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ASYMPTOTIC BEHAVIOR FOR NUMERICAL SOLUTIONS OF A SEMILINEAR PARABOLIC EQUATION WITH A NONLINEAR BOUNDARY CONDITION*

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Abstract. This paper concerns the study of the numerical approximation for the following initial-boundary value problem,

$$\begin{aligned} &u_t = u_{xx} - au^p, \quad 0 < x < 1, \quad t > 0,\\ &u_x(0,t) = 0, \quad u_x(1,t) + bu^q(1,t) = 0, \quad t > 0,\\ &u(x,0) = u_0(x) \geq 0, \quad 0 \leq x \leq 1, \end{aligned}$$

where $a > 0$, $b > 0$ and $p > q > 1$. We show that the solution of a semidiscrete form of the initial value problem above goes to zero as t approaches infinity and give its asymptotic behavior. We provide some numerical experiments that illustrate our analysis.

Key words. semidiscretizations, semilinear parabolic equation, asymptotic behavior, convergence

AMS subject classifications. 35B40, 35B50, 35K60, 65M06

1. Introduction.

Consider the following initial-boundary value problem,

$$u_t = u_{xx} - au^p, \quad 0 < x < 1, \quad t > 0, \tag{1.1}$$

$$u_x(0,t) = 0, \quad u_x(1,t) + bu^q(1,t) = 0, \quad t > 0, \tag{1.2}$$

$$u(x,0) = u_0(x) \geq 0, \quad 0 \leq x \leq 1, \tag{1.3}$$

where $a > 0$, $b > 0$, $p > q > 1$, $u_0 \in C^1([0,1])$, $u_0^{'}(0) = 0$, and $u_0^{'}(1) + bu_0^q(1) = 0$.

The theoretical study of asymptotic behavior of solutions for semilinear parabolic equations has been the subject of investigations of many authors; see [2, 4] and the references cited therein. In particular in [4], when $b = 0$, the authors have shown that the solution u of (1.1)–(1.3) decays to zero as t goes to infinity and satisfies

$$\lim_{t\to\infty} t^{\frac{1}{p-1}} \|u(x,t)\|_\infty = C_*$$

where $C_* = (\frac{1}{b(p-1)})^{\frac{1}{p-1}}$. Similar results have been obtained in [2] in the case where $b > 0$ and $p > q > 1$. Indeed, in this case, it is shown that the solution u decays to zero as t approaches infinity and $\lim_{t\to\infty} t^{\frac{1}{q-1}} \|u(x,t)\|_\infty = C_0$ where $C_0 = (\frac{1}{b(q-1)})^{\frac{1}{q-1}}$.

In this paper we are interesting in the numerical study of (1.1)–(1.3). First, using a semidiscrete form of (1.1)–(1.3), we prove similar results for the semidiscrete solution. Previously, authors have used numerical methods to study the phenomenon of blow-up and the one of extinction; see [1, 3]. This paper is organized as follows. In the Section 2, we give a semidiscrete form of (1.1)–(1.3) and we prove some results about the discrete maximum principle. In Section 3, we show that the semidiscrete solution goes to zeros as t goes to infinity and give its asymptotic behavior. In Section 4, we prove that the semidiscrete scheme converges. Finally, in Section 5, we give some numerical results.

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2. The semidiscrete scheme. In this section, we give some lemmas which will be used later. Let I be a positive integer, and define the grid $x_i = ih$, $0 \le i \le I$, where $h = 1/I$. We approximate the solution u of the problem (1.1)–(1.3) by the solution $U_h(t) = (U_0(t), U_1(t), \dots, U_I(t))^T$ of the semidiscrete equations,

$$\frac{dU_i(t)}{dt} = \delta^2 U_i(t) - a(U_i(t))^p, \quad 0 \le i \le I-1, \quad t > 0, \tag{2.1}$$

$$\frac{dU_I(t)}{dt} = \delta^2 U_I(t) - a(U_I(t))^p - \frac{2b}{h}(U_I(t))^q, \quad t > 0, \tag{2.2}$$

$$U_i(0) = U_i^0 \ge 0, \quad 0 \le i \le I, \tag{2.3}$$

where

$$\delta^2 U_i(t) = \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{h^2}, \quad 1 \le i \le I-1,$$

$$\delta^2 U_0(t) = \frac{2U_1(t) - 2U_0(t)}{h^2}, \quad \delta^2 U_I(t) = \frac{2U_{I-1}(t) - 2U_I(t)}{h^2}.$$

The following lemma is a semidiscrete form of the maximum principle.

