Electronic Transactions on Numerical Analysis.
Volume 30, pp. 187-202, 2008.

ISSN 1068-9613.

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LOW-RANK ITERATIVE METHODS FOR PROJECTED GENERALIZED LYAPUNOV EQUATIONS*

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Abstract. We generalize an alternating direction implicit method and the Smith method for large-scale projected generalized Lyapunov equations. Such equations arise in model reduction of descriptor systems. Low-rank versions of these methods are also presented, which can be used to compute low-rank approximations to the solution of projected generalized Lyapunov equations with low-rank symmetric, positive semidefinite right-hand side. Numerical examples are presented.

Key words. projected generalized Lyapunov equations, alternating direction implicit method, Smith method, low-rank approximation

AMS subject classifications. 65F10, 65F30, 15A22, 15A24,

1. Introduction.

Consider a linear time-invariant descriptor system

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{1.1}$$

where $E,\ A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, $C \in \mathbb{R}^{p,n}$, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input and $y(t) \in \mathbb{R}^p$ is the output. The matrix E may be singular, but the pencil $\lambda E - A$ is assumed to be *regular*, i.e., $\det(\lambda E - A) \not\equiv 0$. Descriptor systems arise in many different applications including electrical circuit simulation, multibody dynamics and spatial discretization of partial differential equations, e.g., [6, 7, 8, 40]. Stability analysis and some control problems for (1.1) are strongly related to the projected generalized continuous-time algebraic Lyapunov equations (GCALEs)

$$EXA^T + AXE^T = -P_l BB^T P_l^T, \qquad X = P_r X P_r^T, \tag{1.2}$$

$$E^T XA + A^T XE = -P_r^T C^T C P_r, \qquad X = P_l^T X P_l \tag{1.3}$$

and the projected generalized discrete-time algebraic Lyapunov equations (GDALEs)

$$AXA^T - EXE^T = Q_l BB^T Q_l^T, \qquad X = Q_r X Q_r^T, \tag{1.4}$$

$$A^T XA - E^T XE = Q_r^T C^T C Q_r, \qquad X = Q_l^T X Q_l, \tag{1.5}$$

where P_l and P_r are the spectral projectors onto the left and right deflating subspaces of $\lambda E - A$ corresponding to the finite eigenvalues, $Q_l = I - P_l$ and $Q_r = I - P_r$ are the spectral projectors onto the left and right deflating subspaces corresponding to the eigenvalue at infinity. Let $\lambda E - A$ be in Weierstrass canonical form

$$E = W \begin{bmatrix} I_q & 0 \\ 0 & N \end{bmatrix} T \qquad \text{and} \qquad A = W \begin{bmatrix} J & 0 \\ 0 & I_{n-q} \end{bmatrix} T, \tag{1.6}$$

where the matrices W and T are nonsingular, J corresponds to the finite eigenvalues of $\lambda E - A$ and N being nilpotent corresponds to the eigenvalue at infinity. The index ν of

*Received March 10, 2006. Accepted for publication March 21, 2008. Published online on August 4, 2008. Recommended by P. Van Dooren. This work was supported by the DFG Research Center MATHEON "Mathematics for key technologies" in Berlin.

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nilpotency of N is the *index* of the pencil $\lambda E - A$. Using (1.6) the projectors P_l and P_r can be represented as

$$P_l = W \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} W^{-1} \qquad \text{and} \qquad P_r = T^{-1} \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} T. \tag{1.7}$$

It has been shown in [34] that if the pencil $\lambda E - A$ is *stable*, i.e., all its finite eigenvalues have negative real part, then the projected GCALEs (1.2) and (1.3) have the unique symmetric, positive semidefinite solutions which define the proper controllability and observability Gramians of the descriptor system (1.1). Furthermore, the projected GDALEs (1.4) and (1.5) have the unique symmetric, positive semidefinite solutions which are the improper controllability and observability Gramians of (1.1). The Gramians play a central role in analysis and control design problems for descriptor systems, such as the characterization of controllability and observability properties, computing $\mathcal{H}_2$ or Hankel norm, minimal and balanced realizations as well as balanced truncation model order reduction [4, 34, 35, 36].

The numerical solution of standard Lyapunov equations with $E = I$ has been the topic of numerous publications [2, 3, 12, 13, 14, 17, 18, 22, 24, 29]. A variety of direct and iterative methods has been proposed there for computing the solutions of such equations, their Cholesky factors or low-rank approximations. The case of nonsingular E has been considered in [5, 9, 15, 16, 21]. Until now, only direct methods have been extended to projected Lyapunov equations [33]. The solutions of (1.2) – (1.5) can be computed by the generalized Schur-Bartels-Stewart or the generalized Schur-Hammarling methods that are based on the preliminary reduction of the pencil $\lambda E - A$ to the generalized Schur form, solution of the generalized Sylvester and Lyapunov equations and back transformation. Since these methods cost $O(n^3)$ operations and require $O(n^2)$ memory location, they can be used only for problems of small or medium size.

Due to the practical importance of the numerical solution of large-scale projected generalized Lyapunov equations that occur in balanced truncation model reduction of descriptor systems [35], the development of iterative methods for such equations is a challenging problem. In this paper we generalize the alternating direction implicit (ADI) method [17, 18, 22] and the Smith method [22, 29] to the projected generalized Lyapunov equations (1.2) and (1.4) with large sparse matrix coefficients. The dual equations (1.3) and (1.5) can be solved in a similar way. Low-rank versions of the ADI and Smith methods are also presented, which can be used to compute low-rank approximations to the solutions of (1.2) and (1.4) with a low-rank right-hand side. Such a problem arises, for example, in model reduction. Note that the number m of columns of the matrix B in (1.2) and (1.4) relates to the number of inputs of the underlying descriptor system (1.1) and it is usually small compared to the state space dimension n of the problem.

A major difficulty in the numerical solution of projected Lyapunov equations is that we need to compute the spectral projectors P_l, P_r or Q_l, Q_r. Fortunately, in many applications such as computational fluid dynamics and constrained structural mechanics, the matrices E and A have some special block structure. As the following examples show, this structure can be used to construct the projectors P_l and P_r in explicit form.

