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A SIMPLIFICATION OF THE LAPLACE METHOD FOR DOUBLE INTEGRALS. APPLICATION TO THE SECOND APPELL FUNCTION*

JOSÉ L. LÓPEZ† AND PEDRO J. PAGOLA†

Abstract. The main difficulties in the Laplace method of asymptotic expansions of double integrals result from a change of variables. Generalizing previous work for simple integrals, we propose a variant of the method for double integrals, which avoids this change of variables and simplifies the computations. The calculation of the coefficients of the asymptotic expansion is remarkably simple. Moreover, the asymptotic sequence is as simple as in the standard Laplace's method: inverse powers of the asymptotic variable. A new asymptotic expansion of the second Appell's function $F_2(a,b,b',c,c';x,y)$ for large b, b', c and c' is given as an illustration.

Key words. asymptotic expansions of integrals, Laplace method for double integrals, second Appell function

AMS subject classifications. 41A60, 41A58, 33C65

1. Introduction.

Consider integrals of the form

$$F(z) \equiv \int\int_{\Omega} e^{-zf(x,y)} g(x,y)dxdy, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded or unbounded convex domain, z is a large positive parameter and $f(x,y)$ and $g(x,y)$ are smooth enough functions in Ω. Long ago Laplace made the observation that the major contribution of the integrand to the integral (1.1) comes from the neighborhoods of the points where $f(x,y)$ attains its smallest value. For instance, if $f(x,y)$ has its minimum value only at a point $(x_0,y_0) \in \Omega^0$, where $f(x,y)$ and $g(x,y)$ are analytic, the gradient of f at that point vanishes, $\nabla f(x_0,y_0) = (0,0)$, and the Hessian matrix of f at that point, $Hf(x_0,y_0)$, is positive definite. Then, Laplace's result is

$$F(z) \sim \frac{\pi g(x_0,y_0)}{z\sqrt{\mathrm{Det}[Hf(x_0,y_0)]}} e^{-zf(x_0,y_0)}, \qquad z \to \infty. \tag{1.2}$$

The right hand side above is the first term of a complete asymptotic expansion that can be obtained in the following way [6, Chap. 8, Sec. 10]. The Hessian matrix $Hf(x_0,y_0)$ can be diagonalized after an orthogonal change of variables. Then, without loss of generality, we may assume that the Taylor expansion of $f(x,y)$ at (x_0,y_0) has the form

$$\begin{aligned} f(x,y) &= f(x_0,y_0) + a(x-x_0)^2 + b(y-y_0)^2 + \cdots \\ &= f(x_0,y_0) + a(x-x_0)^2[1+P(x,y)] + b(y-y_0)^2[1+Q(x,y)], \end{aligned}$$

where $a, b > 0$, $P(x,y)$ and $Q(x,y)$ are analytic at (x_0,y_0) and $P(x_0,y_0) = Q(x_0,y_0) = 0$. Perform in (1.1) the change of integration variables $(x,y) \to (u,v)$ defined by $u = (x-x_0)\sqrt{1+P(x,y)}$, $v = (y-y_0)\sqrt{1+Q(x,y)}$,

$$F(z) = e^{-zf(x_0,y_0)} \int\int_{\Omega'} e^{-z(au^2+bv^2)} h(u,v)dudv, \tag{1.3}$$

where Ω' is the image of Ω under this change of variables,

$$h(u,v) \equiv g(x(u,v),y(u,v))\frac{\partial(x,y)}{\partial(u,v)}, \tag{1.4}$$

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†Departamento de Ingeniería Matemática e Informática, Universidad Pública de Navarra, 31006-Pamplona, Spain (`{jl.lopez,pedro.pagola}@unavarra.es`).

and $\partial(x,y)/\partial(u,v)$ is the Jacobian of the transformation $(x,y)\to(u,v)$. If $h(u,v)$ has a Taylor expansion at $(u,v)=(0,0)$,

$$h(u,v)\sim\sum_{n=0}^{\infty}\sum_{m=0}^{n}c_{m,n-m}u^m v^{n-m}, \tag{1.5}$$

then we can apply Watson's Lemma to the integral (1.3): replace (1.5) in the right hand side of (1.3) and interchange sum and integral [6, Chap. 8, Sec. 10],

$$F(z)\sim e^{-zf(x_0,y_0)}\sum_{n=0}^{\infty}\sum_{m=0}^{n}c_{2m,2n-2m}\Phi_{n,m}(z), \qquad z\to\infty, \tag{1.6}$$

where the coefficients $c_{n,m}$ are the Taylor coefficients of the function $h(u,v)$ at $(u,v)=(0,0)$ (see (1.5)) and the asymptotic sequence $\Phi_{n,m}(z)$ is

$$\Phi_{n,m}(z)\equiv\int\int_{\Omega'}e^{-z(au^2+bv^2)}u^{2m}v^{2n-2m}dudv$$
$$\sim\int_{-\infty}^{\infty}e^{-zau^2}u^{2m}du\int_{-\infty}^{\infty}e^{-zbv^2}v^{2n-2m}dv=\frac{\Gamma(m+1/2)\Gamma(n-m+1/2)}{a^{m+1/2}b^{n-m+1/2}z^{n+1}}.$$

We see that in the standard Laplace's method, the computation of the asymptotic sequence $\Phi_{n,m}(z)$ is straightforward, whereas the computation of the coefficients $c_{n,m}$ is very difficult because of the complexity of the above mentioned change of variable; see the example in [6, Chap. 8, Sec. 10] where the first term of the expansion of a double integral is derived.

A first attempt to simplify this procedure is explored in [2] by means of an example: a double integral representation of the Appell function F_2. For particular functions f and g in the integral (1.1), it is shown that a Taylor expansion of g at the critical point of f is enough to obtain an asymptotic expansion of (1.1) for large z. As it is shown in [2], this idea is very useful to obtain a uniform expansion of F_4, which requires a two-point Taylor expansion of the function g. On the other hand, as the example of the Appell function F_2 shows, the practical applicability of the method requires a not very complicated functional form for f.