LEMMA 2.1. *Let $a_h(t) \in C^0([0,T], \mathbb{R}^{I+1})$ and let $V_h(t) \in C^1([0,T], \mathbb{R}^{I+1})$ such that*

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) + a_i(t) V_i(t) \ge 0, \quad 0 \le i \le I, \quad t \in (0,T),$$
$$V_i(0) \ge 0, \quad 0 \le i \le I.$$

Then we have $V_i(t) \ge 0$ for $0 \le i \le I$, $t \in (0,T)$.

Proof. Let $T_0 < T$ and let $m = \min_{0 \le i \le I, 0 \le t \le T_0} V_i(t)$. Since, $V_h(t)$ is a continuous function on the compact $[0, T_0]$, there exists $t_0 \in [0, T_0]$ such that $m = V_{i_0}(t_0)$ for a certain $i_0 \in \{0, \dots, I\}$. It is not hard to see that

$$\frac{dV_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{V_{i_0}(t_0) - V_{i_0}(t_0 - k)}{k} \le 0, \tag{2.4}$$

$$\delta^2 V_{i_0}(t_0) = \frac{2V_1(t_0) - 2V_0(t_0)}{h^2} \ge 0 \quad \text{if} \quad i_0 = 0, \tag{2.5}$$

$$\delta^2 V_{i_0}(t_0) = \frac{V_{i_0+1}(t_0) - 2V_{i_0}(t_0) + V_{i_0-1}(t_0)}{h^2} \ge 0 \quad \text{if} \quad 1 \le i_0 \le I-1, \tag{2.6}$$

$$\delta^2 V_{i_0}(t_0) = \frac{2V_{I-1}(t_0) - 2V_I(t_0)}{h^2} \ge 0 \quad \text{if} \quad i_0 = I. \tag{2.7}$$

Define the vector $Z_h(t) = e^{\lambda t} V_h(t)$, where λ is large enough that $a_{i_0}(t_0) - \lambda > 0$. A straightforward computation reveals that

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + (a_{i_0}(t_0) - \lambda) Z_{i_0}(t_0) \ge 0. \tag{2.8}$$

We observe from (2.4)–(2.7) that $\frac{dZ_{i_0}(t_0)}{dt} \le 0$ and $\delta^2 Z_{i_0}(t_0) \ge 0$. Using (2.8), we arrive at $(a_{i_0}(t) - \lambda) Z_{i_0}(t_0) \ge 0$, which implies that $Z_{i_0}(t_0) \ge 0$. Therefore $V_{i_0}(t_0) = m \ge 0$ and we have the desired result. ∎

Another form of the maximum principle for semidiscrete equations is the following comparison lemma.

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LEMMA 2.2. *Let* $V_h(t)$, $U_h(t) \in C^1([0,\infty),\mathbb{R}^{I+1})$ *and* $f \in C^0(\mathbb{R}\times\mathbb{R},\mathbb{R})$ *such that for* $t \in (0,\infty)$

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) + f(V_i(t),t) < \frac{dU_i(t)}{dt} - \delta^2 U_i(t) + f(U_i(t),t), \quad 0 \le i \le I, \tag{2.9}$$

$$V_i(0) < U_i(0), \quad 0 \le i \le I. \tag{2.10}$$

Then we have $V_i(t) < U_i(t)$, $0 \le i \le I$, $t \in (0,\infty)$.

