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MINIMAL GERSCHGORIN SETS FOR PARTITIONED MATRICES II. THE SPECTRAL CONJECTURE.*

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Dedicated to Olga Taussky and John Todd, on the occasion of their important birthdays in 1996, for their inspiring work in matrix theory and numerical analysis.

Abstract. In an earlier paper from 1970, entitled "Minimal Gerschgorin sets for partitioned matrices," a Spectral Conjecture, related to norms and spectral radii of special partitioned matrices, was stated, this conjecture being at the heart of the sharpness of the boundaries of the associated minimal Gerschgorin sets under partitioning. In this paper, this Spectral Conjecture is affirmatively settled, and is applied to the sharpness of the minimal Gerschgorin set in the special case when the block-diagonal entries are null matrices. The paper following this article then makes use of the proof of the Spectral Conjecture to obtain the general sharpness of the boundaries of the associated minimal Gerschgorin sets for partitioned matrices.

Key words. minimal Gerschgorin sets, partitioned matrices, monotonicity.

AMS subject classification. 15A18.

1. Introduction and Notations.

For n a positive integer, let $\mathbb{C}^n$ denote the vector space of column vectors $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T$, where $x_j \in \mathbb{C}$ for $j = 1, 2, \cdots, n$. Then, following the notations of [9], by a *partition* π of $\mathbb{C}^n$, we mean a collection of pairwise disjoint nonempty linear subspaces $\{W_i\}_{i=1}^N$ whose direct sum is $\mathbb{C}^n$, i.e.,

$$\mathbb{C}^n = W_1 \dot{+} W_2 \dot{+} \cdots \dot{+} W_N. \tag{1.1}$$

For nonnegative integers $\{r_j\}_{j=0}^N$ with $r_0 := 0 < r_1 < \cdots < r_N := n$ and for the standard column basis vectors $\{\mathbf{e}_j\}_{j=1}^n$ for $\mathbb{C}^n$, i.e.,

$$\mathbf{e}_j = (\delta_{j,1}, \delta_{j,2}, \cdots, \delta_{j,n})^T \quad (j = 1, 2, \cdots, n),$$

where $\delta_{i,j}$ is the Kronecker delta function, we assume, without essential loss of generality, that

$$W_j = span\,\{\mathbf{e}_k : r_{j-1} + 1 \leq k \leq r_j\} \quad (j = 1, 2, \cdots, N), \tag{1.2}$$

and we denote the partition π by $\pi := \{r_j\}_{j=0}^N$. We further set

$$p_j := \dim W_j (= r_j - r_{j-1}) \quad (j = 1, 2, \cdots, N). \tag{1.3}$$

For a given partition π of $\mathbb{C}^n$, define the norm N-tuple

$$\phi := (\phi_1, \phi_2, \cdots \phi_N),$$

where each ϕ_j is a vector norm on W_j $(j = 1, 2, \cdots, N)$. If P_j denotes the projection operator from $\mathbb{C}^n$ to W_j for each j, it is easily verified, with $X_j := \mathbf{P}_j\mathbf{x}$, that

$$\|\mathbf{x}\|_\phi := \max_{1\leq j\leq N} \{\phi_j(X_j)\} \quad (\mathbf{x} \in \mathbb{C}^n) \tag{1.4}$$

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is a vector norm on $\mathbb{C}^n$. Then for any matrix B in $\mathbb{C}^{n,n}$, $\|B\|_\phi$ denotes the induced operator norm of B for the vector norm of (1.4), i.e.,

$$\|B\|_\phi := \sup_{\|\mathbf{x}\|_\phi=1} \|B\mathbf{x}\|_\phi. \tag{1.5}$$

It is convenient to define Φ_π as the collection of all such norm N-tuples $\phi = (\phi_1, \phi_2, \cdots, \phi_N)$, associated with the partition π.

Next, given a matrix A in $\mathbb{C}^{n,n}$ and given a partition $\pi = \{r_j\}_{j=0}^N$ of $\mathbb{C}^n$, we can express the matrix A, partitioned with respect to π, as

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,N} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,N} \\ \vdots & & & \vdots \\ A_{N,1} & A_{N,2} & \cdots & A_{N,N} \end{bmatrix} = [A_{i,j}] \quad (i,j = 1, 2, \cdots, N), \tag{1.6}$$

where each submatrix $A_{i,j}$ represents a linear transformation from W_j into W_i.

To state the Spectral Conjecture of [9], fix a partition π of $\mathbb{C}^n$ and consider any matrix $E = [E_{i,j}]$ in $\mathbb{C}^{n,n}$, partitioned by π, which satisfies

$$E_{i,i} = O \quad (i = 1, 2, \cdots, N). \tag{1.7}$$

Following Sylvester (cf. [5, p. 108]), we shall use throughout the notation that a matrix E, which satisfies (1.7), is *π-invertebrate* (i.e., it has no backbone or spine). For each $\phi \in \Phi_\pi$, define

$$\Gamma_\pi^\phi(E) := \{B = [B_{i,j}] \in \mathbb{C}^{n,n} : B_{i,i} = O\,(i = 1, 2, \cdots, N) \text{ and } \|B\|_\phi = \|E\|_\phi\}, \tag{1.8}$$

and its subset

$$\Gamma_\pi(E) := \bigcap_{\phi \in \Phi_\pi} \Gamma_\pi^\phi(E). \tag{1.9}$$

It is clear from (1.8) and (1.9) that

$$\begin{aligned} \Gamma_\pi(E) = \quad & \{B = [B_{i,j}] \in \mathbb{C}^{n,n} : B_{i,i} = O\,(i = 1, 2, \cdots, N) \text{ and} \\ & \|B\|_\phi = \|E\|_\phi \text{ for all } \phi \in \Phi_\pi\}. \end{aligned} \tag{1.10}$$

If $\rho(C)$ denotes the *spectral radius* of a matrix C in $\mathbb{C}^{m,m}$ (i.e., $\rho(C) = \max\{|\lambda| : \lambda$ is an eigenvalue of $C\}$) and if ω is any vector norm on $\mathbb{C}^m$, it is well-known that $\|C\|_\omega \geq \rho(C)$, where $\|C\|_\omega$ is the induced operator norm of C with respect to the vector norm ω. Hence, it is evident from (1.8) that

$$\rho(\tilde{B}) \leq \|\tilde{B}\|_\phi = \|E\|_\phi \quad \left(\tilde{B} \in \Gamma_\pi^\phi(E)\right),$$

so that

$$\sup_{\tilde{B} \in \Gamma_\pi^\phi(E)} \rho(\tilde{B}) \leq \|E\|_\phi. \tag{1.11}$$

In the same manner, we have

$$\rho(B) \leq \|B\|_\phi = \|E\|_\phi \quad (B \in \Gamma_\pi(E);\ \phi \in \Phi_\pi), \tag{1.12}$$

and, since the right side of (1.12) is independent of $B \in \Gamma_\pi(E)$ and since the left side is independent of $\phi \in \Phi_\pi$, it follows (cf. [9, Theorem 3]) that

$$\sup_{B\in\Gamma_\pi(E)} \rho(B) \le \inf_{\phi\in\Phi_\pi} \|E\|_\phi. \tag{1.13}$$

It was shown in [9] that equality holds in (1.13) in special cases, such as in the case when the partition π is such that (cf. (1.1)) $\dim W_j = 1 \quad (j = 1, 2, \cdots, n)$, where the Perron-Frobenius theory of nonnegative matrices is used to establish equality in (1.13), and in the case when the partitioned matrix E is a weakly cyclic matrix of some index p (cf. [7, p. 39]), the latter result being due to Robert [6]. It was conjectured in [9] that equality always holds in (1.13), and this is called here the

$$\textit{Spectral Conjecture:} \qquad \sup_{B\in\Gamma_\pi(E)} \rho(B) \overset{?}{=} \inf_{\phi\in\Phi_\pi} \|E\|_\phi. \tag{1.14}$$

We show below that this Spectral Conjecture is indeed true, and that a stronger form (to be given below in Theorem 2.1) is valid in general.