EXAMPLE 1.1. Consider the descriptor system (1.1) with

$$E = \begin{bmatrix} E_{11} & E_{12} \\ 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{1.8}$$

where E_{11} is nonsingular. Such a system arises, for example, after linearization and spatial discretization of the Euler equation that describes the flow of a fluid through a supersonic

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diffuser [40]. If the matrix $A_{21}E_{11}^{-1}E_{12} - A_{22}$ is nonsingular, then the pencil $\lambda E - A$ in (1.8) is of index 1, and the projectors P_l and P_r are given by

$$P_l = \begin{bmatrix} I & (A_{12} - A_{11}E_{11}^{-1}E_{12})(A_{21}E_{11}^{-1}E_{12} - A_{22})^{-1} \\ 0 & 0 \end{bmatrix},$$

$$P_r = \begin{bmatrix} I - E_{11}^{-1}E_{12}(A_{21}E_{11}^{-1}E_{12} - A_{22})^{-1}A_{21} & -E_{11}^{-1}E_{12}(A_{21}E_{11}^{-1}E_{12} - A_{22})^{-1}A_{22} \\ (A_{21}E_{11}^{-1}E_{12} - A_{22})^{-1}A_{21} & I + (A_{21}E_{11}^{-1}E_{12} - A_{22})^{-1}A_{22} \end{bmatrix}.$$

EXAMPLE 1.2. Consider the descriptor system (1.1), where E and A have the form

$$E = \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{bmatrix}. \tag{1.9}$$

Such systems arise in spatial discretization of the instationary incompressible Stokes equation [6, 7] and the convection equation [19]. If E_{11} and $A_{21}E_{11}^{-1}A_{12}$ are nonsingular, then the pencil $\lambda E - A$ in (1.9) is of index 2. In this case the spectral projectors P_l and P_r have the following form

$$P_l = \begin{bmatrix} \Pi_l & -\Pi_l A_{11}E_{11}^{-1}A_{12}(A_{21}E_{11}^{-1}A_{12})^{-1} \\ 0 & 0 \end{bmatrix},$$

$$P_r = \begin{bmatrix} \Pi_r & 0 \\ -(A_{21}E_{11}^{-1}A_{12})^{-1}A_{21}E_{11}^{-1}A_{11}\Pi_r & 0 \end{bmatrix},$$

where $\Pi_l = I - A_{12}(A_{21}E_{11}^{-1}A_{12})^{-1}A_{21}E_{11}^{-1}$ is a projector onto the kernel of $A_{21}E_{11}^{-1}$ along the image of A_{12} and $\Pi_r = I - E_{11}^{-1}A_{12}(A_{21}E_{11}^{-1}A_{12})^{-1}A_{21} = E_{11}^{-1}\Pi_l E_{11}$.

EXAMPLE 1.3. The motion of multibody systems with holonomic constraints can be described by nonlinear differential-algebraic equations of the first order [8, 28]. Linearization of these equations around an equilibrium state leads to the descriptor system (1.1) with

$$E = \begin{bmatrix} I & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & I & 0 \\ K & D & -G^T \\ G & 0 & 0 \end{bmatrix}, \tag{1.10}$$

where M is a nonsingular mass matrix, K is a stiffness matrix, D is a damping matrix and G is a matrix of constraints. If G has full row rank, then the pencil $\lambda E - A$ is of index 3, and the spectral projectors P_l and P_r can be computed as

$$P_l = \begin{bmatrix} \Pi & 0 & -\Pi M^{-1}DG_1 \\ -\Pi^T D(I - \Pi) & \Pi^T & -\Pi^T(K + D\Pi M^{-1}D)G_1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$P_r = \begin{bmatrix} \Pi & 0 & 0 \\ -\Pi M^{-1}D(I - \Pi) & \Pi & 0 \\ G_1^T(K\Pi - D\Pi M^{-1}D(I - \Pi)) & G_1^T D\Pi & 0 \end{bmatrix}.$$

Here $G_1 = M^{-1}G^T(GM^{-1}G^T)^{-1}$ and $\Pi = I - M^{-1}G^T(GM^{-1}G^T)^{-1}G = I - G_1 G$ is a projector onto the kernel of G along the image of $M^{-1}G^T$, see [28] for details.

In the following we will assume that the projectors P_l, P_r, Q_l and Q_r are given. Clearly, we do not compute these projectors explicitly. Instead, we use matrix-vector multiplication and linear system solvers if an inverse is required. In Section 2, we present a generalization of the ADI method and its low-rank version for the projected GCALE (1.2). In Section 3, we discuss the numerical solution of the projected Lyapunov equations (1.2) and (1.4) via the (cyclic) Smith method. Section 4 contains some results of numerical experiments.

Throughout the paper the open left half-plane is denoted by $\mathbb{C}^-$. We will denote by $\mathbb{R}^{n,m}$ and $\mathbb{C}^{n,m}$ the spaces of $n \times m$ real and complex matrices, respectively. The real part of a complex number z is denoted by $\mathrm{Re}(z)$. The matrix A^T stands for the transpose of $A \in \mathbb{R}^{n,m}$, A^* denotes the complex conjugate and transpose of $A \in \mathbb{C}^{n,m}$, and $A^{-T} = (A^{-1})^T$. An identity matrix of order n is denoted by I_n or simply I. We will denote by $\|A\|_2$ the spectral matrix norm and by $\|A\|_F$ the Frobenius matrix norm of $A \in \mathbb{C}^{n,m}$.

2. Alternating direction implicit method. The ADI method was originally proposed for linear systems [20] and then used in [17, 18, 22, 38] to solve standard continuous-time Lyapunov equations. The case of nonsingular E has been considered in [16]. In this section, we present a generalization of the ADI method for the projected GCALE (1.2).

Assume that the pencil $\lambda E - A$ is stable. Then the matrix A is nonsingular, and the projected GCALE (1.2) is equivalent to the projected standard Lyapunov equation

$$(A^{-1}E)X + X(A^{-1}E)^T = -P_r A^{-1}BB^T A^{-T} P_r^T, \qquad X = P_r X P_r^T. \tag{2.1}$$

In this case an approximate solution of (1.2) can be computed by the ADI method applied to (2.1). The ADI iteration is given by

$$\begin{aligned} X_k &= (A^{-1}E + \tau_k I)^{-1}(A^{-1}E - \overline{\tau}_k I)X_{k-1}(A^{-1}E - \tau_k I)^T (A^{-1}E + \overline{\tau}_k I)^{-T} \\ &\quad -2\mathrm{Re}(\tau_k)(A^{-1}E + \tau_k I)^{-1} P_r A^{-1}BB^T A^{-T} P_r^T (A^{-1}E + \overline{\tau}_k I)^{-T} \end{aligned} \tag{2.2}$$

with an initial matrix $X_0 = 0$ and the shift parameters $\tau_1, \dots, \tau_k \in \mathbb{C}^-$. It follows from

$$\begin{aligned} P_r(A^{-1}E - \overline{\tau}_k I) &= (A^{-1}E - \overline{\tau}_k I)P_r, \\ P_r(A^{-1}E + \tau_k I)^{-1} &= (A^{-1}E + \tau_k I)^{-1} P_r \end{aligned}$$

that $X_k = P_r X_k P_r^T$, i.e., the second equation in (1.2) and (2.1) is satisfied exactly. The iteration (2.2) can also be written as

$$\begin{aligned} X_k &= (E + \tau_k A)^{-1}(E - \overline{\tau}_k A)X_{k-1}(E - \tau_k A)^T (E + \overline{\tau}_k A)^{-T} \\ &\quad -2\mathrm{Re}(\tau_k)(E + \tau_k A)^{-1} P_l BB^T P_l^T (E + \overline{\tau}_k A)^{-T}. \end{aligned} \tag{2.3}$$

The following propositions give the convergence results for the ADI iteration.

