Electronic Transactions on Numerical Analysis.
Volume 30, pp. 247-257, 2008.

ISSN 1068-9613.

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NUMERICAL BLOW-UP SOLUTIONS FOR SOME SEMILINEAR HEAT EQUATIONS*

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Abstract. This paper concerns the study of the numerical approximation for the following initial-boundary value problem,

$$u_t = u_{xx} + \frac{b}{x}u_x + u^p, \quad x \in (0,1), \quad t \in (0,T),$$
$$u_x(0,t) = 0, \quad u(1,t) = 0, \quad t \in (0,T),$$
$$u(x,0) = u_0(x), \quad x \in [0,1],$$

where $b > 0$ and $p > 1$. We give some conditions under which the solution of a semidiscrete form of the above problem blows up in a finite time and estimate its semidiscrete blow-up time. Under some assumptions, we also show that the semidiscrete blow-up time converges to the continuous blow-up time when the mesh size goes to zero. Finally, we give some numerical results to illustrate our analysis.

Key words. semidiscretizations, discretizations, semilinear heat equations, semidiscrete blow-up time

AMS subject classifications. 35B40, 35K65, 65M06

1. Introduction. In this paper, we consider the following initial-boundary value problem for semilinear heat equation of the form

$$u_t = u_{xx} + \frac{b}{x}u_x + u^p, \quad x \in (0,1), \quad t \in (0,T), \tag{1.1}$$
$$u_x(0,t) = 0, \quad u(1,t) = 0, \quad t \in (0,T), \tag{1.2}$$
$$u(x,0) = u_0(x), \quad x \in [0,1], \tag{1.3}$$

which models the temperature distribution of a large number of physical phenomenon from physics, chemistry and biology. The term u^p represents nonlinear heat generation with $p > 1$ and b is a positive parameter. Here $u_0 \in C^0([0,1])$, $u_0'(0) = 0$, $u_0(1) = 0$, $(0,T)$ is the maximal time interval on which $\|u(x,t)\|_\infty$ is finite where $\|u(x,t)\|_\infty = \max_{0\leq x\leq 1} |u(x,t)|$. The time T may be finite or infinite. When T is infinite, we say that the solution u exists globally. When T is finite, the solution u develops a singularity in a finite time, namely

$$\lim_{t\to T} \|u(x,t)\|_\infty = \infty.$$

In this case, we say that the solution u blows up in a finite time and the time T is called the blow-up time of the solution u.

The theoretical study of blow-up of solutions for semilinear parabolic equations has been the subject of investigations of many authors; see [2, 5, 7, 9, 10], and the references cited therein.

The authors have proved that under some assumptions, the solution of (1.1)–(1.3) blows up in a finite time and the blow-up time is estimated.

Let us notice that if we consider the semilinear heat equation

$$u_t = u_{xx} + u^p, \quad x \in B, \quad t \in (0,T),$$

*Received January 8, 2008. Accepted for publication May 22, 2008. Published online on September 12, 2008. Recommended by O. Widlund.
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with boundary conditions

$$u(x,t)=0, \quad x\in S, \quad t\in(0,T),$$

and initial data

$$u(x,0)=u_0(x)\in \overline{B},$$

where $B=\{x\in\mathbb{R}^n: |x|<1\}$, $S=\{x\in\mathbb{R}^n: |x|=1\}$, the radial symmetric solutions are solutions of (1.1)–(1.3) with $b=1$.

In this paper, we are interested in the numerical study using a semidiscrete form of (1.1)–(1.3). Let I be a positive integer and define the grid $x_i=ih$, $0\le i\le I$, where $h=1/I$. Approximate the solution u of (1.1)–(1.3) by the solution $U_h(t)=(U_0(t),U_1(t),\dots,U_I(t))^T$ of the following semidiscrete equations

$$\frac{dU_0(t)}{dt}=(1+b)\delta^2U_0(t)+U_0^p(t), \quad t\in(0,T_b^h), \tag{1.4}$$

$$\frac{dU_i(t)}{dt}=\delta^2U_i(t)+\frac{b}{ih}\delta^+U_i(t)+U_i^p(t), \quad 1\le i\le I-1, \quad t\in(0,T_b^h), \tag{1.5}$$

$$U_I(t)=0, \quad t\in(0,T_b^h), \tag{1.6}$$

$$U_i(0)=\varphi_i, \quad 0\le i\le I, \tag{1.7}$$

where

$$\delta^2U_0(t)=\frac{2U_1(t)-2U_0(t)}{h^2},$$

$$\delta^2U_i(t)=\frac{U_{i+1}(t)-2U_i(t)+U_{i-1}(t)}{h^2}, \quad 1\le i\le I-1,$$

