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AN OVERLAPPING ADDITIVE SCHWARZ-RICHARDSON METHOD FOR MONOTONE NONLINEAR PARABOLIC PROBLEMS*

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Abstract. We construct and study a scalable overlapping Additive Schwarz-Richardson (ASR) algorithm for monotone nonlinear parabolic problems discretized implicitly in time. At each time step, the Additive Schwarz preconditioner is built using the linear part of the nonlinear operator, partitioning the domain of the problem into overlapping subdomains, solving local problems on these subdomains and solving an additional coarse problem associated with the subdomain mesh. This preconditioner is then applied to the nonlinear operator using a Richardson iteration. We prove first an abstract convergence result and then convergence rate estimates showing the scalability of the ASR algorithm. The results of numerical experiments in the plane confirm the theoretical estimates and illustrate the performance of the one and two-level ASR algorithm and in the presence of discontinuous coefficients in the parabolic operator.

Key words. monotone nonlinear parabolic problems, domain decomposition preconditioners, overlapping additive Schwarz, finite elements, implicit time discretizations

AMS subject classifications. 65M55, 65H05

1. Introduction. In recent years, domain decomposition methods have been extended in different ways to nonlinear problems arising in many application areas. As a first approach, domain decomposition methods provide preconditioners for the Jacobian system in a Newton-like iteration. In this context, Schwarz-type preconditioners have been successfully used by Cai et al. to solve problems from various applied fields, e.g., computational fluid dynamics [11, 17], full potential problems [10], cardiac electrical activity [25], and unsteady nonlinear radiation diffusion [27]. Additive Schwarz methods have been used not only as inner iterations in a Newton-Krylov-Schwarz (NKS) scheme, but also as outer iterations in nested solvers such as ASPIN [12, 1] or in the nonlinear additive Schwarz method by Dryja and Hackbusch [15]. Different approaches have been studied by Lui [19, 20] using the method of monotone iterations for continuous nonlinear elliptic and parabolic PDEs and by Boglaev [4] using monotone Schwarz methods at the discrete level for singularly perturbed reaction-diffusion problems. Extensions of the classical additive and multiplicative Schwarz methods have been studied by Tai and Espedal [31, 32] for convex programming problems and by Badea [2] for constrained minimization problems.

In this paper, we follow instead the work of Cai and Dryja [9] for monotone elliptic problems and extend it to monotone nonlinear parabolic problems. The main idea is to build an overlapping additive Schwarz preconditioner for the linear part of the operator, partitioning the domain of the problem into overlapping subdomains, solving local problems on these subdomains, and solving an additional coarse problem associated with the subdomain mesh. This preconditioner is then applied to the nonlinear operator using a Richardson iteration. The linearity of the preconditioner allows us to employ the main technical tools of the classical abstract theory of additive Schwarz methods (see, e.g., [33]), and prove an abstract convergence result for the resulting iterative method. With this result, we can then obtain precise convergence estimates for the Additive Schwarz-Richardson (ASR) method applied to the time discretization of monotone nonlinear parabolic problems. The two-level ASR method turns out to be scalable and with a convergence rate depending only on the ratio H/δ as in the

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linear case, where H is the subdomain characteristic size and δ the overlap size. Without a coarse space, the one-level ASR method can still have a constant upper bound if the time step size τ is small enough. Otherwise, the convergence rate depends on the ratio $\tau/(H\delta)$ and scalability is lost as in the linear case. In case of generous overlap $\delta = CH$, these estimates agree with the estimates obtained by Cai [7, 8] for linear parabolic problems.

The rest of the paper is organized as follows. In Section 2, we introduce the nonlinear parabolic problem, its main properties and time discretization. In Section 3, we define the ASR method in both its functional and matrix form. An abstract convergence result is given in Section 4, where we prove some technical lemmas leading to the main result of Theorem 4.7. In Section 5, this abstract result is applied to the time discretization of our nonlinear parabolic problem and convergence rate estimates are obtained for both one- and two-level ASR methods. Section 6 concludes the paper with the results of several numerical experiments in the plane, confirming the theoretical results obtained and illustrating the scalability of the ASR method. We also compare the ASR method with the Linearly Implicit Euler method, based on solving an appropriate linear system involving the Jacobian of the nonlinear operator by using GMRES with the Additive Schwarz preconditioner as in ASR and we show that the ASR method is asymptotically less expensive.