PROPOSITION 2.1. *If the pencil $\lambda E - A$ is stable and $\tau_1, \dots, \tau_k \in \mathbb{C}^-$, then the ADI iteration (2.3) converges to the solution X of the projected GCALE (1.2).*

Proof. Let X be a solution of the projected GCALE (1.2). The error matrix $X - X_k$ can be computed from (2.3) recursively as

$$X - X_k = \mathcal{A}_k X \mathcal{A}_k^*, \tag{2.4}$$

where

$$\mathcal{A}_k = P_r(E + \tau_k A)^{-1}(E - \overline{\tau}_k A) \cdot \ldots \cdot (E + \tau_1 A)^{-1}(E - \overline{\tau}_1 A). \tag{2.5}$$

Using the Weierstrass canonical form (1.6) and representation (1.7), we find that

$$\mathcal{A}_k = T^{-1} \begin{bmatrix} J_k & 0 \\ 0 & 0 \end{bmatrix} T \tag{2.6}$$

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with $J_k = (I+\tau_k J)^{-1}(I-\overline{\tau}_k J)\cdot\ldots\cdot(I+\tau_1 J)^{-1}(I-\overline{\tau}_1 J)$. If $\lambda E - A$ is stable and $\tau_1,\ldots,\tau_k \in \mathbb{C}^-$, then all the eigenvalues of $(I+\tau_j J)^{-1}(I-\overline{\tau}_j J)$, $j=1,\ldots,k$, lie inside the unit circle, and, hence, X_k converges toward the solution X. □

PROPOSITION 2.2. *Consider the projected GCALE (1.2). Assume that the pencil $\lambda E - A$ is in Weierstrass canonical form (1.6), where J is diagonal. Then the k-th iterate X_k of the ADI method satisfies the estimate*

$$\|X - X_k\|_2 \le \kappa^2(T)\rho^2(\mathcal{A}_k)\|X\|_2, \tag{2.7}$$

where $\kappa(T) = \|T\|_2\|T^{-1}\|_2$ is the spectral condition number of the right transformation matrix T in (1.6) and $\rho(\mathcal{A}_k)$ is the spectral radius of the matrix $\mathcal{A}_k$ given in (2.5).

Proof. Estimate (2.7) immediately follows from (2.4) and (2.6). □

2.1. Computing the shift parameters. As Proposition 2.2 shows, the convergence rate of the ADI iteration is determined by the spectral radius of the matrix $\mathcal{A}_k$ as in (2.5) and depends strongly on the choice of shift parameters. The minimization of this spectral radius with respect to the parameters $\tau_1,\ldots,\tau_k$ leads to the generalized ADI minimax problem

$$\{\tau_1,\ldots,\tau_k\} = \operatorname*{arg\,min}_{\tau_1,\ldots,\tau_k\in\mathbb{C}^-}\ \max_{t\in \mathrm{Sp}_f(E,A)} \frac{|(1-\overline{\tau}_1 t)\cdot\ldots\cdot(1-\overline{\tau}_k t)|}{|(1+\tau_1 t)\cdot\ldots\cdot(1+\tau_k t)|}, \tag{2.8}$$

where $\mathrm{Sp}_f(E,A)$ denotes the set of finite eigenvalues of the pencil $\lambda E - A$. The computation of the optimal shift parameters is a difficult problem, since the finite eigenvalues of the pencil $\lambda E - A$ are, in general, unknown and expensive to compute. This problem is solved for standard Lyapunov equations with $E = I$ and symmetric A, e.g., [39], while the case of complex eigenvalues of A is still not completely understood; see [18, 27, 32, 39] for some contributions. To compute the suboptimal ADI shift parameters for the standard problem, a heuristic algorithm has been proposed in [22]. This algorithm is based on Arnoldi iterations [25] applied to the matrices A and A^{-1}. It can also be extended to the generalized problem (2.8). Due to the nonsingularity of A this problem is equivalent to

$$\{\tau_1,\ldots,\tau_k\} = \operatorname*{arg\,min}_{\tau_1,\ldots,\tau_k\in\mathbb{C}^-}\ \max_{t\in \mathrm{Sp}(A^{-1}E)\setminus\{0\}} \frac{|(t-\overline{\tau}_1)\cdot\ldots\cdot(t-\overline{\tau}_k)|}{|(t+\tau_1)\cdot\ldots\cdot(t+\tau_k)|}, \tag{2.9}$$

where $\mathrm{Sp}(A^{-1}E)$ denotes the spectrum of the matrix $A^{-1}E$. Thus, the suboptimal ADI shift parameters $\tau_1,\ldots,\tau_k$ can be determined by the heuristic procedure [22, Algorithm 5.1] from a set of largest and smallest (in modulus) non-zero approximate eigenvalues of $A^{-1}E$. A conventional approach for computing largest and smallest eigenvalues of a matrix is to apply an Arnoldi process to this matrix and its inverse, respectively. However, if E is singular, then the inverse of $A^{-1}E$ does not exist. Observe that the reciprocals of the smallest non-zero eigenvalues of $A^{-1}E$ are the largest finite eigenvalues of the pencil $\lambda E - A$. The latter can be determined by an Arnoldi procedure applied to the matrix PA, where

$$P = P_r(EP_r - AQ_r)^{-1} = (P_l E - Q_l A)^{-1}P_l = T^{-1}\begin{bmatrix} I_q & 0 \\ 0 & 0\end{bmatrix}W^{-1},$$

and T, W are the transformation matrices as in (1.6), see [30] for details. Similarly to the projectors P_r and P_l, the matrix P can be obtained in explicit form using the special block structure of the matrices E and A.

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2.2. Low-rank version of the generalized ADI method. Recently, an efficient modification of the ADI method has been proposed to compute low-rank approximations to the solutions of standard Lyapunov equations with large-scale matrix coefficients [17, 22]. This is the *low-rank alternating direction implicit* (LR-ADI) *method*. It was observed that the eigenvalues of the symmetric solutions of Lyapunov equations with low-rank right-hand side generally decay very rapidly, and such solutions may be well approximated by low-rank matrices [1, 10, 23, 31]. A similar result holds for projected generalized Lyapunov equations. In other words, it is possible to find a matrix Z with a small number of columns such that ZZ^T is an approximate solution of the projected GCALE (1.2). The matrix Z is referred to as the *low-rank Cholesky factor* of the solution X of (1.2).

