Electronic Transactions on Numerical Analysis.
Volume 30, pp. 128-143, 2008.
Copyright © 2008, Kent State University.
ISSN 1068-9613.

ETNA
Kent State University
http://etna.math.kent.edu

THE DYNAMICAL MOTION OF THE ZEROS OF THE PARTIAL SUMS OF e^z, AND ITS RELATIONSHIP TO DISCREPANCY THEORY*

RICHARD S. VARGA[†], AMOS J. CARPENTER[‡], AND BRYAN W. LEWIS[§]

Dedicated to Edward B. Saff on his 64th birthday, January 2, 2008.

Abstract. With $s_n(z) := \sum_{k=0}^{n} z^k/k!$ denoting the n-th partial sum of e^z, let its zeros be denoted by $\{z_{k,n}\}_{k=1}^{n}$ for any positive integer n. If θ_1 and θ_2 are any angles with $0 < \theta_1 < \theta_2 < 2\pi$, let Z_{θ_1,θ_2} be the associated sector, in the z-plane, defined by

$$Z_{\theta_1,\theta_2} := \{z \in \mathbb{C} : \theta_1 \le \arg z \le \theta_2\}.$$

If $\#\left(\{z_{k,n}\}_{k=1}^{n} \bigcap Z_{\theta_1,\theta_2}\right)$ represents the number of zeros of $s_n(z)$ in the sector Z_{θ_1,θ_2}, then Szegő showed in 1924 that

$$\lim_{n\to\infty} \frac{\#\left(\{z_{k,n}\}_{k=1}^{n} \bigcap Z_{\theta_1,\theta_2}\right)}{n} = \frac{\phi_2 - \phi_1}{2\pi},$$

where ϕ_1 and ϕ_2 are defined in the text. The associated *discrepancy function* is defined by

$$\mathrm{disc}_n(\theta_1,\theta_2) := \#\left(\{z_{k,n}\}_{k=1}^{n} \bigcap Z_{\theta_1,\theta_2}\right) - n\left(\frac{\phi_2-\phi_1}{2\pi}\right).$$

One of our new results shows, for any θ_1 with $0 < \theta_1 < \pi$, that

$$\mathrm{disc}_n(\theta_1, 2\pi - \theta_1) \sim K \log n, \ \text{ as } \ n \to \infty,$$

where K is a positive constant, depending only on θ_1. Also new in this paper is a study of the *cyclical nature* of $\mathrm{disc}_n(\theta_1,\theta_2)$, as a function of n, when $0 < \theta_1 < \pi$ and $\theta_2 = 2\pi - \theta_1$. An upper bound for the approximate cycle length, in this case, is determined in terms of ϕ_1. All this can be viewed in our *Interactive Supplement*, which shows the dynamical motion of the (normalized) zeros of the partial sums of e^z and their associated discrepancies.

Key words. partial sums of e^z, Szegő curve, discrepancy function

AMS subject classifications. 30C15, 30E15

1. Introduction.

Let $s_n(z) := \sum_{k=0}^{n} z^k/k!$ denote the n-th partial sum of e^z for each positive integer n, and denote the zeros of $s_n(z)$ by $\{z_{k,n}\}_{k=1}^{n}$. Let Z_{θ_1,θ_2} be the sector in the complex plane $\mathbb{C}$, defined by

$$Z_{\theta_1,\theta_2} := \{z \in \mathbb{C} : \theta_1 \le \arg z \le \theta_2\}, \tag{1.1}$$

for angles θ_1 and θ_2 such that $0 < \theta_1 < \theta_2 < 2\pi$. Let $\#\left(\{z_{k,n}\}_{k=1}^{n} \bigcap Z_{\theta_1,\theta_2}\right)$ represent the number of zeros of $s_n(z)$ that lie in the sector Z_{θ_1,θ_2}. A beautiful result of Szegő [7] states that

$$\lim_{n\to\infty} \frac{\#\left(\{z_{k,n}\}_{k=1}^{n} \bigcap Z_{\theta_1,\theta_2}\right)}{n} = \frac{\phi_2 - \phi_1}{2\pi}, \tag{1.2}$$

where if D_∞ is the closed curve (called the Szegő curve) in the unit disk which is given by

$$D_\infty := \{z \in \mathbb{C} : |ze^{1-z}| = 1 \text{ for } |z| \le 1\}, \tag{1.3}$$

*Received April 12, 2007. Accepted for publication January 31, 2008. Published online on May 29, 2008. Recommended by D. Lubinsky.

†Institute for Computational Mathematics, Kent State University, Kent, OH 44242, U.S.A. E-mail: varga@math.kent.edu.

‡Department of Mathematics and Actuarial Science, Butler University, Indianapolis, IN 46208, U.S.A. E-mail: acarpent@butler.edu.

§Rocketcalc, 100 West Crain Avenue, Kent, OH 44240, U.S.A. E-mail: blewis@rocketcalc.com.

ETNA
Kent State University
http://etna.math.kent.edu

then there are unique positive values $r_\infty(\theta_1)$ and $r_\infty(\theta_2)$, in $(0,1)$, such that $r_\infty(\theta_1)e^{i\theta_1}$ and $r_\infty(\theta_2)e^{i\theta_2}$ are points of D_∞ of (1.3). The angles ϕ_j in (1.2) are then defined by

$$\phi_j := \theta_j - r_\infty(\theta_j)\sin\theta_j, \qquad j = 1,2. \tag{1.4}$$

The *discrepancy function* $\mathrm{disc}_n(\theta_1,\theta_2)$, for the zeros in Z_{θ_1,θ_2}, is defined as

$$\mathrm{disc}_n(\theta_1,\theta_2) := \#\left(\{z_{k,n}\}_{k=1}^n \bigcap Z_{\theta_1,\theta_2}\right) - n\left(\frac{\phi_2-\phi_1}{2\pi}\right). \tag{1.5}$$

It follows from (1.2) that the function $\mathrm{disc}_n(\theta_1,\theta_2)$ would behave, at worst, like $o(n)$, as $n\to\infty$. Our first aim in this paper is to show in Theorem 2.5 the sharper result that

$$\mathrm{disc}_n(\theta_1,\theta_2) \sim K\log n, \quad \text{for all } n \text{ sufficiently large}, \tag{1.6}$$

where K is a positive constant depending on θ_1 and θ_2. It is of interest to note that Szegő [7] showed that the associated discrepancy function, but now in the w-plane under the mapping

$$w = ze^{1-z},$$

is bounded as a function of n.

Our second aim is to closely study the cyclical nature of the sequence $\{\mathrm{disc}_n(\theta_1,\theta_2)\}_{n=1}^\infty$. Specifically, for $0<\theta_1<\pi$ and $\theta_2 = 2\pi-\theta_1$, the approximate cyclic length, of what we call the *short-term pattern*, is determined as a function of ϕ_1. (Long-term patterns are also described in Section 3.)

Our third aim in this paper is to illustrate the *dynamical motion* of the zeros of $s_n(nz)$, as n varies, with our *Interactive Supplement* accompanying this paper. This allows the reader to *input* θ_1, in the range $\pi/4\le\theta_1\le 3\pi/4$, where $\theta_2 := 2\pi-\theta_1$, and to *input* n, the degree of $s_n(nz)$, in the range $1\le n\le 200$. The reader's computer then graphs the n zeros of $s_n(nz)$ in the z-plane. On increasing n, one sees the actual "fanning out" of the zeros of $s_n(nz)$, into the upper and lower half-planes of the z-plane. In addition, the discrepancy function, $\mathrm{disc}_n(\theta_1,\theta_2)$, is then displayed to four decimal digits, at each n-th step. This calculation is based on the stored zeros of all polynomials $\{s_n(nz)\}_{n=1}^{200}$, whose zeros were *all* determined to 200 decimal digits.