In this paper we investigate a different simplification of the Laplace's method in which we expand both f and g at the critical point of f and then the complexity of the function f is not a handicap. Moreover, we perform a general analysis for the integral (1.1) and then we apply this analysis to the particular example of the Appell function F_2, obtaining a new expansion of this function. The idea is based in the modification of the Laplace's method for one-dimensional integrals proposed in [3], which simplifies the computation of the coefficients of the expansion without complicating the computation of the asymptotic sequence. It is shown there that, for one-dimensional integrals, a change of variable is not necessary to obtain an asymptotic expansion. Consider the integral

$$\int_a^b e^{-zf(x)}g(x)dx. \tag{1.7}$$

Inspired on the idea of Burkhardt and Perron [1, Chap. 2], it is shown in [3] that it is just necessary to expand both, $f(x)$ and $g(x)$ at the critical point $x_0\in(a,b)$ of $f(x)$ and interchange sum and integral. Suppose that $f'(x_0)=0$ and $f''(x_0)>0$ and write

$$f(x)=f(x_0)+\frac{f''(x_0)}{2}(x-x_0)^2+f_3(x), \qquad g(x)e^{-zf_3(x)}=\sum_{n=0}^{\infty}a_n(z)(x-x_0)^n.$$

Introducing this expansion in (1.7) and replacing sum and integral we find

$$(1.8)\qquad \int_a^b e^{-zf(x)}g(x)dx = e^{-zf(x_0)}\sum_{n=0}^{\infty} a_n(z)\int_a^b e^{-\frac{1}{2}zf''(x_0)(x-x_0)^2}(x-x_0)^n dx$$
$$\sim e^{-zf(x_0)}\sum_{n=0}^{\infty} a_{2n}(z)\frac{2^{n+1/2}\Gamma(n+1/2)}{(f''(x_0)z)^{n+1/2}}.$$

It is shown in [3] that $a_n(z) = \mathcal{O}(z^{-\lfloor n/3 \rfloor})$ as $z \to \infty$ and also that the above computation is not only formal but correct. Because the coefficients a_n depend of z, (1.8) is not a genuine Poincaré expansion, but the terms of the expansion can be grouped to obtain a genuine Poincaré expansion [3].

In this paper we extend the simplifying idea introduced in [3] from simple to double integrals. We will simplify the computation of the coefficients of the expansion without complicating the computation of the asymptotic sequence. This is shown in Section 2. In Section 3 we apply the idea to the second Appell function, obtaining a new asymptotic expansion of this function when four of its variables are large. Section 4 contains some final remarks.

2. The modified Laplace's method. Suppose that $f(x,y)$ has a finite number of isolated absolute minima in $\bar{\Omega}$. (We will not consider here coalescence of minima or poles of the integrand or other situations that require a uniform approach.) Then, by subdividing the domain of integration we may assume, without loss of generality, that $f(x,y)$ has only one absolute minimum in $\bar{\Omega}$. Moreover, by means of a simple translation of the variables (x,y), we may assume that that minimum occurs at $(x,y) = (0,0) \in \bar{\Omega}$. Suppose that both $f(x,y)$ and $g(x,y)$ have a Taylor expansion at $(0,0)$ with a common radius of convergence r. This condition may be relaxed and require only that both $f(x,y)$ and $g(x,y)$ have an asymptotic expansion at $(0,0)$. But for the sake of clarity in the exposition we require the analyticity of f and g at the minimum of f. On the other hand, this is the usual situation in most of the practical examples.

We will consider that the boundary $\partial\Omega$ of Ω is piecewise smooth, that is, it has a finite number of corner points. Moreover, by subdividing Ω, we can consider that the angle between the two tangent lines at the corner points is not greater than $\pi/2$. We will consider that the unique absolute minimum $(0,0)$ of $f(x,y)$ in $\bar{\Omega}$ falls into one of the following three categories (P_1), (P_2) or (P_3) detailed below (see Fig. 2.1):

(P_1) $\vec{\nabla} f(0,0) = (0,0)$ and $Hf(0,0) > 0$, that is, $(0,0)$ is a non-degenerate local minimum of $f(x,y)$. We can choose orthogonal coordinates (x,y) in which $Hf(0,0)$ is diagonal. In these coordinates, the first terms of the Taylor expansion of $f(x,y)$ at $(0,0)$ are

$$f(x,y) = f(0,0) + ax^2 + by^2 + \cdots,$$

with $a = \frac{1}{2}f_{xx}(0,0) > 0$ and $b = \frac{1}{2}f_{yy}(0,0) > 0$.

If $\vec{\nabla} f(0,0) \neq (0,0)$, then the point $(0,0)$ can only be located at $\partial\Omega$. In this case we distinguish two different situations (P_2) and (P_3) detailed below.

(P_2) $\vec{\nabla} f(0,0) \neq (0,0)$ and $(0,0)$ is a smooth point of $\partial\Omega$. Then $\vec{\nabla} f(0,0)$ must point inside Ω. Moreover, $\vec{v} \cdot \vec{\nabla} f(0,0) = 0$ for any vector $\vec{v}$ tangent to $\partial\Omega$ at $(0,0)$. We require for that tangent vector $\vec{v}$ that $\vec{v}^{\,T} \cdot Hf(0,0) \cdot \vec{v} > 0$ [6, Chap. 8, Sec. 2]. We can choose orthogonal coordinates (x,y) such that $x > 0$ for every $(x,y) \in \Omega$, $\vec{v} = (0, v_y)$ and $\vec{\nabla} f(0,0) = (a,0)$, with $a = f_x(0,0) > 0$ and $b = \frac{1}{2}f_{yy}(0,0) > 0$.