$$\delta^+U_i(t)=\frac{U_{i+1}(t)-U_i(t)}{h}, \quad 1\le i\le I-1.$$

Here, $(0,T_b^h)$ is the maximal time interval on which $\|U_h(t)\|_\infty$ is finite where

$$\|U_h(t)\|_\infty=\sup_{0\le i\le I}|U_i(t)|.$$

When T_b^h is finite, we say that the solution $U_h(t)$ blows up in a finite time and the time T_b^h is called the blow-up time of the solution $U_h(t)$. We give some conditions under which the solution of (1.4)–(1.7) blows up in a finite time and estimate its semidiscrete blow-up time. We also show that the semidiscrete blow-up time converges to the theoretical one when the mesh size goes to zero. A similar study has been undertaken in [1, 4, 6, 8]. In [1, 8], the authors have considered the equation (1.1) for $b=0$ with Dirichlet boundary conditions and nonnegative initial data. Numerical methods for heat equations with nonlinear boundary conditions have been described in [4, 6]. In the same way in [2], the numerical extinction has been studied using some discrete and semidiscrete schemes (a solution u extincts in a finite time if it reaches the value zero in a finite time).

The paper is organized as follows. In the Section 2, we give some properties concerning our scheme. In Section 3, under some conditions, we prove that the solution of the semidiscrete problem blows up in a finite time and estimate its semidiscrete blow-up time. In Section 4, we study the convergence of the semidiscrete blow-up time. Finally, in Section 5 we report on some numerical experiments using several discretisations of (1.1)–(1.3).

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2. Properties of the semidiscrete scheme. In this section, we give some results about the discrete maximum principle. The following lemma is a discrete form of the maximum principle.

LEMMA 2.1. *Let* $a_h(t) \in C^0([0,T],\mathbb{R}^{I+1})$ *and let* $V_h(t) \in C^1([0,T],\mathbb{R}^{I+1})$ *such that for* $t \in (0,T)$,

$$
\begin{aligned}
&\frac{dV_0(t)}{dt} - (1+b)\delta^2 V_0(t) + a_0(t)V_0(t) \geq 0,\\
&\frac{dV_i(t)}{dt} - \delta^2 V_i(t) - \frac{b}{ih}\delta^+ V_i(t) + a_i(t)V_i(t) \geq 0, \quad 1 \leq i \leq I-1,\\
&V_I(t) \geq 0,\\
&V_i(0) \geq 0, \quad 0 \leq i \leq I.
\end{aligned} \tag{2.1}
$$

Then we have $V_i(t) \geq 0$, $0 \leq i \leq I$, $t \in (0,T)$.

Proof. Let $T_0 < T$ and let $m = \inf_{0\leq i \leq I\ t\in[0,T_0]} V_i(t)$. Since for $i \in \{0,\dots,I\}$, $V_i(t)$ is a continuous function, there exists $t_0 \in [0,T_0]$ such that $m = V_{i_0}(t_0)$ for a certain $i_0 \in \{0,\dots,I\}$. If $i_0 = I$, we have a contradiction because of (2.1). When i_0 is between 0 and $I-1$, we observe that

$$\frac{dV_{i_0}(t_0)}{dt} = \lim_{k\to 0}\frac{V_{i_0}(t_0) - V_{i_0}(t_0-k)}{k} \leq 0, \tag{2.2}$$

$$\delta^2 V_{i_0}(t_0) = \frac{V_{i_0+1}(t_0) - 2V_{i_0}(t_0) + V_{i_0-1}(t_0)}{h^2} \geq 0 \quad \text{if} \quad 1 \leq i_0 \leq I-1, \tag{2.3}$$

$$\delta^+ V_{i_0}(t_0) = \frac{V_{i_0+1}(t_0) - V_{i_0}(t_0)}{h} \geq 0 \quad \text{if} \quad 1 \leq i_0 \leq I-1, \tag{2.4}$$

$$\delta^2 V_{i_0}(t_0) = \frac{2V_1(t_0) - 2V_0(t_0)}{h^2} \geq 0 \quad \text{if} \quad i_0 = 0. \tag{2.5}$$

