**Electronic Transactions on Numerical Analysis.**
**Volume 30, pp. 291-304, 2008.**

**ISSN 1068-9613.**

**ETNA**
**Kent State University**
**http://etna.math.kent.edu**

# A CONVERGENT ADAPTIVE FINITE ELEMENT METHOD WITH OPTIMAL COMPLEXITY*

ROLAND BECKER†, SHIPENG MAO‡, AND ZHONG-CI SHI‡

**Abstract.** In this paper, we introduce and analyze a simple adaptive finite element method for second order elliptic partial differential equations. The marking strategy depends on whether the data oscillation is sufficiently small compared to the error estimator in the current mesh. If the oscillation is small compared to the error estimator, we mark as many edges such that their contributions to the local estimator are at least a fixed proportion of the global error estimator (bulk criterion for the estimator). Otherwise, we reduce the oscillation by marking sufficiently many elements, such that the oscillations of the marked cells are at least a fixed proportion of the global oscillation (bulk criterion for the oscillation). This marking strategy guarantees a strict reduction of the error augmented by the oscillation term. Both convergence rates and optimal complexity of the adaptive finite element method are established, with an explicit expression of the constants in the estimates.

**Key words.** adaptive finite element method, a posteriori error estimator, convergence rate, optimal computational complexity

**AMS subject classifications.** 65N12, 65N15, 65N30, 65N50

## 1. Introduction.

The analysis of adaptive finite element methods has made important progress in recent years. Up to now, a large amount of work has been performed concerning AFEMs based on a posteriori error estimation for finite element methods, which typically consists of successive loops of the sequence

$$\text{SOLVE} \to \text{ESTIMATE} \to \text{MARK} \to \text{REFINE}.$$

We refer to the review articles of Eriksson et al. [16] and the books of Ainsworth [1], Babuška [2], Verfürth [24] and the references therein.

On the other hand, while these adaptive finite element methods have been shown to be very successful computationally, the theory describing the advantages of such methods over their nonadaptive counterparts is still not complete. Apart from the well-known results in the one dimensional case by Babuška and Vogelius [3], the convergence of AFEMs in the multidimensional case was an open issue before the work by Dörfler [15], which was later extended by Morin, Nochetto and Siebert [19, 20], and more recently by Carstensen and Hoppe for mixed FEM [7] and for nonconforming FEM [8], by Mekchay and Nochetto for general second order linear elliptic PDE [17]. Especially, the importance and necessity of controlling data oscillations and inner nodes are pointed out in [19] and [20].

Another important breakthrough in the theoretical understanding of AFEMs is the estimation of the dimension of the adaptively constructed discrete spaces, first achieved by Binev, Dahmen and DeVore [5] who showed the optimal computational complexity. The key to prove the optimality was the introduction of an additional so-called coarsening step. A further significant improvement has been achieved by Stevenson [22] who shows that the additional coarsening step is not necessary in order to prove optimal complexity. The importance of the above mentioned results lays in the fact that they show optimal complexity of adaptive algorithms in the following sense: if the exact solution can be approximated by

*Received March 31, 2007. Accepted for publication July 9, 2008. Published online on November 21, 2008. Recommended by K. Burrage.

† Laboratoire de Mathématiques Appliquées UMR-CNRS 5142, Université de Pau, 64013 Pau Cedex, France (roland.becker@univ-pau.fr).

‡ Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and System Science, Chinese Academy of Science, PO Box 2719, Beijing, 100080, China ({maosp, shi}@lsec.cc.ac.cn).

a given adaptive method at a certain rate (quotient of accuracy to number of unknowns), the iteratively constructed sequence of meshes will realize this rate up to a constant factor.

In this paper, we present a simple adaptive finite element method for second order elliptic partial differential equations, which is a modification of the MNS algorithm of [19] and [20] by Morin, Nochetto and Siebert. Our modification is motivated by the idea that if the data oscillation term is small compared to the error estimator, it is sufficient to mark elements such that the sum of the local error indicators amounts to a fixed proportion of the global error estimator, otherwise we only need to perform a similar marking strategy for the oscillation term. The adaptive algorithm considered here simplifies the MNS algorithm, but its convergence proof is not obvious. Since in one refinement step we mark elements either according to the error estimator or according to the oscillation term, one cannot expect the oscillation term to be reduced in every iteration as is the case in the MNS algorithm. Therefore, in order to prove convergence of our algorithm, we need to couple the error and oscillation term by an argument similar to [19]. As a novel theoretical result, we prove a contraction property of the error augmented by the data oscillation term. In addition, both convergence rates and optimal complexity of the adaptive finite element method are established by a detailed analysis in the spirit of [19] and [22].

An outline of the remaining parts of the paper is as follows. In Section 2, we introduce the set-up and discretization of the model problem, an a posteriori error estimate for the finite element method and the adaptive algorithm AFEM along with some notations and preliminaries for subsequent use. In Section 3 we present some useful lemmata concerning the a posteriori error estimator and prove the convergence rates and optimal complexity of the adaptive finite element method by a detailed analysis. Finally, some comments and extensions of the results conclude the paper in Section 4.

## 2. A simple adaptive finite element method.

We start this section with some useful notation. Throughout this paper, we adopt the standard conventions for Sobolev spaces (see, e.g., [11]), the norms and seminorms of a function $v$ defined on an open set $G$:

$$\|v\|_{m,G} = \left(\int_G \sum_{|\alpha|\le m} |D^\alpha v|^2\right)^{\frac{1}{2}}, \quad |v|_{m,G} = \left(\int_G \sum_{|\alpha|= m} |D^\alpha v|^2\right)^{\frac{1}{2}}.$$

Let $\Omega \subset R^n$ be a bounded polygonal (polyhedral) domain. We consider the following second order elliptic equations : Find $u \in H_0^1(\Omega)$ such that

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $f \in L^2(\Omega)$.

We denote by $(\cdot,\cdot)_G$ the $L^2(G)$ inner product, and if $G = \Omega$, we drop the index $\Omega$ for simplicity. For any $f \in L^2(\Omega)$, the weak formulation of the problem (2.1) reads as follows:

$$\begin{cases} \text{Find } u \in H_0^1(\Omega), & \text{such that} \\ a(u,v) = (f,v), & \forall\, v \in H_0^1(\Omega) \end{cases} \tag{2.2}$$

with $a(u,v) = \int_\Omega \nabla u \cdot \nabla v\, dx$.