2. Background and statement of results. To study the behavior of the zeros of the partial sums $s_n(z)$ of e^z, it is convenient to study instead the *normalized* partial sums $s_n(nz) := \sum_{k=1}^n (nz)^k/k!$, whose zeros, henceforth denoted by $\{z_{k,n}\}_{k=1}^n$, have the same arguments as the zeros of $s_n(z)$. This leaves $\mathrm{disc}_n(\theta_1,\theta_2)$ in (1.5) unchanged. An application of the Eneström-Kakeya Theorem (see [4, p. 137, Exercise 2], or [6, p.88, Problem 22]) shows that all zeros $\{z_{k,n}\}_{k=1}^n$ of $s_n(nz)$ lie in the unit disk $\Delta := \{z\in\mathbb{C} : |z|\le 1\}$ for any $n\ge 1$. From compactness considerations, there are necessarily accumulation points in Δ for $\cup_{n=1}^\infty\{z_{k,n}\}_{k=1}^n$, and Szegő [7] established that each such accumulation point must lie on the curve D_∞ of (1.3), and, conversely, that each point of D_∞ is an accumulation point of these zeros of $s_n(nz)$. Buckholtz [1] later proved that all zeros of all $s_n(nz)$ lie *outside* of D_∞ for any $n\ge 1$, and that

$$\mathrm{dist}[\{z_{k,n}\}_{k=1}^n; D_\infty] \le \frac{2e}{\sqrt{n}}, \quad \text{for any } n\ge 1, \tag{2.1}$$

where

$$\mathrm{dist}[\{z_{k,n}\}_{k=1}^n; D_\infty] := \max_{1\le k\le n}(\mathrm{dist}[z_{k,n}; D_\infty]),$$

ETNA
Kent State University
http://etna.math.kent.edu

and where $\mathrm{dist}[z_{k,n}; D_\infty] := \min_{z \in D_\infty} |z_{k,n} - z|$. It was shown in [2] that the exponent of $1/2$ for n in (2.1) is the best possible, and that the constant $2e$ can be reduced in (2.1) to $0.636\,657$.[1]

In [2], the following curve was defined for each positive integer n:

$$D_n := \left\{ z \in \mathbb{C} : \begin{array}{l} |ze^{1-z}|^n = \tau_n \sqrt{2\pi n} \left|\frac{1-z}{z}\right|, \\ |z| \leq 1, \text{ and} \\ |\arg z| \geq \cos^{-1}\left(\frac{n-2}{n}\right) \end{array} \right\}, \tag{2.2}$$

where τ_n, defined by

$$\tau_n := \frac{n!}{n^n e^{-n} \sqrt{2\pi n}},$$

is the exact error in Stirling's formula. For calculations of τ_n when n is very large, the following asymptotic series (cf. Henrici [3]) for τ_n can be useful:

$$\tau_n \cong 1 + \frac{n^{-1}}{12} + \frac{n^{-2}}{288} - \frac{139n^{-3}}{51840} + \cdots, \quad \text{as } n \to \infty,$$

and

$$\log \tau_n \cong \frac{n^{-1}}{12} - \frac{n^{-3}}{360} + \frac{n^{-5}}{1260} - \frac{n^{-7}}{1680} + \frac{n^{-9}}{1188} - \cdots, \quad \text{as } n \to \infty.$$

For any fixed δ with $0 < \delta \leq 1$, each D_n curve gives a much better approximation to where the zeros of $s_n(nz)$ lie, in that from [2, Theorem 4],

$$\mathrm{dist}[\{z_{k,n}\}_{k=1}^n \backslash C_\delta; D_n] = O\left(\frac{1}{n^2}\right), \qquad \text{as } n \to \infty, \tag{2.3}$$

where $C_\delta := \{z \in \mathbb{C} : |z - 1| < \delta\}$. The exponent of 2 for n in (2.3) was shown in [2] to be best possible.

As defined in (2.2), the curve D_n is not a closed curve, so we make the following modifications of (2.2). First, as will be explained in the proof of Proposition 2.1 in Section 5, the curve D_n of (2.2) can be extended, for each $n \geq 1$, to the boundary of Δ in two unique points, $e^{i\lambda_n}$ and $e^{-i\lambda_n}$, where $0 < \lambda_n < \pi$ for each $n \geq 1$. Then, the circular arc $\left\{e^{i\sigma} : -\lambda_n \leq \sigma \leq +\lambda_n\right\}$ is annexed to the extended D_n curve, thereby producing the following closed curve $\tilde{D}_n$ in Δ:

$$\tilde{D}_n := \left\{ \begin{array}{l} z \in \mathbb{C} : |ze^{1-z}|^n = \tau_n \sqrt{2\pi n} \left|\frac{1-z}{z}\right|, \\ |z| \leq 1, \text{ and } \lambda_n \leq \arg z \leq 2\pi - \lambda_n \end{array} \right\} \bigcup \left\{e^{i\sigma} : -\lambda_n \leq \sigma \leq +\lambda_n\right\}. \tag{2.4}$$

The curves $\tilde{D}_n$, for $n = 1, 5$, and ∞, are given below in Figure 2.1.

We remark that it can be shown that λ_n can be expressed as the convergent series

$$\lambda_n = \frac{1.704\,097}{n^{1/2}} + \frac{0.280\,778}{n^{3/2}} + O\left(\frac{1}{n^{5/2}}\right), \text{ as } n \to \infty. \tag{2.5}$$

With the above definitions, we state the following proposition, whose proof is sketched in Section 5.

PROPOSITION 2.1. *For each positive integer n, the following are valid:*

[1] Except for discrepancy numbers $\mathrm{disc}_n(\theta_1, \theta_2)$, which are given to four decimal digits in the *Interactive Supplement*, we truncate, in the text, the displayed fractional part of noninteger numbers to six decimal digits.

ETNA
Kent State University
http://etna.math.kent.edu

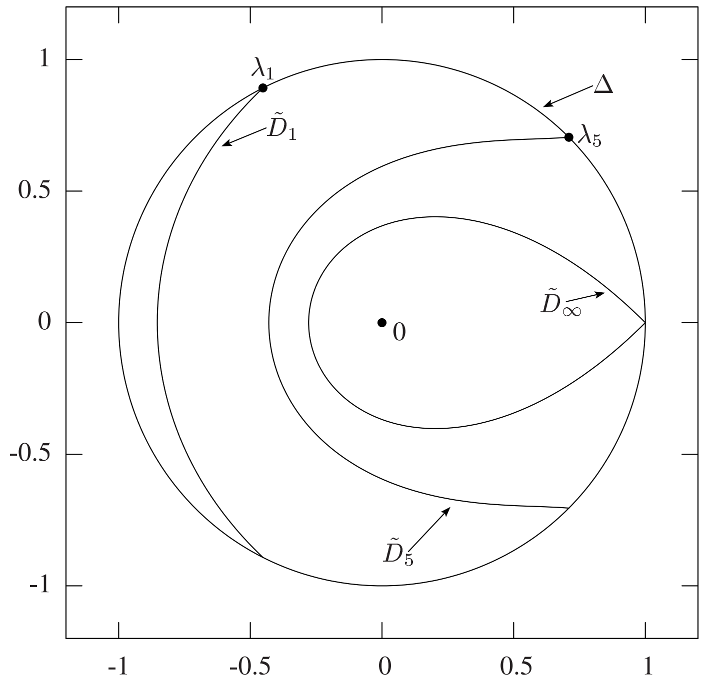

FIG. 2.1. *The curve* $\tilde{D}_n$*, for* $n = 1, 5,$ *and* ∞.

i) *The curve* $\tilde{D}_n$ *is a simple closed curve in* Δ *which is star-shaped with respect to* $z = 0$*;*

ii) *For each* θ *with* $\lambda_n \le \theta \le 2\pi - \lambda_n$*, there is a unique number* $r_n(\theta)$*, with* $0 < r_n(\theta) \le 1$*, such that* $z = r_n(\theta)e^{i\theta}$ *is a point of* $\tilde{D}_n$*, and satisfies*

$$\frac{z(ze^{1-z})^n}{\tau_n\sqrt{2\pi n}(1-z)} = e^{i\Psi_n(\theta)}, \tag{2.6}$$

where $\Psi_n(\theta)$ *is defined, on the interval* $[\lambda_n, 2\pi - \lambda_n]$*, by*

$$\Psi_n(\theta) := n\left[\theta - r_n(\theta)\sin\theta\right] + \theta + \tan^{-1}\left(\frac{r_n(\theta)\sin\theta}{1 - r_n(\theta)\cos\theta}\right); \tag{2.7}$$

iii) $0 < \Psi_n(\lambda_n) < 2\pi$ *for each* $n \ge 1$*;*

iv) *For each fixed* $n \ge 1$*,* $\Psi_n(\theta)$ *is a strictly increasing function of* θ *on* $[\lambda_n, 2\pi - \lambda_n]$*, and for each integer* k *with* $1 \le k \le n$*, there is a unique point* $\hat{z}_{k,n} = \hat{r}_{k,n}e^{i\hat{\theta}_{k,n}}$ *on the curve* $\tilde{D}_n$*, with* $\lambda_n < \hat{\theta}_{k,n} < 2\pi - \lambda_n$*, such that*

$$\Psi_n(\hat{\theta}_{k,n}) = 2\pi k; \tag{2.8}$$

v) *If* $\{z_{k,n}\}_{k=1}^n$ *denotes the (exact) zeros of* $s_n(nz)$*, then, for any* $z_{k,n}$ *not in* $C_\delta := \{z \in \mathbb{C} : |z-1| < \delta\}$*, where* δ *is fixed with* $0 < \delta \le 1$*,*

$$|z_{k,n} - \hat{z}_{k,n}| = O\left(\frac{1}{n^2}\right), \quad as \;\; n \to \infty, \tag{2.9}$$

where the constant, implicit in $O(\frac{1}{n^2})$*, depends only on* δ.