2. Statement of Our Main Result. To state our main result, some additional notation is needed. Given a partition π of $\mathbb{C}^n$ and given a matrix $E = [E_{i,j}]$ in $\mathbb{C}^{n,n}$ which is π- invertebrate (cf. (1.7)), then for any $\phi = (\phi_1, \phi_2, \cdots, \phi_N)$ in Φ_π, define

$$M^\phi_{i,j}(E) := \sup_{\phi_j(X_j)=1} \phi_i(E_{i,j}X_j) \quad (i, j = 1, 2, \cdots, N), \tag{2.1}$$

which is just the operator norm of the linear transformation $E_{i,j}$ from W_j into W_i. The N^2 nonnegative numbers of (2.1) then determine the $N \times N$ *comparison matrix* $\mathcal{M}^\phi(E)$

$$\mathcal{M}^\phi(E) := [M^\phi_{i,j}(E)]. \tag{2.2}$$

For any $\phi = (\phi_1, \phi_2, \cdots, \phi_N)$ in Φ_π and for any $\mathbf{x}$ in $\mathbb{C}^n$ with $\|\mathbf{x}\|_\phi = 1$, it follows from (1.4) and (2.1), with $P_j\mathbf{x} := X_j$, that

$$\begin{aligned}\|E\mathbf{x}\|_\phi &= \max_{1\le i\le N}\left\{\phi_i\left(\sum_{j=1}^N E_{i,j}X_j\right)\right\} \le \max_{1\le i\le N}\left\{\sum_{j=1}^N \phi_i(E_{i,j}X_j)\right\} \\ &\le \max_{1\le i\le N}\left\{\sum_{j=1}^N M^\phi_{i,j}(E)\phi_j(X_j)\right\} \le \left\{\max_{1\le i\le N}\sum_{j=1}^N M^\phi_{i,j}(E)\right\}\left\{\max_{1\le j\le N}\phi_j(X_j)\right\}.\end{aligned}$$

But as $\|\mathbf{x}\|_\phi = 1$ implies from (1.4) that $\max_{1\le j\le N} \phi_j(X_j) = 1$, we have

$$\|E\mathbf{x}\|_\phi \le \max_{1\le i\le N} \sum_{j=1}^N M^\phi_{i,j}(E) = \|\mathcal{M}^\phi(E)\|_\infty,$$

where $\|\mathcal{M}^\phi(E)\|_\infty$ is the operator norm of the matrix $\mathcal{M}^\phi(E)$ of (2.2), induced by the norm ℓ_∞ on $\mathbb{C}^N$. As the above holds for any $\mathbf{x}$ with $\|\mathbf{x}\|_\phi = 1$, we have

$$\|E\|_\phi \le \|\mathcal{M}^\phi(E)\|_\infty,$$

from which it follows, using (1.11) and (1.13), that

$$\sup_{\tilde{B}\in\Gamma_\pi^\phi(E)} \rho(\tilde{B}) \leq \|E\|_\phi \leq \|\mathcal{M}^\phi(E)\|_\infty \quad (\phi\in\Phi_\pi), \tag{2.3}$$

as well as

$$\sup_{B\in\Gamma_\pi(E)} \rho(B) \leq \inf_{\phi\in\Phi_\pi} \|E\|_\phi \leq \inf_{\phi\in\Phi_\pi} \|\mathcal{M}^\phi(E)\|_\infty. \tag{2.4}$$

With this, our main result can be stated as

THEOREM 2.1. *Let π be a partition of $\mathbb{C}^n$, and let E, in $\mathbb{C}^{n,n}$, be π-invertebrate. Then,*

$$\sup_{B\in\Gamma_\pi(E)} \rho(B) = \inf_{\phi\in\Phi_\pi} \|E\|_\phi = \inf_{\phi\in\Phi_\pi} \rho\left(\mathcal{M}^\phi(E)\right) = \inf_{\phi\in\Phi_\pi} \|\mathcal{M}^\phi(E)\|_\infty. \tag{2.5}$$

The first equality of (2.5) thus affirmatively settles the Spectral Conjecture of (1.14), and the final equality of (2.5), involving the infinity norms of associated comparison matrices, is a further, and perhaps unexpected, characterization.

As might be imagined, the Perron-Frobenius theory of nonnegative matrices plays a role in the proof of Theorem 2.1, since the $N\times N$ comparison matrix $\mathcal{M}^\phi(E)$ of (2.2) is a nonnegative matrix.

3. Proof of Theorem 2.1. Given the partition π of $\mathbb{C}^n$ and given a matrix $A = [A_{i,j}]$ in $\mathbb{C}^{n,n}$ which is partitioned by π, let $\mathcal{G}_\pi(A)$ denote the block-directed graph of A, i.e., given N distinct vertices $\{v_j\}_{j=1}^N$, there is an arc from v_i to v_j whenever $A_{i,j}\not\equiv O\,(i,j=1,2,\cdots,N)$. This block-directed graph $\mathcal{G}_\pi(A)$ is said to be *strongly connected* if, for any two vertices v_i and v_j $(i,j=1,2,\cdots,N)$, there is a path, consisting of abutting arcs, from vertex v_i to vertex v_j. In this case, $A=[A_{i,j}]$ is said to be π-*block irreducible*, and we similarly say that A is π-*block reducible* if $\mathcal{G}_\pi(A)$ is not strongly connected. As is readily seen, A is π-block irreducible (π-block reducible) if and only its $N\times N$ comparison matrix (cf. (2.2.)) $\mathcal{M}^\phi(A)$ is irreducible (reducible) for *each* ϕ in Φ_π, in the standard terminology of, say, [7, p. 19 and p. 45]. (We remark that the notion here of π-block irreducibility *differs* from the notion of π-irreducibility of [9].)

PROPOSITION 3.1. *Let π be a partition of $\mathbb{C}^n$, and let $A\in\mathbb{C}^{n,n}$. If A is π-block irreducible, then, given any ϕ in Φ_π, there is a positive vector $\mathbf{u}=(u_1,u_2,\cdots,u_N)^T$ with $u_j>0$ for $j=1,2,\cdots,N$, such that the norm N-tuple, defined by*

$$\tilde{\phi} := \left(\frac{\phi_1}{u_1}, \frac{\phi_2}{u_2}, \cdots, \frac{\phi_N}{u_N}\right), \tag{3.1}$$

satisfies

$$\rho\left(\mathcal{M}^\phi(A)\right) = \|\mathcal{M}^{\tilde{\phi}}(A)\|_\infty = \rho\left(\mathcal{M}^{\tilde{\phi}}(A)\right). \tag{3.2}$$

If A is π-block reducible, then, given any $\epsilon>0$, there is a norm N-tuple $\tilde{\phi}\in\Phi_\pi$, such that

$$\|\mathcal{M}^{\tilde{\phi}}(A)\|_\infty \leq \rho\left(\mathcal{M}^\phi(A)\right)+\epsilon, \ \textit{where}\ \rho\left(\mathcal{M}^{\tilde{\phi}}(A)\right) = \rho\left(\mathcal{M}^\phi(A)\right). \tag{3.3}$$