Let $\mathcal{T}_H$ be a conforming regular triangulation of $\Omega$ and let $V_H$ denote the finite element space of piecewise linear functions over $\mathcal{T}_H$. We denote by $V^H$ the space of continuous piecewise linear functions over $\mathcal{T}_H$, and let $V_0^H$ be the subspace of functions of $V^H$ that vanish at the boundary $\partial\Omega$. Let $u_H$ denote the solution of the discrete problem

$$\begin{cases} \text{Find } u_H \in V_0^H, & \text{such that} \\ a(u_H,v_H) = (f,v_H), & \forall\, v_H \in V_0^H. \end{cases} \tag{2.3}$$

ETNA
Kent State University
http://etna.math.kent.edu



We shall not discuss the step SOLVE which deserves a separate investigation. We assume that the solutions of the finite-dimensional problems can be generated to any accuracy to accomplish this in optimal space and time complexity. Multigrid-like methods are well-known to achieve this goal, cf. [4, 26].

We denote by $\mathcal{E}_H$ the set of edges (or faces in 3D) of the triangulation $\mathcal{T}_H$ that do not belong to the boundary $\partial\Omega$ of the domain $\Omega$. For $E \in \mathcal{E}_H$, $H_E$ denotes the diameter of $E$ and the domain $\omega_E$ is the union of the two elements in $\mathcal{T}_H$ sharing $E$. For any $K \in \mathcal{T}_H$, $H_K$ stands for its diameter and the domain $\omega_K$ is the union of the adjacent elements in $\mathcal{T}_H$.

Subtracting (2.2) from (2.3) and integrating by parts yields

$$a(u-u_H,v) = \sum_{K\in\mathcal{T}_H}(f, v-\mathcal{I}_H v) + \sum_{E\in\mathcal{E}_H}\int_E J_E(v-\mathcal{I}_H v)ds, \forall v \in H_0^1(\Omega).$$

Here and below, $J_E = [[\nabla u_H]]_E \cdot \nu$ represents the jump of flux across side $E$ which is independent of the orientation of the unit normal $\nu$, and $\mathcal{I}_H$ denotes the Clément interpolation operator [12]. It plays an important role in the analysis of the reliability, which is well established in the literature; see, for example, [9].

Let $\eta_E$ be the local error indicator associated with edge $E \in \mathcal{E}_H$ which is defined as

$$\eta_E(u_H) := \left(\sum_{K\in\omega_E}\|H_K f\|^2_{0,K} + \|H_E^{\frac{1}{2}} J_E\|^2_{0,E}\right)^{1/2}.$$

For any given subset $\mathcal{F}_H \subseteq \mathcal{E}_H$ and $\mathcal{S}_H \subseteq \mathcal{T}_H$, we define

$$\eta(u_H,\mathcal{F}_H) := \left(\sum_{E\in\mathcal{F}_H}\eta_E^2(u_H)\right)^{1/2}$$

and

$$\operatorname{osc}(f,\mathcal{S}_H) := \left(\sum_{K\in\mathcal{S}_H}\|H_K(f-f_H)\|^2_{0,K}\right)^{1/2},$$

where $f_H$ denotes a piecewise constant approximation of $f$ on $\mathcal{T}_H$. If $f \in L^2(\Omega)$, its value on $K$ is the mean value of $f$ over $K$.

The following upper and lower bounds are well known; see, e.g., [1] and [24].

LEMMA 2.1 (Upper bound). *There exists a constant $C_1 > 0$ depending only on the minimum angle of $\mathcal{T}_H$ such that*

$$|u-u_H|^2_{1,\Omega} \le C_1\eta^2(u_H,\mathcal{T}_H). \tag{2.4}$$

LEMMA 2.2 (Lower bound). *There exist two constants $C_2, C_3 > 0$ depending only on the minimum angle of $\mathcal{T}_H$ such that, for any $E \in \mathcal{E}_H$,*

$$\eta_E^2(u_H) \le C_2\sum_{K\in\omega_E}|u-u_H|^2_{1,K} + C_3 osc^2(f,\omega_E). \tag{2.5}$$

Summing up all $E \in \mathcal{E}_H$ in (2.5) we have

$$\eta^2(u_H,\mathcal{E}_H) \le (n+1)C_2|u-u_H|^2_{1,\Omega} + (n+1)C_3\operatorname{osc}^2(f,\mathcal{T}_H). \tag{2.6}$$

We note that we can assume without loss of generality $C_2 \geq C_3$.

In practice, both the local error estimator $\eta(u_H, \mathcal{F}_H)$ and the oscillation term $\operatorname{osc}(f, \mathcal{S}_H)$ should be used in the MARK step of the algorithm. The precise way they are used in the MARK step influences the convergence of the AFEM; see [19] and [20]. What is more, it also influences the optimality of the AFEM. Therefore, the MARK step plays a key role in AFEMs and should be designed properly.

As for the REFINE step, we need to carefully choose the rule for dividing the marked triangles such that the family of meshes obtained by this refinement rule is conforming and shape regular. In addition, we need to control the number of elements added in order to ensure the overall optimality of the refinement procedure. In this article, we shall use the *newest vertex bisection* technique. We refer to [5, 18, 22] for details of this algorithm and restrict ourselves to list the following properties used later on.

LEMMA 2.3. *Let $\mathcal{T}_{h_k}, k = 0, \ldots n$ be a sequence of locally refined triangulations created by the newest vertex algorithm, starting from the initial mesh $\mathcal{T}_{h_0}$. Let $\mathcal{M}_k, k = 0, \ldots n-1$ be the collection of all marked triangles in step $k$. Let $\mathcal{N}(\mathcal{T})$ denote the number of elements of a triangulation $\mathcal{T}$. Then $\mathcal{T}_{h_n}$ is uniformly shape regular and the shape regularity of $\mathcal{T}_{h_n}$ only depends on that of $\mathcal{T}_{h_0}$ and furthermore,*

$$\mathcal{N}(\mathcal{T}_{h_n}) \leq \mathcal{N}(\mathcal{T}_{h_0}) + C_0 \sum_{k=0}^{n-1} \mathcal{N}(\mathcal{M}_k). \tag{2.7}$$

REMARK 2.4. The result (2.7) was first proved by Binev, Dahmen and DeVore [5] in the 2D triangular case and generalized by Stenvenson [23] to the case of general n-simplices.

Another important rule which appears in the REFINE step is *the interior node property*. Let $\mathcal{T}_h$ be a refinement of the triangulation $\mathcal{T}_H$. We say that the refinement satisfies the interior node property if each element of the marked set $\mathcal{M}_h$ to be refined, as well as each of its edges, contains a node of $\mathcal{T}_h$ in its interior. In fact, the interior node property is also a necessary condition for the error reduction of adaptive linear finite element methods; see [19] for an example which shows that if the refinement does not produce interior nodes, the error may not change.