Proof. Define the vector $Z_h(t) = U_h(t) - V_h(t)$. Let t_0 be the first $t > 0$ such that $Z_h(t) > 0$ for $t \in [0,t_0)$, but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0,\dots,I\}$. We observe that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k\to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0-k)}{k} \le 0.$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \ge 0 \quad \text{if} \quad 1 \le i_0 \le I-1,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} \ge 0 \quad \text{if} \quad i_0 = 0,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_{I-1}(t_0) - 2Z_I(t_0)}{h^2} \ge 0 \quad \text{if} \quad i_0 = I,$$

which implies that $\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + f(U_{i_0}(t_0),t_0) - f(V_{i_0}(t_0),t_0) \le 0$. But this inequality contradicts (2.9). ◻

3. Asymptotic behavior. In this section, we show that the solution U_h of (2.1)–(2.3) goes to zero as t approaches infinity and give its asymptotic behavior. To start, let us show that the solution $U_h(t)$ decays uniformly to zero by the following

THEOREM 3.1. *The solution* $U_h(t)$ *of (2.1)–(2.3) goes to zero as* $t \to \infty$ *and we have the following estimates,*

$$0 \le \|U_h(t)\|_\infty \le \frac{1}{(\|U_h(0)\|_\infty^{1-p} + b(p-1)t)^{\frac{1}{p-1}}} \quad \text{for} \quad t \in [0,+\infty).$$

Proof. Introduce the function $\alpha(t)$ defined as follows,

$$\alpha(t) = \frac{1}{(\|U_h(0)\|_\infty^{1-p} + b(p-1)t)^{\frac{1}{p-1}}},$$

and let W_h the vector such that $W_i(t) = \alpha(t)$. It is not hard to see that

$$\frac{dW_i(t)}{dt} - \delta^2 W_i(t) + (W_i(t))^p = 0, \quad 0 \le i \le I-1, \quad t \in (0,T),$$

$$\frac{dW_I(t)}{dt} - \delta^2 W_I(t) + (W_I(t))^p + \frac{2b}{h}(W_I(t))^q \ge 0, \quad t \in (0,T),$$

$$W_i(0) \ge U_i(0), \quad 0 \le i \le I,$$

where $(0,T)$ is the maximal time interval on which $\|U_h(t)\|_\infty < \infty$. Setting $Z_h(t) = W_h(t) - U_h(t)$ and using the mean value theorem, we find that

$$\frac{dZ_i(t)}{dt} - \delta^2 Z_i(t) + p(\theta_i(t))^{p-1} Z_i(t) = 0, \quad 0 \le i \le I-1, \quad t \in (0,T),$$

$$\frac{dZ_I(t)}{dt} - \delta^2 Z_I(t) + (p(\theta_I(t))^{p-1} + \frac{2b}{h}(\theta_I(t))^{q-1}) Z_I(t) \ge 0, \quad t \in (0,T),$$

$$Z_i(0) \ge 0, \quad 0 \le i \le I,$$

where θ_i is an intermediate value between $U_i(t)$ and $W_i(t)$. Apply Lemma 2.1, to obtain $0 \leq U_h(t) \leq W_h(t)$ for $t \in (0,T)$. If $T < \infty$ we have

$$\|U_h(T)\|_\infty \leq \frac{1}{(\|U_h(0)\|_\infty^{1-p} + b(p-1)T)^{\frac{1}{p-1}}} < \infty,$$

which leads to a contradiction. Hence $T = \infty$ and we have the desired result. □

The statement of the main result of this section is the following

THEOREM 3.2. *Let U_h be the solution of (2.1)–(2.3). Then we have*

$$\lim_{t\to\infty} t^{\frac{1}{q-1}}\|U_h(t)\|_\infty = C_0,$$

where $C_0 = (\frac{1}{b(q-1)})^{\frac{1}{q-1}}$.

The proof of Theorem 3.2 is based on the following lemmas. The function

$$\mu(x) = -\lambda(C_0+x) + b(C_0+x)^q,$$

where $C_0 = (\frac{1}{b(q-1)})^{\frac{1}{q-1}}$ and $\lambda = \frac{1}{q-1}$, is crucial in the proofs of the lemmas below. The following result show that the solution $U_h(t)$ of (2.1)–(2.3) is bounded from above by a function which decays to zero.

LEMMA 3.3. *Let U_h be the solution of (2.1)–(2.3). For any $\varepsilon > 0$, there exist positive times T and τ such that*

$$U_i(t+\tau) \leq (C_0+\varepsilon)(t+T)^{-\lambda} + \varphi_i(t+T)^{-\lambda-1}, \quad 0 \leq i \leq I,$$

where $\varphi_i = -\frac{b}{2}(C_0+\varepsilon)^q i^2 h^2$.