Define the vector $Z_h(t) = e^{\lambda t}V_h(t)$ where λ is such that $a_{i_0}(t_0) - \lambda > 0$. A straightforward computation reveals that

$$\frac{dZ_{i_0}(t_0)}{dt} - (1+b)\delta^2 Z_{i_0}(t_0) + (a_{i_0}(t_0) - \lambda)Z_{i_0}(t_0) \geq 0 \quad \text{if} \quad i_0 = 0, \tag{2.6}$$

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) - \frac{b}{ih}\delta^+ Z_{i_0}(t_0) + (a_{i_0}(t_0) - \lambda)Z_{i_0}(t_0) \geq 0 \quad \text{if} \quad 1 \leq i_0 \leq I-1. \tag{2.7}$$

We observe from (2.2)–(2.5) that

$$\frac{dZ_{i_0}(t_0)}{dt} \leq 0, \quad \delta^2 Z_{i_0}(t_0) \geq 0 \quad \textit{and} \quad \delta^+ Z_{i_0}(t_0) \geq 0.$$

Using (2.6)–(2.7), we arrive at $(a_{i_0}(t) - \lambda)Z_{i_0}(t_0) \geq 0$, which implies that $Z_{i_0}(t_0) \geq 0$. Therefore $V_{i_0}(t_0) = m \geq 0$ and we have the desired result. □

Another version of the discrete maximum principle is the following comparison lemma.

LEMMA 2.2. *Let* $V_h(t)$, $U_h(t) \in C^1([0,T],\mathbb{R}^{I+1})$ *and* $f \in C^0(\mathbb{R}\times\mathbb{R},\mathbb{R})$ *such that for* $t \in (0,T)$

$$\frac{dV_0(t)}{dt} - (1+b)\delta^2 V_0(t) + f(V_0(t),t) < \frac{dU_0(t)}{dt} - (1+b)\delta^2 U_0(t) + f(U_0(t),t), \tag{2.8}$$

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) - \frac{b}{ih}\delta^+ V_i(t) + f(V_i(t),t) \tag{2.9}$$

$$< \frac{dU_i(t)}{dt} - \delta^2 U_i(t) - \frac{b}{ih}\delta^+ U_i(t) + f(U_i(t),t), \quad 1 \le i \le i-1,$$

$$V_I(t) < U_I(t), \tag{2.10}$$

$$V_i(0) < U_i(0), \quad 0 \le i \le I. \tag{2.11}$$

Then we have $V_h(t) < U_h(t)$, $t \in (0,T)$.

Proof. Define the vector $Z_h(t) = U_h(t) - V_h(t)$. Let t_0 be the first $t > 0$ such that $Z_h(t) > 0$ for $t \in [0, t_0)$, but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \ldots, I\}$. If $i_0 = I$, we have a contradiction because of (2.10). If i_0 is between 0 and $I-1$, we observe that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k\to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0-k)}{k} \le 0.$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \ge 0 \quad \text{if} \quad 1 \le i_0 \le I-1,$$

$$\delta^+ Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - Z_{i_0}(t_0)}{h} \ge 0 \quad \text{if} \quad 1 \le i_0 \le I-1,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} \ge 0 \quad \text{if} \quad i_0 = 0,$$

which implies that

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) - \frac{b}{i_0 h}\delta^+ Z_{i_0}(t_0) + f(U_{i_0}(t_0),t_0) - f(V_{i_0}(t_0),t_0) \le 0 \quad \text{if} \quad 1 \le i_0 \le I-1,$$

$$\frac{dZ_{i_0}(t_0)}{dt} - (1+b)\delta^2 Z_{i_0}(t_0) + f(U_{i_0}(t_0),t_0) - f(V_{i_0}(t_0),t_0) \le 0 \quad \text{if } i_0 = 0.$$

But these inequalities contradict (2.8) and (2.9). ☐

To finish this section, let us state a result on the operators δ^2 and δ^+.

LEMMA 2.3. *Let* $U_h \in \mathbb{R}^{I+1}$ *such that* $U_h \ge 0$. *Then we have*

$$\delta^2 U_i^p \ge pU_i^{p-1}\delta^2 U_i, \quad 1 \le i \le I-1,$$
$$\delta^+ U_i^p \ge pU_i^{p-1}\delta^+ U_i, \quad 1 \le i \le I-1.$$