2. Continuous and discrete nonlinear parabolic problems. We consider a polyhedral domain $\Omega \subset R^d,\ d = 2, 3$, with Lipschitz continuous boundary $\partial\Omega$, the spaces

$$V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_1 \subset \partial\Omega,\ meas(\Gamma_1) > 0\},\ \ L^2(\Omega),$$

and the nonlinear form $b : H^1(\Omega) \times H^1(\Omega) \longrightarrow \mathbf{R}$ satisfying the following properties:

1. b is *Lipschitz continuous*: $\exists L > 0$ such that, $\forall v,\ w,\ z \in H^1(\Omega)$, $|b(v, z) - b(w, z)| \leq L||v - w||_{H^1(\Omega)} \cdot ||z||_{H^1(\Omega)}$;
2. b is *bounded*: $\exists C > 0$ such that $|b(v, w)| \leq C(1 + ||v||_{H^1(\Omega)})||w||_{H^1(\Omega)}$, $\forall v, w \in H^1(\Omega)$;
3. b is *hemicontinuous*: $\forall u, v, w \in H^1(\Omega)$, the function $\alpha \longrightarrow b(u + \alpha v, w)$ is continuous;
4. b is *strictly monotone*: $b(v, v-w) - b(w, v-w) \geq 0,\ \forall v, w \in H^1(\Omega)$, and equality holds only for $v = w$;
5. b is linear in its second argument: $b(v, \sum_{i=1}^{n} \alpha_i w_i) = \sum_{i=1}^{n} \alpha_i b(v, w_i), \forall v, w_i \in H^1(\Omega)$, $\forall \alpha_i \in \mathbf{R},\ i = 1, \ldots, n$;
6. $b(v, v) \geq c||v||^2_{H^1(\Omega)} - c_0||v||_{H^1(\Omega)} - c_1||v||^2_{L^2(\Omega)} - c_2,\ \forall v \in H^1(\Omega)$, where $c > 0, c_0 > 0, c_1 \geq 0, c_2 \geq 0$ are constants.

We consider the following nonlinear parabolic problem: given $u_0 \in L^2(\Omega)$ and $f \in L^2((0, T); V^*)$, find $u \in W \equiv \{u \in L^2((0, T); V),\ u' \in L^2((0, T); V^*)\}$, such that

$$\text{(2.1)} \qquad \begin{cases} < u'(t), w > + b(u(t), w) = < f(t), v >,\ \forall t \in (0, T) \setminus E_w,\ \forall w \in V, \\ u(0) = u_0, \end{cases}$$

where $E_w \subset (0, T)$ is a set of measure zero that depends on the function w.

The continuous problem (2.1) is discretized implicitly in time by the backward Euler method and in space by the finite element method. We suppose for simplicity that the time interval $(0, T)$ is discretized with a uniform time step $\tau = T/M$ into M equal subintervals and that the domain Ω is discretized with a regular finite element triangulation $\mathcal{T}_h$ with mesh size h. The associated piecewise linear finite element space V_h is defined by

$$V_h = \{v|v \text{ is continuous on } \overline{\Omega},\ v|_k \text{ is linear } \forall k \in \mathcal{T}_h,\ v = 0 \text{ on } \overline{\Gamma}_1\}.$$

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We denote by $u_h^m \in V_h$ the finite element approximation of a function $u \in V$ at time $t^m = m\tau$ and let $f^m = \frac{1}{\tau}\displaystyle\int_{t_{m-1}}^{t_m} f(t)dt$. We then obtain the following fully discrete problem: given an arbitrary sequence $\{u_h^0\} \subset L^2(\Omega)$ of approximations of u^0, such that $\lim\limits_{h\to 0} ||u_h^0 - u^0|| = 0$, find $u_h^m \in V_h$, such that

$$\left(\frac{u_h^m - u_h^{m-1}}{\tau}, v\right) + b(u_h^m, v) =< f^m, v >, \ \forall v \in V_h. \tag{2.2}$$

Results on the existence and uniqueness of the solution of the discrete and continuous parabolic problems can be found, e.g., in [35, Theorems 45.3 and 46.4], respectively. The convergence of the discrete solution to the continuous one is presented in [35, Theorems 46.4 and 47.1].