Proof. As remarked above, if A is π-block irreducible, its comparison matrix, $\mathcal{M}^{\phi}(A)$, is an irreducible nonnegative matrix for each $\phi \in \Phi_\pi$. From the Perron-Frobenius theory of nonnegative irreducible matrices (cf. [7, p. 30]), there is a column vector $\mathbf{u} = (u_1, u_2, \cdots, u_N)^T$ with positive components (dependent on ϕ) such that

$$\mathcal{M}^{\phi}(A)\mathbf{u} = \rho\left(\mathcal{M}^{\phi}(A)\right)\mathbf{u}.$$

With $D := diag(u_1, u_2, \cdots, u_N)$, so that D is a nonsingular nonnegative matrix, the above equation can be written as

$$\mathcal{M}^{\phi}(A)D\boldsymbol{\xi} = \rho(\mathcal{M}^{\phi}(A))D\boldsymbol{\xi}, \text{ where } \boldsymbol{\xi} := (1, 1, \cdots, 1)^T \in \mathbb{R}^N,$$

so that

$$D^{-1}\mathcal{M}^{\phi}(A)D\boldsymbol{\xi} = \rho\left(\mathcal{M}^{\phi}(A)\right)\boldsymbol{\xi}. \tag{3.4}$$

This implies that the row sums of the nonnegative matrix $D^{-1}\mathcal{M}^{\phi}(A)D$ are all equal to $\rho(\mathcal{M}^{\phi}(A))$. As is well known (cf. [7, p. 15]), this implies

$$\|D^{-1}\mathcal{M}^{\phi}(A)D\|_\infty = \rho\left(\mathcal{M}^{\phi}(A)\right). \tag{3.5}$$

From (3.1), the norm N-tuple $\tilde{\phi}$, which is clearly an element of Φ_π, can be directly verified to satisfy

$$\mathcal{M}^{\tilde{\phi}}_{i,j}(A) = \frac{u_j}{u_i}\mathcal{M}^{\phi}_{i,j}(A) \quad (i, j = 1, 2, \cdots, N),$$

so that its associated comparison matrix, $\mathcal{M}^{\tilde{\phi}}(A)$, is given by

$$\mathcal{M}^{\tilde{\phi}}(A) = D^{-1}\mathcal{M}^{\phi}(A)D. \tag{3.6}$$

Hence, (3.5) becomes

$$\|\mathcal{M}^{\tilde{\phi}}(A)\|_\infty = \rho\left(\mathcal{M}^{\phi}(A)\right),$$

which is the first desired equality of (3.2). Then, since $\mathcal{M}^{\tilde{\phi}}(A)$ and $\mathcal{M}^{\phi}(A)$ are similar from (3.6), we have $\rho(\mathcal{M}^{\phi}(A)) = \rho(\mathcal{M}^{\tilde{\phi}}(A))$, which gives the last desired equality of (3.2).

For the case that A in $\mathbb{C}^{n,n}$ is π-reducible, the proof of (3.3), which is omitted, is based on the notion of the *normal reduced form* (cf. [7, p. 46]) of the reducible matrix $\mathcal{M}^{\phi}(A)$, and on ϵ-scalings of this matrix (cf. Householder [2, p. 46]). (A detailed proof of (3.3) can be found in Krautstengl [4]). □

Next, given a partition π of $\mathbb{C}^n$ and given a matrix A in $\mathbb{C}^{n,n}$, consider the non-negative numbers, $\alpha(A)$ and $\beta(A)$, defined by

$$\alpha(A) := \inf_{\phi \in \Phi_\pi} \|\mathcal{M}^{\phi}(A)\|_\infty \text{ and } \beta(A) := \inf_{\phi \in \Phi_\pi} \rho\left(\mathcal{M}^{\phi}(A)\right). \tag{3.7}$$

Since $\|\mathcal{M}^{\phi}(A)\|_\infty \geq \rho(\mathcal{M}^{\phi}(A))$ for any $\phi \in \Phi_\pi$, it is evident that

$$\alpha(A) \geq \beta(A). \tag{3.8}$$

On the other hand, using appropriate sequences of norm N-tuples, it is a straightforward consequence of Proposition 3.1 that equality holds in (3.8) in all cases, which we state as

COROLLARY 3.2. *For any partition π of $\mathbb{C}^n$ and for any A in $\mathbb{C}^{n,n}$, partitioned by π,*

$$\inf_{\phi\in\Phi_\pi} \rho\left(\mathcal{M}^\phi(A)\right) = \inf_{\phi\in\Phi_\pi} \|\mathcal{M}^\phi(A)\|_\infty. \tag{3.9}$$

We remark, on comparing (2.4) and (3.9), that to establish Theorem 2.1, it suffices to show that

$$\sup_{B\in\Gamma_\pi(E)} \rho(B) \stackrel{?}{=} \inf_{\phi\in\Phi_\pi} \rho\left(\mathcal{M}^\phi(B)\right). \tag{3.10}$$

Next, given a partition π of $\mathbb{C}^n$, we set

$$\begin{aligned}\Phi_{\pi,\infty} := \{\psi = (\psi_1,\psi_2,\cdots,\psi_N) : \psi_i(\mathbf{u}) = \|S_i\mathbf{u}\|_\infty \text{ for all } \mathbf{u}\in W_i,\\ \text{where } S_i \text{ is a nonsingular matrix in } \mathbb{C}^{p_i,p_i}\},\end{aligned} \tag{3.11}$$

so that

$$\Phi_{\pi,\infty} \subset \Phi_\pi. \tag{3.12}$$

It is obvious that $\Phi_{\pi,\infty}$ is a *proper* subset of Φ_π if and only if $\max_{1\le i\le N} p_i > 1$. For $\psi = (\psi_1,\psi_2,\cdots,\psi_N) \in \Phi_{\pi,\infty}$, its associated nonsingular matrices $\{S_i\}_{i=1}^N$, from (3.11), then define the nonsingular block-diagonal matrix

$$\mathcal{S} := diag(S_1,S_2,\cdots,S_N) \in \mathbb{C}^{n,n}, \tag{3.13}$$

and, conversely, each such nonsingular block diagonal matrix in (3.13), with $S_i \in \mathbb{C}^{p_i,p_i}$ $(i = 1,2,\cdots,N)$, defines a norm N-tuple in $\Phi_{\pi,\infty}$. With the definitions of (2.1) and (3.11), we note, for any $\psi = (\psi_1,\psi_2,\cdots,\psi_N)$ in $\Phi_{\pi,\infty}$ and for any matrix $B = [B_{i,j}]$ in $\mathbb{C}^{n,n}$, which is partitioned by π, that

$$M^\psi_{i,j}(B) = \|S_iB_{i,j}S_j^{-1}\|_\infty \quad (i,j = 1,2,\cdots,N). \tag{3.14}$$