Proof. Using Taylor's expansion, we get

$$\delta^2 U_i^p = pU_i^{p-1}\delta^2 U_i + (U_{i+1}-U_i)^2 \frac{p(p-1)}{2h^2}\theta_i^{p-2} + (U_{i-1}-U_i)^2 \frac{p(p-1)}{2h^2}\eta_i^{p-2}, \quad 1 \le i \le I-1,$$

$$\delta^2 U_0^p = pU_0^{p-1}\delta^2 U_0 + (U_1 - U_0)^2 \frac{p(p-1)}{h^2}\theta_i^{p-2},$$

$$\delta^+ U_i^p = pU_i^{p-1}\delta^2 U_i + (U_{i+1}-U_i)^2 \frac{p(p-1)}{2h^2}\chi_i^{p-2}, \quad 1 \le i \le I-1,$$

where θ_i and χ_i are intermediate values between U_i and U_{i+1}, η_i is an intermediate value between U_i and U_{i-1}. Use the fact that U_h is nonnegative to complete the rest of the proof. ☐

3. Semidiscrete blow-up solutions. In this section under some assumptions, we show that the solution of the semidiscrete problem blows up in a finite time and estimate its semidiscrete blow-up time.

THEOREM 3.1. *Suppose that there exists a positive integer A such that the initial data at (1.7) satisfies*

$$(1+b)\delta^2 \varphi_0 + \varphi_0^p \ge A\varphi_0^p,$$

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$$\delta^2\varphi_i + \frac{b}{ih}\delta^+\varphi_i + \varphi_i^p \geq A\varphi_i^p, \quad 1 \leq i \leq I-1. \tag{3.1}$$

Then the solution $U_h(t)$ of (1.4)–(1.7) blows up in a finite time T_b^h and we have the following estimate

$$T_b^h \leq \frac{1}{A}\frac{\|\varphi_h\|_\infty^{1-p}}{(p-1)}.$$

Proof. Since $(0, T_b^h)$ be the maximal time interval on which $\|U_h(t)\|_\infty$ is finite, our aim is to show that T_b^h is finite and obeys the above inequality. Introduce the vector J_h such that

$$J_i = \frac{dU_i}{dt} - AU_i^p, \quad 0 \leq i \leq I.$$

A straightforward calculation gives

$$\frac{dJ_0}{dt} - (1+b)\delta^2 J_0 = \frac{d}{dt}(\frac{dU_0}{dt} - (1+b)\delta^2 U_0) - ApU_0^{p-1}\frac{dU_0}{dt} + A(1+b)\delta^2 U_0^p,$$
$$\frac{dJ_i}{dt} - \delta^2 J_i = \frac{d}{dt}(\frac{dU_i}{dt} - \delta^2 U_i) - ApU_i^{p-1}\frac{dU_i}{dt} + A\delta^2 U_i^p, \quad 1 \leq i \leq I-1.$$

From Lemma 2.3, we have $\delta^2 U_i^p \geq pU_i^{p-1}\delta^2 U_i$ which implies that

$$\frac{dJ_0}{dt} - (1+b)\delta^2 J_0 \geq \frac{d}{dt}(\frac{dU_0}{dt} - (1+b)\delta^2 U_0) - ApU_0^{p-1}(\frac{dU_0}{dt} - (1+b)\delta^2 U_0),$$
$$\frac{dJ_i}{dt} - \delta^2 J_i \geq \frac{d}{dt}(\frac{dU_i}{dt} - \delta^2 U_i) - ApU_i^{p-1}(\frac{dU_i}{dt} - \delta^2 U_i), \quad 1 \leq i \leq I-1.$$

Use (1.4)–(1.5) to obtain

$$\frac{dJ_0}{dt} - (1+b)\delta^2 J_0 \geq pU_0^{p-1}J_0,$$
$$\frac{dJ_i}{dt} - \delta^2 J_i \geq pU_i^{p-1}J_i + \frac{b}{ih}(\delta^+\frac{dU_i}{dt} - ApU_i^{p-1}\delta^+ U_i), \quad 1 \leq i \leq I-1.$$

Taking into account the expression of $J_h(t)$, we get $\delta^+ J_i = \delta^+\frac{dU_i}{dt} - A\delta^+ U_i^p$, which implies that $\delta^+\frac{dU_i}{dt} = \delta^+ J_i + A\delta^+ U_i^p$. From Lemma 2.3, we arrive at $\delta^+\frac{dU_i}{dt} \geq \delta^+ J_i + AU_i^{p-1}\delta^+ U_i$. Therefore, we get $\delta^+\frac{dU_i}{dt} - AU_i^{p-1}\delta^+ U_i \geq \delta^+ J_i$ and due to (3.1), we discover that

$$\frac{dJ_i}{dt} - \delta^2 J_i - \frac{b}{ih}\delta^+ J_i \geq pU_i^{p-1}J_i, \quad 1 \leq i \leq I-1.$$