Next, it would appear that to precisely determine $\#\left(\{z_{k,n}\}_{k=1}^{n} \bigcap Z_{\theta_1,\theta_2}\right)$, one would need to know many very precise zeros $\{z_{k,n}\}_{k=1}^{n}$ of $s_n(nz)$. This, for n large, would be a daunting task! However, we give a very accurate estimate of $\#\left(\{z_{k,n}\}_{k=1}^{n} \bigcap Z_{\theta_1,\theta_2}\right)$, which avoids finding any zeros of $s_n(nz)$. This estimate is stated below in Proposition 2.2, after some preliminary definitions are introduced.

Fixing any θ_1 with $0 < \theta_1 < \pi$, consider any positive integer n such that $\lambda_n \leq \theta_1$, where we see from (2.5) that this inequality holds for all n sufficiently large. Then, from part **ii)** of Proposition 2.1, there is a unique number $r_n(\theta_1)$, with $0 < r_n(\theta_1) < 1$, such that $z := r_n(\theta_1)\, e^{i\theta_1}$ is a point of $\tilde{D}_n$ which satisfies (2.6). With $\Psi_n(\theta_1)$ defined in (2.7), and, with the following notation[2]:

$$\langle\langle x \rangle\rangle := \text{greatest integer } < x, \text{ for any real } x,$$

it follows, from the strictly increasing nature of Ψ_n from part **iv)** of Proposition 2.1, that

$$\langle\langle \Psi_n(\theta_1)/2\pi \rangle\rangle \geq 0. \tag{2.10}$$

This brings us to the statement of

PROPOSITION 2.2. *Given any θ_1 with $0 < \theta_1 < \pi$, the number of zeros of $s_n(nz)$ in the sector $-\theta_1 \leq \arg z \leq +\theta_1$ is approximately*

$$2\langle\langle \Psi_n(\theta_1)/2\pi \rangle\rangle, \tag{2.11}$$

so that, by symmetry,

$$\#\left(\{z_{k,n}\}_{k=1}^{n} \bigcap Z_{\theta_1,2\pi-\theta_1}\right) \doteq n - 2\langle\langle \Psi_n(\theta_1)/2\pi \rangle\rangle. \tag{2.12}$$

The proof of Proposition 2.2 is given in Section 5.

To illustrate now the result of (2.12) of Proposition 2.2, suppose that $\theta_1 = \pi/2$, and we choose $n = 98$, and $n = 99$. Then, from part **ii)** of Proposition 2.1, $r_n(\pi/2)$ and $\Psi_n(\pi/2)$ are numerically determined to be

$$\begin{cases} r_{98}(\pi/2) = 0.384\,136, & \text{and } \Psi_{98}(\pi/2)/2\pi = 18.816\,919, \text{ and} \\ r_{99}(\pi/2) = 0.383\,989, & \text{and } \Psi_{99}(\pi/2)/2\pi = 19.008\,074. \end{cases} \tag{2.13}$$

From (2.12), this gives

$$\begin{cases} \#(\{z_{k,98}\}_{k=1}^{98} \bigcap Z_{\pi/2,3\pi/2}) & \doteq & 98 - 2\langle\langle \Psi_{98}(\pi/2)/2\pi \rangle\rangle = 98 - 2(18) = 62, \text{ and} \\ \#(\{z_{k,99}\}_{k=1}^{99} \bigcap Z_{\pi/2,3\pi/2}) & \doteq & 99 - 2\langle\langle \Psi_{99}(\pi/2)/2\pi \rangle\rangle = 99 - 2(19) = 61. \end{cases} \tag{2.14}$$

Because we have all the zeros of $\{s_n(nz)\}_{n=1}^{200}$ to an accuracy of 200 decimal digits, it turns out that the final numbers of (2.14), i.e., 62 and 61, are *exactly* the number of the zeros of $s_{98}(98z)$ and $s_{99}(99z)$, respectively, in the sector $\pi/2 \leq \theta_1 \leq 3\pi/2$, without having directly determined any zeros of $s_n(nz)$. In addition, in the symmetric sector case of $\theta_1 = \dfrac{\pi}{2}$ and $\theta_2 = \dfrac{3\pi}{2}$, it follows from (1.4) that $\phi_1 = \dfrac{\pi}{2} - \dfrac{1}{e}$ and $\phi_2 = \dfrac{3\pi}{2} + \dfrac{1}{e}$, so that

$$\frac{\phi_2 - \phi_1}{2\pi} = \frac{1}{2} + \frac{1}{e\pi} = 0.617\,099,$$

[2]Note that this is not the *floor function* $\lfloor x \rfloor$, which is defined as the greatest integer $\leq x$.

ETNA
Kent State University
http://etna.math.kent.edu

which gives us from (1.5) and (2.14) that

$$\text{disc}_{98}\left(\frac{\pi}{2},\frac{3\pi}{2}\right) = +1.524\,233, \text{ and } \text{disc}_{99}\left(\frac{\pi}{2},\frac{3\pi}{2}\right) = -0.092\,866. \tag{2.15}$$

The two numerical discrepancies of (2.15) *agree* with the rounded numbers, in these cases, of the *Interactive Supplement* of this paper.

We remark that using the expression in (2.11) to estimate the number of zeros of $s_n(nz)$, in the sector $-\theta_1 \le \arg z \le +\theta_1$, is generally very accurate, but it is evident that this estimate can be *faulty* when $\Psi_n(\theta_1)/2\pi$ is exceedingly close to an integer, and this can change the estimate in (2.12) by ± 2. This will be considered in more detail in Section 5.

Our next result gives an equivalent representation for $\text{disc}_n(\theta_1, 2\pi - \theta_1)$, where its proof is given in Section 5.

PROPOSITION 2.3. *Given any θ_1 with $0 < \theta_1 < \pi$, assume that $\theta_2 = 2\pi - \theta_1$, the symmetric case. Then, for any positive integer n,*

$$\begin{aligned}\text{disc}_n(\theta_1,\theta_2) = &\left\{\frac{\Psi_n(\theta_1)}{\pi} - 2\langle\langle\frac{\Psi_n(\theta_1)}{2\pi}\rangle\rangle\right\}\\ &-\left\{\frac{\theta_1}{\pi} + \frac{1}{\pi}\tan^{-1}\left(\frac{r_n(\theta_1)\sin\theta_1}{1-r_n(\theta_1)\cos\theta_1}\right)\right\}\\ &+\left\{\frac{n\sin\theta_1}{\pi}\left[r_n(\theta_1) - r_\infty(\theta_1)\right]\right\}.\end{aligned} \tag{2.16}$$

We remark that each of the three quantities in braces, in (2.16), can be seen to be positive. For example, the first term in braces in (2.16) can be seen, using (2.10), to satisfy

$$0 < \left\{\frac{\Psi_n(\theta_1)}{\pi} - 2\langle\langle\frac{\Psi_n(\theta_1)}{2\pi}\rangle\rangle\right\} \le 2, \text{ any } \theta_1 \text{ with } 0<\theta_1<\pi, \text{ any } n \ge 1. \tag{2.17}$$

Next, we have the result of Proposition 2.4, whose proof is again given in Section 5.