In these coordinates, the first terms of the Taylor expansion of $f(x,y)$ at $(0,0)$ are

$$f(x,y) = f(0,0) + ax + by^2 + \cdots.$$

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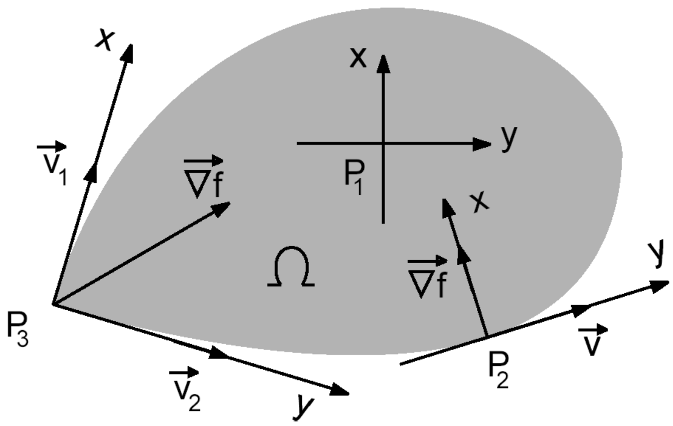

FIG. 2.1. *The contour $\partial\Omega$ of the domain Ω has only one corner point P_3. In the case (P_1) the minimum $(0,0)$ can be located at any point of $\bar{\Omega}$. In the cases (P_2) and (P_3) the minimum $(0,0)$ must be located in $\partial\Omega$.*

(P_3) $\vec{\nabla} f(0,0) \neq (0,0)$ and the point $(0,0)$ is a corner point of $\partial\Omega$. In this case, the boundary $\partial\Omega$ has two tangent vectors $\vec{v}_1$ and $\vec{v}_2$ at $(0,0)$ with $\vec{v}_1 \cdot \vec{\nabla} f(0,0) > 0$ and $\vec{v}_2 \cdot \vec{\nabla} f(0,0) > 0$. We can choose orthogonal coordinates (x,y) with $x > 0$ and $y > 0$ for every $(x,y) \in \Omega$ such that $a = f_x(0,0) > 0$ and $b = f_y(0,0) > 0$. In these coordinates, the first terms of the Taylor expansion of $f(x,y)$ at $(0,0)$ are

$$f(x,y) = f(0,0) + ax + by + \cdots.$$

We will not consider here other possible types of absolute minima such us degenerate relative minima, curves of minima, etc., [6, Chap. 8].

We define $f_1(x,y) \equiv f(x,y) - f_0(x,y)$, with

$$f_0(x,y) = f(0,0) + ax^\alpha + by^\beta \tag{2.1}$$

and the values of a, b, α, β and $s \equiv \beta/\alpha$ listed in Table 2.1.

TABLE 2.1
The possible values of a, b, α, β and s considered in (2.1) depend on the case (P_1), (P_2) or (P_3) under consideration and are always positive.

Case	a	b	α	β	s
(P_1)	$\frac{1}{2}f_{xx}(0,0)$	$\frac{1}{2}f_{yy}(0,0)$	2	2	1
(P_2)	$f_x(0,0)$	$\frac{1}{2}f_{yy}(0,0)$	1	2	2
(P_3)	$f_x(0,0)$	$f_y(0,0)$	1	1	1

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The function $e^{-zf_1(x,y)}$ has a Taylor expansion at $(x,y)=(0,0)$,

$$e^{z[ax^\alpha+by^\beta]}=\sum_{n=0}^{\infty}\frac{z^n}{n!}\left[ax^\alpha+by^\beta\right]^n=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{z^n}{m!(n-m)!}[ax^\alpha]^m[by^\beta]^{n-m}.$$

We write, for a given $r>0$,

$$e^{-zf(x,y)}=\sum_{n=0}^{\infty}\sum_{m=0}^{\lfloor n/s\rfloor}A_{m,n-sm}(z)x^my^{n-sm},\qquad x^2+y^2<r^2,$$

with

$$A_{m,p}(z)\equiv\frac{1}{m!p!}\frac{\partial^m}{\partial x^m}\frac{\partial^p}{\partial y^p}e^{-zf(x,y)}\Big|_{(x,y)=(0,0)}.\tag{2.2}$$

Also, the function $g(x,y)$ has a Taylor expansion at $(x,y)=(0,0)$,

$$g(x,y)=\sum_{n=0}^{\infty}\sum_{m=0}^{\lfloor n/s\rfloor}g_{m,n-sm}x^my^{n-sm},\qquad x^2+y^2<r^2,$$

with

$$g_{m,p}\equiv\frac{1}{m!p!}\frac{\partial^m}{\partial x^m}\frac{\partial^p}{\partial y^p}g(x,y)\Big|_{(x,y)=(0,0)}.$$

Define $h(x,y;z)\equiv e^{-zf_1(x,y)}g(x,y)$. Then,

$$\begin{aligned}h(x,y;z)=&\left[\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{z^n}{m!(n-m)!}[ax^\alpha]^m[by^\beta]^{n-m}\right]\times\left[\sum_{n=0}^{\infty}\sum_{m=0}^{\lfloor n/s\rfloor}A_{m,n-sm}(z)x^my^{n-sm}\right]\\&\times e^{zf(0,0)}\left[\sum_{n=0}^{\infty}\sum_{m=0}^{\lfloor n/s\rfloor}g_{m,n-sm}x^my^{n-sm}\right]=\sum_{n=0}^{\infty}\sum_{m=0}^{\lfloor n/s\rfloor}h_{m,n-sm}(z)x^my^{n-sm},\end{aligned}\tag{2.3}$$

with

$$h_{m,n-sm}(z)\equiv e^{zf(0,0)}\sum_{k=0}^{\lfloor n/\beta\rfloor}\sum_{j=\text{Max}\{0,k-\lfloor n/\beta-m/\alpha\rfloor\}}^{\text{Min}\{k,\lfloor m/\alpha\rfloor\}}\frac{z^ka^jb^{k-j}}{j!(k-j)!}H_{m-\alpha j,n-\beta k-sm+\beta j}(z)\tag{2.4}$$

and

$$H_{m,n-sm}(z)\equiv\sum_{k=0}^{n}\sum_{j=\text{Max}\{0,m-\lfloor(n-k)/s\rfloor\}}^{\text{Min}\{m,\lfloor k/s\rfloor\}}A_{j,k-sj}(z)g_{m-j,n-k-sm+sj},\tag{2.5}$$

empty sums being understood as zero.