Obviously, $J_I(t) = 0$ and the hypotheses on the initial data ensure that $J_h(0) \geq 0$. It follows from Lemma 2.1 that $J_h(t) \geq 0$ for $t \in (0, T_b^h)$, which implies that $\frac{dU_i}{dt} \geq AU_i^p$, $0 \leq i \leq I$. This estimation may be rewritten as follows

$$\frac{dU_i}{U_i^q} \geq Adt, \quad 0 \leq i \leq I.$$

Integrating this inequality over (t, T_b^h), we find that

$$T_b^h - t \leq \frac{1}{A}\frac{(U_i(t))^{1-p}}{(p-1)}, \tag{3.2}$$

which implies that

$$T_b^h \leq \frac{1}{A} \frac{\|\varphi_h\|_\infty^{1-p}}{(p-1)}.$$

Therefore T_b^h is finite and the proof is complete. ∎

REMARK 3.2. The inequalities (3.2) imply that

$$T_b^h - t_0 \leq \frac{1}{A} \frac{\|U_h(t_0)\|_\infty^{1-p}}{(p-1)} \quad \text{if} \quad 0 < t_0 < T_b^h$$

and there exists a constant $C > 0$ such that

$$U_i(t) \leq \frac{C}{(T_b^h - t)^{\frac{1}{p-1}}}, \quad 0 \leq i \leq I.$$

THEOREM 3.3. *Let $U_h(t)$ be the solution of (1.4)–(1.7). Then we have*

$$T_b^h \geq \frac{\|\varphi_h\|_\infty^{1-p}}{(p-1)}.$$

Proof. Let i_0 be such that $U_{i_0}(t) = \max_{0 \leq i \leq I} U_i(t)$. It is not hard to see that

$$\delta^2 U_{i_0}(t) = \frac{U_{i_0+1}(t) - 2U_{i_0}(t) + U_{i_0-1}(t)}{h^2} \leq 0 \quad \text{if} \quad 1 \leq i_0 \leq I-1,$$
$$\delta^+ U_{i_0}(t) = \frac{U_{i_0+1}(t) - U_{i_0}(t)}{2h} \leq 0 \quad \text{if} \quad 1 \leq i_0 \leq I-1,$$
$$\delta^2 U_{i_0}(t) = \frac{2U_1(t) - 2U_0(t)}{h^2} \leq 0 \quad \text{if} \quad i_0 = 0.$$

We deduce that $\frac{dU_{i_0}}{dt} \leq U_{i_0}^p$, which implies that $\frac{dU_{i_0}}{U_{i_0}^p} \leq dt$. Integrating this inequality over $(0, T_b^h)$, we obtain

$$T_b^h \geq \frac{(U_{i_0}(0))^{1-p}}{(p-1)}.$$

Use the fact that $U_{i_0}(0) = \|\varphi_h\|_\infty$ to complete the rest of the proof. ∎

4. Convergence of the semidiscrete blow-up time. In this section, under some assumptions, we show that the blow-up time for the solution of the semidiscrete problem converges to the blow-up time for the solution of the continuous problem when the mesh size tends to zero. In order to prove this result, firstly we show that for each fixed time interval $[0, T]$ where the solution u of (1.1)–(1.3) is defined, the solution $U_h(t)$ of (1.4)–(1.7) approximates u, when the mesh parameter h goes to zero by the following theorem.

THEOREM 4.1. *Assume that (1.1)–(1.3) has a solution $u \in C^{3,1}([0,1] \times [0,T])$ and the initial condition at (1.7) satisfies*

$$\|\varphi_h - u_h(0)\|_\infty = o(1) \quad \text{as} \quad h \to 0, \tag{4.1}$$

where $u_h(t) = (u(x_0,t), ..., u(x_I,t))^T$. Then, for h sufficiently small, the problem (1.4)–(1.7) has a unique solution $U_h \in C^1([0,T], \mathbb{R}^{I+1})$ such that

$$\max_{0 \leq t \leq T} \|U_h(t) - u_h(t)\|_\infty = O(\|\varphi_h - u_h(0)\|_\infty + h) \quad \text{as} \quad h \to 0.$$

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Proof. Since $u \in C^{3,1}$, there exist positive constants K and M such that

$$\|u\|_\infty \leq K, \quad p(K+1)^{p-1} \leq M. \tag{4.2}$$

The problem (1.4)–(1.7) has for each h, a unique solution $U_h \in C^1([0,T_q^h),\mathbb{R}^{I+1})$. Let $t(h)$ the greatest value of $t > 0$ such that

$$\|U_h(t) - u_h(t)\|_\infty < 1 \quad \text{for} \quad t \in (0, t(h)). \tag{4.3}$$