We are now in the position to present our adaptive algorithm AFEM. Similar adaptive mesh adaptation algorithms have been presented in the literature [19, 22]. The new ingredient in Algorithm 1 is the introduction of an adaptive marking strategy, which compares the oscillation term with the estimator in each step of the iteration. Depending on this comparison, only the dominant term is used for local refinement. The makes an importance difference with the algorithms known before. Since for many practical applications, the oscillation term can be expected to be significantly smaller, the algorithm will practically be driven by the estimator. In the recent technical reports [10, 14, 21], published after submission of the present article, the authors also try to overcome the drawback of the original MNS algorithm. However, they do not consider an adaptive marking strategy, which allows us here to prove quasi-optimal convergence behavior.

Finally, we comment on the choice of the constants in Algorithm 1. According to our analysis, the constant $\gamma$ has to be chosen small enough. A theoretical value ensuring geometrical convergence is given in Theorem 3.5; see (3.5) below. An additional condition for the choice of $\alpha$ is necessary in order to guarantee optimal complexity in Theorem 3.7; see (3.27) below. It is clear that such a condition has to be imposed, since the choice of $\alpha = 1$ corresponds to global refinement in each step of the algorithm.

**3. Convergence and optimality of AFEM.** In this section we shall prove the convergence and optimality of the algorithm developed in Section 2. The techniques are adapted

ETNA
Kent State University
http://etna.math.kent.edu



**Algorithm 1 (AFEM)**

(0) Select parameters $0 < \alpha, \theta, \gamma < 1$ and an initial mesh $\mathcal{T}_0$, and set $k = 0$.
(1) Solve the discrete system (2.3) on $\mathcal{T}_k$ for the finite element solution $u_k$.
(2) Compute the a posteriori error estimator $\eta(u_k, \mathcal{T}_k)$ and oscillation term $\mathrm{osc}(f, \mathcal{T}_k)$. If $\eta(u_k, \mathcal{T}_k) \leq \epsilon$, then stop.
(3) i) If $\mathrm{osc}^2(f, \mathcal{T}_k) < \gamma\, \eta^2(u_k, \mathcal{T}_k)$ mark the minimal edge set $\mathcal{F}_k$ of $\mathcal{E}_k$ such that

$$\eta^2(u_k, \mathcal{F}_k) \geq \alpha\, \eta^2(u_k, \mathcal{E}_k). \tag{2.8}$$

Define the marked elements $\mathcal{M}_k = \bigcup_{E \in \mathcal{F}_k} \omega_E$.

ii) Otherwise choose the marked elements set $\mathcal{M}_k$ of $\mathcal{T}_k$ to be set of elements with the minimal cardinality such that

$$\mathrm{osc}^2(f, \mathcal{M}_k) \geq \theta\, \mathrm{osc}^2(f, \mathcal{T}_k). \tag{2.9}$$

(4) Let $\mathcal{T}_{k+1}$ be the refinement of $\mathcal{T}_k$ (in the case i), the refinement should satisfy the *interior node property*).
(5) Set $k := k + 1$ and go to step (1).

from [5, 17, 19, 22]. For completeness we include some results established in the mentioned references without proofs.

The convergence analysis starts from the orthogonality relation between $u - u_H$ and $u_h - u_H$, the so-called Pythagoras equality, which follows immediately from the Galerkin orthogonality.

LEMMA 3.1 (Galerkin orthogonality). *Let $\mathcal{T}_h$ be a refinement of the triangulation $\mathcal{T}_H$ such that $V^H \subset V^h$, suppose $u_H, u_h$ are then the discrete finite element solutions over $\mathcal{T}_H$ and $\mathcal{T}_h$, respectively. Then the following relation holds:*

$$|u - u_h|^2_{1,\Omega} = |u - u_H|^2_{1,\Omega} - |u_h - u_H|^2_{1,\Omega}. \tag{3.1}$$

The following local bound for the estimator in terms of the local difference between two Galerkin solutions up to a local oscillation term plays a key role in the convergence analysis of AFEM.

LEMMA 3.2. *Let $\mathcal{T}_h$ be a refinement of the triangulation $\mathcal{T}_H$ such that $V^H \subset V^h$, if for any $E \in \mathcal{E}_H$, both $E$ and $K \in \omega_E$ satisfy the interior node property, then we have*

$$\eta^2_E(u_H) \leq C_4 \sum_{K \in \omega_E} |u_h - u_H|^2_{1,K} + C_5 osc^2(f, \omega_E). \tag{3.2}$$

As mentioned in the previous section, a successful convergent AFEM should include the so-called oscillation reduction. This idea was developed by Morin, Nochetto and Siebert [19, 20], and is stated as follows.

LEMMA 3.3 (Oscillation reduction). *Let $0 < \sigma < 1$ be the reduction factor of element size associated with one refinement step. Given $0 < \theta < 1$, let $\widehat{\alpha} := 1 - (1 - \sigma^2)\theta$. Let $\mathcal{M}_H$ be a subset of $T_H$ such that*

$$osc^2(f, \mathcal{M}_H) \geq \theta osc^2(f, \mathcal{T}_H).$$

*If $T_h$ is a triangulation obtained from $T_H$ by refining at least every element in $\mathcal{M}_H$, then the following data oscillation reduction occurs:*

$$osc^2(f, \mathcal{T}_h) \leq \widehat{\alpha} osc^2(f, \mathcal{T}_H).$$

ETNA
Kent State University
http://etna.math.kent.edu



The following lemma deals with a localized version of the upper bound for the difference between two Galerkin solutions with respect to two different partitions, which was proved by Stevenson [22].

LEMMA 3.4. *Let $C_1$ be the constant in Lemma 2.1. Then there exists a subset $\mathcal{F}_H \subset \mathcal{E}_H$, such that*

$$|u_h - u_H|^2_{1,\Omega} \le C_1 \eta^2(u_H, \mathcal{F}_H) \tag{3.3}$$

*and*

$$\mathcal{N}(\mathcal{F}_H) \le C_6(\mathcal{N}(\mathcal{T}_h) - \mathcal{N}(\mathcal{T}_H)). \tag{3.4}$$

Based on Lemmas 2.1, 2.2 and Lemmas 3.1,3.2, 3.3, we are now in a position to prove the convergence of Algorithm 1 developed in the last section. Since they are of importance in the choice of the parameters employed in Algorithm 1 and since the Lemmata are given without proofs, we add some comments on the involved constants. Except $C_0$, they all depend on the minimal angle condition. To be more precise, constants $C_4$ and $C_5$ depend on Verfürth's inverse estimate and could be determined by an eigenvalue problem. The constants $C_1$ and $C_6$ depend on the Clément operator; see [9, 25]. The constant $C_0$ depends on the details of the refinement algorithm; see [5, 23].