Proof. Introduce the vector W_h defined as follows,

$$W_i(t) = (C_0+\varepsilon)t^{-\lambda} + \varphi_i t^{-\lambda-1}, \quad 0 \leq i \leq I.$$

A straightforward computation reveals that

$$\begin{aligned}
\frac{dW_i}{dt} - \delta^2 W_i + aW_i^p &= -\lambda(C_0+\varepsilon)t^{-\lambda-1} - (\lambda+1)t^{-\lambda-2}\varphi_i \\
&\quad + at^{-\lambda p}((C_0+\varepsilon) + \varphi_i t^{-1})^p - t^{-\lambda-1}\delta^2\varphi_i, \\
\frac{dW_I}{dt} - \delta^2 W_I + aW_I^p + \frac{2b}{h}W_I^q &= -\lambda(C_0+\varepsilon)t^{-\lambda-1} + at^{-\lambda p}((C_0+\varepsilon)+\varphi_I t^{-1})^p \\
&\quad - (\lambda+1)t^{-\lambda-2}\varphi_I - t^{-\lambda-1}\delta^2\varphi_I \\
&\quad + \frac{2b}{h}t^{-\lambda-1}((C_0+\varepsilon)+\varphi_I t^{-1})^q,
\end{aligned}$$

because $\lambda q = \lambda + 1$. From the mean value theorem, we get

$$(C_0+\varepsilon+\varphi_I t^{-1})^q = (C_0+\varepsilon)^q + \chi_I(t)t^{-1},$$

where $\chi_I(t)$ is a bounded function. We deduce that

$$\begin{aligned}
\frac{dW_i}{dt} - \delta^2 W_i + aW_i^p &= t^{-\lambda-1}(\mu(\varepsilon) - (\lambda+1)t^{-1}\varphi_i \\
&\quad + at^{-\lambda p+\lambda+1}(C_0+\varepsilon+t^{-1}\varphi_i)^p), \\
\frac{dW_I}{dt} - \delta^2 W_I + aW_I^p + \frac{2b}{h}W_I^q &= t^{-\lambda-1}(-\lambda(\mu(\varepsilon) \\
&\quad + at^{-\lambda p+\lambda+1}(C_0+\varepsilon+t^{-1}\varphi_I)^p + \frac{2b}{h}(C_0+\varepsilon)^q \\
&\quad + \frac{2b}{h}\chi_I t^{-1}).
\end{aligned}$$

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We observe that $\mu(0) = 0$ and $\mu^{'}(0) = 1$. This implies that for $\varepsilon > 0$, $\mu(\varepsilon) > 0$. We also see that $-\lambda p + \lambda + 1 = \frac{q-p}{q-1} < 0$. We deduce that there exists a positive time T such that

$$\begin{aligned}
&\frac{dW_i}{dt} - \delta^2 W_i + aW_i^p > 0,\\
&\frac{dW_I}{dt} - \delta^2 W_I + aW_I^p + \frac{2b}{h} W_I^q > 0,\\
&W_i(T) > \frac{T^{-\lambda} C_0}{2}.
\end{aligned}$$

Since from Theorem 3.1 $U_h(t)$ goes to zero as t tends to infinity, there exists $\tau > T$ such that $U_i(\tau) < \frac{T^{-\lambda} C_0}{2} < W_i(T)$. Introduce the vector $Z_h(t)$ such that $Z_h(t) = U_h(t + \tau - T)$. A routine computation yields

$$\begin{aligned}
&\frac{dZ_i}{dt} - \delta^2 Z_i + aZ_i^p > 0, \quad 0 \le i \le I-1, \quad t \ge T,\\
&\frac{dZ_I}{dt} - \delta^2 Z_I + aZ_I^p + \frac{2b}{h} Z_I^q(t) > 0, \quad t \ge T,\\
&Z_i(T) = U_i(\tau) < W_i(T).
\end{aligned}$$

