\kappa$ is defined as $\mathbb{T}\backslash\{m\}$ if $\mathbb{T}$ has a left-scattered maximum m, and as $\mathbb{T}$ otherwise.

DEFINITION 2.1. *A function $f : \mathbb{T} \to \mathbb{R}$ is called Δ-differentiable at a point $t \in \mathbb{T}^\kappa$ if there exists a real number d and for a given ϵ there is a neighborhood U of t such that $|f(\sigma(t)) - f(s) - d\,(\sigma(t) - s)| < \epsilon|\sigma(t) - s|$ for all $s \in U$.* The number d is denoted by $f^\Delta(t)$. As usual, f is said to be differentiable on (a, b) if it is differentiable at every point $t \in (a, b)^\kappa$. It can be shown that $f^\Delta(t) = f'(t)$ if $\mathbb{T} = \mathbb{R}$, and $f^\Delta(t) = \Delta f(t) := f(t + 1) - f(t)$ if $\mathbb{T} = \mathbb{Z}$. In fact, it is possible to show that

$$f^\Delta(t) = \lim_{s \to t} \frac{f(s) - f(t)}{s - t} \quad \text{if } \mu(t) = 0, \qquad f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} \quad \text{if } \mu(t) > 0,$$

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and if $f, g : \mathbb{T} \to \mathbb{R}$ are Δ-differentiable, then so are $f+g$, $f-g$, fg, f/g. Moreover, $f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t)$ holds on any arbitrary time scale.

DEFINITION 2.2. *A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous, if it is continuous at every right-dense point and if the left-sided limit exists at every left-dense point. A function $g(t,x) : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is called rd-continuous if $g(t, x(t)) : \mathbb{T} \to \mathbb{R}$ is so.*

The set of all rd-continuous functions defined on $\mathbb{T}$ is denoted by $\mathbb{C}_{rd}(\mathbb{T})$. The space of functions that are differentiable with rd-continuous derivatives is denoted by $\mathbb{C}^1_{rd}(\mathbb{T})$. It is noteworthy to mention that every rd-continuous function f has an antiderivative F. As in the continuous case, a function F is called an antiderivative of f on $\mathbb{T}$ if $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}^{\kappa}$.

DEFINITION 2.3. *A function $f : \mathbb{T} \to \mathbb{R}$ is called regressive if $1 + \mu(t) f(t) \neq 0$ for all $t \in \mathbb{T}^{\kappa}$. The set of all rd-continuous and regressive functions defined on $\mathbb{T}$ is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T})$.*

In this study, we also employ a notion of a generalized exponential function on an arbitrary time scale. The definition below is extracted from [2].

DEFINITION 2.4 (Exponential function). *Let $p \in \mathcal{R}$ and $t_0 \in \mathbb{T}$. The unique solution of the initial value problem $x^{\Delta} = p(t)\,x$, $x(t_0) = 1$ is called the generalized exponential function and is denoted by $e_p(\cdot, t_0)$.* Let $\lambda \in \mathbb{R}$. It turns out that if $\mathbb{T} = \mathbb{R}$, then $e_{\lambda}(t, t_0) = e^{\lambda(t - t_0)}$, and if $\mathbb{T} = \mathbb{Z}$ and $\lambda \neq -1$, then $e_{\lambda}(t, t_0) = (1+\lambda)^{t - t_0}$.

For an extensive list of properties and detailed discussion on $e_p(\cdot, t_0)$ we refer to [2].

Another useful tool is a variation of parameters formula for first order linear nonhomogeneous dynamic equations which can be stated as follows.

THEOREM 2.5 (Theorem 2.77, [2]). *Suppose that $p \in \mathcal{R}$ and $f \in \mathbb{C}_{rd}(\mathbb{T})$. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$. Then the unique solution of $y^{\Delta} = p(t)\,y + f(t)$, $y(t_0) = y_0$ is given by*

$$y(t) = e_p(t, t_0)[y_0 + \int_{t_0}^{t} e_p(t_0, \sigma(\tau)) f(\tau)\, \Delta\tau].$$

In this set up,

$$y_p = e_p(t, t_0) \int_{t_0}^{t} e_p(t_0, \sigma(\tau)) f(\tau)\, \Delta\tau$$

is becomes a particular solution of the dynamic equation

$$y^{\Delta} = p(t)\,y + f(t).$$

In case $p(t) = \lambda$ and $f(t) = e_{\lambda}(t, t_0)$, it follows that $y_p = e_{\lambda}(t, t_0) \displaystyle\int_{t_0}^{t} \frac{1}{1 + \lambda\mu(\tau)}\, \Delta\tau$. As special cases, this expression results in the well-known particular solutions $y_p = te^{\lambda t}$ if $\mathbb{T} = \mathbb{R}$ and $y_p = t(1+\lambda)^t$ if $\mathbb{T} = \mathbb{Z}$.