Equation (2.4) is an explicit formula for the coefficients $h_{m,p}(z)$ of the Taylor expansion of $h(x,y;z)$ at $(x,y)=(0,0)$, although the asymptotic behavior of $h_{m,p}(z)$ when $z\to\infty$ is not clear from that formula. We obtain below a different formula for $h_{m,p}(z)$ which shows its asymptotic behavior (although it is less explicit). The asymptotic behaviour of the coefficients $h_{m,p}(z)$ depends on the order of the first term of the Taylor expansion of $f_1(x,y)$ at $(x,y)=(0,0)$. From the definition of $f_1(x,y)$ we have that, for $x^2+y^2<r^2$,

$$f_1(x,y) = \sum_{n=\beta+1}^{\infty}\sum_{m=0}^{\lfloor n/s\rfloor} f_{m,n-sm}x^m y^{n-sm} =$$

$$\left\{\begin{array}{rl} f_{3,0}x^3+f_{2,1}x^2y+f_{1,2}xy^2+f_{0,3}y^3+\cdots & \text{in case } (P_1)\\ f_{1,1}xy+f_{0,3}y^3+f_{2,0}x^2+f_{1,2}xy^2+f_{0,4}y^4+\cdots & \text{in case } (P_2)\\ f_{2,0}x^2+f_{1,1}xy+f_{0,2}y^2+\cdots & \text{in case } (P_3)\end{array}\right.$$

with

$$f_{m,p}\equiv \frac{1}{p!m!}\frac{\partial^m}{\partial x^m}\frac{\partial^p}{\partial y^p}f(x,y)\bigg|_{(x,y)=(0,0)}.$$

Therefore, for $x^2+y^2<r^2$,

$$e^{-zf_1(x,y)} = 1+\sum_{n=1}^{\infty}\frac{(-z)^n}{n!}\left[f_1(x,y)\right]^n = \sum_{n=0}^{\infty}\sum_{m=0}^{\lfloor n/s\rfloor} c_{m,n-sm}(z)x^m y^{n-sm}, \tag{2.6}$$

with $c_{0,0}(z)=1$ and, for $m=0,1,2,\ldots$,

$$\left\{\begin{array}{rl} c_{m,1-m}(z)=c_{m,2-m}(z)=0 \ \text{ and } \ c_{m,n-m}(z)=\mathcal{O}(z^{\lfloor n/3\rfloor}) & \text{for } n\geq 3 \ \text{ in } (P_1)\\ c_{0,1}(z)=c_{m,2-2m}(z)=0, c_{m,n-2m}(z)=\mathcal{O}(z^{\text{Min}\{\lfloor n/3\rfloor,\lfloor(n-m)/2\rfloor\}}) & \text{for } n\geq 3 \ \text{ in } (P_2)\\ c_{m,1-m}(z)=0 \ \text{ and } \ c_{m,n-m}(z)=\mathcal{O}(z^{\lfloor n/2\rfloor}) & \text{for } n\geq 2 \ \text{ in } (P_3).\end{array}\right.$$

Coefficients $h_{m,n-sm}(z)$ are defined in (2.4), but they may also be written in the form

$$h_{m,n-sm}(z) = \sum_{k=0}^{n}\ \sum_{j=\text{Max}\{0,m-\lfloor(n-k)/s\rfloor\}}^{\text{Min}\{m,\lfloor k/s\rfloor\}} c_{j,k-sj}(z)g_{m-j,n-k-sm+sj} \tag{2.7}$$

and we have, as $z\to\infty$,

$$h_{m,n-sm}(z) = \left\{\begin{array}{ll} \mathcal{O}\left(z^{\lfloor n/3\rfloor}\right) & \text{in case } (P_1)\\ \mathcal{O}\left(z^{\lfloor n/3\rfloor}\right) & \text{in case } (P_2)\\ \mathcal{O}\left(z^{\lfloor n/2\rfloor}\right) & \text{in case } (P_3).\end{array}\right. \tag{2.8}$$

With these preliminaries, we can write the integral (1.1) in the form

$$F(z) = e^{-zf(0,0)}\int\int_{\Omega} e^{-z(ax^\alpha+by^\beta)}h(x,y;z)dxdy. \tag{2.9}$$

The Taylor series in the right hand side of (2.3) converges uniformly and absolutely to the function $h(x,y;z)$ in the disk $x^2+y^2<r^2$. Therefore, if the integration region Ω is contained in that disk, we can replace that expansion in (2.9) and interchange sum and integral,

$$F(z) = e^{-zf(0,0)}\sum_{n=0}^{\infty}\sum_{m=0}^{\lfloor n/s\rfloor} h_{m,n-sm}(z)\Phi_{m,n-sm}(z), \tag{2.10}$$

with

$$\Phi_{n,p}(z) \equiv \int\int_\Omega e^{-z(ax^\alpha+by^\beta)}x^m y^p dxdy.$$

After the change of variables $x \to z^{-1/\alpha}x$ and $y \to z^{-1/\beta}y$ we find

$$\Phi_{m,n-sm}(z) = z^{-\frac{m+1}{\alpha}-\frac{n-sm+1}{\beta}} \int\int_{\Omega'} e^{-ax^\alpha}x^m e^{-by^\beta}y^{n-sm}dxdy, \tag{2.11}$$

where Ω' is the image of Ω after the above mentioned change of variables; see Fig. 2.2. We have

$$\left|\int\int_{\Omega'} e^{-ax^\alpha-by^\beta}x^m y^{n-sm}dxdy\right| \le \begin{cases} 4\int_0^\infty e^{-ax^2}x^m\,dx\int_0^\infty e^{-by^2}y^{n-m}\,dy & \text{in } (P_1)\\ 2\int_0^\infty e^{-ax}x^m\,dx\int_0^\infty e^{-by^2}y^{n-2m}\,dy & \text{in } (P_2)\\ \int_0^\infty e^{-ax}x^m\,dx\int_0^\infty e^{-by}y^{n-m}\,dy & \text{in } (P_3).\end{cases} \tag{2.12}$$