In analogy with the definition of $\Gamma_\pi(E)$ of (1.10) where $E = [E_{i,j}]$ in $\mathbb{C}^{n,n}$ is π-invertebrate, we also set

$$\begin{aligned}\Gamma_{\pi,\infty}(E) := \ &\{B = [B_{i,j}] \in \mathbb{C}^{n,n} : B_{i,i} = O\ (i = 1,2,\cdots,N)\\ &\text{and } \|B\|_\psi = \|E\|_\psi \text{ for } all\ \psi\in\Phi_{\pi,\infty}\},\end{aligned} \tag{3.15}$$

which gives

$$\Gamma_{\pi,\infty}(E) \supset \Gamma_\pi(E).$$

However, a close examination of the constructions in the proof of Theorem 4.3 of [9], based on ℓ_∞-type norms, shows in fact that

$$\Gamma_{\pi,\infty}(E) = \Gamma_\pi(E). \tag{3.16}$$

Hence, the inequalities of (2.4) can be expressed, using the equalities of (3.9) and (3.16), as

$$\sup_{B\in\Gamma_{\pi,\infty}(E)} \rho(B) \le \inf_{\phi\in\Phi_\pi} \|E\|_\phi \le \inf_{\phi\in\Phi_\pi} \rho\left(\mathcal{M}^\phi(E)\right) \le \inf_{\psi\in\Phi_{\pi,\infty}} \rho\left(\mathcal{M}^\psi(E)\right), \tag{3.17}$$

where the last inequality follows from the inclusion of (3.12). Thus, to establish Theorem 2.1, it now suffices to show that

$$\sup_{B\in\Gamma_{\pi,\infty}(E)} \rho(B) \stackrel{?}{=} \inf_{\psi\in\Phi_{\pi,\infty}} \rho\left(\mathcal{M}^\psi(E)\right). \tag{3.18}$$

For notational convenience, we also define, for any $\phi \in \Phi_\pi$,

$$\begin{aligned}\Omega_\pi^\phi(E) := \quad & \{B=[B_{i,j}] \in \mathbb{C}^{n,n} : B_{i,i} = O \text{ and } M_{i,j}^\phi(B) = M_{i,j}^\phi(E) \\ & (i,j=1,2,\cdots,N)\}.\end{aligned} \tag{3.19}$$

Proposition 3.3. *Let π be a partition of $\mathbb{C}^n$, and let E, in $\mathbb{C}^{n,n}$, be π-invertebrate. Then for any $\psi = (\psi_1, \psi_2, \cdots, \psi_N) \in \Phi_{\pi,\infty}$, there is a matrix $\tilde{B} \in \Omega_\pi^\psi(E)$ such that*

$$\rho(\tilde{B}) = \rho(\mathcal{M}^\psi(E)). \tag{3.20}$$

Proof. For any $\psi \in \Phi_{\pi,\infty}$, let $\hat{B} = [\hat{B}_{i,j}]$ in $\mathbb{R}^{n,n}$, partitioned by π, be defined by means of

$$\hat{B}_{i,j} := M_{i,j}^\psi(E)\cdot Y_{i,j} \quad (i,j=1,2,\cdots,N), \tag{3.21}$$

where the $M_{i,j}^\psi(E)$'s are the scalars of (2.1), and where the block submatrix $Y_{i,j}$, a matrix in $\mathbb{R}^{p_i,p_j}$, is defined by

$$\begin{cases} Y_{i,j} := I_{p_i} \text{ if } i=j, \\ Y_{i,j} := \left[\dfrac{I_{p_j}}{O_{p_i-p_j,p_j}}\right] \text{ if } p_i > p_j, \text{ and } Y_{i,j} := \left[I_{p_i}|O_{p_i,p_j-p_i}\right] \text{ if } p_i < p_j; \end{cases} \tag{3.22}$$

here, I_r denotes the $r\times r$ identity matrix and $O_{s,t}$ denotes the null matrix in $\mathbb{R}^{s,t}$. Note that each submatrix $Y_{i,j}$ has a *unit main diagonal*, i.e.,

$$[Y_{i,j}]_{k,k} = 1 \quad (k=1,2,\cdots,\min(p_i;p_j)),$$

with all remaining entries of $Y_{i,j}$ being zero. Note that since $E_{i,i} = O$, so that $M_{i,i}^\psi(E) = 0$, then (3.21) implies that $\hat{B}_{i,i} = O$ for all $i = 1,2,\cdots N$. In addition, it is evident from (3.21) and (3.22) that the partitioned matrix $\hat{B} = [\hat{B}_{i,j}]$, of (3.21), is a nonnegative matrix, as is each of its powers, $(\hat{B})^k, k=1,2,\cdots$.

We next bound the powers $(\hat{B})^k =: [\hat{B}_{i,j}^{(k)}]$, for all $k = 1,2,\cdots$. With $[(\mathcal{M}^\psi(E))^k]_{i,j}$ denoting the (i,j)-th entry of the matrix $(\mathcal{M}^\psi(E))^k$ in $\mathbb{R}^{N,N}$, consider the nonnegative matrix $C^{(k)}$, partitioned by π, where

$$C^{(k)} := [C_{i,j}^{(k)}] \in \mathbb{R}^{n,n} \quad (k=1,2,\cdots), \tag{3.23}$$

and where its submatrices, $C_{i,j}^{(k)}$, are defined by

$$C_{i,j}^{(k)} := \left[\left(\mathcal{M}^{\psi}(E)\right)^{k}\right]_{i,j} \cdot Y_{i,j} \quad (i,j = 1,2,\cdots,N;\ k = 1,2,\cdots). \tag{3.24}$$

In other words, each submatrix $C_{i,j}^{(k)}$ is the nonnegative submatrix $Y_{i,j}$ of (3.22), multiplied by the nonnegative scalar $[(\mathcal{M}^{\psi}(E))^k]_{i,j}$. We claim that, as nonnegative matrices,

$$\hat{B}_{i,j}^{(k)} \leq C_{i,j}^{(k)} \quad (i,j = 1,2,\cdots,N;\ k = 1,2,\cdots), \tag{3.25}$$

and we claim, moreover, that *equality* holds in (3.25) in the $(1,1)$-entry and in all (zero) off-diagonal entries of these submatrices, for *all* $i,j = 1,2,\cdots,N$, and for *all* $k = 1,2,\cdots$. The key for seeing this is to verify, inductively, that any matrix product

$$\prod_{j=1}^{m} Y_{s_{j-1},s_j} := Y_{s_0,s_1} \cdot Y_{s_1,s_2} \cdots Y_{s_{m-1},s_m} \quad (\text{in } \mathbb{R}^{s_0,s_m})$$

has, with the notations of (3.22), the form

$$\prod_{j=1}^{m} Y_{s_{j-1},s_j} = \left[\begin{array}{c|c} I_r & O_{r,s_m-r} \\ \hline O_{s_0-r,r} & O_{s_0-r,s_m-r} \end{array}\right], \text{ where } r := \min_{0\leq j\leq m} \{p_{s_j}\}. \tag{3.26}$$

Hence, the matrix product of (3.26) has exactly r (diagonal) entries which are unity, with all other entries zero. Note that as r of (3.26) is at least unity, the $(1,1)$-entry of any such matrix product is *always* unity. Consequently, from the definition of (3.22) we have

$$\prod_{j=1}^{m} Y_{s_{j-1},s_j} \leq Y_{s_0,s_m},$$

where equality necessarily holds at least in the $(1,1)$-entry, as well as trivially in all (zero) off-diagonal entries. Because the matrix products of (3.26) occur naturally in the powers of $\hat{B}$, then, on taking into account the nonnegative multipliers $M_{i,j}^{\psi}(E)$ in (3.21) and $[(\mathcal{M}^{\psi}(E))^k]_{i,j}$ in (3.24), the claims for (3.25) follow.