3. Langenhop inequality. In 1960, C. E. Langenhop [8] proved a version of the following theorem when $\mathbb{T} = \mathbb{R}$. Recently, the same theorem was proved by Zafer [12] in the case when $\mathbb{T} = \mathbb{Z}$. It is also noteworthy to mention that in [11], Theorem 2.3.2 is given as a discrete version of Langenhop's inequality. Here we unify the previous results by use of time scale methods.

THEOREM 3.1. *Let $t_0 \in \mathbb{T}$. Suppose that $g : \mathbb{R} \to \mathbb{R}$ is nondecreasing, $g(u)$ is rd-continuous, and $g(u) > 0$ for $u > 0$. Let $u : \mathbb{T} \to \mathbb{R}$ and $v : \mathbb{T} \to \mathbb{R}_+$ be rd-continuous. If*

$$u(t) \geq u(s) - \int_{s}^{t} v(r) g(u(r))\, \Delta r, \quad t, s \in [t_0, T], \tag{3.1}$$

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then

$$u(t) \geq w(t,s), \quad t,s \in [t_0,T]$$

where w is the maximal solution of

$$w^{\Delta} = -v(t)g(w), \quad w(s,s) = u(s). \tag{3.2}$$

If $g(u) = u$, then

$$u(t) \geq u(s)e_{-v}(t,s), \quad t,s \in [t_0,T].$$

Proof. Fix $t \in (t_0,T]$ and for $s \in [t_0,t]$ define

$$y(s) = u(t) + \int_s^t v(r)g(u(r))\Delta r. \tag{3.3}$$

It is clear that

$$y^{\Delta}(s) + v(s)g(u(s)) = 0 \tag{3.4}$$

and

$$y(t) = u(t).$$

Moreover, from (3.1) and (3.3) we have

$$y(s) \geq u(s), \quad s \in [t_0,t]. \tag{3.5}$$

Using (3.5) in (3.4), we obtain

$$y^{\Delta}(t) \geq -v(t)g(y(t)).$$

In view of the theory developed in [5] for maximal-minimal solutions on time scales and by comparison with (3.2), (see also [4, 7]), we have

$$y(t) \geq w(t,s), \quad t,s \in [t_0,T]. \tag{3.6}$$

Employing (3.5) and (3.6), we obtain the desired conclusion. ☐

It is remarkable that the conclusion of the theorem remains valid in the limit as s tends to t_0, but if s is fixed as t_0 in (3.1), then as was shown by Langenhop for the case $\mathbb{T} = \mathbb{R}$ the estimate is no longer true.

4. Bounds on the norm of solutions. Let $z : \mathbb{T} \to \mathbb{C}^m$ and $f(t,z(t))$ be rd-continuous. We shall consider the first order system

$$z^{\Delta} = f(t,z), \quad t \geq t_0 \tag{4.1}$$

Let us assume that for some norm in C^m, which we shall denote by $|.|$, the function f satisfies

$$|f(t,z)| \leq v(t)g(|z|), \quad t \geq t_0, \quad z \in \mathbb{C}^m, \tag{4.2}$$

where

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(a) $v : \mathbb{T} \to \mathbb{R}_+$ is rd-continuous, $1 - \mu(t)v > 0$;
(b) $g(u)$ is rd-continuous and nondecreasing for $u \geq 0$, and strictly positive for $u > 0$.

It follows from (4.1) and (4.2) that

$$|z(t)| \leq |z(s)| + \int_s^t v(r)g(|z(r)|)\,\Delta r \tag{4.3}$$

and

$$|z(t)| \geq |z(s)| - \int_s^t v(r)g(|z(r)|)\,\Delta r \tag{4.4}$$

for all $s, t \in [t_0, \infty)$.

The main results of this section are as follows.

THEOREM 4.1. *If $z(t)$ is solution of (4.1) such that $z(t_0) = A$, then*

$$|z(t)| \leq w_1(t) \tag{4.5}$$

and

$$|z(t)| \geq w_2(t) \tag{4.6}$$

for all $t \geq t_0$, where w_1 is a minimal solution of

$$w^\Delta = v(t)g(w(t)), \quad w(t_0) = |A|$$

and w_2 is a maximal solution of

$$w^\Delta = -v(t)g(w(t)), \quad w(t_0) = |A|.$$

THEOREM 4.2. *Let $g(u) = u$. If $z(t)$ is solution of (4.1), then for all $t \geq t_0$,*

$$|A|\, e_{-v}(t, t_0) \leq |z(t)| \leq |A|\, e_v(t, t_0). \tag{4.7}$$

Upper bounds in (4.5) and (4.7) are obtained from (4.3) by applying Bihari and Gronwall type inequalities on time Scales, (see [2, 7]), respectively. Lower bounds in (4.6) and (4.7), however, are new and follow from (4.4) on using Theorem 3.1.