The relation (4.1) implies that $t(h) > 0$ for h sufficiently small. Let $t^*(h) = \min\{t(h), T\}$. By the triangle inequality, we obtain

$$\|U_h(t)\|_\infty \leq \|u_h(t)\|_\infty + \|U_h(t) - u_h(t)\|_\infty \quad \text{for} \quad t \in (0, t^*(h)),$$

which implies that

$$\|U_h(t)\|_\infty \leq 1 + K \quad \text{for} \quad t \in (0, t^*(h)). \tag{4.4}$$

Let $e_h(t) = U_h(t) - u_h(t)$ be the error of discretization. Since $u \in C^{3,1}$, using Taylor's expansion, we have for $t \in (0, t^*(h))$,

$$\frac{de_0(t)}{dt} - (1+b)\delta^2 e_0(t) = p\xi_0^{p-1} e_0(t) + o(h),$$
$$\frac{de_i(t)}{dt} - \delta^2 e_i(t) - \frac{b}{ih}\delta^+ e_i(t) = p\xi_i^{p-1} e_i(t) + o(h),$$

where ξ_i is an intermediate value between $U_i(t)$ and $u(x_i,t)$. Using (4.2) and (4.4), there exists a positive constant M such that

$$\frac{de_0(t)}{dt} - (1+b)\delta^2 e_0(t) \leq M|e_0(t)| + Mh,$$
$$\frac{de_i(t)}{dt} - \delta^2 e_i(t) - \frac{b}{ih}\delta^+ e_i(t) \leq M|e_i(t)| + Mh, \quad 1 \leq i \leq I-1.$$

Introduce the vector $Z_h(t)$ such that

$$Z_i(t) = e^{(M+1)t}(\|\varphi_h - u_h(0)\|_\infty + Mh), \quad 0 \leq i \leq I.$$

A straightforward calculation reveals that

$$\frac{dZ_0(t)}{dt} - (1+b)\delta^2 Z_0(t) > M|Z_0(t)| + Mh,$$
$$\frac{dZ_i(t)}{dt} - \delta^2 Z_i(t) - \frac{b}{ih}\delta^+ Z_i(t) > M|Z_i(t)| + Mh, \quad 1 \leq i \leq I-1,$$
$$Z_I(t) > e_I(t).$$

It follows from Lemma 2.2 that $Z_h(t) > e_h(t)$ for $t \in (0, t^*(h))$. In the same way, we also prove that $Z_h(t) > -e_h(t)$ for $t \in (0, t^*(h))$, which implies that

$$\|U_h(t) - u_h(t)\|_\infty \leq e^{(M+1)t}(\|\varphi_h - u_h(0)\|_\infty + Mh), \quad t \in (0, t^*(h)).$$

Let us show that $t^*(h) = T$. Suppose that $T > t(h)$. From (4.3), we obtain

$$1 = \|U_h(t(h)) - u_h(t(h))\|_\infty \leq e^{(M+1)T}(\|\varphi_h - u_h(0)\|_\infty + Mh).$$

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Since the term on the right hand side of the above inequality tends to zero as h goes to zero, we deduce that $1 \leq 0$, which is impossible. Consequently $t^*(h) = T$, and the proof is complete. □

Now, we are in a position to prove the main theorem of this section.

THEOREM 4.2. *Suppose that the problem (1.1)–(1.3) has a solution u which blows up in a finite time T_b such that $u \in C^{3,1}([0,1] \times [0,T_b))$ and the initial data at (1.7) satisfies*

$$\|\varphi_h - u_h(0)\|_\infty = o(1) \quad as \quad h \to 0.$$

Under the assumptions of Theorem 3.1, the problem (1.4)–(1.7) has a solution $U_h(t)$ which blows up in a finite time T_b^h and

$$\lim_{h\to 0} T_b^h = T_b.$$

Proof. Let $\varepsilon > 0$. There exists a positive constant N such that

$$\frac{1}{A}\frac{x^{1-p}}{(p-1)} \leq \frac{\varepsilon}{2} < \infty \quad \text{for} \quad x \in [N, +\infty). \tag{4.5}$$