As an example illustrating the inequalities of (3.25), consider the partition $\pi := \{0,2,5,7\}$ of $\mathbb{C}^{7,7}$, so that $p_1 = \dim W_1 = 2$, $p_2 = \dim W_2 = 3$, and $p_3 = \dim W_3 = 2$. If we have, say,

$$\mathcal{M}^{\psi}(E) = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 0 & 3 \\ 2 & 1 & 0 \end{bmatrix},$$

then $\hat{B}$ in $\mathbb{R}^{7,7}$ is given from (3.21) by

$$\hat{B} = \left[\begin{array}{cc|ccc|cc} 0 & 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ \hline 3 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 \end{array}\right].$$

It can be verified in this case that

$$(\mathcal{M}^{\psi}(E))^5 = \begin{bmatrix} 204 & 124 & 236 \\ 372 & 192 & 372 \\ 236 & 124 & 204 \end{bmatrix}, \quad (\hat{B})^5 = \left[\begin{array}{rr|rrr|rr} 204 & 0 & 124 & 0 & 0 & 236 & 0 \\ 0 & 204 & 0 & 124 & 0 & 0 & 236 \\ \hline 372 & 0 & 192 & 0 & 0 & 372 & 0 \\ 0 & 372 & 0 & 192 & 0 & 0 & 372 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 236 & 0 & 124 & 0 & 0 & 204 & 0 \\ 0 & 236 & 0 & 124 & 0 & 0 & 204 \end{array}\right],$$

while

$$C^{(5)} = \left[\begin{array}{rr|rrr|rr} 204 & 0 & 124 & 0 & 0 & 236 & 0 \\ 0 & 204 & 0 & 124 & 0 & 0 & 236 \\ \hline 372 & 0 & 192 & 0 & 0 & 372 & 0 \\ 0 & 372 & 0 & 192 & 0 & 0 & 372 \\ 0 & 0 & 0 & 0 & 192 & 0 & 0 \\ \hline 236 & 0 & 124 & 0 & 0 & 204 & 0 \\ 0 & 236 & 0 & 124 & 0 & 0 & 204 \end{array}\right].$$

This illustrates the inequalities of (3.25) and the subsequent claims.

The inequalities of (3.25) can be used as follows. Because equality always holds in (3.25) in at least the $(1,1)$-entry of each submatrix, the trace of $(\hat{B})^k$ then satisfies the inequalities

$$\text{(3.27)} \qquad \operatorname{tr}\left[\left(\mathcal{M}^{\psi}(E)\right)^k\right] \le \operatorname{tr}\left[(\hat{B})^k\right] \le d \cdot \operatorname{tr}\left[\left(\mathcal{M}^{\psi}(E)\right)^k\right] \quad (k = 1, 2, \cdots),$$

where $d := \max_{1 \le i \le N} p_i$ (where $p_i := \dim W_i$). As d is independent of k, then taking k-th roots gives

$$\text{(3.28)} \qquad \overline{\lim_{k\to\infty}} \left\{\operatorname{tr}\left[\left(\mathcal{M}^{\psi}(E)\right)^k\right]\right\}^{1/k} = \overline{\lim_{k\to\infty}} \left\{\operatorname{tr}\left[(\hat{B})^k\right]\right\}^{1/k}.$$

But as $\mathcal{M}^{\psi}(E)$, in $\mathbb{R}^{N,N}$, and $\hat{B}$, in $\mathbb{R}^{n,n}$, are nonnegative matrices, it is essentially well-known (and this can be found in Kingman [3]) that the quantities in (3.28) are, respectively, $\rho(\mathcal{M}^{\psi}(E))$ and $\rho(\hat{B})$; whence,

$$\text{(3.29)} \qquad \rho\left(\mathcal{M}^{\psi}(E)\right) = \rho(\hat{B}).$$

(For a recent more general result, involving traces of powers of matrices and spectral radii, see Fiedler and Pták [1, Prop. 3.7].)

Next, since $\psi = (\psi_1, \psi_2, \cdots, \psi_N)$ in $\Phi_{\pi,\infty}$ defines (cf. (3.13)) the nonsingular block-diagonal matrix $\mathcal{S} = diag(S_1, S_2, \cdots, S_N)$ in $\mathbb{C}^{n,n}$, set

$$\text{(3.30)} \qquad \tilde{B} := \mathcal{S}^{-1}\hat{B}\mathcal{S}, \text{ and write } \tilde{B} = [\tilde{B}_{i,j}],$$

where the last expression in (3.30) denotes the partitioning of $\tilde{B}$ with respect to π. As $\tilde{B}$ is similar to $\hat{B}$ from (3.30), it follows from (3.29) that

$$\text{(3.31)} \qquad \rho\left(\mathcal{M}^{\psi}(E)\right) = \rho(\tilde{B}).$$

To complete the proof of Proposition 3.3, it remains to show that $\tilde{B} \in \Omega^{\psi}_{\pi,\infty}(E)$. From (3.30), the submatrices $\tilde{B}_{i,j}$, of $\tilde{B}$, and $\hat{B}_{i,j}$, of $\hat{B}$, are related through

$$\tilde{B}_{i,j} = S_i^{-1}\hat{B}_{i,j}S_j \quad (i,j = 1,2,\cdots,N), \tag{3.32}$$

so that from (3.14) and (3.32),

$$\begin{aligned} M^{\psi}_{i,j}(\tilde{B}) &= \|S_i\tilde{B}_{i,j}S_j^{-1}\|_\infty = \|S_i(S_i^{-1}\hat{B}_{i,j}S_j)S_j^{-1}\|_\infty = \|\hat{B}_{i,j}\|_\infty \\ &= M^{\psi}_{i,j}(E)\cdot\|Y_{i.j}\|_\infty, \end{aligned}$$

the last equality following from (3.21). On using the definition of $Y_{i,j}$ of (3.22), it is easily seen that $\|Y_{i,j}\|_\infty = 1$ in all cases (for square or rectangular submatrices). Hence,

$$M^{\psi}_{i,j}(\tilde{B}) = M^{\psi}_{i,j}(E) \quad (i,j = 1,2,\cdots,N),$$

proving (cf. (3.19)) that $\tilde{B} \in \Omega^{\psi}_{\pi}(E)$. □

As a consequence of Proposition 3.3, we next have

PROPOSITION 3.4. *Let π be a partition of $\mathbb{C}^n$, and let E, in $\mathbb{C}^{n,n}$, be π-invertebrate. Then for any $\psi \in \Phi_{\pi,\infty}$,*

$$\sup_{B\in\Omega^{\psi}_{\pi}(E)} \rho(B) = \rho(\mathcal{M}^{\psi}(E)). \tag{3.33}$$

Proof. Fixing $\psi \in \Phi_{\pi,\infty}$, we first show that

$$\sup_{B\in\Omega^{\psi}_{\pi}(E)} \rho(B) \le \rho\left(\mathcal{M}^{\psi}(E)\right). \tag{3.34}$$