5. Oscillation of damped second order delay dynamic equations. We shall consider the oscillatory behavior of solutions of second order delay equations of the form

$$x^{\Delta\Delta}(t) + q(t)\frac{[x^\Delta(t)]^2}{[x^\Delta(t) + x^\Delta(\sigma(t))]} + p(t)[x(g(t))]^3 = 0. \tag{5.1}$$

We restrict our attention to solutions of (5.1) which exist on some ray $[t_0, \infty)$, the interval being understood in time scale sense. A solution $x(t)$ is called oscillatory if it is neither eventually positive nor eventually negative.

With regards to (5.1) the following conditions are assumed to hold:
(a) $g : \mathbb{T} \to \mathbb{R}_+$ is differentiable, nondecreasing, $g(t) \leq t$ and $\lim_{t\to\infty} g(t) = \infty$;
(b) $p : \mathbb{T} \to \mathbb{R}_+$ and $q : \mathbb{T} \to \mathbb{R}$ are rd-continuous, there is an rd-continuous function $m : \mathbb{T} \to \mathbb{R}_+$ such that $1 - \mu m > 0$ and $q(t) \leq m(t)$ for all $t \in [0, \infty)$.

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(c) $\lim_{t\to\infty} Q(t,T) < \infty$ for any fixed $T \in [0,\infty)$, where

$$Q(t,T) = \int_T^t \sqrt{e_{-m}(s,T)}\,\Delta s;$$

(d) The inequality

$$y^{\Delta}(t) + q(t)\frac{[y(t)]^2}{[y(t)+y(\sigma(t))]} < 0 \tag{5.2}$$

has no oscillatory solution.

Note that if $\mathbb{T} = \mathbb{R}$ then (d) holds without imposing any condition at all. Indeed, if $y(t_*) = 0$, then $y'(t_*) < -q(t_*)y(t_*)/2 = 0$ which means that $y(t)$ cannot have a zero larger than t_*.

LEMMA 5.1. *Let* (a)–(d) *hold, and $x(t)$ be an eventually positive solution of (5.1). Then there is a $T \geq t_0$ such that $x^{\Delta}(t)$ is of constant sign for $t \in [T,\infty)$ and*

$$x(t) \geq -Q(\infty,t)x^{\Delta}(t), \quad t \in [T,\infty). \tag{5.3}$$

Proof. Suppose that $x(t)$ is eventually positive. Clearly, there exist $t_1 \in \mathbb{T}$ such that $x(t) > 0$ and $x(g(t)) > 0$ on $[t_1,\infty)$. Furthermore, $x^{\Delta}(t)$ is not oscillatory; since, otherwise, $y = x^{\Delta}(t)$ is an oscillatory solution of (5.2), contradicting (d). We may assume that $x^{\Delta}(t) < 0$, $x^{\Delta}(\sigma(t)) < 0$ on $[T,\infty)$. The case where $x^{\Delta}(t)$ is eventually positive can be handled in a similar manner. Multiplying (5.1) by $x^{\Delta}(t) + x^{\Delta}(\sigma(t))$ and integrating over $[T,t]$ we see that

$$\int_T^t [x^{\Delta}(s) + x^{\Delta}(\sigma(s))]x^{\Delta\Delta}(s)\,\Delta s + \int_T^t q(s)[x^{\Delta}(s)]^2\,\Delta s > 0,$$

from which we get

$$[x^{\Delta}(t)]^2 \geq [x^{\Delta}(T)]^2 - \int_T^t m(s)[x^{\Delta}(s)]^2\,\Delta s.$$

Hence, by the Langenhop inequality,

$$[x^{\Delta}(t)]^2 \geq [x^{\Delta}(T)]^2 e_{-m}(t,T).$$

Clearly, this inequality results in

$$x^{\Delta}(t) \leq x^{\Delta}(T)\sqrt{e_{-m}(t,T)}.$$

Integrating the above inequality over $[T,t]$, we have

$$x(t) \leq x(T) + x^{\Delta}(T)Q(t,T)$$

and hence

$$x(t) \geq -x^{\Delta}(t)Q(\infty,T). \qquad \square$$

THEOREM 5.2. *In addition to* (a)–(d) *assume that*

$$1 - \mu(t)q(t)\frac{Q(\infty,\sigma(t))}{Q(\infty,t)+Q(\infty,\sigma(t))} > 0, \tag{5.4}$$