THEOREM 3.5 (Convergence of AFEM). *Let $\{\mathcal{V}^k\}_{k\ge 0}$ be a sequence of nested finite element spaces generated by algorithm* AFEM *and let $\{u_k\}_{k\ge 0}$ be the corresponding sequence of finite element solutions. Assume that*

$$0 < \gamma < \gamma^* := \frac{\alpha}{(n+1)C_2[(n+1)C_1C_5 + \alpha C_3]}. \tag{3.5}$$

*Then there exist constants $\beta > 0$ and $0 < \rho < 1$, depending only on the shape regularity of meshes, the data, the dimension $n$, the parameters $\alpha, \theta, \gamma$ used by* AFEM*, such that for any two consecutive iterates $k$ and $k+1$ we have*

$$|u - u_{k+1}|^2_{1,\Omega} + \beta osc^2(f, \mathcal{T}_{k+1}) \le \rho\Big(|u - u_k|^2_{1,\Omega} + \beta osc^2(f, \mathcal{T}_k)\Big). \tag{3.6}$$

*Therefore, algorithm* AFEM *converges with a linear rate $\rho$, namely*

$$|u - u_k|^2_{1,\Omega} + \beta osc^2(f, \mathcal{T}_k) \le C^* \rho^k, \tag{3.7}$$

*where $C^* := |u - u_0|^2_{1,\Omega} + \beta osc^2(f, \mathcal{T}_0)$. The reduction rate is:*

$$\rho = 1 - \frac{(1-\mu)\alpha}{(n+1)C_1C_4}, \tag{3.8}$$

*with $\mu$ defined through (3.16), (3.18), and (3.20) below. The value of $\beta$ is defined by $\beta = \max(\beta_1, \beta_2)$ with $\beta_1$ and $\beta_2$ deined below in (3.15) and (3.17), respectively.*

*Proof.* We treat the two possible cases of the algorithm separately. First consider the case $\mathrm{osc}^2(f, \mathcal{T}_k) < \gamma\eta^2(u_k, \mathcal{E}_k)$. By Lemma 2.1, Lemma 3.2 and the marking strategy (2.8), we have

$$\begin{aligned} |u - u_k|^2_{1,\Omega} &\le C_1\eta^2(u_k, \mathcal{E}_k) \le \frac{C_1}{\alpha}\eta^2(u_k, \mathcal{F}_k) \\ &\le \frac{(n+1)C_1}{\alpha}\Big(C_4|u_{k+1} - u_k|^2_{1,\Omega} + C_5 \mathrm{osc}^2(f, \mathcal{T}_k)\Big), \end{aligned}$$

which implies that

$$|u_{k+1} - u_k|_{1,\Omega}^2 \geq \frac{\alpha}{(n+1)C_1C_4}|u - u_k|_{1,\Omega}^2 - \frac{C_5}{C_4}\mathrm{osc}^2(f, \mathcal{T}_k). \tag{3.9}$$

Let $\beta > 0$ be a constant to be chosen in the subsequent analysis. Thanks to the Galerkin orthogonality (3.1), one can prove

$$\begin{aligned}
&|u - u_{k+1}|_{1,\Omega}^2 + \beta\mathrm{osc}^2(f, \mathcal{T}_{k+1})\\
&\leq |u - u_k|_{1,\Omega}^2 - |u_k - u_{k+1}|_{1,\Omega}^2 + \beta\mathrm{osc}^2(f, \mathcal{T}_k)\\
&\leq \left(1 - \frac{\alpha}{(n+1)C_1C_4}\right)|u - u_k|_{1,\Omega}^2 + \left(\beta + \frac{C_5}{C_4}\right)\mathrm{osc}^2(f, \mathcal{T}_k).
\end{aligned}$$

Introducing another constant $0 < b < 1$ and using the lower bound (2.6), we get

$$\begin{aligned}
&|u - u_{k+1}|_{1,\Omega}^2 + \beta\mathrm{osc}^2(f, \mathcal{T}_{k+1})\\
&\leq \left(1 - \frac{\alpha}{(n+1)C_1C_4}\right)|u - u_k|_{1,\Omega}^2\\
&+ \gamma b\left(\beta + \frac{C_5}{C_4}\right)\eta^2(u_k, \mathcal{E}_k) + (1-b)\left(\beta + \frac{C_5}{C_4}\right)\mathrm{osc}^2(f, \mathcal{T}_k)\\
&\leq \left(1 - \frac{\alpha}{(n+1)C_1C_4} + (n+1)bC_2\gamma\left(\beta + \frac{C_5}{C_4}\right)\right)|u - u_k|_{1,\Omega}^2\\
&+ \left((1-b)\left(\beta + \frac{C_5}{C_4}\right) + (n+1)bC_3\gamma\left(\beta + \frac{C_5}{C_4}\right)\right)\mathrm{osc}^2(f, \mathcal{T}_k).
\end{aligned} \tag{3.10}$$

In view of (3.10), in order to prove (3.6), we select the two constants $\beta$ and $b$ such that

$$\begin{aligned}
&(1-b)\left(\beta + \frac{C_5}{C_4}\right) + (n+1)bC_3\gamma\left(\beta + \frac{C_5}{C_4}\right)\\
&\leq \left(1 - \frac{\alpha}{(n+1)C_1C_4} + (n+1)bC_2\gamma\left(\beta + \frac{C_5}{C_4}\right)\right)\beta
\end{aligned} \tag{3.11}$$

and

$$\left(1 - \frac{\alpha}{(n+1)C_1C_4} + (n+1)bC_2\gamma\left(\beta + \frac{C_5}{C_4}\right)\right) < 1. \tag{3.12}$$

For the sake of our analysis, we can select another parameter $\mu \in (0,1)$, and $b$ is chosen such that

$$b = \frac{\mu\alpha}{(n+1)^2C_1C_2C_4\gamma\left(\beta + \frac{C_5}{C_4}\right)}, \tag{3.13}$$

which implies that the error reduction rate is

$$\rho := 1 - \frac{(1-\mu)\alpha}{(n+1)C_1C_4}. \tag{3.14}$$

Substituting (3.13) into (3.11) and rearranging, we obtain

$$-\frac{\alpha}{(n+1)C_1C_4}(1-\mu)\beta \geq \frac{C_5}{C_4} - \frac{\mu\alpha}{(n+1)C_1C_4}\left(\frac{1}{(n+1)C_2\gamma} - C_3\right),$$

which implies

$$\beta \le \beta_1(\mu) := \frac{-(n+1)C_1C_5 + \mu\alpha\left(\frac{1}{(n+1)C_2\gamma} - C_3\right)}{(1-\mu)\alpha} \tag{3.15}$$

if we choose $\mu$ such that

$$\mu > \mu_1^* := \frac{(n+1)C_1C_5}{\alpha\left(\frac{1}{(n+1)C_2\gamma} - C_3\right)}. \tag{3.16}$$

Note that $\mu_1^* < 1$ under the assumption that $0 < \gamma < \gamma^*$.