A low-rank version of the generalized ADI iteration (2.3) can be derived analogously to the standard case [17, 22]. First of all note that the matrix X_k in (2.3) is Hermitian, positive semidefinite, and the Cholesky factor Z_k of $X_k = Z_k Z_k^*$ has the form

$$\begin{aligned} Z_k &= [\, \sqrt{-2\mathrm{Re}(\tau_k)}(E+\tau_k A)^{-1}P_l B, \quad (E+\tau_k A)^{-1}(E-\overline{\tau}_k A)Z_{k-1}\,] \\ &= [\, \alpha_k S_k P_l B, \quad \alpha_{k-1} S_k R_k S_{k-1} P_l B, \quad \ldots, \quad \alpha_1 S_k R_k \cdots R_2 S_1 P_l B\,], \end{aligned}$$

where $\alpha_j = \sqrt{-2\mathrm{Re}(\tau_j)}$, $S_j = (E+\tau_j A)^{-1}$ and $R_j = E - \overline{\tau}_j A$. Taking into account that

$$S_k A S_j = S_j A S_k, \qquad R_k A^{-1} R_j = R_j A^{-1} R_k, \qquad S_k R_j = A^{-1} R_j S_k A$$

for k, $j = 1, 2, \ldots$, the matrix Z_k can be rewritten as

$$Z_k = [\, B_0, F_{k-1}B_0, \;\; F_{k-2}F_{k-1}B_0, \;\; \ldots, \;\; F_1 F_2 \cdots F_{k-1} B_0\,], \tag{2.10}$$

where $B_0 = \sqrt{-2\mathrm{Re}(\tau_k)}(E+\tau_k A)^{-1}P_l B = \sqrt{-2\mathrm{Re}(\tau_k)}P_r(E+\tau_k A)^{-1}B$ and

$$F_j = \sqrt{\frac{\mathrm{Re}(\tau_j)}{\mathrm{Re}(\tau_{j+1})}}\, S_j R_{j+1} = \sqrt{\frac{\mathrm{Re}(\tau_j)}{\mathrm{Re}(\tau_{j+1})}}\, \big(I - (\tau_j + \overline{\tau}_{j+1})(E+\tau_j A)^{-1}A\big)\,.$$

If we reenumerate the shift parameters in reverse order, then we obtain the following algorithm for computing the low-rank Cholesky factor of the solution of (1.2).

ALGORITHM 2.1. *The generalized LR-ADI method for the projected GCALE.*

INPUT: E, A, $P_l \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, *shift parameters* $\tau_1, \ldots, \tau_{k_{\max}} \in \mathbb{C}^-$.
OUTPUT: *A low-rank Cholesky factor* Z_k *of the solution* $X \approx Z_k Z_k^*$ *of* (1.2).
1. $Z^{(1)} = \sqrt{-2\mathrm{Re}(\tau_1)}\,(E+\tau_1 A)^{-1}P_l B, \quad Z_1 = Z^{(1)}$.
2. FOR $k = 2, 3, \ldots$

$$Z^{(k)} = \sqrt{\frac{\mathrm{Re}(\tau_k)}{\mathrm{Re}(\tau_{k-1})}}\, \big(I - (\overline{\tau}_{k-1} + \tau_k)(E+\tau_k A)^{-1}A\big)\, Z^{(k-1)}, \tag{2.11}$$

$$Z_k = [\, Z_{k-1}, \;\; Z^{(k)}\,].$$

END FOR

The ADI iteration can be stopped as soon as a *normalized residual norm* given by

$$\eta(Z_k) = \frac{\|EZ_kZ_k^*A^T + AZ_kZ_k^*E^T + P_lBB^TP_l^T\|_F}{\|P_lBB^TP_l^T\|_F} \tag{2.12}$$

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satisfies the condition $\eta(Z_k) \le tol$ with a user-defined tolerance tol or a stagnation of normalized residual norms is observed. If the number of shift parameters is smaller than the number of iterations required to attain a prescribed tolerance, then we reuse these parameters in a cyclic manner. Note that computing the normalized residuals $\eta(Z_k)$ even via the efficient methods proposed in [22, 27] can still be quite expensive for large-scale problems. It should also be noted that for ill-conditioned problems, the small residual norm does not imply that the error in the computed solution is also small; see [33].

If $X_k = Z_k Z_k^*$ converges to the solution of (1.2), then

$$\lim_{k\to\infty} Z^{(k)} (Z^{(k)})^* = \lim_{k\to\infty} (X_k - X_{k-1}) = 0.$$

Therefore, just as in the standard case [17], the stopping criterion in Algorithm 2.1 can also be based on the condition $\|Z^{(k)}\| \le tol$ or $\|Z^{(k)}\|/\|Z_k\| \le tol$ with some matrix norm $\|\cdot\|$.

REMARK 2.3. The matrices $(E + \tau_k A)^{-1}$ in Algorithm 2.1 do not have to be computed explicitly. Instead, we solve linear systems of the form $(E + \tau_k A)x = P_l b$ either by computing (sparse) LU factorizations and forward/backward substitutions or by using iterative Krylov subspace methods [26]. In the latter case the generalized LR-ADI method has the memory complexity $O(k_{ADI}mn)$ and costs $O(k_{\mathrm{ls}} k_{ADI}mn)$ flops, where k_{ADI} is the number of outer ADI iterations and k_{ls} is the number of inner linear solver iterations. This method becomes efficient for large-scale sparse Lyapunov equations only if $k_{\mathrm{ls}} k_{ADI} m$ is much smaller than n.

REMARK 2.4. In exact arithmetic the matrices Z_k satisfy $Z_k = P_r Z_k$ and, hence, the second equation in (1.2) is fulfilled for the low-rank approximation $Z_k Z_k^T$. However, in finite precision arithmetic a drift-off effect may occur. In this case we need to project $Z^{(k)}$ onto the image of P_r by pre-multiplication with P_r. In order to limit the additional computation cost we can do this, for example, at every second or third iteration step.

REMARK 2.5. If E is nonsingular, but ill-conditioned with respect to inversion, then Algorithm 2.1 may provide a better result than the classical LR-ADI method [16, 17, 22] applied to the matrices $E^{-1}A$ and $E^{-1}B$.

Note that if at least one of the shift parameters is complex, then the low-rank Cholesky factor Z_k may be complex although the solution X of (1.2) is real. As in the standard case [17, 22], in order to keep the real factors, we can take the complex shift parameters in complex conjugate pairs $\{\tau_k, \tau_{k+1}\}$ with $\tau_{k+1} = \overline{\tau}_k$ and compute the iterates Z_k as follows

$$Z_k = [\, Z_{k-1}, \;\; Z^{(k)} \,] \qquad \text{if} \qquad \tau_k \;\text{ is real} \tag{2.13}$$

and

$$\begin{cases} Z_k = [\, Z_{k-1}, \;\; Z_1^{(k)} \,] \\ Z_{k+1} = [\, Z_k, \;\; Z_2^{(k)} \,] \end{cases} \qquad \text{if} \qquad \begin{cases} \tau_k \;\text{ is complex} \\ \tau_{k+1} = \overline{\tau}_k \end{cases}. \tag{2.14}$$

Here $Z_1^{(k)} = \sqrt{2}\,|\tau_k|(E + \tau_{k+1}A)^{-1}AZ^{(k)}$, $Z_2^{(k)} = \sqrt{2}\,(E + \tau_{k+1}A)^{-1}EZ^{(k)}$ and $Z^{(k)}$ is as in (2.11). One can show that (2.13) and the double step (2.14) provide the real Cholesky factors of the solution of the projected GCALE (1.2).