PROPOSITION 2.4. *Given any fixed θ_1 with $0 < \theta_1 < \pi$,*

$$\frac{n\sin\theta_1}{\pi}\left[r_n(\theta_1) - r_\infty(\theta_1)\right] \sim \frac{\log(2\pi n)}{2\pi}\left(\frac{r_\infty(\theta_1)\sin\theta_1}{1-r_\infty(\theta_1)\cos\theta_1}\right), \text{ as } n\to\infty. \tag{2.18}$$

Then, because of the properties of the terms in braces in (2.16), we have

THEOREM 2.5. *Given any θ_1 with $0 < \theta_1 < \pi$, assume $\theta_2 = 2\pi - \theta_1$. Then,*

$$\text{disc}_n(\theta_1,\theta_2) \sim K\log n, \quad as \quad n\to\infty, \tag{2.19}$$

where $K > 0$ is dependent only on θ_1.

To numerically illustrate here the result of (2.19) we have, in the case $\theta_1 = \frac{\pi}{2}$, that, as shown in Section 5,

$$\frac{n\sin\theta_1}{\pi}\left[r_n\left(\frac{\pi}{2}\right) - r_\infty\left(\frac{\pi}{2}\right)\right] = \frac{n}{\pi}\left[r_n\left(\frac{\pi}{2}\right) - \frac{1}{e}\right] \sim \frac{\log(2\pi n)}{2e\pi}, \tag{2.20}$$

as $n\to\infty$. This means that the last term in braces in (2.16) tends slowly to $+\infty$ as $n\to\infty$, while the other two terms in braces in (2.16) can be seen to be bounded. More concretely, we have that for $\theta_1 = \frac{\pi}{2}$,

$$\frac{n}{\pi}\left[r_n\left(\frac{\pi}{2}\right) - \frac{1}{e}\right] = 12.356\,575, \text{ for } n = 10^{90}. \tag{2.21}$$

ETNA
Kent State University
http://etna.math.kent.edu

3. The interesting oscillations of $disc_n(\theta_1, \theta_2)$ in the symmetric case. One of the most intriging results, from this research, is that actual calculations of $\{\mathrm{disc}_n\,(\theta_1, \theta_2)\}_{n=1}^{\infty}$, in the symmetric case, produce *patterns* of two distinct types, which likely could not have been conjectured purely from theoretical results. For the symmetric case, these patterns can be classified as

$$\left\{ \begin{array}{l} \textit{short-term} \text{ patterns of increases of the } \mathrm{disc}_n(\theta_1, \theta_2), \text{ and} \\ \textit{long-term} \text{ patterns of increases or decreases of the } \mathrm{disc}_n(\theta_1, \theta_2). \end{array} \right. \tag{3.1}$$

Both of these patterns can be immediately seen from our *Interactive Supplement*, which was written in Java by our third author. On setting $\theta_1 = \pi/2$, one sees, at the bottom of the screen, a *short-term pattern* of a sequence of four or five successive increases in $\mathrm{disc}_n(\pi/2, 3\pi/2)$, where the increases at each step are approximately 0.3829, followed by a *long-term pattern*, in which the short-term patterns are successively slightly increasing or slightly decreasing from step to step. This can also be seen to be the case in other choices of θ_1, as well. We remark that these short-term and long-term patterns are valid only for symmetric sectors.

Our next theoretical result here has to do with the short-term patterns.

THEOREM 3.1. *Given any θ_1 with $0 < \theta_1 < \pi$, assume that $\theta_2 = 2\pi - \theta_1$, the symmetric sector case, and let ϕ_1 be determined from (1.4). Then, the length $\hat{\ell}$ of each short-term pattern is at most*

$$\hat{\ell} := 1 + \left\lfloor \frac{2\pi}{\phi_1} \right\rfloor, \tag{3.2}$$

for all n sufficiently large, where the floor function $\lfloor x \rfloor$ is defined as the greatest integer $\leq x$.

As an example of the result of (3.2), consider the case of $\theta_1 = \frac{\pi}{4}$ and $\theta_2 = \frac{7\pi}{4}$. It follows from (1.4) that $r_\infty(\pi/4) = 0.538\,278$, and $\phi_1 = 0.404\,778$. In this case, $\frac{2\pi}{\phi_1} = 15.522\,544$, so that from (3.2),

$$\hat{\ell} = 1 + \left\lfloor \frac{2\pi}{\phi_1} \right\rfloor = 16.$$

In this case, the short term pattern consists of at most 16 steps. This can be seen, from our *Interactive Supplement*, with $\theta_1 = \frac{\pi}{4}$, to be correct. Similarly, for $\theta_1 = \frac{\pi}{2}$ and $\theta_2 = \frac{3\pi}{2}$, $r_\infty(\pi/2) = \frac{1}{e} = 0.367\,879$, and $\phi_1 = 1.202\,916$, so that $\frac{2\pi}{\phi_1} = 5.223\,291$, so that

$$\hat{\ell} = 1 + \left\lfloor \frac{2\pi}{\phi_1} \right\rfloor = 6.$$

In this case, the short term pattern consists of at most 6 steps. Again, this can be verified from our *Interactive Supplement*.

4. Extensions. With θ_1 satisfying $0 < \theta_1 < \pi$, and with $\theta_2 := 2\pi - \theta_1$, we have considered only *symmetric sectors* in the previous sections, and we now extend these results to general sectors Z_{θ_1,θ_2}, of (1.1), where $0 < \theta_1 < \theta_2 < 2\pi$. Note however that since the zeros, of the real polynomial $s_n(nz)$, occur in conjugate complex pairs, we may assume, without loss of generality, that $0 < \theta_1 < \pi$. Then, θ_2 either satisfies $\pi \leq \theta_2 < 2\pi$, or $0 < \theta_1 < \theta_2 \leq \pi$.

With $\tilde{\theta}_1 := 2\pi - \theta_1$ and $\tilde{\theta}_2 := 2\pi - \theta_2$, we consider the following three cases.

Case 1. $0 < \theta_1 < \pi$, $\pi \leq \theta_2 < 2\pi$, and $0 < \theta_1 < \tilde{\theta}_2 \leq \pi$, which is shown in Figure 4.1. It is then geometrically evident, from Figure 4.1, that

$$\#\left(\{z_{k,n}\}_{k=1}^{n} \bigcap Z_{\theta_1,\theta_2}\right) = \frac{\#\left(\{z_{k,n}\}_{k=1}^{n} \bigcap Z_{\theta_1,\tilde{\theta}_1}\right) + \#\left(\{z_{k,n}\}_{k=1}^{n} \bigcap Z_{\tilde{\theta}_2,\theta_2}\right)}{2},$$

ETNA
Kent State University
http://etna.math.kent.edu

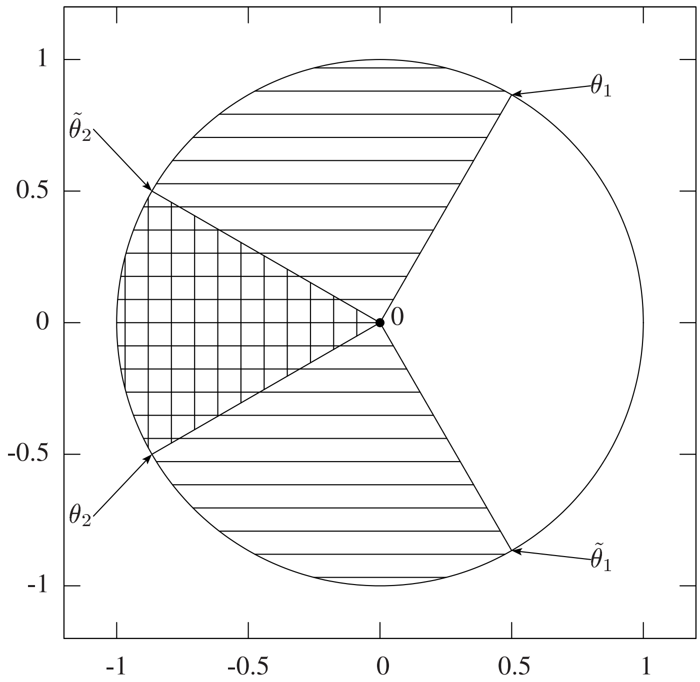

FIG. 4.1. *Case 1:* $0 < \theta_1 < \pi$, $\pi \le \theta_2 < 2\pi$, *and* $0 < \theta_1 < \tilde{\theta}_2 \le \pi$.