Now, let us consider the case $\operatorname{osc}^2(f, \mathcal{T}_k) \ge \gamma\eta^2(u_k, \mathcal{E}_k)$, then the marking strategy (2.9) will be adopted. Let $0 < a < 1$ be a constant to be chosen suitably. By Lemmas 3.3 and 2.1, we have

$$\begin{aligned}
&|u - u_{k+1}|_{1,\Omega}^2 + \beta\operatorname{osc}^2(f, \mathcal{T}_{k+1}) \\
&= (1-a)|u - u_{k+1}|_{1,\Omega}^2 + a|u - u_{k+1}|_{1,\Omega}^2 + \beta\operatorname{osc}^2(f, \mathcal{T}_{k+1}) \\
&\le (1-a)|u - u_{k+1}|_{1,\Omega}^2 + aC_1\eta^2(u_k, \mathcal{E}_k) + \beta\widehat{\alpha}\operatorname{osc}^2(f, \mathcal{T}_k) \\
&\le (1-a)|u - u_k|_{1,\Omega}^2 + \left(\frac{aC_1}{\gamma} + \beta\widehat{\alpha}\right)\operatorname{osc}^2(f, \mathcal{T}_k).
\end{aligned}$$

We will choose the constant $a$ such that the error contraction in the second case is also $\rho$, that is to say,

$$a = \frac{(1-\mu)\alpha}{(n+1)C_1C_4}.$$

Then in order to prove (3.6), it is sufficient that the constant $\beta$ satisfies

$$\frac{aC_1}{\gamma} + \beta\widehat{\alpha} \le (1-a)\beta,$$

which implies

$$\beta \ge \beta_2(\mu) := \frac{\frac{C_1}{\gamma}(1-\mu)\alpha}{(1-\widehat{\alpha})(n+1)C_1C_4 - (1-\mu)\alpha} \tag{3.17}$$

under the assumption that

$$\mu > \mu_2^* := 1 - \frac{(1-\widehat{\alpha})(n+1)C_1C_4}{\alpha}. \tag{3.18}$$

Let us now discuss the selection of the value of $\mu$. If we select a fixed value for $\mu$ and set $\beta = \max\{\beta_1, \beta_2\}$, (3.6) will be obtained. In view of (3.15) and (3.17), the proper value of $\beta$ can be found if and only if

$$\beta_2(\mu) \le \beta_1(\mu),$$

which is equivalent to

$$f(\mu) := \lambda_1\mu^2 + \lambda_2\mu + \lambda_3 \ge 0,$$

where

$$\begin{cases} \lambda_1 := \alpha^2 \left( \dfrac{1}{(n+1)C_2\gamma} - C_3 \right) - \dfrac{C_1\alpha}{\gamma}, \\ \lambda_2 := \alpha \left( \dfrac{1}{(n+1)C_2\gamma} - C_3 \right) \Big( (1-\widehat{\alpha})(n+1)C_1C_4 - \alpha \Big) \\ \qquad\qquad\qquad\qquad\qquad\qquad -(n+1)C_1C_5\alpha + \dfrac{2C_1\alpha}{\gamma}, \\ \lambda_3 := (n+1)C_1C_5 \Big( \alpha - (1-\widehat{\alpha})(n+1)C_1C_4 \Big) - \dfrac{C_1\alpha}{\gamma}. \end{cases}$$

It can be checked that

$$f(1) = (1-\widehat{\alpha})(n+1)C_1C_4 \left( \alpha \left( \frac{1}{(n+1)C_2\gamma} - C_3 \right) - (n+1)C_1C_5 \right) > 0. \tag{3.19}$$

By the continuity of the function $f$ we know that there exists a constant $0 < \mu_3^* < 1$ such that $f(\mu_3^*) \geq 0$. Then the value of $\mu$ can be selected such that

$$\max\{\mu_1^*, \mu_2^*, \mu_3^*\} < \mu < 1. \tag{3.20}$$

Thus we have proved (3.6). Since (3.7) is a direct consequence of (3.6), the proof of the theorem is complete. □

For the proof of the optimal complexity of algorithm AFEM, we introduce some notation from nonlinear approximation theory, developed in [5, 6, 13, 22]. Let $\mathcal{H}_N$ be the set of all triangulations $\mathcal{T}$ which are obtained by refinement of a regular initial triangulation $\mathcal{T}_0$ and the cardinality of which satisfy $\mathcal{N}(\mathcal{T}) \leq N$. For a given triangulation, the associated finite element approximation of the problem (2.3) is denoted by $u_{\mathcal{T}}$. Next we define the approximation class

$$\mathcal{W}^s := \Big\{ (u, f) \in (H_0^1(\Omega), L^2(\Omega)) : \|(u, f)\|_{\mathcal{W}^s} < +\infty \Big\},$$

with

$$\|(u, f)\|_{\mathcal{W}^s} := \sup_{N \geq \mathcal{N}(\mathcal{T}_0)} N^s \inf_{\mathcal{T} \in H_N} \Big( |u - u_{\mathcal{T}}|_{1,\Omega}^2 + \mathrm{osc}^2(f, \mathcal{T}) \Big).$$

We say that an adaptive finite element method realizes optimal convergence rates if whenever $(u, f) \in \mathcal{W}^s$, it produces the approximation $u_k$ with respect to the triangulation $\mathcal{T}_k$ such that

$$|u - u_k|_{1,\Omega} \leq C\mathcal{N}(\mathcal{T}_k)^{-s}.$$

First, we estimate the number of elements added in one single refinement step.

LEMMA 3.6. *Let $\{\mathcal{V}^k\}_{k \geq 0}$ be a sequence of nested finite element spaces produced by algorithm* AFEM *and let $\{u_k\}_{k \geq 0}$ be the corresponding sequence of finite element solutions. Assume that $0 < \gamma < \gamma^*$,*

$$C_1C_2\alpha + C_3\gamma < \frac{1}{n+1}, \tag{3.21}$$

*and $(u, f) \in \mathcal{W}^s$. Then there exists a constant $C_1^*$, depending only on the shape regularity of the initial mesh, the data, the dimension $n$, the parameters $\alpha, \theta, \gamma$ used by* AFEM, *and $\mathcal{N}(\mathcal{T}_0)$, such that*

$$\mathcal{N}(\mathcal{M}_k) \leq C_1^* \Big( |u - u_k|_{1,\Omega}^2 + osc^2(f, \mathcal{T}_k) \Big)^{-1/s} \tag{3.22}$$

*with*

$$C_1^* := \max\Big\{(n+1)C_6\lambda_1^{-1/s}, \lambda_2^{-1/s}\Big\}\big\|(u,f)\big\|_{\mathcal{W}^s}^{1/s},$$