For $B = [B_{i,j}]$ in $\Omega^{\psi}_{\pi}(E)$, let $\sigma(B)$ denote the set of all eigenvalues of B, and let $\lambda \in \sigma(B)$. Hence, $B\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x} \neq \mathbf{0}$, which we can express, with the partition π, as $\lambda X_i = \sum_{j=1}^{N} B_{i,j}X_j$ $(i = 1,2,\cdots,N)$. On applying the norms ψ_i, this gives

$$\begin{aligned} |\lambda|\cdot\psi_i(X_i) &= \psi_i\left(\sum_{j=1}^{N} B_{i,j}X_j\right) \le \sum_{j=1}^{N}\psi_i\left(B_{i,j}X_j\right) \\ &\le \sum_{j=1}^{N} M^{\psi}_{i,j}(B)\psi_j(X_j) = \sum_{j=1}^{N} M^{\psi}_{i,j}(E)\psi_j(X_j), \end{aligned} \tag{3.35}$$

for all $i = 1,2,\cdots,N$, the last equality following from the fact (cf. (3.19)) that $B \in \Omega^{\psi}_{\pi}(E)$. With the notation of (2.2), the inequalities in (3.35) can be simply expressed in matrix form as

$$|\lambda|\cdot\begin{bmatrix}\psi_1(X_1)\\ \psi_2(X_2)\\ \vdots\\ \psi_N(X_N)\end{bmatrix} \le \mathcal{M}^{\psi}(E)\cdot\begin{bmatrix}\psi_1(X_1)\\ \psi_2(X_2)\\ \vdots\\ \psi_N(X_N)\end{bmatrix}, \tag{3.36}$$

where $\mathcal{M}^{\psi}(E)$ is a nonnegative matrix in $\mathbb{R}^{N,N}$, and where the $\psi_i(X_i)$'s are nonnegative numbers with $\max_{1\le i\le N} \psi_i(X_i) > 0$. It is a familiar consequence (cf. [7, p. 47]) of the Perron-Frobenius theory of nonnegative matrices that the inequalities of (3.36) imply that

$$|\lambda| \le \rho\left(\mathcal{M}^{\psi}(E)\right).$$

As the above inequality holds for any $\lambda \in \sigma(B)$ and for any $B \in \Omega_{\pi}^{\psi}(E)$, we have the inequality of (3.34). But on applying (3.20) of Proposition 3.3, there is a $\tilde{B} \in \Omega_{\pi,\infty}^{\psi}(E)$ with $\rho(\tilde{B}) = \rho(\mathcal{M}^{\psi}(E))$, which then gives the desired case of equality in (3.33). □

As an immediate consequence of (3.33) of Proposition 3.4, we have

Corollary 3.5. *Let π be a partition of $\mathbb{C}^n$ and let E, in $\mathbb{C}^{n,n}$, be π-invertebrate. Then,*

$$\inf_{\psi\in\Phi_{\pi,\infty}} \left\{ \sup_{B\in\Omega_{\pi}^{\psi}(E)} \rho(B) \right\} = \inf_{\psi\in\Phi_{\pi,\infty}} \left\{ \rho\left(\mathcal{M}^{\psi}(E)\right) \right\}. \tag{3.37}$$

On recalling the string of inequalities in (3.17), we now have from (3.37) that

$$\sup_{B\in\Gamma_{\pi,\infty}(E)} \rho(B) \le \inf_{\phi\in\Phi_{\pi}} \|E\|_{\phi} \le \inf_{\psi\in\Phi_{\pi,\infty}} \rho\left(\mathcal{M}^{\psi}(E)\right) = \inf_{\psi\in\Phi_{\pi,\infty}} \left\{ \sup_{B\in\Omega_{\pi}^{\psi}(E)} \rho(B) \right\}. \tag{3.38}$$

Thus, in order to finally complete the proof of Theorem 2.1, it suffices from (3.38) to establish

$$\sup_{B\in\Gamma_{\pi,\infty}(E)} \rho(B) \stackrel{?}{=} \inf_{\psi\in\Phi_{\pi,\infty}} \left\{ \sup_{B\in\Omega_{\pi}^{\psi}(E)} \rho(B) \right\}. \tag{3.39}$$

With the definition of the set $\Omega_{\pi}^{\phi}(E)$ of (3.19), we next set

$$\Omega_{\pi}(E) := \bigcap_{\phi\in\Phi_{\pi}} \Omega_{\pi}^{\phi}(E) = \bigcap_{\psi\in\Phi_{\pi,\infty}} \Omega_{\pi}^{\psi}(E), \tag{3.40}$$

the second equality following from the constructions in the proof of Theorem 4.3 of [9]. It also follows from Theorem 4.3 of [9] that $B = [B_{i,j}]$ is in $\Omega_{\pi}(E)$ if and only if, for each pair of integers (k,ℓ) with $1 \le k,\ell \le N$, there exists a real parameter $\theta_{k,\ell}$ (with $0 \le \theta_{k,\ell} \le 2\pi$) such that

$$B_{k,\ell} = \exp(i\theta_{k,\ell}) \cdot E_{k,\ell} \quad (k,\ell = 1,2,\cdots,N). \tag{3.41}$$

But then, from the definition of $\Gamma_{\pi,\infty}(E)$ in (3.15) and from (3.16), we see that

$$\Gamma_{\pi,\infty}(E) = \Omega_{\pi}(E). \tag{3.42}$$

Thus, with (3.42), we can equivalently express (3.39) (for what is needed to complete the proof of Theorem 2.1) as

$$\sup_{B\in\Omega_{\pi}(E)} \rho(B) \stackrel{?}{=} \inf_{\psi\in\Phi_{\pi,\infty}} \left\{ \sup_{B\in\Omega_{\pi}^{\psi}(E)} \rho(B) \right\}. \tag{3.43}$$

We consider first the quantity on the left in (3.43). Since the diagonal blocks of E by hypothesis satisfy $E_{i,i} = O\,(i = 1, 2, \cdots, N)$, each $B = [B_{i,j}]$ in $\Omega_\pi(E)$ is then determined from (3.41) by $N^2 - N$ real parameters $\theta_{k,\ell}$ $(0 \le \theta_{k,\ell} \le 2\pi)$. As $\rho(B)$ is a continuous function of these parameters, compactness considerations show that there is a $\underline{B}$ in $\Omega_\pi(E)$ for which

$$\sup_{B \in \Omega_\pi(E)} \rho(B) = \rho(\underline{B}). \tag{3.44}$$

It is also important to point out from (3.41) that $B \in \Omega_\pi(E)$ implies $e^{i\eta} B \in \Omega_\pi(E)$ for every real η. Hence, for the spectra $\sigma(\Omega_\pi(E))$ of all matrices in $\Omega_\pi(E)$, i.e.,

$$\sigma\left(\Omega_\pi(E)\right) := \bigcup_{B \in \Omega_\pi(E)} \left(\sigma(B)\right), \tag{3.45}$$

we note that if $\lambda \in \sigma(B)$ for some B in $\Omega_\pi(E)$, then the entire circle $\{z \in \mathbb{C} : |z| = |\lambda|\}$ is contained in $\sigma(\Omega_\pi(E))$. In particular, this means that

$$\{z \in \mathbb{C} : |z| = \rho(\underline{B})\} \subset \sigma\left(\Omega_\pi(E)\right) \subset \{z \in \mathbb{C} : |z| \le \rho(\underline{B})\}. \tag{3.46}$$

Next, let ψ be an arbitrary, but fixed, element of $\Phi_{\pi,\infty}$, and let B^ψ, in $\Omega_\pi^\psi(E)$, be such that

$$\rho(B^\psi) = \sup\left\{\rho(B) : B \in \Omega_\pi^\psi(E)\right\}. \tag{3.47}$$

The existence of such a B^ψ follows directly from (3.20) of Proposition 3.3 and (3.33) of Proposition 3.4. As above, we similarly have

$$e^{i\eta} B^\psi \in \Omega_\pi^\psi(E) \text{ for every real } \eta,$$

and this implies, as in (3.46), that

$$\left\{z \in \mathbb{C} : |z| = \rho(B^\psi)\right\} \subset \sigma\left(\Omega_\pi^\psi(E)\right) \subset \left\{z \in \mathbb{C} : |z| \le \rho(B^\psi)\right\}. \tag{3.48}$$