3. An Additive Schwarz-Richardson algorithm. Since in the rest of the paper we only consider discrete functions, for simplicity, we drop the indices h and m and denote by u both the finite element approximation $u = \sum_{j=1}^{n} u_j\phi_j$ of the continuous solution in the finite element basis $\{\phi_j,\ j = 1,\dots,n\}$ of V_h, and its vector representation $u = [u_1,\dots,u_n]^T$. Problem (2.2) can then be written as the nonlinear algebraic system,

$$B(u) = \hat{g}, \tag{3.1}$$

where

$$\begin{array}{lll} B(u) = [B_1,\dots,B_n]^T, & \text{with} & B_j = (u,\phi_j) + \tau b(u,\phi_j), \\ \hat{g} = [g_1,\dots,g_n]^T, & \text{with} & g_j = \tau < f^m, \phi_j > + (u_h^{m-1}, \phi_j). \end{array}$$

Following the abstract Schwarz theory, presented, e.g., in [33], we consider a family of subspaces $V_i \subset V_h, i = 0,\dots,N$, and interpolation (or extension) operators $R_i^T : V_i \longrightarrow V_h$, and assume that V_h admits the decomposition

$$V_h = \sum_{i=0}^{N} R_i^T V_i.$$

We suppose that there exists a symmetric, continuous and coercive bilinear form $a : V \times V \longrightarrow \mathbf{R}$, such that

$$b(u,v) = a(u,v) + \tilde{b}(u,v), \tag{3.2}$$

with $\tilde{b}$ a nonlinear form, monotone and Lipschitz continuous with Lipschitz constant $\tilde{L}$. Nonlinear forms $b(u,v)$ with such a structure arise, e.g., in the field of computational electrocardiology, where research on parallel solvers for the associated nonlinear parabolic reaction-diffusion models (known as monodomain and bidomain models) is currently very active; see, e.g., [25, 13, 30, 22, 24, 21, 23, 28].

Since the bilinear form

$$a_\tau(u,v) = (u,v) + \tau a(u,v)$$

defines a scalar product on V, we can introduce local symmetric, positive definite bilinear forms $\tilde{a}_{\tau,\,i} : V_i \times V_i \longrightarrow \mathbf{R}$, and we make the standard three assumptions of the abstract Schwarz theory (see [33] for more details):

- *stable decomposition:* there exists a constant C_0, such that every $u \in V_h$ admits a decomposition $u = \sum_{i=0}^{N} R_i^T u_i,\ u_i \in V_i,\ i = 0, \ldots, N$ that satisfies

$$\sum_{i=0}^{N} \tilde{a}_{\tau,i}(u_i, u_i) \leq C_0^2 a_\tau(u, u); \tag{3.3}$$

- *strengthened Cauchy-Schwarz inequality:* $\exists\ \epsilon_{ij} \in [0, 1]\ i, j = 1, \ldots, N$, such that $\forall u_i \in V_i,\ \forall u_j \in V_j$

$$|a_\tau(R_i^T u_i, R_j^T u_j)| \leq \epsilon_{ij} a_\tau(R_i^T u_i, R_i^T u_i)^{1/2} a_\tau(R_j^T u_j, R_j^T u_j)^{1/2}; \tag{3.4}$$

(we denote by $\rho(\mathcal{E})$ the spectral radius of the matrix $\mathcal{E} = (\epsilon_{ij})$)
- *local stability:* there exists $\omega > 0$, such that

$$a_\tau(R_i^T u_i, R_i^T u_i) \leq \omega \tilde{a}_{\tau,i}(u_i, u_i),\ \forall u_i \in V_i,\ 0 \leq i \leq N. \tag{3.5}$$

We define the "projection"-like operators $\widetilde{Q}_i : V_h \longrightarrow V_i$ by

$$\tilde{a}_{\tau,i}(\widetilde{Q}_i(u), v_i) = (u, R_i^T v_i) + \tau b(u, R_i^T v_i),\ \forall v_i \in V_i, u \in V_h, \tag{3.6}$$

and their extensions $Q_i : V_h \longrightarrow R_i^T V_i \subset V_h$ defined by

$$Q_i(u) = R_i^T \widetilde{Q}_i(u) \ \text{ and } \ Q(u) = \sum_{i=0}^{N} Q_i(u).$$

Let $\widetilde{A}_{\tau,i} \equiv (\tilde{a}_{\tau,i}(\phi_j, \phi_l))_{j,l}$ be the matrix representation of the local bilinear form $\tilde{a}_{\tau,i}$.