*where $\lambda_1$ and $\lambda_2$ are defined by (3.24) and (3.26), respectively.*

*Proof.* We split the proof into two cases as in the proof of Theorem 3.5. Let us consider the first case, i.e., $\operatorname{osc}^2(f,\mathcal{T}_k) < \gamma\,\eta^2(u_k,\mathcal{E}_k)$. Suppose $\lambda_1 \in (0,1)$ is a fixed constant to be chosen appropriately in the subsequent analysis. Let $\mathcal{T}_k^*$ be a triangulation refined from $\mathcal{T}_0$ with minimal number of elements such that

$$|u - u_{\mathcal{T}_k^*}|_{1,\Omega}^2 \le \lambda_1\Big(|u-u_k|_{1,\Omega}^2 + \operatorname{osc}^2(f,\mathcal{T}_k)\Big).$$

Then by the definition of the norm $\|\cdot\|_{\mathcal{W}^s}$,

$$\mathcal{N}(\mathcal{T}_k^*) \le \lambda_1^{-1/s}\Big(|u-u_k|_{1,\Omega}^2 + \operatorname{osc}^2(f,\mathcal{T}_k)\Big)^{-1/s}\big\|(u,f)\big\|_{\mathcal{W}^s}^{1/s}.$$

Let us choose $\mathcal{T}_k'$ as the refinement of $\mathcal{T}_k$ with minimal number of elements such that $V_k^* \subset V_k'$ and thus

$$|u - u_{\mathcal{T}_k'}|_{1,\Omega}^2 \le |u - u_{\mathcal{T}_k^*}|_{1,\Omega}^2 \le \lambda_1\Big(|u-u_k|_{1,\Omega}^2 + \operatorname{osc}^2(f,\mathcal{T}_k)\Big). \tag{3.23}$$

Note that by the definition of $\mathcal{T}_k'$ there holds

$$\begin{aligned}\mathcal{N}(\mathcal{T}_k') - \mathcal{N}(\mathcal{T}_k) &\le \mathcal{N}(\mathcal{T}_k^*)\\ &\le \lambda_1^{-1/s}\Big(|u-u_k|_{1,\Omega}^2 + \operatorname{osc}^2(f,\mathcal{T}_k)\Big)^{-1/s}\big\|(u,f)\big\|_{\mathcal{W}^s}^{1/s}.\end{aligned}$$

In the following we shall bound $\mathcal{N}(\mathcal{T}_{k+1}) - \mathcal{N}(\mathcal{T}_k)$ by $\mathcal{N}(\mathcal{T}_k') - \mathcal{N}(\mathcal{T}_k)$ to obtain the desired results. In view of Lemma 3.4, there exists a subset $\mathcal{F}_k^* \subset \mathcal{E}_k$ such that

$$|u_k - u_{\mathcal{T}_k'}|_{1,\Omega}^2 \le C_1\eta^2(u_k,\mathcal{F}_k^*)$$

and

$$\mathcal{N}(\mathcal{F}_k^*) \le C_6(\mathcal{N}(\mathcal{T}_k') - \mathcal{N}(\mathcal{T}_k)).$$

Then, if the value of $\lambda_1$ is chosen as

$$\lambda_1 := \frac{\frac{1}{n+1} - C_3\gamma - C_1C_2\alpha}{\frac{1}{n+1} - C_3\gamma + C_2\gamma}, \tag{3.24}$$

we have by the Galerkin orthogonality (3.1), (3.23) and (2.6)

$$\begin{aligned}\eta^2(u_k,\mathcal{F}_k^*) &\ge \frac{|u_k - u_{\mathcal{T}_k'}|_{1,\Omega}^2}{C_1} = \frac{|u-u_k|_{1,\Omega}^2 - |u-u_{\mathcal{T}_k'}|_{1,\Omega}^2}{C_1}\\ &\ge \frac{(1-\lambda_1)|u-u_k|_{1,\Omega}^2 - \lambda_1\operatorname{osc}^2(f,\mathcal{T}_k)}{C_1}\\ &\ge \frac{1}{C_1}\left[\frac{(1-\lambda_1)}{(n+1)C_2}\eta^2(u_k,\mathcal{E}_k) - \left(\lambda_1 + \frac{(1-\lambda_1)C_3}{C_2}\right)\operatorname{osc}^2(f,\mathcal{T}_k)\right]\\ &\ge \frac{\eta^2(u_k,\mathcal{E}_k)}{C_1C_2}\left[\frac{1}{n+1} - C_3\gamma - \lambda_1\left(\frac{1}{n+1} + \gamma(C_2-C_3)\right)\right].\end{aligned}$$

ETNA
Kent State University
http://etna.math.kent.edu



The denominator in (3.24) is positive due to our former assumption $C_2 \geq C_3$. Assumption (3.21) leads to $\lambda_1 < 1$.

With the choice of $\lambda_1$ in (3.24) we get

$$\eta^2(u_k, \mathcal{F}_k^*) \geq \alpha \eta^2(u_k, \mathcal{E}_k).$$

Since in the marking strategy we choose the minimal edge set $\mathcal{F}_k \subset \mathcal{E}_k$ such that (2.8) holds, and we conclude that

$$\begin{aligned}
\mathcal{N}(\mathcal{M}_k) &\leq (n+1)\mathcal{N}(\mathcal{F}_k) \leq (n+1)\mathcal{N}(\mathcal{F}_k^*) \\
&\leq (n+1)C_6(\mathcal{N}(\mathcal{T}_k') - \mathcal{N}(\mathcal{T}_k)) \\
&\leq (n+1)^2 C_6 \lambda_1^{-1/s} \big\|(u,f)\big\|_{\mathcal{W}^s}^{1/s} \\
&\quad \Big(|u-u_k|_{1,\Omega}^2 + \mathrm{osc}^2(f,\mathcal{T}_k)\Big)^{-1/s}.
\end{aligned} \tag{3.25}$$

Next we turn to the case $\mathrm{osc}^2(f,\mathcal{T}_k) \geq \gamma \eta^2(u_k, \mathcal{E}_k)$. Similar to the first case, suppose that $\lambda_2 \in (0,1)$ is a fixed constant and $\mathcal{T}_k^*$ be a triangulation refined from $\mathcal{T}_0$ with minimal number of elements such that

$$\mathrm{osc}^2(f,\mathcal{T}_k^*) \leq \lambda_2 \Big(|u-u_k|_{1,\Omega}^2 + \mathrm{osc}^2(f,\mathcal{T}_k)\Big)$$

and

$$\mathcal{N}(\mathcal{T}_k^*) \leq \lambda_2^{-1/s} \Big(|u-u_k|_{1,\Omega}^2 + \mathrm{osc}^2(f,\mathcal{T}_k)\Big)^{-1/s} \big\|(u,f)\big\|_{\mathcal{W}^s}^{1/s}.$$