and, on using the definition of (1.5), it can be verified that

$$\mathrm{disc}_n(\theta_1, \theta_2) = \left[\mathrm{disc}_n(\theta_1, \tilde{\theta}_1) + \mathrm{disc}_n(\tilde{\theta}_2, \theta_2)\right]/2. \tag{4.1}$$

Case 2. $0 < \theta_1 < \pi$, $\pi < \theta_2 < 2\pi$, and $0 < \tilde{\theta}_2 < \theta_1 < \pi$, which is shown in Figure 4.2. Similarly, we obtain

$$\mathrm{disc}_n(\theta_1, \theta_2) = \left[\mathrm{disc}_n(\tilde{\theta}_2, \theta_2) + \mathrm{disc}_n(\theta_1, \tilde{\theta}_1)\right]/2, \tag{4.2}$$

The final case to be considered is

Case 3. $0 < \theta_1 < \theta_2 \le \pi$, which is shown in Figure 4.3, and it similarly follows that

$$\mathrm{disc}_n(\theta_1, \theta_2) = \left[\mathrm{disc}_n(\theta_1, \tilde{\theta}_1) - \mathrm{disc}_n(\theta_2, \tilde{\theta}_2)\right]/2, \tag{4.3}$$

Thus, we have shown how the general function $\mathrm{disc}_n(\theta_1, \theta_2)$ can be expressed in terms of symmetric sectors. This will be used below to extend the result of Theorem 2.5, on symmetric sectors, to general sectors. Its proof is given in Section 5.

THEOREM 4.1. *Given any angles* θ_1 *and* θ_2 *with* $0 < \theta_1 < \theta_2 < 2\pi$, *then,*

$$\mathrm{disc}_n(\theta_1, \theta_2) \sim K \log n, \quad as \ \ n \to \infty, \tag{4.4}$$

where $K > 0$ *is dependent only on* θ_1 *and* θ_2.

ETNA
Kent State University
http://etna.math.kent.edu

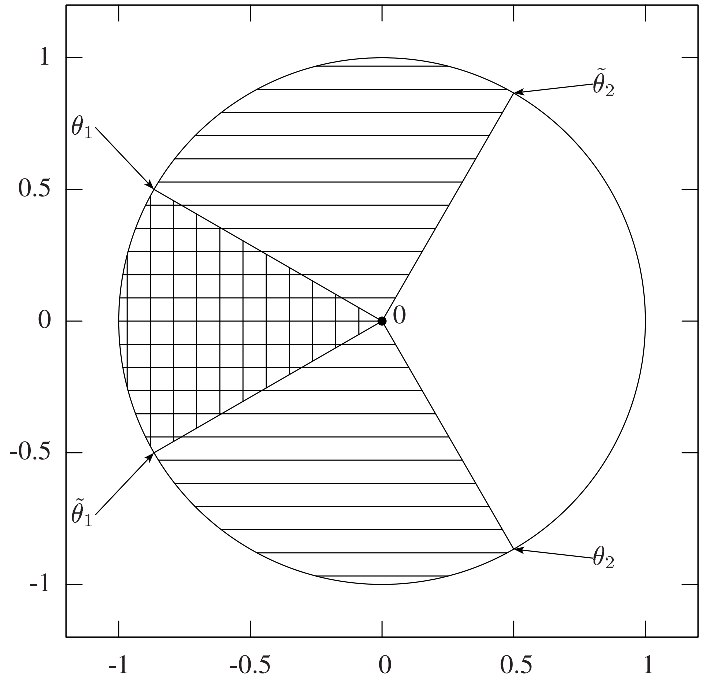

FIG. 4.2. *Case 2:* $0 < \theta_1 < \pi$, $\pi < \theta_2 < 2\pi$, *and* $0 < \tilde{\theta}_2 < \theta_1 < \pi$.

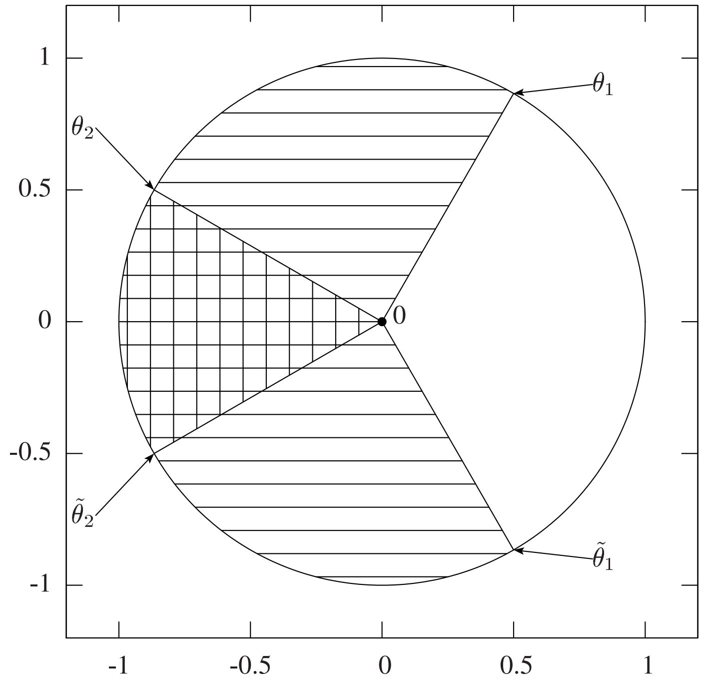

FIG. 4.3. *Case 3:* $0 < \theta_1 < \theta_2 \leq \pi$.

ETNA
Kent State University
http://etna.math.kent.edu

5. Proofs. *Proof of Proposition 2.1.* For each positive integer n, let λ_n be the largest positive number such that $e^{i\lambda_n}$ is a point of $\tilde{D}_n$, i.e., from (2.4),

$$e^{n(1-\cos\lambda_n)} = \tau_n\sqrt{2\pi n}(2-2\cos\lambda_n)^{1/2} = 2\tau_n\sqrt{\pi n}(1-\cos\lambda_n)^{1/2}. \tag{5.1}$$

With $v_n := 2n(1-\cos\lambda_n)$, the squaring of the expression in (5.1) gives

$$e^{v_n} = 2\pi\tau_n^2 v_n. \tag{5.2}$$

Then, the largest of the two positive solutions of (5.2), called v_n, can be expressed as the convergent expression

$$v_n = 2.903\,948 + \frac{0.254\,204}{n} - \frac{0.005\,843}{n^2} + O\left(\frac{1}{n^3}\right), \text{ as } n \to \infty. \tag{5.3}$$

From (5.2) and $v_n := 2n(1-\cos\lambda_n)$, it can then be verified that

$$\lambda_n = \cos^{-1}\left\{1-\frac{v_n}{2n}\right\} = \frac{1.704\,097}{\sqrt{n}} + \frac{0.280\,778}{n^{3/2}} + O\left(\frac{1}{n^{5/2}}\right), \text{ as } n \to \infty, \tag{5.4}$$

which was stated in (2.5). Then, improving slightly on the discussion in [2, Section 3], it follows that, for any θ with $\lambda_n < \theta < 2\pi - \lambda_n$, there is a unique positive $r_n(\theta) < 1$ such that $r_n(\theta)e^{i\theta}$ is a point on $\tilde{D}_n$. Thus, having annexed the circular arc $\{z = e^{i\theta} : -\lambda_n \leq \theta \leq +\lambda_n\}$ to form the curve $\tilde{D}_n$, then $\tilde{D}_n$ is a simple closed curve in Δ which is star-shaped with respect to $z = 0$, giving part **i)** of Proposition 2.1. Part **ii)** of Proposition 2.1 then follows from equations (3.12) and (3.13) of [2].

It can be verified from (5.4) and (2.7) that $\Psi_1(\lambda_1) = 4.022\,922$ radians, and that $\Psi_n(\lambda_n)$ is strictly decreasing in n to $\pi/2 = 1.570\,796$, as $n \to \infty$. Thus,

$$\Psi_n(\lambda_n) < 2\pi = 6.283\,185, \qquad \text{for each } n \geq 1, \tag{5.5}$$

from which part **iii)** of Proposition 2.1 follows.