PROPOSITION 2.6. *Let E, $A \in \mathbb{R}^{n,n}$ and $B \in \mathbb{R}^{n,m}$. Assume that $\lambda E - A$ is stable and that the complex shift parameters appear in complex conjugate pairs $\{\tau_k, \tau_{k+1} = \overline{\tau}_k\}$. Then the sequence of the matrices Z_k in* (2.13), (2.14) *is real. Furthermore, for complex τ_k, we have*

$$Z_{k+1}Z_{k+1}^T = [\, Z_{k-1}, \;\; Z^{(k)}, \;\; Z^{(k+1)} \,][\, Z_{k-1}, \;\; Z^{(k)}, \;\; Z^{(k+1)} \,]^*.$$

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Proof. We prove the first part of this proposition by induction on k. If τ_1 is real, then

$$Z_1 = Z^{(1)} = \sqrt{-2\tau_1}(E+\tau_1 A)^{-1}P_l B$$

is real. If τ_1 is complex and $\tau_2 = \overline{\tau}_1$, then we have

$$\begin{aligned} Z_1 &= Z_1^{(1)} = 2\sqrt{-\mathrm{Re}(\tau_1)}\,|\tau_1|(E+\overline{\tau}_1 A)^{-1}A(E+\tau_1 A)^{-1}P_l B, \\ Z_2 &= [\,Z_1,\ 2\sqrt{-\mathrm{Re}(\tau_1)}(E+\overline{\tau}_1 A)^{-1}E(E+\tau_1 A)^{-1}P_l B\,]. \end{aligned} \tag{2.15}$$

Next we show that the matrices $(E+\overline{\tau}A)^{-1}A(E+\tau A)^{-1}$ and $(E+\overline{\tau}A)^{-1}E(E+\tau A)^{-1}$ are real for any complex τ. Since the pencil $\lambda E - A$ is stable, the matrix A is nonsingular. Therefore,

$$\begin{aligned} (E+\overline{\tau}A)^{-1}A(E+\tau A)^{-1} &= \big((E+\tau A)A^{-1}(E+\overline{\tau}A)\big)^{-1} \\ &= (EA^{-1}E + 2\mathrm{Re}(\tau)E + |\tau|^2 A)^{-1}. \end{aligned}$$

Clearly, the inverse of a real matrix is also real. Furthermore, we have that

$$(E+\overline{\tau}A)^{-1}E(E+\tau A)^{-1} = A^{-1}E(E+\overline{\tau}A)^{-1}A(E+\tau A)^{-1}$$

is real. Thus, Z_1 and Z_2 in (2.15) are both real.

Assume now that the matrix Z_k is real, where k is the index of a real parameter or of the second element in a complex conjugate pair $\{\tau_{k-1}, \tau_k = \overline{\tau}_{k-1}\}$. Then the matrix

$$\begin{aligned} \hat{Z}^{(k)} &= (E-\overline{\tau}_k A)Z^{(k)} \\ &= \sqrt{-2\mathrm{Re}(\tau_k)}(E-\overline{\tau}_k A)(E+\tau_k A)^{-1}\cdot\ldots\cdot(E-\overline{\tau}_1 A)(E+\tau_1 A)^{-1}P_l B \end{aligned}$$

is real. For real τ_{k+1}, we obtain that

$$Z^{(k+1)} = \sqrt{\frac{\mathrm{Re}(\tau_{k+1})}{\mathrm{Re}(\tau_k)}}(E+\tau_{k+1}A)^{-1}\hat{Z}^{(k)}$$

is real, and, hence, $Z_{k+1} = [\,Z_k,\ Z^{(k+1)}\,]$ is also real. If τ_{k+1} is complex, then

$$\begin{aligned} Z_1^{(k+1)} &= |\tau_{k+1}|\sqrt{\frac{2\mathrm{Re}(\tau_{k+1})}{\mathrm{Re}(\tau_k)}}(E+\overline{\tau}_{k+1}A)^{-1}A(E+\tau_{k+1}A)^{-1}\hat{Z}^{(k)}, \\ Z_2^{(k+1)} &= \sqrt{\frac{2\mathrm{Re}(\tau_{k+1})}{\mathrm{Re}(\tau_k)}}(E+\overline{\tau}_{k+1}A)^{-1}E(E+\tau_{k+1}A)^{-1}\hat{Z}^{(k)} \end{aligned}$$

are real. Therefore, $Z_{k+1} = [\,Z_k,\ Z_1^{(k+1)}\,]$ and $Z_{k+2} = [\,Z_{k+1},\ Z_2^{(k+1)}\,]$ are also real.

Further, for complex τ_k, we have $Z^{(k+1)} = \big(I - 2\tau_{k+1}(E+\tau_{k+1}A)^{-1}A\big)\,Z^{(k)}$ and

$$\begin{aligned} Z_{k+1}Z_{k+1}^T &= Z_{k-1}Z_{k-1}^T + Z_1^{(k)}\big(Z_1^{(k)}\big)^T + Z_2^{(k)}\big(Z_2^{(k)}\big)^T \\ &= Z_{k-1}Z_{k-1}^T + 2\,Z^{(k)}\big(Z^{(k)}\big)^* \\ &\quad -2\tau_{k+1}(E+\tau_{k+1}A)^{-1}AZ^{(k)}\big(Z^{(k)}\big)^* - 2\overline{\tau}_{k+1}Z^{(k)}\big(Z^{(k)}\big)^*A^T(E+\overline{\tau}_{k+1}A)^{-T} \\ &\quad +4\,|\tau_{k+1}|^2(E+\tau_{k+1}A)^{-1}AZ^{(k)}\big(Z^{(k)}\big)^*A^T(E+\overline{\tau}_{k+1}A)^{-T} \\ &= Z_{k-1}Z_{k-1}^T + Z^{(k)}\big(Z^{(k)}\big)^* + Z^{(k+1)}\big(Z^{(k+1)}\big)^* \\ &= [\,Z_{k-1},\ Z^{(k)},\ Z^{(k+1)}\,][\,Z_{k-1},\ Z^{(k)},\ Z^{(k+1)}\,]^*. \quad \square \end{aligned}$$