Let $\mathcal{T}_k'$ be the refinement of $\mathcal{T}_k$ with minimal number of elements such that $V_k^* \subset V_k'$ and then

$$\mathrm{osc}^2(f,\mathcal{T}_k') \leq \mathrm{osc}^2(f,\mathcal{T}_k^*) \leq \lambda_2 \Big(|u-u_k|_{1,\Omega}^2 + \mathrm{osc}^2(f,\mathcal{T}_k)\Big)$$

and

$$\mathcal{N}(\mathcal{T}_k') - \mathcal{N}(\mathcal{T}_k) \leq \lambda_2^{-1/s} \Big(|u-u_k|_{1,\Omega}^2 + \mathrm{osc}^2(f,\mathcal{T}_k)\Big)^{-1/s} \big\|(u,f)\big\|_{\mathcal{W}^s}^{1/s}.$$

Let $\mathcal{M}_K^* := \{K | K \in \mathcal{T}_k; K \in \mathcal{T}_k'\}$. Then by Lemma 2.1, we have

$$\begin{aligned}
\mathrm{osc}^2(f,\mathcal{T}_k) &\geq \frac{1}{\lambda_2}\Big(\mathrm{osc}^2(f,\mathcal{T}_k') - \lambda_2 |u-u_k|_{1,\Omega}^2\Big) \\
&\geq \frac{1}{\lambda_2}\Big(\mathrm{osc}^2(f,\mathcal{T}_k') - \lambda_2 C_1 \eta^2(u_k,\mathcal{E}_k)\Big) \\
&\geq \left(\frac{1}{\lambda_2} - \frac{C_1}{\gamma}\right) \mathrm{osc}^2(f,\mathcal{T}_k') \\
&\geq \left(\frac{1}{\lambda_2} - \frac{C_1}{\gamma}\right) \mathrm{osc}^2(f,\mathcal{M}_K^*) \\
&= \left(\frac{1}{\lambda_2} - \frac{C_1}{\gamma}\right) \Big(\mathrm{osc}^2(f,\mathcal{T}_K) - \mathrm{osc}^2(f,\mathcal{T}_K \setminus \mathcal{M}_K^*)\Big).
\end{aligned}$$

then if the value of $\lambda_2$ is chosen as

$$\lambda_2 := \frac{1-\theta}{1+\frac{C_1(1-\theta)}{\gamma}}, \tag{3.26}$$

we get

$$\mathrm{osc}^2(f, \mathcal{T}_K \setminus \mathcal{M}_K^*) \geq \theta \mathrm{osc}^2(f, \mathcal{T}_k).$$

Since in the marking strategy we choose the minimal edge set $\mathcal{M}_k \subset \mathcal{T}_k$ such that (2.9) holds, and we conclude that

$$\begin{aligned}\mathcal{N}(\mathcal{M}_k) &\leq \mathcal{N}(\mathcal{T}_K \setminus \mathcal{M}_K^*)\\ &\leq (\mathcal{N}(\mathcal{T}_k') - \mathcal{N}(\mathcal{T}_k))\\ &\leq \lambda_2^{-1/s} \|(u,f)\|_{\mathcal{W}^s}^{1/s} \Big( |u-u_k|_{1,\Omega} a^2 + \mathrm{osc}^2(f, \mathcal{T}_k)\Big)^{-1/s},\end{aligned}$$

which, together with (3.25) implies the desired result. ☐

Now, we can prove the optimality of the algorithm AFEM.

THEOREM 3.7 (Optimal complexity of AFEM). *Let $\{\mathcal{V}^k\}_{k\geq 0}$ be a sequence of nested finite element spaces produced by algorithm* AFEM *and let $\{u_k\}_{k\geq 0}$ be the corresponding sequence of finite element solutions. Assume that $(u,f) \in \mathcal{W}^s$, $\gamma$ satisfies $0 < \gamma < \gamma^*$ (defined in (3.5)), and further that $\alpha$ satisfies*

$$C_1 C_2 \alpha + C_3 \gamma < \frac{1}{n+1}. \tag{3.27}$$

*Then there exists a constant $C_2^*$, such that*

$$|u-u_k|_{1,\Omega}^2 + osc^2(f, \mathcal{T}_k) \leq C_2^* \Big(\mathcal{N}(\mathcal{T}_k) - \mathcal{N}(\mathcal{T}_0)\Big)^{-s} \tag{3.28}$$

*with* $C_2^* := \max\{1, \beta\} \left( \frac{C_0 C_1^* \left(1-\rho^{\frac{k}{s}}\right)}{\rho^{-1/s}-1} \right)^s$.

*In addition there exists another constant $C_3^*$ such that for any $\epsilon > 0$ the following holds. Let $N$ be the first index such that $\eta(u_N, \mathcal{E}_N) \leq \epsilon$. Then we have*

$$\mathcal{N}(\mathcal{T}_N) - \mathcal{N}(\mathcal{T}_0) \leq C_3^* \epsilon^{-2/s} \tag{3.29}$$

*with* $C_3^* := C_0 C_1^* \min\left\{1, \frac{1}{\beta}\right\}^{-1/s} \frac{1-\rho^{N/s}}{\rho^{-1/s}-1} \left( \frac{\min\left\{1, \frac{C_2 \beta}{C_3}\right\}}{(n+1)C_2}\right)^{-1/s}$.

*Proof*. In view of (3.22) in Lemma 3.6, for any $0 \leq i \leq k$, there holds

$$\mathcal{N}(\mathcal{M}_i) \leq C_1^* \min\left\{1, \frac{1}{\beta}\right\}^{-1/s} \Big( |u-u_i|_{1,\Omega}^2 + \beta \mathrm{osc}^2(f, \mathcal{T}_i)\Big)^{-1/s},$$

together with

$$\Big( |u-u_i|_{1,\Omega}^2 + \beta \mathrm{osc}^2(f, \mathcal{T}_i)\Big)^{-1/s} \leq \rho^{\frac{k-i}{s}} \Big( |u-u_k|_{1,\Omega}^2 + \beta \mathrm{osc}^2(f, \mathcal{T}_k)\Big)^{-1/s}$$

ETNA
Kent State University
http://etna.math.kent.edu



obtained from (3.6) in Theorem 3.5, we have

$$
\begin{aligned}
\mathcal{N}(\mathcal{T}_k) - \mathcal{N}(\mathcal{T}_0) &\le C_0 \sum_{i=0}^{k-1} \mathcal{N}(\mathcal{M}_i) \\
&\le C_0 C_1^* \min\left\{1, \frac{1}{\beta}\right\}^{-1/s} \sum_{i=0}^{k-1} \Big( |u-u_i|_{1,\Omega}^2 + \beta \mathrm{osc}^2(f,\mathcal{T}_i) \Big)^{-1/s} \\
&\le C_0 C_1^* \min\left\{1, \frac{1}{\beta}\right\}^{-1/s} \left( \sum_{i=0}^{k-1} \rho^{\frac{k-i}{s}} \right) \Big( |u-u_k|_{1,\Omega}^2 + \beta \mathrm{osc}^2(f,\mathcal{T}_k) \Big)^{-1/s} \\
&\le C_0 C_1^* \min\left\{1, \frac{1}{\beta}\right\}^{-1/s} \frac{1-\rho^{\frac{k}{s}}}{\rho^{-1/s}-1} \Big( |u-u_k|_{1,\Omega}^2 + \beta \mathrm{osc}^2(f,\mathcal{T}_k) \Big)^{-1/s},
\end{aligned}
\tag{3.30}
$$

which implies (3.28).