Since u blows up at the time T_b, then there exists T_1 such that $|T_1 - T_b| \leq \frac{\varepsilon}{2}$ and $\|u(x,t)\|_\infty \geq 2N$ for $t \in [T_1, T_b]$. Letting $T_2 = \frac{T_1+T_b}{2}$, we see that $\sup_{t\in[0,T_2]} |u(x,t)| < \infty$. It follows from Theorem 4.1 that the problem (1.4)–(1.7) has a solution $U_h(t)$ which obeys $\sup_{t\in[0,T_2]} |U_h(t) - u_h(t)|_\infty \leq N$. Applying the triangle inequality, we get $\|U_h(t)\|_\infty \geq \|u_h(t)\|_\infty - \|U_h(t) - u_h(t)\|_\infty$, which leads to $\|U_h(t)\|_\infty \geq N$ for $t \in [0,T_2]$. From Theorem 3.1, $U_h(t)$ blows up at the time T_b^h. We deduce from Remark 3.2 and (4.5) that

$$|T_b^h - T_b| \leq |T_b^h - T_2| + |T_2 - T_b| \leq \frac{\varepsilon}{2} + \frac{1}{A}\frac{\|U_h(T_2)\|_\infty^{1-p}}{(p-1)} \leq \varepsilon,$$

and we have the desired result. □

5. Numerical experiments. In this section, we study the phenomenon of blow-up, using full discrete schemes (explicit and implicit) of (1.1)–(1.3). Firstly, we approximate the solution u of (1.1)–(1.3) by the solution $U_h^{(n)} = (U_0^n, U_1^n, \ldots, U_I^n)^T$ of the following explicit scheme

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = (1+b)\delta^2 U_0^{(n)} + (U_0^{(n)})^p, \tag{5.1}$$

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \delta^2 U_i^{(n)} + \frac{b}{ih}(\frac{U_{i+1}^{(n)} - U_i^{(n)}}{h}) + (U_i^{(n)})^p, \quad 1 \leq i \leq I-1, \tag{5.2}$$

$$U_I^{(n)} = 0, \tag{5.3}$$

$$U_i^{(0)} = 20\cos(ih\frac{\pi}{2}), \quad 0 \leq i \leq I, \tag{5.4}$$

where $n \geq 0$,

$$\Delta t_n = \min\{\frac{h^2}{2+b}, h^2\|U_h^{(n)}\|_\infty^{1-p}\},$$

$$\delta^2 U_0^{(n)} = \frac{2U_1^{(n)} - 2U_0^{(n)}}{h^2},$$

$$\delta^2 U_i^{(n)} = \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2}, \quad 1 \leq i \leq I-1.$$

Let us notice that the restriction on the time step guarantees the positivity of the discrete solution.

Secondly, approximate the solution u of (1.1)–(1.3) by the solution $U_h^{(n)}$ of the following implicit scheme

$$\frac{U_0^{(n+1)}-U_0^{(n)}}{\Delta t_n} = (1+b)\delta^2 U_0^{(n+1)} + (U_0^{(n)})^p, \tag{5.5}$$

$$\frac{U_i^{(n+1)}-U_i^{(n)}}{\Delta t_n} = \delta^2 U_i^{(n+1)} + \frac{b}{ih}\Big(\frac{U_{i+1}^{(n+1)}-U_i^{(n+1)}}{h}\Big) + (U_i^{(n)})^p, \quad 1\le i\le I-1, \tag{5.6}$$

$$U_I^{(n+1)} = 0, \tag{5.7}$$

$$U_i^{(0)} = 20\cos(ih\frac{\pi}{2}), \quad 0\le i\le I, \tag{5.8}$$

where $\Delta t_n = h^2\|U_h^{(n)}\|_\infty^{1-p}$. The above equations may be rewritten in the following form

$$(1+2\frac{\Delta t_n}{h^2})U_0^{(n+1)} - 2\frac{\Delta t_n}{h^2}U_1^{(n+1)} = U_0^{(n)} + \Delta t_n (U_0^{(n)})^p, \tag{5.9}$$

$$\begin{aligned} -\frac{\Delta t_n}{h^2}U_{i-1}^{(n+1)} + (1+2\frac{\Delta t_n}{h^2}+\frac{b\Delta t_n}{ih^2})U_i^{(n+1)} - (\frac{\Delta t_n}{h^2}+\frac{b\Delta t_n}{ih^2})U_{i+1}^{(n+1)} \\ = U_i^{(n)} + \Delta t_n (U_i^{(n)})^p, \quad 1\le i\le I-1, \end{aligned} \tag{5.10}$$