We deduce from Lemma 2.2 that $Z_h(t) \le W_h(t)$. That is,

$$U_i(t + \tau - T) \le W_i(t) \quad \text{for} \quad t \ge T,$$

which leads us to the result. □

LEMMA 3.4. *Let U_h be the solution of (2.1)–(2.3). For any $\varepsilon > 0$, there exists a positive time τ such that*

$$U_i(t+1) \ge (C_0 - \varepsilon)(t+\tau)^{-\lambda} + \varphi_i (t+\tau)^{-\lambda-1}, \quad 0 \le i \le I.$$

Proof. Introduce the vector V_h such that

$$V_i(t) = (C_0 - \varepsilon)t^{-\lambda} + \varphi_i t^{-\lambda-1}.$$

A direct calculation yields

$$\begin{aligned}
\frac{dV_i}{dt} - \delta^2 V_i + aV_i^p &= -\lambda(C_0 - \varepsilon)t^{-\lambda-1} - (\lambda+1)t^{-\lambda-2}\\
&+ a((C_0 - \varepsilon)t^{-\lambda} + \varphi_i t^{-\lambda-1})^p\\
&= t^{-\lambda-1}(-\lambda(C_0 - \varepsilon) - (\lambda+1)t^{-1}\\
&+ a(C_0 - \varepsilon + \varphi_i t^{-1})^p),
\end{aligned}$$

$$\begin{aligned}
\frac{dV_I}{dt} - \delta^2 V_I + aV_I^p + \frac{2b}{h} V_I^q &= -\lambda(C_0 - \varepsilon)t^{-\lambda-1} - (\lambda+1)t^{-\lambda-2}\varphi_I\\
&+ at^{-\lambda p}(C_0 - \varepsilon + \varphi_I t^{-1})^p - t^{-\lambda-1}\delta^2 \varphi_I\\
&+ 2\frac{b}{h} t^{-\lambda-1}(C_0 - \varepsilon + \varphi_I t^{-1})^q,
\end{aligned}$$

because $q\lambda = \lambda + 1$. From the mean value theorem, we have $(C_0 - \varepsilon + \varphi_I t^{-1})^q = (C_0 - \varepsilon)^p + \chi_I(t)t^{-1}$, where $\chi_I(t)$ is a bounded function. Using this equality, we deduce that

$$\frac{dV_i}{dt} - \delta^2 V_i + aV_i^p = t^{-\lambda-1}(\mu(-\varepsilon) - (\lambda+1)\varphi_i t^{-1} + \chi_i t^{-1}),$$
$$\frac{dV_I}{dt} - \delta^2 V_I + aV_I^p + \frac{2b}{h}V_I^q = t^{-\lambda-1}(\mu(-\varepsilon) - (\lambda+1)t^{-1}$$
$$+\xi_i t^{-1} + \frac{2b}{h}t^{-q\lambda+\lambda+1}(C_0 - \varepsilon + t^{-1})^q).$$

Obviously, $-q\lambda + \lambda + 1 < 0$. Also, since $\mu(0) = 0$ and $\mu^{'}(0) = 1$, it is easy to see that $\mu(-\varepsilon) < 0$. Hence there exists a positive time T such that

$$\frac{dV_i}{dt} - \delta^2 V_i + aV_i^p < 0, \quad 0 \leq i \leq I-1, \quad t \in [T, +\infty),$$
$$\frac{dV_I}{dt} - \delta^2 V_I + aV_I^p + \frac{2b}{h}V_I^q < 0, \quad t \in [T, +\infty).$$

Since $V_h(t)$ goes to zero as t approaches infinity, there exists $\tau > \max(T, 1)$ such that $V_h(\tau) < U_h(1)$. Setting $X_h(t) = V_h(t + \tau - 1)$, we observe that

$$\frac{dX_i}{dt} - \delta^2 X_i + aX_i^p > 0, \quad 0 \leq i \leq I-1, \quad t \geq T,$$
$$\frac{dX_I}{dt} - \delta^2 X_I + aX_I^p + \frac{2b}{h}X_I^q > 0, \quad t \geq T,$$
$$X_i(1) = V_i(\tau) < U_i(1), \quad 0 \leq i \leq I.$$

We deduce from Lemma 2.2 that

$$U_i(t) \geq V_i(t + \tau - 1) \quad \text{for} \quad t \geq 1,$$

which leads us to the result. □

Now, we are in a position to prove the main result.