Next, for any fixed $n \geq 1$, it was stated in [2, p. 118] that $\Psi_n(\theta)$ is a strictly increasing function of θ on the interval $\left[\cos^{-1}\left(\frac{n-2}{n}\right), 2\pi - \cos^{-1}\left(\frac{n-2}{n}\right)\right]$, where the end-points of this interval come from the definition of the curve D_n in (2.2). Recalling that the curve $\tilde{D}_n$ of (2.4) is just an extension of the curve D_n to the boundary of Δ, the proof from [2] similarly shows that $\Psi_n(\theta)$ is a strictly increasing function of θ on the longer interval $[\lambda_n, 2\pi - \lambda_n]$, where λ_n is the largest number such that $e^{i\lambda_n}$ is a point of $\tilde{D}_n$, and this gives **iv)** of Proposition 2.1.

Finally, the proof of **v)** of Proposition 2.1 again comes directly from [2, p. 118], completing the proof. □

Proof of Proposition 2.2. With Proposition 2.1, assume that $\lambda_n \leq \theta_1 < \pi$, and let $z = r_n(\theta_1)e^{i\theta_1}$ be the associated unique point of $\tilde{D}_n$, i.e.,

$$\left|\frac{z\left(ze^{1-z}\right)^n}{\tau_n\sqrt{2\pi n}(1-z)}\right| = 1,$$

which means solving the following equation for $r_n(\theta_1)$:

$$\frac{r_n(\theta_1)\left[r_n(\theta_1)e^{1-r_n(\theta_1)\cos\theta_1}\right]^n}{\tau_n\sqrt{2\pi n}\left\{1-2r_n(\theta_1)\cos\theta_1 + r_n^2(\theta_1)\right\}^{1/2}} = 1. \tag{5.6}$$

Then from (2.7), compute $\Psi_n(\theta_1)$, as well as $\langle\langle\Psi_n(\theta_1)/2\pi\rangle\rangle$. This latter number then estimates the number of zeros of $s_n(nz)$ in the sector $0 \leq \theta < \theta_1$, and, as $s_n(nz)$, with positive

ETNA
Kent State University
http://etna.math.kent.edu

coefficients, has no zeros on the positive real axis, the even number $2\langle\langle\Psi_n(\theta_1)/2\pi\rangle\rangle$ estimates the total number of zeros of $s_n(nz)$ in the symmetric sector $-\theta_1 \leq \theta \leq \theta_1$. Thus, $n - 2\langle\langle\Psi_n(\theta_1)/2\pi\rangle\rangle$ estimates the total number of zeros of $s_n(nz)$ in the complementary symmetric sector $[\theta_1, 2\pi - \theta_1]$, which is stated in (2.12). $\square$

This leads us to the

Proof of Proposition 2.3. From (2.12) and (1.5), we have that

$$\operatorname{disc}_n(\theta_1, 2\pi - \theta_1) = n - 2\langle\langle\frac{\Psi_n(\theta_1)}{2\pi}\rangle\rangle - n\left(\frac{\phi_2 - \phi_1}{2\pi}\right). \tag{5.7}$$

Since $\theta_2 = 2\pi - \theta_1$, it follows from (1.3) and (1.4) that $r_\infty(\theta_2) = r_\infty(\theta_1)$ and $\phi_2 = 2\pi - \phi_1$. Thus, $\dfrac{\phi_2 - \phi_1}{2\pi} = 1 - \dfrac{\phi_1}{\pi}$. Substituting this in (5.7) gives

$$\operatorname{disc}_n(\theta_1, 2\pi - \theta_1) = \frac{n\phi_1}{\pi} - 2\langle\langle\frac{\Psi_n(\theta_1)}{2\pi}\rangle\rangle. \tag{5.8}$$

Next, we can rewrite (2.7) as

$$\Psi_n(\theta_1) = n\left[\theta_1 - r_\infty(\theta_1)\sin\theta_1\right] + \theta_1 + \tan^{-1}\left(\frac{r_n(\theta_1)\sin\theta_1}{1 - r_n(\theta_1)\cos\theta_1}\right) + n\sin\theta_1\left(r_\infty(\theta_1) - r_n(\theta_1)\right),$$

which from (1.4) gives

$$\Psi_n(\theta_1) = n\phi_1 + \theta_1 + \tan^{-1}\left(\frac{r_n(\theta_1)\sin\theta_1}{1 - r_n(\theta_1)\cos\theta_1}\right) + n\sin\theta_1\left(r_\infty(\theta_1) - r_n(\theta_1)\right),$$

or equivalently,

$$n\phi_1 = \Psi_n(\theta_1) - \theta_1 - \tan^{-1}\left(\frac{r_n(\theta_1)\sin\theta_1}{1 - r_n(\theta_1)\cos\theta_1}\right) - n\sin\theta_1\left(r_\infty(\theta_1) - r_n(\theta_1)\right). \tag{5.9}$$

Substituting the above expression for $n\phi_1/\pi$ in (5.8) then gives

$$\begin{aligned} \operatorname{disc}_n(\theta_1, 2\pi - \theta_1) = \; & \left\{\frac{\Psi_n(\theta_1)}{\pi} - 2\langle\langle\frac{\Psi_n(\theta_1)}{2\pi}\rangle\rangle\right\} \\ & - \left\{\frac{\theta_1}{\pi} + \frac{1}{\pi}\tan^{-1}\left(\frac{r_n(\theta_1)\sin\theta_1}{1 - r_n(\theta_1)\cos\theta_1}\right)\right\} \\ & + \left\{\frac{n\sin\theta_1}{\pi}\left[r_n(\theta_1) - r_\infty(\theta_1)\right]\right\}, \end{aligned} \tag{5.10}$$

which gives the desired result of (2.16) of Proposition 2.3. $\square$

As previously remarked, the terms in the three braces of (5.10) are all positive. Moreover, the first term in brackets satisfies, from the definition in (2.10), the inequalities of (2.17), for any $n \geq 1$.

We next turn to the

Proof of Proposition 2.4. Given any fixed θ_1 with $0 < \theta_1 < \pi$, the quantity in the braces of (2.18) satisfies

$$\frac{n\sin\theta_1}{\pi}\left[r_n(\theta_1) - r_\infty(\theta_1)\right] > 0, \quad \text{for any } n \geq 1, \tag{5.11}$$

since $r_n(\theta_1) > r_\infty(\theta_1)$ for any $n \geq 1$. Next, set

$$M_n(\theta_1) := \frac{n\sin\theta_1}{\pi}\left[r_n(\theta_1) - r_\infty(\theta_1)\right]. \tag{5.12}$$

It follows from (5.6) that

$$\begin{aligned} r_n(\theta_1)\left\{r_n(\theta_1)e^{1-r_n(\theta_1)\cos\theta_1}\right\}^n &= \tau_n\sqrt{2\pi n}\left\{1-2r_n(\theta_1)\cos\theta_1+r_n^2(\theta_1)\right\}^{1/2}, \\ \text{with } r_\infty(\theta_1)e^{1-r_\infty(\theta_1)\cos\theta_1} &= 1. \end{aligned} \tag{5.13}$$

Setting

$$r_n(\theta_1) := r_\infty(\theta_1)+\delta_n(\theta_1), \tag{5.14}$$

so that $\delta_n(\theta_1)>0$ for all $n\geq 1$, then the first equation of (5.13) can be expressed as

$$\left(1+\frac{\delta_n(\theta_1)}{r_\infty(\theta_1)}\right)\left\{\left(1+\frac{\delta_n(\theta_1)}{r_\infty(\theta_1)}\right)e^{-\delta_n(\theta_1)\cos\theta_1}\right\}^n = \tau_n\sqrt{2\pi n}\,\frac{\left\{1-2r_n(\theta_1)\cos\theta_1+r_n^2(\theta_1)\right\}^{1/2}}{r_\infty(\theta_1)}. \tag{5.15}$$

On taking logarithms and dividing by n, we have

$$\log\left(1+\frac{\delta_n(\theta_1)}{r_\infty(\theta_1)}\right)-\delta_n(\theta_1)\cos\theta_1 = \frac{\log(2\pi n)}{2n} + \text{ lower order terms in } n. \tag{5.16}$$