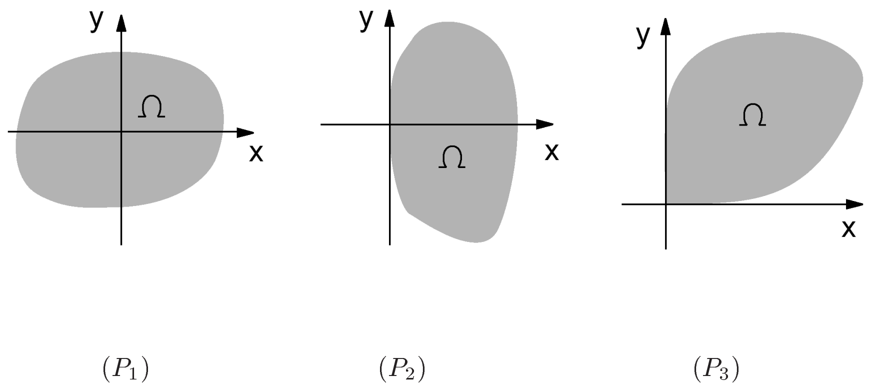

FIG. 2.2. *(P_1) The domain Ω' is the domain Ω expanded a factor $\sqrt{z}$. Then, asymptotically, $\Omega' \to \mathbb{R}^2$. (P_2) The domain Ω' is the domain Ω expanded a factor z in the x-direction and by a factor $\sqrt{z}$ in the y-direction. Then, $\Omega' \subset (-\infty,\infty)\times[0,\infty)$. (P_3) The domain Ω' is the domain Ω expanded a factor z. The angle at the corner points is not greater than $\pi/2$ and then $\Omega' \subset [0,\infty)\times[0,\infty)$.*

In the three cases, these integrals are finite and independent of z. Then, in general, $\Phi_{m,n-sm}(z) = \mathcal{O}(z^{-(m+1)/\alpha-(n-sm+1)/\beta})$. Although we can be more accurate in the cases (P_1) and (P_2). In the case (P_1) we have that $\Omega' \sim \mathbb{R}^2$ (see Fig. 2.2) and from (2.11) we see that if m or n are odd then $\Phi_{m,n-m}(z) = \mathcal{O}(z^{-p})$ for any $p \in \mathbb{N}$. In the case (P_2) we have that $\Omega' \sim [0,\infty)\times\mathbb{R}$ (see Fig. 2.2), and from (2.11) we see that if n is odd then $\Phi_{m,n-2m}(z) = \mathcal{O}(z^{-p})$ for any $p \in \mathbb{N}$. Therefore, we have

$$\begin{cases} \Phi_{2m,2n-2m}(z) = \mathcal{O}(z^{-n-1}) & \text{in case } (P_1)\\ \Phi_{m,2n-2m}(z) = \mathcal{O}(z^{-n-3/2}) & \text{in case } (P_2)\\ \Phi_{m,n-m}(z) = \mathcal{O}(z^{-n-2}) & \text{in case } (P_3).\end{cases}$$

We can write

$$F(z) = e^{-zf(0,0)} \sum_{n=0}^{\infty} \Lambda_n(z), \qquad \Lambda_n(z) \equiv \sum_{m=0}^{\lfloor n/s \rfloor} h_{m,n-sm}(z) \Phi_{m,n-sm}(z). \tag{2.13}$$

In cases (P_1) and (P_2), we have $\Lambda_{2n+1}(z) = \mathcal{O}\left(z^{-p}\right)$ for any $p \in \mathbb{N}$ and

$$\begin{cases} \Lambda_{2n}(z) = \mathcal{O}(z^{\lfloor 2n/3 \rfloor - n - 1}) & \text{in case } (P_1) \\ \Lambda_{2n}(z) = \mathcal{O}(z^{\lfloor 2n/3 \rfloor - n - 3/2}) & \text{in case } (P_2) \\ \Lambda_{n}(z) = \mathcal{O}(z^{\lfloor n/2 \rfloor - n - 2}) & \text{in case } (P_3). \end{cases}$$

Then, (2.13) is not a genuine Poincaré expansion. But we can group the terms of (2.13) in such a way that we get a genuine Poincaré expansion,

$$F(z) = e^{-zf(0,0)} \sum_{n=0}^{\infty} \Psi_n(z), \tag{2.14}$$

with $\Psi_0(z) = \Lambda_0(z)$ and, for $n = 1, 2, 3, \ldots$,

$$\Psi_n(z) \equiv \sum_{k=(\beta+1)n-\beta}^{(\beta+1)n} \Lambda_{\beta k}(z). \tag{2.15}$$

In the case (P_1) we have $\Psi_n(z) = \mathcal{O}\left(z^{-n-1}\right)$, in the case (P_2) we have $\Psi_n(z) = \mathcal{O}\left(z^{-n-3/2}\right)$ and, in the case, (P_3) we have $\Psi_n(z) = \mathcal{O}\left(z^{-n-2}\right)$. Observe that we have grouped the terms of the sum $\sum_{n=0}^{\infty} \Lambda_n(z)$ of (2.13) in blocks $\Psi_n(z)$ of two (in the case (P_3)) or three (in the cases (P_1) and (P_2)) terms $\Lambda_k(z)$.

If the integration domain Ω is not contained in the disk $D = \left\{(x,y); x^2 + y^2 < r^2\right\}$ of convergence of the Taylor series of $h(x,y;z)$ at $(x,y) = (0,0)$, we cannot replace the expansion (2.3) in (2.9) and interchange the sum and the integral as we did to get equation (2.10). Equality (2.10) does not hold but the right hand side of (2.10) is still an asymptotic expansion of $F(z)$. To see this, divide the integration domain Ω in two pieces, the one contained in the disk D : $\Omega_{in} \equiv \Omega \bigcap D$ and its complement not contained in D : $\Omega_{out} \equiv \Omega \setminus \Omega_{in}$. We can write (1.1) in the form

$$F(z) = \int\int_{\Omega_{in}} e^{-zf(x,y)} g(x,y) dx dy + \int\int_{\Omega_{out}} e^{-zf(x,y)} g(x,y) dx dy. \tag{2.16}$$

As the point $(x,y) = (0,0)$ is the only absolute minimum of $f(x,y)$ in Ω, $\exists\, \epsilon > 0$ such that $f(x,y) \geq f(0,0) + \epsilon$ for any $(x,y) \in \Omega_{out}$. Then we have

$$\int\int_{\Omega_{out}} e^{-zf(x,y)} g(x,y) dx dy = e^{-z(f(0,0)+\epsilon)} \int\int_{\Omega_{out}} e^{-z(f(x,y)-f(0,0)-\epsilon)} g(x,y) dx dy.$$