Proof of Theorem 3.2. From Lemma 3.3 and Lemma 3.4, we deduce

$$(C_0 - \varepsilon) \leq \lim_{t\to\infty} \inf(\frac{U_i(t)}{t^\lambda}) \leq \lim_{t\to\infty} \sup(\frac{U_i(t)}{t^\lambda}) \leq (C_0 + \varepsilon),$$

for any $\varepsilon > 0$ and the proof is complete. □

4. Convergence. In this section, we will show that for each fixed time interval $[0, T]$ where u is defined, the solution $U_h(t)$ of (2.1)–(2.3) approximates u, when the mesh parameter h goes to zero.

THEOREM 4.1. *Assume that (1.1)–(1.3) has a solution $u \in C^{4,1}([0,1] \times [0,T])$ and the initial condition at (2.3) satisfies*

$$\|U_h^0 - u_h(0)\|_\infty = o(1) \quad as \quad h \to 0, \tag{4.1}$$

where $u_h(t) = (u(x_0,t), ..., u(x_I,t))^T$. Then, for h sufficiently small, the problem (2.1)–(2.3) has a unique solution $U_h \in C^1([0,T], \mathbb{R}^{I+1})$ such that

$$\max_{0\leq t\leq T} \|U_h(t) - u_h(t)\|_\infty = O(\|U_h^0 - u_h(0)\|_\infty + h^2) \quad as \quad h \to 0.$$

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Proof. Since $u \in C^{4,1}$, there exist two positive constants K and L be such that

$$\frac{2\|u_{xxx}\|_\infty}{3} \leq \frac{K}{2}, \quad \frac{\|u_{xxxx}\|_\infty}{12} \leq \frac{K}{2}, \quad \|u\|_\infty \leq K, \quad ap(K+1)^{p-1} \leq L, \tag{4.2}$$
$$2q(K+1)^{q-1} \leq L.$$

The problem (2.1)–(2.3) has for each h, a unique solution $U_h \in C^1([0, T_q^h), \mathbb{R}^{I+1})$. Let $t(h)$ the greatest value of $t > 0$ such that

$$\|U_h(t) - u_h(t)\|_\infty < 1 \quad \text{for} \quad t \in (0, t(h)). \tag{4.3}$$

The relation (4.1) implies that $t(h) > 0$ for h sufficiently small. Let $t^*(h) = \min\{t(h), T\}$. By the triangle inequality, we obtain

$$\|U_h(t)\|_\infty \leq \|u(x,t)\|_\infty + \|U_h(t) - u_h(t)\|_\infty \quad \text{for} \quad t \in (0, t^*(h)),$$

which implies that

$$\|U_h(t)\|_\infty \leq 1 + K \quad \text{for} \quad t \in (0, t^*(h)). \tag{4.4}$$

Let $e_h(t) = U_h(t) - u_h(x,t)$ be the error of discretization. Using Taylor's expansion, we have for $t \in (0, t^*(h))$,

$$\frac{de_i(t)}{dt} - \delta^2 e_i(t) = \frac{h^2}{12} u_{xxxx}(\widetilde{x}_i, t) - ap\xi_i^{p-1} e_i(t),$$
$$\frac{de_I(t)}{dt} - \delta^2 e_I(t) = \frac{2}{h} q\theta_I^{q-1} e_I + \frac{2h^2}{3} u_{xxx}(\widetilde{x}_I, t) + \frac{h^2}{12} u_{xxxx}(\widetilde{x}_I, t) - ap\xi_I^{p-1} e_I(t),$$

where $\theta_I \in (U_I(t), u(x_I, t)$ and $\xi_i \in (U_i(t), u(x_i, t)$. Using (4.2) and (4.4), we arrive at

$$\frac{de_i(t)}{dt} - \delta^2 e_i(t) \leq L|e_i(t)| + Kh^2, \quad 0 \leq i \leq I-1,$$
$$\frac{de_I(t)}{dt} - \frac{(2e_{I-1}(t) - 2e_I(t))}{h^2} \leq \frac{L|e_I(t)|}{h} + L|e_I(t)| + Kh^2.$$