Hence, for n large, we see that $\delta_n(\theta_1)$ is small and positive, so that

$$\log\left(1+\frac{\delta_n(\theta_1)}{r_\infty(\theta_1)}\right) = \frac{\delta_n(\theta_1)}{r_\infty(\theta_1)} + \text{ lower order terms.}$$

This gives from (5.16) that

$$\delta_n(\theta_1)\left\{\frac{1}{r_\infty(\theta_1)}-\cos\theta_1\right\} = \frac{\log(2\pi n)}{2n} + \text{ lower order terms in } n,$$

which we can write as

$$\delta_n(\theta_1)\sim\frac{\log(2\pi n)}{2n}\left(\frac{r_\infty(\theta_1)}{1-r_\infty(\theta_1)\cos\theta_1}\right), \quad \text{as } n\to\infty. \tag{5.17}$$

Thus, from (5.12) and (5.14),

$$M_n(\theta_1)\sim\frac{\log(2\pi n)}{2\pi}\left(\frac{r_\infty(\theta_1)\sin\theta_1}{1-r_\infty(\theta_1)\cos\theta_1}\right), \quad \text{as } n\to\infty, \tag{5.18}$$

so that $M_n(\theta_1)$ is unbounded as $n\to\infty$. $\square$

We remark that for n very large, the *accuracy* of the approximation of (5.18) is also *very large*. We estimate that the result of (2.21), for $n = 10^{90}$, is accurate to over 80 decimal digits!

Proof of Theorem 2.5. This is an easy consequence of Propositions 2.3 and 2.4. The first term in braces of (5.10) always lies in the interval $(0,2]$, from (2.17). Next, the negative second term in (5.10), for $0<\theta_1<\pi$, clearly always lies in the interval $[-\frac{3}{2},0]$, since θ_1/π, by hypothesis, lies in $(0,1)$, and, because the argument of the next term in turn is always positive, then this term can be no more than $\frac{1}{2}$. Hence, as the third term in braces of (2.16) tends to $+\infty$, as $n\to\infty$, then (2.19) of Theorem 2.5 follows. $\square$

Proof of Theorem 3.1. Given any θ_1 with $0 < \theta_1 < \pi$, assume that $\theta_2 = 2\pi - \theta_1$, so that the sector Z_{θ_1,θ_2} of (1.1) is *symmetric* about the real axis. To estimate the number of zeros of $s_n(nz)$ in Z_{θ_1,θ_2}, we use the fact that the numbers $\{\hat{z}_{k,n}\}_{k=1}^{n} \setminus C_\delta$, are, from (2.9), close to the actual zeros $\{z_{k,n}\}_{k=1}^{n}$ of $s_n(nz)$. In particular, consider the unique points $\left\{\hat{z}_{k,n} = \hat{r}_{k,n}e^{i\hat{\theta}_{k,n}}\right\}_{k=1}^{n}$ of $\tilde{D}_n$, for which (cf. (2.8))

$$\Psi_n(\hat{\theta}_{k,n}) := 2\pi k, \quad \text{for all } 1 \le k \le n, \tag{5.19}$$

so that, from (2.6),

$$\frac{\hat{z}_{k,n}(\hat{z}_{k,n}e^{1-\hat{z}_{k,n}})^n}{\tau_n\sqrt{2\pi n}(1-\hat{z}_{k,n})} = 1. \tag{5.20}$$

Then, in place of the n points $\left\{\hat{z}_{k,n} = \hat{r}_{k,n}e^{i\hat{\theta}_{k,n}}\right\}_{k=1}^{n}$, we consider the following n *uniformly spaced* (in angle) *points*, defined as

$$|\hat{w}_{k,n}| = 1, \text{ with } \arg \hat{w}_{k,n} = \frac{2\pi k}{n+1}, \text{ for all } 1 \le k \le n. \tag{5.21}$$

We remark that asking if the approximate zero $\hat{z}_{k,n}$ of $s_n(nz)$ is in the sector Z_{θ_1,θ_2} is *equivalent* to asking if $\hat{w}_{k,n}$ of (5.20) satisfies

$$\frac{\Psi_n(\theta_1)}{n+1} \le \arg \hat{w}_{k,n} \le 2\pi - \frac{\Psi_n(\theta_1)}{n+1}. \tag{5.22}$$

We further remark that, as $\hat{w}_{k,n}$ can be expressed as

$$\hat{w}_{k,n} = \hat{z}_{k,n}e^{1-\hat{z}_{k,n}}\left(\frac{e^{\hat{z}_{k,n}-1}}{\tau_n\sqrt{2\pi n}(1-\hat{z}_{k,n})}\right)^{1/(n+1)},$$

we directly see, on letting $n \to \infty$, how the Szegő curve D_∞ of (1.3) plays a major role in the result of (1.2). We also show, in Figure 5.1, the numbers $\{\hat{w}_{k,16}\}_{k=1}^{16}$ from (5.21).

Next, we order the approximate zeros $\{\hat{z}_{k,n}\}_{k=1}^{n}$ of $s_n(nz)$ by their increasing arguments, i.e.,

$$0 < \arg \hat{z}_{1,n} < \arg \hat{z}_{2,n} < \cdots < \arg \hat{z}_{n,n} < 2\pi. \tag{5.23}$$

The "fanning out" of the exact zeros of $s_n(nz)$, above and below the real axis as n increases, as can be seen in more detail in the *Interactive Supplement*, implies that, in the closed upper half-plane,

$$\arg \hat{z}_{k,n} > \arg \hat{z}_{k,n+1}, \text{ for all } 1 \le k \le \frac{n+1}{2}, \tag{5.24}$$

and that, in the open lower half-plane, we have the reverse:

$$\arg \hat{z}_{k,n} < \arg \hat{z}_{k,n+1}, \text{ for all } \frac{n+1}{2} < k \le n. \tag{5.25}$$

As the nonreal zeros of the real polynomial $s_n(nz)$ occur in conjugate complex pairs, it is sufficient to consider the *motion* (with respect to n) of the zeros, only in the upper half-plane of (5.24). Now, as the approximate zeros $\{\hat{z}_{k,n}\}_{k=1}^{n}$ of $s_n(nz)$, were derived (cf. (5.24))

ETNA
Kent State University
http://etna.math.kent.edu

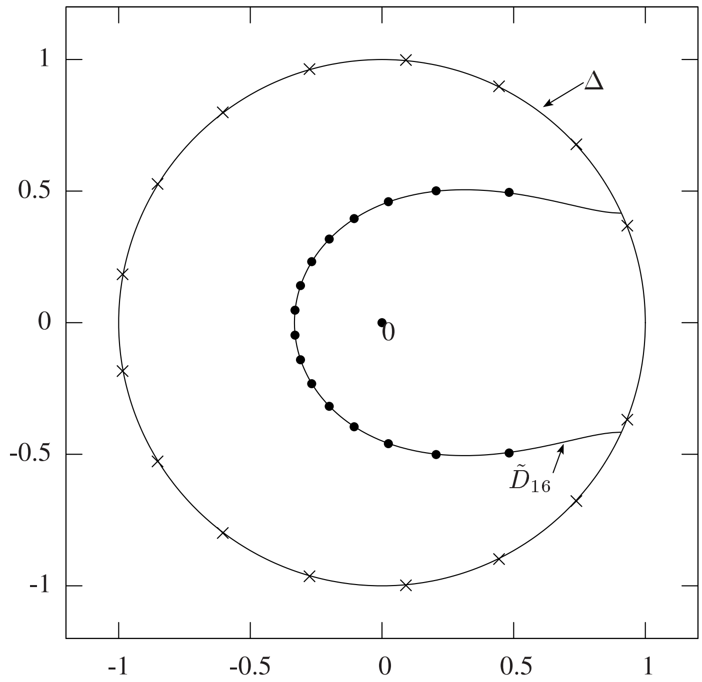

FIG. 5.1. $\{\hat{w}_{k,16}\}_{k=1}^{16}$ *as* $\times$*'s and the zeros of* $s_{16}(16z)$ *as* $\bullet$*'s.*

as specific points $\hat{z}_{k,n} = \hat{r}_{k,n} e^{i\hat{\theta}_{k,n}}$ of the curve $\tilde{D}_n$, then the following analog of (5.24) necessarily holds, i.e., from (5.21),

$$\arg \hat{w}_{k,n} = \frac{2\pi k}{n+1} > \frac{2\pi k}{n+2} = \arg \hat{w}_{k,n+1}, \text{ for all } 1 \leq k \leq \frac{n+1}{2}. \tag{5.26}$$