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Observe that (2.13) and (2.14) involve, in general, complex matrix operations. Complex arithmetic can be avoided if we rewrite (2.13) and (2.14) as

$$\begin{aligned} Z^{(k)} &= \sqrt{\frac{\mathrm{Re}(\tau_k)}{\mathrm{Re}(\tau_{k-1})}}(E+\tau_k A)^{-1}\hat{Z}^{(k-1)}, \\ Z_k &= [\,Z_{k-1},\;\; Z^{(k)}\,], \quad \hat{Z}^{(k)} = (E-\overline{\tau}_k A)Z^{(k)} \end{aligned}$$

for real τ_k and

$$\begin{aligned} Z_1^{(k)} &= |\tau_k|\sqrt{\frac{2\mathrm{Re}(\tau_k)}{\mathrm{Re}(\tau_{k-1})}}(EA^{-1}E+2\mathrm{Re}(\tau_k)E+|\tau_k|^2A)^{-1}\hat{Z}^{(k-1)}, \\ Z_2^{(k)} &= \sqrt{\frac{2\mathrm{Re}(\tau_k)}{\mathrm{Re}(\tau_{k-1})}}A^{-1}E(EA^{-1}E+2\mathrm{Re}(\tau_k)E+|\tau_k|^2A)^{-1}\hat{Z}^{(k-1)} = \frac{1}{|\tau_k|}A^{-1}EZ_1^{(k)}, \\ Z_{k+1} &= [\,Z_{k-1},\;\; Z_1^{(k)},\;\; Z_2^{(k)}\,], \\ \hat{Z}^{(k+1)} &= \sqrt{\frac{\mathrm{Re}(\tau_k)}{\mathrm{Re}(\tau_{k-1})}}(EA^{-1}E-2\mathrm{Re}(\tau_k)E+|\tau_k|^2A)(EA^{-1}E+2\mathrm{Re}(\tau_k)E+|\tau_k|^2A)^{-1}\hat{Z}^{(k-1)} \end{aligned}$$

for a complex conjugate pair $\{\tau_k, \tau_{k+1} = \overline{\tau}_k\}$. A drawback of this procedure is that the inverse of $EA^{-1}E+2\mathrm{Re}(\tau_k)E+|\tau_k|^2A$ is required. Solving linear systems with this matrix is usually more expensive than with $E+\tau_k A$.

3. Smith method. For any parameter $\tau \in \mathbb{C}^-$, the projected GCALE (1.2) is equivalent to the projected discrete-time Lyapunov equation

$$\mathcal{A}X\mathcal{A}^* - X = -P_r\mathcal{B}\mathcal{B}^*P_r^T, \qquad X = P_rXP_r^T, \tag{3.1}$$

where

$$\begin{aligned} \mathcal{A} &= (E+\tau A)^{-1}(E-\overline{\tau}A) = I - 2\mathrm{Re}(\tau)(E+\tau A)^{-1}A, \\ \mathcal{B} &= \sqrt{-2\mathrm{Re}(\tau)}\,(E+\tau A)^{-1}B. \end{aligned}$$

Note that if the pencil $\lambda E - A$ is stable, then P_r is the spectral projector onto the invariant subspace of the matrix $\mathcal{A}$ corresponding to the eigenvalues inside the unit circle. In this case the Smith iteration

$$X_0 = P_r\mathcal{B}\mathcal{B}^*P_r^T, \qquad X_k = P_r\mathcal{B}\mathcal{B}^*P_r^T + \mathcal{A}X_{k-1}\mathcal{A}^* \tag{3.2}$$

can be used to compute an approximate solution of (3.1); see [29]. The number of iterations required for a desired accuracy in X_k depends on the parameter τ. Note that the Smith method (3.2) is, in fact, the generalized ADI iteration (2.3) with a single parameter $\tau = \tau_1 = \ldots = \tau_k$.

A modification of the Smith method has been proposed in [22] for computing a low-rank Cholesky factor of the solution of standard Lyapunov equations with a low-rank right-hand side. This version of the Smith method is based on the LR-ADI iteration with ℓ shift parameters applied in a cyclic manner and referred to as the *low-rank cyclic Smith* (LR-Smith(ℓ)) *method*. It can be generalized for the projected GCALE (1.2) as follows: first one determines Z_ℓ using the generalized LR-ADI method with the shift parameters $\tau_1, \ldots, \tau_\ell$ and then solve the discrete-time Lyapunov equation

$$\mathcal{A}_\ell X \mathcal{A}_\ell^* - X = -Z_\ell Z_\ell^*,$$

where $\mathcal{A}_\ell$ is as in (2.5) with $k=\ell$. In summary, we have the following algorithm to compute the low-rank Cholesky factor of the solution of the projected GCALE (1.2).

ALGORITHM 3.1. *The generalized LR-Smith(ℓ) method for the projected GCALE.*

INPUT: E, A, $P_l, P_r \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, *shift parameters* $\tau_1,\ldots,\tau_\ell \in \mathbb{C}^-$.
OUTPUT: *A low rank Cholesky factor* $Z_{k\ell}$ *of the solution* $X \approx Z_{k\ell}Z_{k\ell}^*$ *of* (1.2).
1. *Compute* Z_ℓ *using Algorithm* 2.1 *and set* $Z^{(1,\ell)} = Z_\ell$.
2. FOR $k = 2,3,\ldots$
$Z^{(k,0)} = Z^{(k-1,\ell)}$,
FOR $j=1,\ldots,\ell$

$$Z^{(k,j)} = \big(I - 2\mathrm{Re}(\tau_j)(E+\tau_jA)^{-1}A\big)Z^{(k,j-1)}, \tag{3.3}$$

END FOR
$Z_{k\ell} = [\,Z_{(k-1)\ell},\;\; P_rZ^{(k,\ell)}\,]$.
END FOR

Note that if Z_ℓ is real and the complex shift parameters appear in complex conjugate pairs, then the matrices $Z_{k\ell}$ are also real.

REMARK 3.1. The generalized LR-Smith(ℓ) method is equivalent to the generalized LR-ADI method with cyclically repeated shift parameters. However, comparing (2.11) and (3.3), where $Z^{(k-1)} \in \mathbb{C}^{n,m}$ and $Z^{(k,j-1)} \in \mathbb{C}^{n,\ell m}$, respectively, one can see that Algorithm 2.1 is more efficient than Algorithm 3.1.

REMARK 3.2. At every iteration step in the generalized LR-ADI and LR-Smith(l) methods the number of columns of the approximate solution factors Z_k and $Z_{k\ell}$ grows by m and ℓm, respectively. To keep the low-rank structure in the Cholesky factors in case of large m and/or slow convergence, we can replace the iterate by its low-rank approximation computed via the updated singular value decomposition; see [11] for details.

Consider now the projected GDALE (1.4). If the matrix A is nonsingular, then equation (1.4) is equivalent to the projected discrete-time Lyapunov equation

$$X - (A^{-1}E)X(A^{-1}E)^T = Q_rA^{-1}BB^TA^{-T}Q_r^T, \qquad X = Q_rXQ_r^T. \tag{3.4}$$

Note that Q_r is the spectral projector onto the invariant subspace of the matrix $A^{-1}E$ corresponding to the zero eigenvalues. In this case $Q_rA^{-1}E = A^{-1}EQ_r$ is nilpotent with the index of nilpotency ν that is equal to the index of the pencil $\lambda E - A$. The unique solution of the projected Lyapunov equation (3.4) is given by

$$X = \sum_{k=0}^{\nu-1}(A^{-1}E)^kQ_rA^{-1}BB^TA^{-T}Q_r^T((A^{-1}E)^T)^k.$$

Thus, the Cholesky factor Y of the solution $X = YY^T$ of (3.4) and also of the projected GDALE (1.4) has the form

$$Y = [\,Q_rA^{-1}B,\; A^{-1}EQ_rA^{-1}B,\; \ldots,\; (A^{-1}E)^{\nu-1}Q_rA^{-1}B\,].$$

It can be computed by the following algorithm that is a generalization of the Smith method for the projected GDALE (1.4).