The last integral above is of the order $\mathcal{O}(1)$ when $z \to \infty$ and then, the integral in Ω_{out} in (2.16) is exponentially small compared with the integral in Ω_{in} and we have

$$\begin{aligned} F(z) &= \int\int_{\Omega_{in}} e^{-zf(x,y)} g(x,y) dx dy + \mathcal{O}\left(e^{-z(f(0,0)+\epsilon)}\right) \\ &= e^{-zf(0,0)} \left[\int\int_{\Omega_{in}} e^{-z(ax^\alpha + by^\beta)} h(x,y;z) dx dy + \mathcal{O}\left(e^{-\epsilon z}\right) \right]. \end{aligned}$$

Now, we can proceed with the above integral over Ω_{in} as we proceeded before in the case in which Ω was contained in D. We obtain

$$F(z) = e^{-zf(0,0)} \left[\sum_{n=0}^{\infty} \Psi_n(z) + \text{exponentially small terms} \right], \tag{2.17}$$

where $\Psi_n(z)$ is given in (2.15).

OBSERVATION 2.1. The coefficients $A_{m,n-sm}(z)$ defined in (2.2) are polynomials in z multiplied by $e^{-zf(0,0)}$. From (2.4) and (2.5), the terms $h_{m,n-sm}(z)$ are also polynomials in z and, from (2.8), they are polynomials of degree at most $\lfloor n/(\beta+1) \rfloor$. From (2.15) we see that the asymptotic sequence $\Psi_n(z)$ is a sum of 3 negative powers of z in the cases (P_1) and (P_2) and of 2 negative powers of z in the case (P_3). Therefore, the expansion (2.14) is a rearrangement of the standard Laplace's expansion and vice versa. This means that the coefficients of the standard Laplace's expansion can be explicitly obtained from the coefficients of (2.14) after an appropriate rearrangement.

We can summarize the above discussion in the following theorem.

THEOREM 2.2. *Let the functions $f(x,y)$ and $g(x,y)$ in (1.1) be continuous in Ω, with $\Omega \in \mathbb{R}^2$ a convex bounded or unbounded domain and $\partial\Omega$ piecewise smooth. The possible corner points of Ω have interior angles not greater than $\pi/2$. Suppose that the integral (1.1) exists for $z \geq z_0$. Let $(0,0)$ be the unique absolute minimum of $f(x,y)$ in $\bar{\Omega}$ and let $f(x,y)$ and $g(x,y)$ be analytic at $(x,y)=(0,0)$. Let that minimum fall into one of the three categories (P_1), (P_2) and (P_3) explained at the beginning of Section 2. Let Ω' be the image of Ω by the transformation $(x,y) \to (z^{-1/\alpha}x, z^{-1/\beta}y)$, with α and β given in Table 2.1, as well as a and b and s. Then,*

$$F(z) \sim e^{-zf(0,0)} \sum_{n=0}^{\infty} \Psi_n(z), \qquad \text{as} \qquad z \to \infty, \tag{2.18}$$

with $\Psi_0(z) = h_{0,0}(z)\Phi_{0,0}(z)$ and, for $n = 1,2,3,\ldots$,

$$\Psi_n(z) \equiv \sum_{k=(\beta+1)n-\beta}^{(\beta+1)n} \sum_{m=0}^{k} h_{\alpha m,\beta k-\beta m}(z)\Phi_{\alpha m,\beta k-\beta m}(z). \tag{2.19}$$

For $n = 0,1,2,\ldots$ and $m = 0,1,2,\ldots,n$,

$$\Phi_{\alpha m,\beta(k-m)}(z) = z^{-k-1/\alpha-1/\beta} \int\int_{\Omega'} e^{-ax^{\alpha}-by^{\beta}} x^{\alpha m} y^{\beta(k-m)} dx dy,$$

$$h_{\alpha m,\beta(k-m)}(z) \equiv e^{zf(0,0)} \sum_{l=0}^{k} \sum_{j=\mathrm{Max}\{0,l+m-k\}}^{\mathrm{Min}\{l,m\}} \frac{z^l a^j b^{l-j}}{j!(l-j)!} H_{\alpha(m-j),\beta(k-l-m+j)}(z),$$

with

$$H_{m,n-sm}(z) \equiv \sum_{k=0}^{n} \sum_{j=\mathrm{Max}\{0,m-\lfloor (n-k)/s \rfloor\}}^{\mathrm{Min}\{m,\lfloor k/s \rfloor\}} A_{j,k-sj}(z) g_{m-j,n-k-sm+sj}.$$

In these formulas, $A_{j,k-sj}(z)$ and $g_{j,k-sj}$ are the Taylor coefficients at $(x,y)=(0,0)$ of $e^{-zf(x,y)}$ and $g(x,y)$ respectively: for a given $r>0$ and $x^2+y^2<r^2$,

$$e^{-zf(x,y)}=\sum_{n=0}^{\infty}\sum_{m=0}^{\lfloor n/s\rfloor}A_{m,n-sm}(z)x^m y^{n-sm},\qquad g(x,y)=\sum_{n=0}^{\infty}\sum_{m=0}^{\lfloor n/s\rfloor}g_{m,n-sm}x^m y^{n-sm}.$$

When $z\to\infty$, the terms of the expansion (2.18) are of the order $\Psi_n(z)=\mathcal{O}\left(z^{-n-1}\right)$ in the case (P_1), $\Psi_n(z)=\mathcal{O}\left(z^{-n-3/2}\right)$ in the case (P_2) and $\Psi_n(z)=\mathcal{O}\left(z^{-n-2}\right)$ in the case (P_3).