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$$\int^{\infty} p(t)\,\Delta t = \infty \tag{5.5}$$

and

$$\int^{\infty} \frac{p(t)Q(\infty,t)Q(\infty,\sigma(t))}{r(t)}\left[\frac{Q(\infty,t)}{r(t)} + \frac{Q(\infty,\sigma(t))}{r(\sigma(t))}\right]\Delta t = \infty, \tag{5.6}$$

where r is a solution of

$$r^{\Delta} = q(t)\frac{Q(\infty,\sigma(t))}{Q(\infty,t)+Q(\infty,\sigma(t))}r^{\sigma}. \tag{5.7}$$

Then every solution of (5.1) is oscillatory.

Proof. We may assume that $x(t)$ is eventually positive, since a similar argument holds when $x(t)$ is eventually negative. By (d) $x^{\Delta}(t)$ is either eventually positive or eventually negative.

Case 1. $x^{\Delta}(t) > 0$ *on* $[t_1,\infty)$ *for some large* t_1. From (5.1) we have

$$x^{\Delta\Delta}(t) + p(t)[x(g(t))]^3 \le 0, \tag{5.8}$$

Integrating (5.8) over $[t_1, t]$ we see that for some constant $k > 0$,

$$x^{\Delta}(t) - x^{\Delta}(t_1) + k\int_{t_1}^{t} p(s)\,\Delta s \le 0,$$

which clearly contradicts (5.5).

Case 2. $x^{\Delta}(t) < 0$ *on* $[t_1,\infty)$ *for some large* t_1. Let r be a solution of (5.7) satisfying $r(t_1) = 1$. In view of (5.4) we see that the function r is positive for all $t \ge t_1$. In fact, it can be expressed in terms of the exponential function. It follows from (5.1) that

$$(rx^{\Delta})^{\Delta} = x^{\Delta}\left[r^{\Delta} - r^{\sigma}q\frac{x^{\Delta}}{x^{\Delta}+x^{\Delta\sigma}}\right] - r^{\sigma}p[x(g(t))]^3.$$

Employing (5.3) leads to

$$(rx^{\Delta})^{\Delta} \le x^{\Delta}\left[r^{\Delta} - r^{\sigma}q\frac{x^{\sigma}/Q}{x/Q + x^{\sigma}/Q^{\sigma}}\right] - r^{\sigma}p[x(g(t))]^3,$$

from which, on using the nonincreasing nature of $x(t)$ as well as (5.7), we have

$$(rx^{\Delta})^{\Delta} \le x^{\Delta}\left[r^{\Delta} - r^{\sigma}q\frac{1/Q}{1/Q+1/Q^{\sigma}}\right] - r^{\sigma}p[x(g(t))]^3 \le -r^{\sigma}p\,x^3,$$

and so

$$\begin{aligned}(-(rx^{\Delta})^{-2})^{\Delta} &= \frac{(rx^{\Delta})^{\Delta}}{rx^{\Delta}r^{\sigma}x^{\Delta\sigma}}\left[\frac{1}{rx^{\Delta}} + \frac{1}{r^{\sigma}x^{\Delta\sigma}}\right]\\ &\ge -\frac{p\,x^3}{rx^{\Delta}x^{\Delta\sigma}}\left[\frac{1}{rx^{\Delta}} + \frac{1}{r^{\sigma}x^{\Delta\sigma}}\right].\end{aligned}$$

Employing (5.3), it is not difficult to see from the last inequality that

$$(-(rx^{\Delta})^{-2})^{\Delta} \ge \frac{p\,Q^2\,Q^{\sigma}}{r^2} + \frac{p\,Q\,[Q^{\sigma}]^2}{rr^{\sigma}}. \tag{5.9}$$

Integrating (5.9) from $t_2 \geq t_1$ to t and using $-(r(t)x^\Delta(t))^{-2} < 0$, we have

$$\int_{t_2}^{t} \left[\frac{p\, Q^2\, Q^\sigma}{r^2} + \frac{p\, Q\, [Q^\sigma]^2}{r r^\sigma} \right] \Delta t < (r(t_2)x^\Delta(t_2))^{-2},$$

which obviously contradicts (5.6). ∎

It is possible to prove a similar theorem when the term $[x(g(t))]^3$ in (5.1) is replaced by $|x(g(t))|^{\alpha-1}x(g(t))$, where $\alpha > 1$. In this situation, one has to be careful though, with tedious calculations coming into picture that arise from the complicated nature of the time scale calculus. Since our focus was to illustrate an application of the Langenhop inequality on time scales, for simplicity we have only considered a particular case, namely $\alpha = 3$.

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