The proof of (3.29) is obvious. In fact, the lower bound (2.6) gives

$$
\Big( |u-u_k|_{1,\Omega}^2 + \beta \mathrm{osc}^2(f,\mathcal{T}_k) \Big)^{-1/s} \le \left( \frac{\min\left\{1, \frac{C_2\beta}{C_3}\right\}}{(n+1)C_2} \right)^{-1/s} \eta^{-\frac{2}{s}}(u_k, \mathcal{E}_k),
\tag{3.31}
$$

then the desired result can be obtained by (3.31) and (3.30). □

**4. Conclusion.** We have presented a new adaptive finite element method, which is a variant of the algorithm of Morin/Nochetto/Siebert. The novelty lies in the treatment of the data oscillation term, which is only used for refinement if it is big compared to the error estimator. We have proved geometrical convergence of the error augmented by the data oscillation term and optimal complexity in the sense of nonlinear approximation theory. The dependence of our results on all involved constants is described.

## REFERENCES


[1] M. AINSWORTH AND J. T. ODEN, *A Posteriori Error Estimation in Finite Element Analysis*, Pure and Applied Mathematics, Wiley-Interscience, John Wiley and Sons, New York, 2000.

[2] I. BABUŠKA AND T. STROUBOULIS, *The Finite Element Method and its Reliability*, Oxford University Press, Oxford, 2001.

[3] I. BABUŠKA AND M. VOGELIUS, *Feedback and adaptive finite element solution of one dimensional boundary value problems*, Numer. Math., 44 (1984), pp. 75–102.

[4] R. BECKER, C. JOHNSON, AND R. RANNACHER, *Adaptive error control for multigrid finite element methods*, Computing, 55 (1995), pp. 271–288.

[5] P. BINEV, W. DAHMEN, AND R. DEVORE, *Adaptive finite element methods with convergence rates*, Numer. Math., 97 (2004), pp. 219–268.

[6] P. BINEV, W. DAHMEN, R. DEVORE, AND P. PETRUCHEV, *Approximation classes for adaptive methods*, Serdica Math. J., 28 (2002), pp. 391–416.

[7] C. CARSTENSEN AND R. H. W. HOPPE, *Error reduction and convergence for an adaptive mixed finite element method*, Math.Comp., 75 (2006), pp. 1033–1042.

[8] C. CARSTENSEN AND R. H. W. HOPPE, *Convergence analysis of an adaptive nonconforming finite element methods*, Numer. Math., 103 (2006), pp. 251–266.

[9] C. CARSTENSEN AND S. A. FUNKEN, *Constants in Clément-interpolation error and residual based a posteriori error estimates in finite element methods*, East-West J. Numer. Math. 8 (2000), pp. 153–175.

[10] J. CASCON, C. KREUZER, R. NOCHETTO, AND K. SIEBERT, *Quasi-optimal convergence rate for an adaptive finite element method*, SIAM J. Numer. Anal., 46 (2008), pp. 2524–2550.

[11] P. G. CIARLET, *The Finite Element method for Elliptic Problems*, North-Holland, Amsterdam, 1978.

[12] P. CLÉMENT, *Approximation by finite element functions using local regularization*, Rev. Française Automat. Informat. Recherche Opérationnelle Sér. RAIRO Analyse Numérique, 9 (1975), pp. 77–84.

[13] R. DEVORE, *Nonlinear approximation*, in Acta Numerica 1998, Vol. 7, A. Iserles, ed., Cambridge University Press, Cambridge, 1998, pp. 51–150.

[14] L. DIENING AND C. KREUZER, *Linear convergence of an adaptive finite element method for the p-Laplacian equation*, SIAM J. Numer. Anal., 46 (2008), pp. 614–638.

[15] W. DÖRFLER, *A convergent adaptive algorithm for Poisson's equation*, SIAM J. Numer. Anal. 33 (1996), pp. 1106–1124.

[16] K. ERIKSSON, D. ESTEP, P. HANSBO, AND C. JOHNSON, *Introduction to adaptive methods for differential equations*, in Acta Numerica 1995, A. Iserles, ed., Cambridge University Press, Cambridge, 1995, pp. 105–158.

[17] K. MEKCHAY AND R. H. NOCHETTO, *Convergence of adaptive finite element methods for general second order linear elliptic PDE*, SIAM J. Numer. Anal., 43 (2005), pp. 1803–1827.

[18] W. F. MITCHELL, *A comparison of adaptive refinement techniques for elliptic problems*, ACM Trans. Math. Software, 15 (1989), pp. 326–347.

[19] P. MORIN, R. H. NOCHETTO, AND K. G. SIEBERT, *Data oscillation and convergence of adaptive FEM*, SIAM J. Numer. Anal. 38 (2000), pp. 466–488.

[20] P. MORIN, R. H. NOCHETTO, AND K. G. SIEBERT, *Convergence of adaptive finite element methods*, SIAM Rev., 44 (2002), pp. 631–658.

[21] P. MORIN, K. G. SIEBERT, AND A. VEESER, *A basic convergence result for conforming adaptive finite elements*, Math. Models Methods Appl. Sci., 18 (2008), pp. 707–737.

[22] R. STEVENSON, *Optimality of a standard adaptive finite element method*, Found. Comput. Math., 7 (2007), pp. 245–269.

[23] R. STEVENSON, *The completion of locally refined simplicial partitions created by bisection*, Math. Comp., 77 (2008), pp. 227–241.

[24] R. VERFÜRTH,*A Review of a Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques*, Wiley-Teubner, Chichester, 1996.

[25] R. VERFÜRTH, *Error estimates for some quasi-interpolation operators*, M2AN Math. Model. Numer. Anal., 33 (1999), pp. 695–713.

[26] H. WU AND Z. CHEN, *Uniform convergence of multigrid V-cycle on adaptively refined finite element meshes for second order elliptic problems*, Sci. China Ser. A, 49 (2006), pp. 1405–1429.