In addition, we see from (2.1) that

$$\hat{z}_{k,n} \in Z_{\theta_1,\theta_2} \text{ if and only if } \frac{\Psi_n(\theta_1)}{n+1} \leq \arg \hat{w}_{k,n} \leq \frac{\Psi_n(\theta_2)}{n+1}, \tag{5.27}$$

or, equivalently from (5.21),

$$\hat{z}_{k,n} \in Z_{\theta_1,\theta_2} \text{ if and only if } \frac{\Psi_n(\theta_1)}{n+1} \leq \frac{2\pi k}{n+1} \leq \frac{\Psi_n(\theta_2)}{n+1}. \tag{5.28}$$

We note, for n an odd positive integer, say $n = 2m+1$, that from (5.21) we have

$$\arg \hat{w}_{m+1,2m+1} = \frac{2\pi(m+1)}{2m+2} = \pi. \tag{5.29}$$

This means that $s_{2m+1}((2m+1)z)$, which has exactly one (negative) real zero, corresponds to the point $\hat{w}_{m+1,2m+1}$, which is also real and negative, from (5.21).

Next, suppose that $\hat{z}_{k,n}$ is *exactly* on the boundary of the symmetric sector Z_{θ_1,θ_2} in the upper half-plane, i.e.,

$$\arg \hat{w}_{k,n} = \frac{2\pi k}{n+1} = \frac{\Psi_n(\theta_1)}{n+1}, \text{ where } 1 \leq k \leq \frac{n+1}{2}. \tag{5.30}$$

If k satisfies $1 \leq k \leq \frac{n-1}{2}$, then

$$\arg \hat{w}_{k,n} < \arg \hat{w}_{k+1,n} = \frac{2\pi(k+1)}{n+1} \leq \pi, \tag{5.31}$$

which implies that $\hat{z}_{k+1,n}$ is also in the upper half-plane of Z_{θ_1,θ_2}. It is then evident, from (5.31), that the numbers $\{\hat{w}_{k+1,m}\}_{m \geq n}$ are all in the upper half-plane, with strictly decreasing arguments, as m increases. (This is the analog, in the w-plane, of the "fanning out" of the zeros of $s_n(nz)$, in the upper half-plane.)

Next, what we seek, from this fanning out of the numbers $\{\hat{w}_{k+1,m}\}_{m \geq n}$, is the smallest nonnegative integer ℓ so that $\hat{z}_{k+1,n}, \hat{z}_{k+1,n+1}, \cdots, \hat{z}_{k+1,n+\ell}$ are all *in* the upper half-plane of Z_{θ_1,θ_2}, while $\hat{z}_{k+1,n+\ell+1}$ is *out* of this sector. This implies from (5.30) that

$$\frac{2\pi(k+1)}{n+\ell+1} \geq \left(\frac{\Psi_n(\theta_1)}{n+1} = \frac{2\pi k}{n+1}\right) > \frac{2\pi(k+1)}{n+\ell+2}, \tag{5.32}$$

so that

$$\ell \leq \frac{n+1}{k} < \ell + 1. \tag{5.33}$$

From (5.30), we can write these inequalities as

$$\ell \leq \frac{2\pi}{\Psi_n(\theta_1)/(n+1)} < \ell + 1. \tag{5.34}$$

Next, it follows from (2.7) and (1.4) that

$$\lim_{n\to\infty} \frac{\Psi_n(\theta_1)}{n+1} = \theta_1 - r_\infty(\theta_1)\sin\theta_1 =: \phi_1, \tag{5.35}$$

where $r_\infty(\theta_1)$ lies on D_∞. Then, as (5.30) implies that

$$\frac{n+1}{k} = \frac{2\pi}{\Psi_n(\theta_1)/(n+1)}, \tag{5.36}$$

we see, assuming that n is large, from (5.35) and (2.7) that

$$\frac{\Psi_n(\theta_1)}{n+1} \approx \phi_1.$$

Moreover, (5.36), coupled with the last inequality of (5.33), gives approximately that

$$\ell \leq \frac{2\pi}{\phi_1} \leq \ell + 1. \tag{5.37}$$

Thus, (5.37) says that $2\pi/\phi_1$ is a good approximation of the positive ℓ in (5.33), when n is large, but, as the *Interactive Supplement* shows, it can also be quite good for small n as well. Furthermore, this implies that the *maximum* nonnegative integer ℓ such that $\hat{z}_{k+1,n+\ell} \notin Z_{\theta_1,\theta_2}$ when $\hat{z}_{k,n} = \theta_1$, is given approximately by

$$1 + \left\lfloor \frac{2\pi}{\phi_1} \right\rfloor, \tag{5.38}$$

which is *independent* of n, and this is the desired result of (3.2) of Theorem 3.1. $\square$

We return to the assumption that $\hat{z}_{k,n}$ is *exactly* on the boundary of Z_{θ_1,θ_2} in the upper half-plane. Suppose now that $\hat{z}_{k,n}$ is outside the sector Z_{θ_1,θ_2}, while $\hat{z}_{k+1,n}$ is in Z_{θ_1,θ_2}. This gives the inequalities

$$\arg \hat{w}_{k+1,n} > \frac{\Psi_n(\theta_1)}{n+1} > \arg \hat{w}_{k,n}.$$

Then on seeking the smallest nonnegative integer h such that $\hat{z}_{k+1,n+h}$ is in Z_{θ_1,θ_2}, while $\hat{z}_{k+1,n+h+1}$ is not, it similarly follows that $h \leq \ell$, where ℓ satisfies (5.33).

Proof of Theorem 4.1. The results of (4.1) - (4.3) show in these cases how $\mathrm{disc}_n(\theta_1, \theta_2)$ can be expressed in terms of discrepancies for symmetric sectors. In particular, for any θ_1 with $0 < \theta_1 < \pi$, with $\theta_2 := 2\pi - \theta_1$, it follows from Propositions 2.3 and 2.4 that

$$\mathrm{disc}_n(\theta_1, \theta_2) \sim \frac{\log(2\pi n)}{2\pi} \left[\frac{r_\infty(\theta_1)\sin\theta_1}{1 - r_\infty(\theta_1)\cos\theta_1} \right], \quad \text{as } n \to \infty. \tag{5.39}$$

But, it can be verified that the function

$$\frac{r_\infty(\theta)\sin\theta}{1 - r_\infty(\theta)\cos\theta}, \quad \text{defined on } (0, 2\pi), \tag{5.40}$$

is *strictly decreasing*, from +1 to -1, in θ, so that, from (4.1) - (4.3), we see, in all cases, that $\mathrm{disc}_n(\theta_1, \theta_2) \sim K \log n$, where $K > 0$ is dependent only on θ_1 and θ_2. □

Acknowledgment. The authors thank Professor Thomas Kriecherbauer for his sage comments and correction.

REFERENCES

[1] J. D. BUCKHOLTZ, *A characterization of the exponential series, Part II*, Amer. Math. Monthly, 73 (1966), pp. 121–123.

[2] A. J. CARPENTER, R. S. VARGA, AND J. WALDVOGEL, *Asymptotics for the zeros of the partial sums of e^z*. I., Rocky Mountain J. Math., 21 (1991), pp. 99–120.

[3] P. HENRICI, *Applied and Computational Complex Analysis*, Vol. 2, John Wiley and Sons, New York, 1977.

[4] M. MARDEN, *Geometry of Polynomials*, Math. Surveys, No. 3, Amer. Math. Soc., Providence, RI, 1966.

[5] D. J. NEWMAN AND T. J. RIVLIN, *The zeros of the partial sums of the exponential function*, J. Approx. Theory, 5 (1972), pp. 405–412.

[6] G. PÓLYA AND G. SZEGŐ, *Aufgaben und Lehrsätze aus der Analysis*, Erster Band, Springer, Berlin, 1954. Problem 22 of Section III, p. 88.

[7] G. SZEGŐ, *Über eine Eigenschaft der Exponentialreihe*, Sitzungsber. Berlin Math. Ges., 23 (1924), pp. 50–64.