If Ω is a rectangle then $\Omega'\sim\mathbb{R}^2$ in the case (P_1), $\Omega'\sim\mathbb{R}\times[0,\infty)$ in the case (P_2) and $\Omega'\sim[0,\infty)\times[0,\infty)$ in the case (P_3). In any case, up to exponentially small terms,

$$\Phi_{\alpha m,\beta(k-m)}(z)\sim\frac{\Gamma\left(m+1/\alpha\right)}{a^{m+1/\alpha}}\frac{\Gamma\left(k-m+1/\beta\right)}{b^{k-m+1/\beta}}\frac{1}{z^{k+1/\alpha+1/\beta}}.$$

3. The Appell function $F_2(c,u,u',v,v';s,t)$ for large u, u', v and v'. The second Appell's hypergeometric function is a generalization of the Gauss hypergeometric function ${}_2F_1$ and it is defined by the double series [5, p. 789],

$$F_2(c,u,u',v,v';s,t)\equiv\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(c)_{m+n}(u)_m(u')_n}{(v)_m(v')_n m!n!}s^m t^n,\qquad |s|+|t|<1.$$

A double integral representation of the second Appell's function is given in [4],

$$F_2(c,u,u',v,v';s,t)=\frac{\Gamma(v)\Gamma(v')}{\Gamma(u)\Gamma(v-u)\Gamma(u')\Gamma(v'-u')}I_2(c,u,u',v,v';s,t),\tag{3.1}$$

with

$$I_2(c,u,u',v,v';s,t)\equiv\int_0^1 dx\int_0^1 dy\,x^{u-1}(1-x)^{v-u-1}y^{u'-1}(1-y)^{v'-u'-1}(1-xs-yt)^{-c},\tag{3.2}$$

$s,t\in\mathbb{C}$, $s+t\notin[1,\infty)$, $v>u>0$ and $v'>u'>0$. We define the asymptotic parameter $z\equiv u-1$ and the following constants γ, δ, and λ,

$$\gamma\equiv\frac{u'-1}{z},\qquad \delta\equiv\frac{v-u-1}{z}\qquad \lambda\equiv\frac{v'-u'-1}{z}.$$

Using these relations, (3.2) can be written in the standard form (1.1),

$$I_2(c,u,u',v,v';s,t)=\int_0^1 dx\int_0^1 dy\,e^{-z\,f(x,y)}g(x,y),\tag{3.3}$$

with

$$f(x,y)=-\log x-\delta\log(1-x)-\gamma\log y-\lambda\log(1-y)\quad\text{and}\quad g(x,y)=(1-xs-yt)^{-c}.$$

We consider $\gamma,\delta,\lambda>0$ and fixed and $z>0$ large, which means that u,u',v and v' are large and of the same order. The functions $f(x,y)$ and $g(x,y)$ are differentiable on $\Omega=(0,1)\times(0,1)$. The unique absolute minimum of $f(x,y)$ in $\bar{\Omega}$ is the point

$$(x_0,y_0)=\left(\frac{1}{1+\delta},\frac{\gamma}{\gamma+\lambda}\right)\in\Omega,$$

which is also the unique relative minimun of $f(x,y)$ in Ω and falls into the case (P_1) considered in the previous section. With the notation used there we have

$$a = \frac{1}{2} f_{xx}(x_0, y_0) = \frac{(1+\delta)^3}{2\delta}, \qquad b = \frac{1}{2} f_{yy}(x_0, y_0) = \frac{(\gamma+\lambda)^3}{2\gamma\lambda},$$

and $h(x,y,z) = e^{-z\, f_1(x,y)} g(x,y)$, with

$$f_1(x,y) = f(x,y) - f(x_0,y_0) - a(x-x_0)^2 - b(y-y_0)^2.$$

Both functions $e^{z\, f_1(x,y)}$ and $g(x,y)$ have Taylor expansion at (x_0,y_0). On the one hand,

$$g(x,y) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{s^m\, t^{n-m}\, \Gamma(n+c)}{m!\,(n-m)!\,(1-sx_0-ty_0)^{c+n}\,\Gamma(c)} (x-x_0)^m (y-y_0)^{n-m}.$$

On the other hand, we can write $e^{-z\, f_1(x,y)} = e^{-z\, f_1(x_0,y_0)} F(x,z) G(y,z)$, with $F(x,z) = x^z (1-x)^{\delta z} e^{az(x-x_0)^2}$ and $G(y,z) = y^{\gamma z}(1-y)^{\lambda z} e^{bz(y-y_0)^2}$. The functions F and G satisfy the following differential equations in the variables x and y, respectively,

$$\begin{aligned} x_0 x(1-x) F'(x,z) &= z(x-x_0)\,[2ax_0 x(1-x) - 1]\, F(x,z), \\ y_0 y(1-y) G'(y,z) &= z(y-y_0)\,[2by_0 y(1-y) - \gamma]\, G(y,z). \end{aligned}$$

Substituting the McLauring series $F(x,z) = \sum_{n=0}^{\infty} f_n(z)(x-x_0)^n$ into the first equation and $G(y,z) = \sum_{m=0}^{\infty} g_m(z)(y-y_0)^m$ into the second one, we obtain

$$f_0(z) = 1, \qquad f_1(z) = f_2(z) = 0, \qquad f_3(z) = z\frac{(\delta-1)(\delta+1)^4}{3\,\delta^2},$$

$$f_{n+4}(z) = \frac{(1-\delta^2)(n+3)}{\delta(n+4)} f_{n+3} + \frac{(1+\delta)^2(n+2)}{\delta(n+4)} f_{n+2} - \frac{(1-\delta)(1+\delta)^4}{\delta^2(n+4)} z f_{n+1} - \frac{(1+\delta)^5}{\delta^2(n+4)} z f_n,$$

and

$$g_0(z) = 1, \qquad g_1(z) = g_2(z) = 0, \qquad g_3(z) = z\frac{(\lambda-\gamma)(\gamma+\lambda)^4}{3\gamma^2\lambda^2},$$

$$g_{m+4}(z) = \frac{(\gamma^2-\lambda^2)(m+3)}{\gamma\lambda(m+4)} g_{m+3} + \frac{(\gamma+\lambda)^2(m+2)}{\gamma\lambda(m+4)} g_{m+2} - \frac{(\gamma-\lambda)(\gamma+\lambda)^4}{\gamma^2\lambda^2(m+4)} z g_{m+1} - \frac{(\gamma+\lambda)^5}{\gamma^2\lambda^2(m+4)} z g_m.$$