ALGORITHM 3.2. *The generalized Smith method for the projected GDALE.*

INPUT: *E, A, $Q_r \in \mathbb{R}^{n,n}$ and $B \in \mathbb{R}^{n,m}$.*
OUTPUT: *A Cholesky factor Y_ν of the solution $X = Y_\nu Y_\nu^T$ of* (1.4).
1. $Y^{(1)} = Q_r A^{-1} B, \quad Y_1 = Y^{(1)}$;
2. FOR $k = 2, 3, \ldots, \nu$
$$Y^{(k)} = A^{-1} E Y^{(k-1)}, \quad Y_k = [\, Y_{k-1}, \;\; Y^{(k)}\,].$$
END FOR

Note that if the index ν of the pencil $\lambda E - A$ is unknown, then the iteration in Algorithm 3.2 can be stopped as soon as $\|Y^{(k)}\| \le tol$ or $\|Y^{(k)}\|/\|Y_k\| \le tol$ with some matrix norm $\|\cdot\|$ and a tolerance tol. If we want to compute the solution of (1.4) as accurate as possible, we should set tol to the machine precision. To avoid the drift-off of the columns of Y_ν from the image of Q_r, the matrices $Y^{(k)}$ should be pre-multiplied with Q_r after some iteration steps.

In the case of singular A, we can consider the Lyapunov equation

$$X - QEXEQ^T = QBB^TQ^T \tag{3.5}$$

with $Q = Q_r(EP_r + AQ_r)^{-1} = (P_l E + Q_l A)^{-1} Q_l$. This equation has the same solution as the projected GDALE (1.4), see [41]. Therefore, the Cholesky factor Y_ν of the solution $X = Y_\nu Y_\nu^T$ of (1.4) can also be computed by applying the Smith method to equation (3.5). Similarly to the projectors Q_r and Q_l, the matrix Q can be constructed in explicit form using the special block structure of E and A.

4. Numerical examples. In this section we present some results of numerical experiments. Computations were done on IBM RS 6000 44P Modell 270 with machine precision $\varepsilon \approx 2.22 \times 10^{-16}$ using MATLAB 7. In all examples the suboptimal ADI shift parameters were computed as described in Section 2.1.

EXAMPLE 4.1. Consider the 2D instationary Stokes equation that describes the flow of an incompressible fluid in a domain. The spatial discretization of this equation by the finite difference method on a uniform staggered grid leads to the descriptor system (1.1) with the matrices E and A as in (1.9), where $E_{11} = I$, $A_{11} = A_{11}^T$ and $A_{21} = A_{12}^T$. In our experiments the state space dimension of the problem is $n = 11040$, and the matrix $B \in \mathbb{R}^{n,1}$ is chosen at random. Note that E and A are symmetric and E is positive semidefinite. In this case the finite eigenvalues of $\lambda E - A$ are real.

In Figure 4.1 we present the normalized residual norm $\eta(Z_k)$ as in (2.12) and the ratio $\zeta(Z_k) = \|Z^{(k)}\|_F / \|Z_k\|_F$ for the generalized LR-ADI method with $\ell = 10$ real shift parameters. The spectral radius of the matrix $\mathcal{A}_{10}$ as in (2.5) is $\rho(\mathcal{A}_{10}) = 0.01$. One can see that the generalized LR-ADI method converges fast, and the solution of the projected GCALE (1.2) can be approximated quite accurately by a matrix of rank 30. The normalized residual norm $\eta(Z_k)$ stagnates on a relatively small level, which is caused by round-off errors. Note that $\zeta(Z_k)$ does not decrease monotonically and more iteration steps are required to achieve $\zeta(Z_k) \le tol$ than $\eta(Z_k) \le tol$.

Figure 4.2 shows the convergence history of the normalized residual norm for the generalized LR-ADI method and the generalized LR-Smith(10) method versus the iteration number. As expected, both methods give similar results.

Furthermore, we computed the full rank Cholesky factor $Y_2 \in \mathbb{R}^{n,2}$ of the solution $X = Y_2 Y_2^T$ of the projected GDALE (1.4) using Algorithm 3.2. The Frobenius norms of the

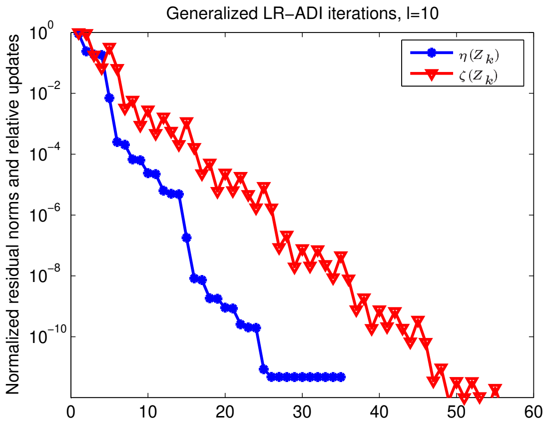

FIG. 4.1. *Example 4.1: convergence history for the generalized LR-ADI method.*

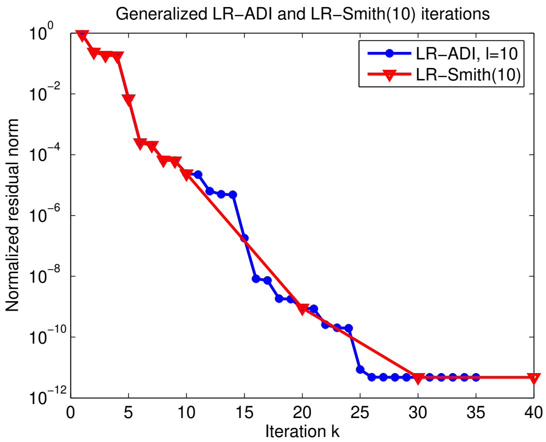

FIG. 4.2. *Example 4.1: normalized residuals for the generalized LR-ADI and LR-Smith(10) methods.*

update matrices are $\|Y^{(1)}\|_F = 3.874 \times 10^5$, $\|Y^{(2)}\|_F = 1.295 \times 10^2$ and $\|Y^{(3)}\|_F \le \varepsilon$. This is not surprising because the pencil $\lambda E - A$ is of index 2.

EXAMPLE 4.2. Consider the descriptor system (1.1) with non-symmetric matrices E and A as in (1.9) that has been obtained by the finite element discretization of a convection problem with a boundary control, see [19] for details. The problem has the state space dimension $n = 2909$, and the matrix $B = [\,0,\; B_2^T\,]^T \in \mathbb{R}^{n,1}$ results from the boundary control.

In Figure 4.3 we present the convergence history in terms of the normalized residual norms for the generalized LR-ADI method and the generalized LR-Smith(11) method. One can see that the solution of the projected GCALE (1.2) can be approximated by a matrix of rank 35. The spectral radius of $\mathcal{A}_{11}$ is $\rho(\mathcal{A}_{11}) = 0.018$.