Consider the function $z(x,t) = e^{((M+1)t + Cx^2)}(\|U_h^0 - u_h(0)\|_\infty + Qh^2)$, where M, C, Q are constants which will be determined later. We get

$$z_t(x,t) - z_{xx}(x,t) = (M + 1 - 2C - 4C^2x^2) z(x,t),$$
$$z_x(0,t) = 0, \; z_x(1,t) = 2Cz(1,t),$$
$$z(x,0) = e^{Cx^2}(\|U_h^0 - u_h(0)\|_\infty + Qh).$$

By a semidiscretization of the above problem, we may choose M, C, Q large enough that

$$\frac{dz(x_i,t)}{dt} > \delta^2 z(x_i,t) + L|z(x_i,t)| + Kh^2, \quad 0 \leq i \leq I-1,$$
$$\frac{dz(x_I,t)}{dt} > \delta^2 z(x_I,t) + \frac{L}{h}|z(x_I,t)| + L|z(x_I,t)| + Kh^2,$$
$$z(x_i, 0 > e_i(0), \quad 0 \leq i \leq I.$$

It follows from Lemma 2.2 that

$$z(x_i,t) > e_i(t) \quad \text{for} \quad t \in (0, t^*(h)), \quad 0 \leq i \leq I.$$

In the same way, we also prove that

$$z(x_i,t) > -e_i(t) \quad \text{for} \quad t \in (0,t^*(h)), \quad 0 \le i \le I,$$

which implies that

$$\|U_h(t)-u_h(t)\|_\infty \le e^{(Mt+C)}(\|U_h^0-u_h(0)\|_\infty + Qh^2), \quad t \in (0,t^*(h)).$$

Let us show that $t^*(h)=T$. Suppose that $T > t(h)$. From (4.3), we obtain

$$1 = \|U_h(t(h))-u_h(t(h))\|_\infty \le e^{(MT+C)}(\|U_h^0-u_h(0)\|_\infty + Qh^2).$$

Since the right hand side of the above inequality goes to zero as h goes to zero, we deduce that $1 \le 0$, which is impossible. Consequently $t^*(h)=T$, and we obtain the desired result. ∎

5. Numerical results. In this section, we give some numerical results. First, we approximate the solution $u(x,t)$ of (1.1)–(1.3) by the solution $U_h^{(n)} = (U_0^n, U_1^n, \ldots, U_I^n)^T$ of the following explicit scheme,

$$\begin{aligned}
\frac{U_i^{(n+1)}-U_i^{(n)}}{\Delta t} &= \delta^2 U_i^{(n)} - a(U_i^{(n)})^{p-1}U_i^{(n+1)}, \quad 0 \le i \le I-1,\\
\frac{U_I^{(n+1)}-U_I^{(n)}}{\Delta t} &= \delta^2 U_I^{(n)} - a(U_I^{(n)})^{p-1}U_I^{(n+1)} - \frac{2b}{h}(U_I^{(n)})^{q-1}U_I^{(n+1)},\\
U_i^{(0)} &= \phi_i > 0, \quad 0 \le i \le I,
\end{aligned}$$

where $n \ge 0$, $\Delta t \le \frac{h^2}{2}$. Let us remark that the restriction on the time step guarantees the positivity of the discrete solution $U_h^{(n)}$.

Now, approximate the solution $u(x,t)$ of (1.1)–(1.3) by the solution $U_h^{(n)}$ of the following implicit scheme,

$$\begin{aligned}
\frac{U_i^{(n+1)}-U_i^{(n)}}{\Delta t} &= \delta^2 U_i^{(n+1)} - (U_i^{(n)})^{p-1}U_i^{(n+1)}, \quad 1 \le i \le I-1,\\
\frac{U_I^{(n+1)}-U_I^{(n)}}{\Delta t} &= \delta^2 U_I^{(n+1)} - a(U_I^{(n)})^{p-1}U_I^{(n+1)} - \frac{2b}{h}(U_I^{(n)})^{p-1}U_I^{(n+1)},\\
U_i^{(0)} &= \phi_i > 0, \quad 0 \le i \le I,
\end{aligned}$$

where $n \ge 0$.