Then, we can write

$$e^{-z\, f_1(x,y)} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} A_{m,n-m}(z)(x-x_0)^m\,(y-y_0)^{n-m}, \qquad A_{n,m}(z) = g_n(z) f_m(z).$$

Applying Theorem 2.2, we have that

$$I_2(c,u,u',v,v';s,t) \sim \left[\frac{\delta^\delta}{(1+\delta)^{\delta+1}} \left(\frac{\gamma}{\gamma+\lambda}\right)^\gamma \left(\frac{\lambda}{\gamma+\lambda}\right)^\lambda\right]^z \sum_{k=0}^{\infty} \Psi_n(z) \quad \text{as} \quad z \to \infty, \tag{3.4}$$

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with $\Psi_0(z)=\dfrac{2\pi\sqrt{\gamma\delta\lambda}}{z\sqrt{(1+\delta)^3(\gamma+\lambda)^3}}\left(1-\dfrac{s}{1+\delta}-\dfrac{\gamma t}{\gamma+\lambda}\right)^{-c}=\mathcal{O}(z^{-1})$, and

$$\Psi_n(z)=\sum_{k=3n-2}^{3n}\sum_{m=0}^{k}h_{2m,2k-2m}(z)\phi_{2m,2k-2m}(z)=\mathcal{O}(z^{-n-1}),\qquad n=1,2,3,\ldots.$$

According to Theorem 2.2, $h_{m,k-m}(z)$ are the Taylor coefficients of the function $h(x,y,z)=e^{-z\,f_1(x,y)}g(x,y)$ at $(x,y)=(x_0,y_0)$ that may be explicitly computed. But the formulas are quite long and we do not reproduce them here. On the other hand, according to the last formula of Theorem 2.2,

$$\phi_{2m,2k-2m}(z)\sim\frac{2^{k+1}\delta^{m+1/2}(\gamma\lambda)^{k-m+1/2}}{(1+\delta)^{3m+3/2}(\gamma+\lambda)^{3k-3m+3/2}z^{k+1}}\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(k-m+\frac{1}{2}\right).$$

The first term of the asymptotic expansion of $F_2(c,u,u',v,v';s,t)$ for large u,u',v,v' is

$$F_2(c,u,u',v,v';s,t)=\left(1-s\frac{u}{v}-t\frac{u'}{v'}\right)^{-c}+\mathcal{O}\left(\frac{1}{u}\right).$$

The first two terms are

$$F_2(c,u,u',v,v';s,t)=$$

$$\frac{2\,\pi\,\Gamma(v)\Gamma(v')(v-u-1)^{v-u-1/2}(v'-u'-1)^{v'-u'-1/2}(u-1)^{u-1/2}(u'-1)^{u'-1/2}}{\Gamma(u)\Gamma(v-u)\Gamma(u')\Gamma(v'-u')(v-2)^{v-1/2}(v'-2)^{v'-1/2}\left(1-s\frac{u-1}{v-2}-t\frac{u'-1}{v'-2}\right)^{c}}\times$$

$$\left\{1+\frac{1}{12}\left[\frac{1}{u}+\frac{1}{v-u}+\frac{13}{v}+\frac{1}{u'}+\frac{1}{v'-u'}+\frac{13}{v'}\right]-\frac{c}{\left(1-s\frac{u}{v}-t\frac{u'}{v'}\right)}\left[s\frac{2u-v}{v^2}+t\frac{2u'-v'}{v'^2}\right]\right.$$

$$\left.+\frac{c(c+1)}{2\left(1-s\frac{u}{v}-t\frac{u'}{v'}\right)^2}\left[s^2\frac{(v-u)v'^2}{u^3v}+t\frac{u'(v'-u')v^2}{u^4v'}\right]+\mathcal{O}\left(\frac{1}{u^2}\right)\right\}.$$

Table 3.1 shows a numerical experiment illustrating the accuracy of this approximation.

TABLE 3.1
Relative errors in the approximation (3.4) of $F_2(c,u,u',v,v';s,t)$ for $c=1$, $s=1/4$, $t=-1/4$, $n=0,1,2$ and different values of of u,u',v,v' using Mathematica with 16 digits of working precision.

	$u=u'=50$ $v=v'=100$	$u=u'=100$ $v=v'=200$	$u=u'=200$ $v=v'=400$	$u=u'=350$ $v=v'=700$	$u=u'=600$ $v=v'=1200$
$n=0$	0.01500460	0.00742222	0.00369139	0.00210457	0.00122612
$n=1$	0.00020752	0.00005076	0.00001255	4.0807×10^{-6}	1.3850×10^{-6}
$n=2$	8.4379×10^{-6}	5.0857×10^{-7}	3.1216×10^{-8}	3.3026×10^{-9}	3.8077×10^{-10}

4. Concluding remarks. The standard Laplace's method of asymptotic expansion of integrals requires a change of variables which makes the computation of the coefficients of the expansion a very complicated task. This complication is harder for double integrals. In general, only the first few terms of the expansion are computed explicitly in practical examples. We have proposed a different method which makes the computation of the complete

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expansion straightforward. With this new procedure, the phase function $f(x,y)$ is always a polynomial of degree at most two in x and y. On the other hand, the remaining integrand $h(x,y,z)$ is not a function only of the integration variables (x,y), but also of the asymptotic variable z. As a consequence of this fact, not only the asymptotic sequence $\Phi_{m,n-sm}(z)$, but also the coefficients of the expansion $h_{m,n-sm}(z)$, depend of z. The asymptotic expansion obtained in this way, $\sum_n \sum_m h_{m,n-sm}(z)\Phi_{m,n-sm}(z)$, is not a genuine Poincaré-type expansion. Nevertheless, the terms of the expansion $\sum_m h_{m,n-sm}(z)\Phi_{m,n-sm}(z)$ can be grouped in new terms $\Psi_n(z)$ and the new asymptotic expansion $\sum_n \Psi_n(z)$ is a genuine Poincaré expansion, $\Psi_n(z) = \mathcal{O}(z^{-n-p})$ as $z \to \infty$.

We have applied the method to the example of the second Appell function and obtained complete asymptotic expansions of this function when some of its parameters are large.

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