$$U_I^{(n+1)} = 0. \tag{5.11}$$

The equalities (5.9)–(5.11) lead us to the linear system below

$$A_h^{(n)} V_h^{(n+1)} = F_h^{(n)},$$

where $A_h^{(n)}$ is a $I\times I$ tridiagonal matrix defined as follows

$$A_h^{(n)} = \begin{bmatrix} a_0 & b_0 & 0 & \cdots & 0 \\ c_1 & a_1 & b_1 & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & b_{I-2} \\ 0 & \cdots & 0 & c_{I-1} & a_{I-1} \end{bmatrix}$$

with

$$\begin{aligned} a_0 &= 1+2\frac{\Delta t_n}{h^2}(1+b), \\ a_i &= 1+2\frac{\Delta t_n}{h^2}(1+b)+\frac{b\Delta t_n}{ih^2}, \quad i=1,\dots,I-1, \\ b_0 &= -2\frac{\Delta t_n}{h^2}(1+b), \quad b_i = -(2\frac{\Delta t_n}{h^2}+\frac{b\Delta t_n}{ih^2}), \quad i=1,...,I-2, \\ c_i &= -\frac{\Delta t_n}{h^2}, \quad i=1,\dots,I-1, \\ F_i^n &= U_i^n + \Delta t_n (U_i^n)^p, \quad 0\le i\le I-1, \\ V_h^{(n+1)} &= \left(U_0^{(n+1)}, U_1^{(n+1)}, \dots, U_{I-1}^{(n+1)}\right)^T. \end{aligned}$$

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It is not hard to see that

$$(A_h^n)_{ii} > 0, \quad (A_h^n)_{ij} < 0 \quad i \neq j, \quad (A_h^n)_{ii} > \sum_{i \neq j} |(A_h^n)_{ij}|.$$

These inequalities imply that the linear system has a unique solution for $n \geq 0$ and the discrete solution is nonnegative. For the proof, see for instance [3]

We need the following definition.

DEFINITION 5.1. *We say that the solution $U_h^{(n)}$ of (5.1)–(5.4) or (5.5)–(5.8) blows up in a finite time if $\lim_{n\to+\infty} \|U_h^{(n)}\| = +\infty$ and the series $\sum_{n=0}^{+\infty} \Delta t_n$ converges. The quantity $\sum_{n=0}^{+\infty} \Delta t_n$ is called the numerical blow-up time of the solution $U_h^{(n)}$.*

In the Tables 5.1–5.6, we present the numerical blow-up times, values of n, the CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128. We take for the numerical blow-up time $T^n = \sum_{j=0}^{n-1} \Delta t_j$ which is computed at the first time when $\Delta t_n = |T^{n+1} - T^n| \leq 10^{-16}$. The order(s) of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

For the numerical values, we take, $U_i^{(0)} = 20\cos(\frac{\pi}{2} ih)$, $p = 2$, and $\tau = h^2$.

First case: $b = 0$

TABLE 5.1
Numerical blow-up times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method.

I	T^n	n	CPU_t	s
16	0.056343	7191	16	-
32	0.056231	27359	69	-
64	0.056210	103908	560	2.31
128	0.055203	405086	7052	1.68

TABLE 5.2
Numerical blow-up times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the first implicit Euler method.

I	T^n	n	CPU_t	s
16	0.056376	7191	20	-
32	0.056240	27360	113	-
64	0.056213	103908	1460	2.22
128	0.056207	406009	20746	1.85

Second case: $b = 1$

TABLE 5.3
Numerical blow-up times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method.

I	T^n	n	CPU_t	s
16	0.067710	7406	13	-
32	0.066986	28264	70s	-
64	0.066527	107691	594	0.76
128	0.06630	409301	7203	1.02

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TABLE 5.4
Numerical blow-up times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the first implicit Euler method.

I	T^n	n	CPU_t	s
16	0.067802	7407	18	-
32	0.067008	28265	109	-
64	0.066532	107692	1506	0.84
128	0.066303	409302	21000	1.05

Third case: $b = 2$

TABLE 5.5
Numerical blow-up times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method.

I	T^n	n	CPU_t	s
16	0.090365	7765	15	-
32	0.089372	29778	71	-
64	0.087243	114065	630	0.85
128	0.086658	435971	8580	0.95

TABLE 5.6
Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the first implicit Euler method.

I	T^n	n	CPU_t	s
16	0.090575	7767	19	-
32	0.0884238	29780	115	-
64	0.087226	114067	1718	0.85
128	0.086636	435972	23051	1.07

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