The above equations may be rewritten in the following form,

$$\begin{aligned}
U_i^{(n)} &= -\frac{\Delta t}{h^2}U_{i-1}^{(n+1)} + (1+2\frac{\Delta t}{h^2} + a\Delta t(U_i^{(n)})^{p-1})U_i^{(n+1)} - \frac{\Delta t}{h^2}U_{i+1}^{(n+1)},\\
U_0^{(n)} &= -\frac{2\Delta t}{h^2}U_1^{(n+1)} + (1+2\frac{\Delta t}{h^2} + a\Delta t(U_0^{(n)})^{p-1})U_0^{(n+1)},\\
U_I^{(n)} &= -\frac{2\Delta t}{h^2}U_{I-1}^{(n+1)} + (1+2\frac{\Delta t}{h^2} + a\Delta t(U_I^{(n)})^{p-1} + \frac{2b}{h}\Delta t|U_I^{(n)}|^{q-1})U_I^{(n+1)},
\end{aligned}$$

which gives the following linear system,

$$A^{(n)}U_h^{(n+1)} = U_h^{(n)},$$

where $A^{(n)}$ is a tridiagonal matrix defined as follows,

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$$A^{(n)} = \begin{bmatrix} d_0 & \frac{-2\Delta t}{h^2} & 0 & 0 & \cdots & 0 & 0 \\ \frac{-\Delta t}{h^2} & d_1 & \frac{-\Delta t}{h^2} & 0 & \cdots & 0 & 0 \\ 0 & \frac{-\Delta t}{h^2} & d_2 & \frac{-\Delta t}{h^2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{-\Delta t}{h^2} & d_{I-2} & \frac{-\Delta t}{h^2} & 0 \\ 0 & 0 & 0 & \cdots & \frac{-\Delta t}{h^2} & d_{I-1} & \frac{-\Delta t}{h^2} \\ 0 & 0 & 0 & \cdots & 0 & \frac{-2\Delta t}{h^2} & d_I \end{bmatrix},$$

with $d_i = 1 + 2\frac{\Delta t}{h^2} + a\Delta t(U_i^{(n)})^{p-1}$ for $0 \leq i \leq I-1$ and

$$d_I = 1 + 2\frac{\Delta t}{h^2} + a\Delta t|U_I^{(n)}|^{p-1} + \frac{2b}{h}\Delta t(U_I^{(n)})^{q-1}.$$

Let us remark that the matrix $A^{(n)}$ satisfies the following properties

$$A_{ii}^{(n)} > 0, \quad A_{ij}^{(n)} < 0 \quad \text{for} \quad i \neq j, \quad |A_{ii}^{(n)}| > \sum_{i \neq j} |A_{ij}^{(n)}|.$$

These properties imply that U_h^n exists for any n and $U_h^{(n)} \geq 0$; see, for instance, [3].

We let $p = 3$, $q = 2$, $a = 1$, $b = 1$, $U_i^0 = 0.8 + 0.8\cos(\pi hi)$ and $\Delta t = \frac{h^2}{2}$. In Tables 5.1 and 5.2, in the rows, we give the first n for which

$$\|n\Delta t U_h^{(n)} - 1\|_\infty < \varepsilon = 10^{-2},$$

the corresponding time $t_n = n\Delta t$, the CPU time, and the order(s) of method computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

TABLE 5.1
Numerical time, number of iterations, CPU time (seconds), and order of the approximations obtained with the implicit Euler method

I	t_n	n	$CPU\ time$	s
16	301.1953	154211	43	-
32	303.1289	620807	319	-
64	303.1140	2483229	1017	2.16
128	303.1177	9932910	9237	2.02
256	303.1186	39731654	13515	2.04

TABLE 5.2
Numerical time, number of iterations, CPU time (seconds), and order of the approximations obtained with the explicit Euler method

I	t_n	n	$CPU\ time$	s
16	301.1953	154211	43	-
32	303.1289	620807	319	-
64	303.1140	2483229	1017	2.16
128	303.1177	9932910	9237	2.02
256	303.1186	39731654	13515	2.04

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