The solution of the projected GDALE (1.4) has been computed in factored form $X = Y_2 Y_2^T$ with the full rank Cholesky factor $Y_2 \in \mathbb{R}^{n,2}$. The Frobenius norms of the update matrices are $\|Y^{(1)}\|_F = 2.397$, $\|Y^{(2)}\|_F = 1.357 \times 10^{-3}$ and $\|Y^{(3)}\|_F \le \varepsilon$.

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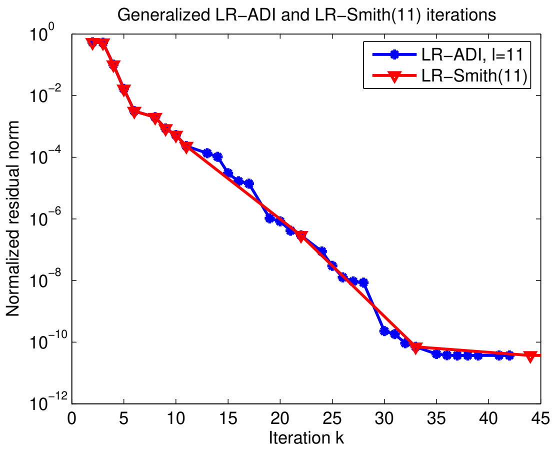

FIG. 4.3. *Example 4.2: normalized residuals for the generalized LR-ADI and LR-Smith(11) methods.*

EXAMPLE 4.3. Consider a damped mass-spring system with g masses, see [37, Section 3.9]. The i-th mass of weight m_i is connected to the $(i+1)$-st mass by a spring and a damper with constants k_i and d_i, respectively, and also to the ground by another spring and damper with constants δ_i and κ_i, respectively. Additionally, we assume that the first mass is connected to the last one by a rigid bar and it can be influenced by a control. The vibration of this system is described by the descriptor system (1.1) with the matrices E and A as in (1.10). For $g = 5000$, we obtain a problem of the state space dimension $n = 10001$ with $B \in \mathbb{R}^{n,1}$. The system parameters are $m_1 = \ldots = m_g = 100$ and

$$\begin{array}{ll} k_1 = \ldots = k_{g-1} = k = 2, & \kappa_1 = \ldots = \kappa_g = \kappa = 4, \\ d_1 = \ldots = d_{g-1} = d = 3, & \delta_1 = \ldots = \delta_g = \delta = 7. \end{array}$$

Figure 4.4 shows the normalized residual norms $\eta(Z_k)$ and the relative updates $\zeta(Z_k)$ for the generalized LR-ADI method with $\ell = 12$ complex ADI shift parameters. We see that the low-rank Cholesky factor Z_k of the solution of the projected GCALE (1.2) computed with the stopping criterion $\zeta(Z_k) \le tol$ has about twice more columns than those computed with the stopping criterion $\eta(Z_k) \le tol$. Since the spectral radius $\rho(\mathcal{A}_{12}) = 0.003$ is very small, the generalized LR-ADI iteration converges very fast.

Furthermore, we change the damping constants to $d = 0.3$ and $\delta = 0.7$ that results in $\rho(\mathcal{A}_{12}) = 0.718$. Figure 4.5 shows that in this case the normalized residual norms $\eta(Z_k)$ decay quite slowly compared to the previous experiment with $\rho(\mathcal{A}_{12}) = 0.003$. This example demonstrates that the convergence rate of the generalized LR-ADI method strongly depends on the spectral radius of the iteration matrix in (2.5).

Finally, we investigate the impact of the growing number of columns in $B \in \mathbb{R}^{n,m}$ on the convergence of the LR-ADI method. Assuming that the first m masses are influenced by controls, we obtain $B = [\, e_{g+1}, \ldots, e_{g+m}\,]$, where e_j denotes the j-th column of I_{2g+1}. Figure 4.6 shows the normalized residual norms for the generalized LR-ADI method applied to the problems with $m = 1, 5$ and 10 inputs. One can see that for different m the convergence is about the same.

5. Conclusion. In this paper we have discussed the numerical solution of large-scale projected generalized Lyapunov equations that arise, for example, in model reduction of descriptor systems. We have presented the generalized low-rank alternating direction implicit

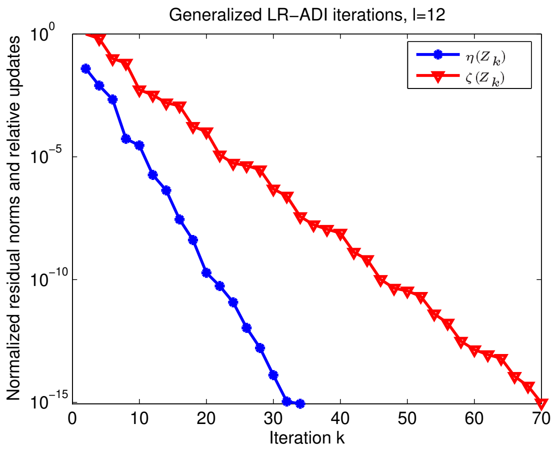

FIG. 4.4. *Example 4.3: convergence history for the generalized LR-ADI method.*

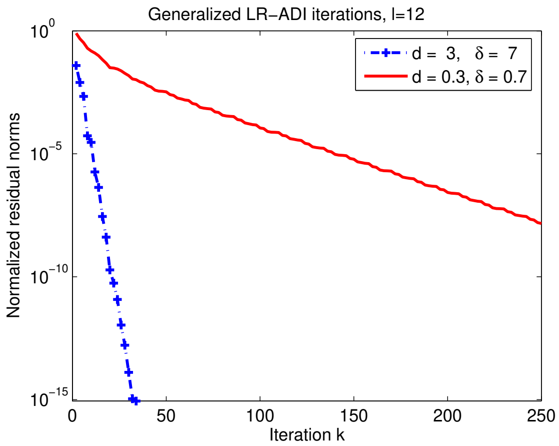

FIG. 4.5. *Example 4.3: convergence history for the generalized LR-ADI method for different sets of damping parameters.*

method and the generalized low-rank cyclic Smith method for computing low-rank approximations to the solutions of these equations. The efficiency of these methods has been demonstrated by numerical experiments.

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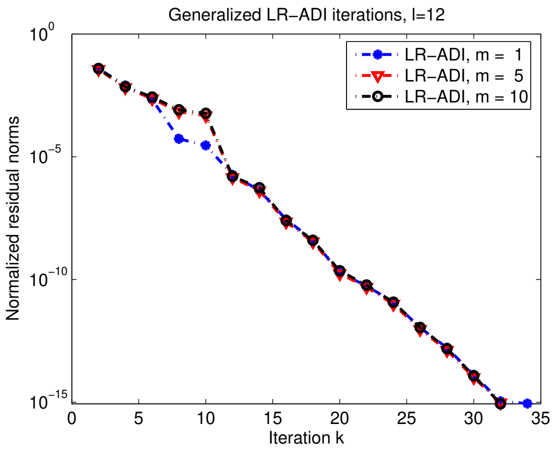

FIG. 4.6. *Example 4.3: normalized residual norms for different number of inputs.*

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