Also, since $\Omega_\pi(E) \subset \Omega_\pi^\psi(E)$, it is evident that

$$\sup_{B \in \Omega_\pi(E)} \{\rho(B)\} \le \sup_{B \in \Omega_\pi^\psi(E)} \{\rho(B)\},$$

so that (cf. (3.44) and (3.47))

$$\rho(\underline{B}) \le \rho(B^\psi). \tag{3.49}$$

Next, we establish

PROPOSITION 3.6. *Let π be a partition of $\mathbb{C}^n$ and let E, in $\mathbb{C}^{n,n}$, be π-invertebrate. Then for any $\psi \in \Phi_{\pi,\infty}$, there is a matrix $B(\alpha)$ in $\mathbb{C}^{n,n}$, whose entries depend continuously on the parameter α (where $\alpha \in [0,1]$), with $B(\alpha)$ in $\Omega_\pi^\psi(E)$ for all $\alpha \in [0,1]$, with $B(0) = \underline{B}$ and with $B(1) = B^\psi$ (where $\underline{B}$ and B^ψ respectively satisfy* (3.44) *and* (3.47)*).*

Proof. Given $\psi \in \Phi_{\pi,\infty}$, let $\mathcal{S} = diag(S_1, S_2, \cdots, S_N)$ be the block-diagonal matrix in $\mathbb{C}^{n,n}$ of (3.13), associated with ψ. As $\underline{B}$ and B^ψ are both in $\Omega_\pi^\psi(E)$, then

$$M_{k,\ell}^\psi(\underline{B}) = M_{k,\ell}^\psi(B^\psi) = M_{k,\ell}^\psi(E) \quad (k, \ell = 1, 2, \cdots, N), \tag{3.50}$$

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where (cf. (3.14)) for a matrix $C = [C_{i,j}]$ in $\mathbb{C}^{n,n}$ which is partitioned by π,

$$M_{k,\ell}^{\psi}(C) := \|S_k C_{k,\ell} S_\ell^{-1}\|_\infty \quad (k, \ell = 1, 2, \cdots, N). \tag{3.51}$$

What is to be established here is that, for each integer pair (k, ℓ) from $(k, \ell = 1, 2, \cdots, N)$, there exists a matrix $A_{k,\ell}(\alpha)$ in $\mathbb{C}^{p_k, p_\ell}$, whose entries depend continuously on the parameter α for $\alpha \in [0, 1]$, such that

$$\|A_{k,\ell}(\alpha)\|_\infty = \mu \text{ for } \textit{all } \alpha \in [0,1] \left(\text{where } \mu := \mu_{k,\ell} := M_{k,\ell}^{\psi}(E)\right), \tag{3.52}$$

with the additional requirements that

$$A_{k,\ell}(0) := S_k \underline{B}_{k,\ell} S_\ell^{-1} \text{ and } A_{k,\ell}(1) := S_k B_{k,\ell}^{\psi} S_\ell^{-1}. \tag{3.53}$$

Once $A_{k,\ell}(\alpha)$ is determined, $B_{k,\ell}(\alpha)$ is then defined (cf. (3.51)) by

$$B_{k,\ell}(\alpha) := S_k^{-1} A_{k,\ell}(\alpha) S_\ell \quad (k, \ell = 1, 2, \cdots, N;\ \alpha \in [0, 1]), \tag{3.54}$$

which in turn determines the desired matrix $B(\alpha) = [B_{k,\ell}(\alpha)]$ in $\mathbb{C}^{n,n}$ of Proposition 3.6.

Fixing the integer pair (k, ℓ), let the vector norm ω on $\mathbb{C}^{p_k \cdot p_\ell}$ be defined, for $\mathbf{x} = [x_1, x_2, \cdots, x_{p_k \cdot p_\ell}]^T$ in $\mathbb{C}^{p_k \cdot p_\ell}$, by $\omega(\mathbf{x}) := \max_{1 \le i \le p_k} \left\{ \sum_{j=1}^{p_\ell} \left| x_{(i-1)p_\ell + j} \right| \right\}$. Clearly, there is an obvious 1-1 relationship between matrices in $\mathbb{C}^{p_k, p_\ell}$ and column vectors in $\mathbb{C}^{p_k \cdot p_\ell}$, and, when the entries of $C_{k,\ell} = [\tau_{i,j}]$ in $\mathbb{C}^{p_k, p_\ell}$ are read in lexicographical (English) order to form a column vector in $\mathbb{C}^{p_k \cdot p_\ell}$, it turns out that the ω-norm of this vector coincides with the operator norm $\|C_{k,\ell}\|_\infty$ of $C_{k,\ell}$. From (3.50)-(3.53), we see that the matrices $A_{k,\ell}(0)$ and $A_{k,\ell}(1)$ in $\mathbb{C}^{p_k, p_\ell}$, when regarded as vectors in $\mathbb{C}^{p_k \cdot p_\ell}$, both lie on the sphere

$$\mathcal{B}_\mu := \{\mathbf{x} \in \mathbb{C}^{p_k \cdot p_\ell} : \omega(\mathbf{x}) = \mu\},$$

where μ is defined in (3.52) Also, it is evident that there is a parameterization $\alpha, \alpha \in [0, 1]$, of any *shortest* path on the sphere $\mathcal{B}_\mu$ from $A_{k,\ell}(0)$ to $A_{k,\ell}(1)$. But then, each point of this path on $\mathcal{B}_\mu$ determines a matrix $A_{k,\ell}(\alpha)$ in $\mathbb{C}^{p_k, p_\ell}$ which necessarily satisfies (3.52) and (3.53), and which in turn determines the matrix $B(\alpha)$ in Proposition 3.6. ☐

Recalling that the eigenvalues of a square matrix depend continuously on its entries and recalling that if $B \in \Omega_\pi(E)$, then so is $e^{i\eta} B$ for η real, we have from Proposition 3.6 the useful consequence of

Corollary 3.7. *With the hypotheses of Proposition 3.6, let ψ be an arbitrary element of $\Phi_{\pi,\infty}$. Then, the following annulus is in $\sigma(\Omega_\pi^{\psi}(E))$:*

$$\left\{ z \in \mathbb{C} : \sup_{B \in \Omega_\pi(E)} \rho(B) \le |z| \le \sup_{B \in \Omega_\pi^{\psi}(E)} \rho(B) \right\} \subset \sigma\left(\Omega_\pi^{\psi}(E)\right). \tag{3.55}$$

This brings us to

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PROPOSITION 3.8. *With the previous notations, and with the hypotheses of Proposition 3.6, we have*

$$\inf_{\psi\in\Phi_{\pi,\infty}}\left\{\sup_{B\in\Omega_\pi^\psi(E)}\rho(B)\right\}=\sup_{B\in\Omega_\pi(E)}\rho(B). \tag{3.56}$$

Proof. Let $\tau:=\sup_{B\in\Omega_\pi(E)}\rho(B)$, and let $r(\psi):=\sup_{B\in\Omega_\pi^\psi}\rho(B)$ for each $\psi\in\Phi_{\pi,\infty}$. By (3.55) of Corollary 3.7, we know that

$$\{z\in\mathbb{C}:\tau\le|z|\le r(\psi)\}\subset\sigma\left(\Omega_\pi^\psi(E)\right)\ \text{ for any }\psi\in\Phi_{\pi,\infty}. \tag{3.57}$$

Taking intersections in (3.57) over all $\psi\in\Phi_{\pi,\infty}$ and using (3.40) gives

$$\left\{z\in\mathbb{C}:\tau\le|z|\le\inf_{\psi\in\Phi_{\pi,\infty}}r(\psi)\right\}\subset\sigma\left(\Omega_\pi(E)\right). \tag{3.58}$$

On the other hand, we have from the definition of τ that

$$\sigma\left(\Omega_\pi(E)\right)\subset\Delta(0,\tau):=\{z\in\mathbb{C}:|z|\le\tau\}. \tag{3.59}$$

On combining (3.58) and (3.59), we deduce that

$$\left\{z\in\mathbb{C}:\tau\le|z|\le\inf_{\psi\in\Phi_{\pi,\infty}}r(\psi)\right\}\subset\Delta(0,\tau), \tag{3.60}$$

which clearly geometrically implies that $\tau=\inf_{\psi\in\Phi_{\pi,\infty}}r(\psi)$, and this establishes (3.56) of Proposition 3.8. Then, recalling the comments concerning (3.43), we also see that Proposition 3.8 in fact implies the truth of Theorem 2.1. ◻

4. Sharpness of the Minimal Gerschgorin Set for Block Partitioned Matrices Having Null Diagonal Blocks. In the previous sections, we considered the Spectral Conjecture and its affirmative solution, subject to the constraint (cf. (1.7)) that the matrix E in $\mathbb{C}^{n,n}$ is π-invertebrate. Although the general treatment of the sharpness of minimal Gerschgorin sets for block partitioned matrices will be treated in [10] *without* this constraint, it is worthwhile (and easy) to now treat the associated minimal Gerschgorin sets and their sharpness for this constrained case. (The proof here of sharpness is different from the sharpness proof to be given in [10].)

As in the previous sections, let π be any partition of $\mathbb{C}^n$, and assume that the matrix E in $\mathbb{C}^{n,n}$ is π-invertebrate. For any norm N-tuple ψ in $\Phi_{\pi,\infty}$ and its associated nonsingular block-diagonal matrix $S=diag(S_1,S_2,\cdots,S_N)$ in $\mathbb{C}^{n,n}$ from (3.13), define (cf. (2.2))

$$G_\pi^\psi(E):=\left\{z\in\mathbb{C}:|z|\le\|\mathcal{M}^\psi(E)\|_\infty\right\}. \tag{4.1}$$

Then, as in (3.19), we have

$$\begin{aligned}\Omega_\pi^\psi(E):=\quad&\{B=[B_{i,j}]\in\mathbb{C}^{n,n}:B_{i,i}=O\text{ and}\\&M_{i,j}^\psi(B)=M_{i,j}^\psi(E)\,(i,j=1,2,\cdots,N)\Big\}.\end{aligned} \tag{4.2}$$

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If

$$\sigma\left(\Omega_\pi^\psi(E)\right) := \bigcup_{B\in\Omega_\pi^\psi} \sigma(B),$$

we easily obtain, as a consequence of Proposition 3.4, the result of

PROPOSITION 4.1. *Let π be a partition of $\mathbb{C}^n$ and let E, in $\mathbb{C}^{n,n}$, be π-invertebrate. Then for any $\psi \in \Phi_{\pi,\infty}$,*

$$\sigma\left(\Omega_\pi^\psi(E)\right) \subset G_\pi^\psi(E). \tag{4.3}$$

We next set

$$G_\pi(E) := \bigcap_{\psi\in\Phi_{\pi,\infty}} G_\pi^\psi(E), \tag{4.4}$$

and we *define* $G_\pi(E)$ of (4.4) to be the *minimal Gerschgorin set for* E, with respect to the partition π, where E in $\mathbb{C}^{n,n}$ is assumed to be π-invertebrate. (We remark that in the case when the partition π is such that $\dim W_j = 1$ $(j = 1, 2, \cdots, n)$, the definition of (4.4) reduces to the original minimal Gerschgorin set of [8] for the matrix $E = [e_{i,j}]$ with $e_{i,i} = 0$ for $i = 1, 2, \cdots, n$.) Of course, with (3.9) of Corollary 3.2, it follows that the minimal Gerschgorin set of (4.4) can also be expressed as

$$G_\pi(E) = \left\{ z \in \mathbb{C} : |z| \leq \inf_{\psi\in\Phi_{\pi,\infty}} \rho\left(\mathcal{M}^\psi(E)\right) \right\}. \tag{4.5}$$

Hence, $G_\pi(E)$ is a closed disk, with center 0, in the complex plane.

With $\sigma\left(\Omega_\pi(E)\right)$ as defined in (3.45) and on taking intersections over all $\psi \in \Phi_{\pi,\infty}$ in (4.3), we obtain the following analogue of Proposition 4.1.

PROPOSITION 4.2. *Let π be a partition of $\mathbb{C}^n$ and let E, in $\mathbb{C}^{n,n}$, be π-invertebrate. Then,*

$$\sigma\left(\Omega_\pi(E)\right) \subset G_\pi(E). \tag{4.6}$$

Now, we have the question of whether the inclusion in (4.6) of Proposition 4.2 is sharp. But, with our previous results, this is easily answered in Theorem 4.3 below. We remark that since $G_\pi(E)$ of (4.5) is a closed disk with center 0, its *boundary*, denoted by $\partial G_\pi(E)$, is thus given by

$$\partial G_\pi(E) := \left\{ z \in \mathbb{C} : |z| = \inf_{\psi\in\Phi_{\pi,\infty}} \rho\left(\mathcal{M}^\psi(E)\right) \right\}. \tag{4.7}$$

THEOREM 4.3. *Let π be a partition of $\mathbb{C}^n$ and let E, in $\mathbb{C}^{n,n}$, be π invertebrate. Then,*

$$\partial G_\pi(E) \subset \sigma\left(\Omega_\pi(E)\right) \subset G_\pi(E). \tag{4.8}$$

Proof. Clearly, $\partial G_\pi(E)$ of (4.7) can be equivalently expressed (from (2.5) of Theorem 2.1, (3.16), and (3.42)) as

$$\partial G_\pi(E) = \left\{ z \in \mathbb{C} : |z| = \sup_{B \in \Omega_\pi(E)} \rho(B) \right\}. \tag{4.9}$$

But since the compactness argument in the proof of Proposition 4 shows that there is a $\tilde{B}$ in $\Omega_\pi(E)$ with $\rho(\tilde{B}) = \sup_{B \in \Omega_\pi(E)} \rho(B)$ and since $e^{i\theta}\tilde{B}$ is in $\Omega_\pi(E)$ for any real θ, then each point of the circle $|z| = \rho(\tilde{B}) = \sup_{B \in \Omega_\pi(E)} \rho(B)$ is an eigenvalue of some matrix in $\Omega_\pi(E)$; whence, $\partial G_\pi(E) \subset \sigma\left(\Omega_\pi(E)\right)$. $\square$

5. Concluding Remarks. There are two items, concerning Theorem 2.1 and its proof, which are worthy of mention. First, it is possible that the hypothesis (1.7), namely, that $E = [E_{i,j}]$ in $\mathbb{C}^{n,n}$ is π-invertebrate, is *not* really essential to the truth of Theorem 2.1, though our proof of Theorem 2.1 makes strong use of this hypothesis. Second, for those raised on finite-dimensional norm constructions, it would seem that there might be a more direct proof of Theorem 2.1, based on constructions of an appropriate equilibrated convex body in W_i to define a unit ball and norm in W_i, for each $i = 1, 2, \cdots, N$. This can indeed be done, but we were not able, by this approach, to obtain the intriguing string of equalities in (2.5) of Theorem 2.1. This is why we opted here for the proof given in Section 3.

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