LEMMA 3.1. *The matrix form of $Q(u)$ is*

$$Q(u) = \mathcal{M}^{-1} B(u), \tag{3.7}$$

where $\mathcal{M} = \left(\sum_{i=0}^{N} R_i^T \widetilde{A}_{\tau,i}^{-1} R_i \right)^{-1}$.

Proof. Let $\{\psi_i^k,\ k = 1, \ldots, n_i\}$ be a basis of the local subspace V_i. Then $\widetilde{Q}_i(u) \in V_i$ can be written as $\widetilde{Q}_i(u) = \sum_l \widetilde{Q}_i(u)^l \psi_i^l$ and the two terms in (3.6) become

$$\begin{aligned} \tilde{a}_{\tau,i}(\widetilde{Q}_i(u), v_i) &= \tilde{a}_{\tau,i}(\sum_l \widetilde{Q}_i(u)^l \psi_i^l, \sum_k v_i^k \psi_i^k) = \sum_l \sum_k \widetilde{Q}_i(u)^l \tilde{a}_{\tau,i}(\psi_i^l, \psi_i^k) v_i^k \\ &= \sum_k v_i^k \sum_l \widetilde{A}_{\tau,i}^{kl} \widetilde{Q}_i(u)^l = v_i^T \widetilde{A}_{\tau,i} \widetilde{Q_i}(u), \end{aligned}$$

and

$$\begin{aligned} (u, R_i^T v_i) + \tau b(u, R_i^T v_i) &= (u, \sum_{l=1}^{n} (R_i^T v_i)^l \phi_l) + \tau b(u, \sum_{l=1}^{n} \left(R_i^T v_i\right)^l \phi_l) \\ &= \sum_{l=1}^{n} (R_i^T v_i)^l \left[(u, \phi_l) + \tau b(u, \phi_l)\right] = \sum_{l=1}^{n} (R_i^T v_i)^l B_l \\ &= (R_i^T v_i)^T B(u). \end{aligned}$$

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The definition of $\widetilde{Q}_i(u)$ implies

$$v_i^T \widetilde{A}_{\tau,i} \widetilde{Q}_i(u) = (R_i^T v_i)^T B(u), \ \forall v_i \in V_i, \ u \in V_h,$$

and consequently

$$\widetilde{A}_{\tau,i} \widetilde{Q}_i(u) = R_i B(u), \qquad \forall u \in V_h.$$

The matrix form of $\widetilde{Q}_i(u)$ is then

$$\widetilde{Q}_i(u) = \widetilde{A}_{\tau,i}^{-1} R_i B(u).$$

Since $Q_i(u) = R_i^T \widetilde{Q}_i(u)$, the matrix form of $Q_i(u)$ is

$$Q_i(u) = R_i^T \widetilde{A}_{\tau,i}^{-1} R_i B(u).$$

Hence,

$$Q(u) = \sum_{i=0}^{N} Q_i(u) = \sum_{i=0}^{N} R_i^T \widetilde{A}_{\tau,i}^{-1} R_i B(u) = \mathcal{M}^{-1} B(u),$$

where $\mathcal{M} = \left(\sum_{i=0}^{N} R_i^T \widetilde{A}_{\tau,i}^{-1} R_i \right)^{-1}$. $\square$

REMARK 3.2. Since the matrix $\mathcal{M}$ is symmetric and positive definite, it defines the $\mathcal{M}$-norm $||u||_{\mathcal{M}} = (u^T \mathcal{M} u)^{1/2}$.

We define $\breve{g}_i = Q_i(u^*)$, where u^* is the exact solution of (3.1), i.e: $B(u^*) = \hat{g}$. We consider the nonlinear system

$$Q(u) = \breve{g}, \tag{3.8}$$

where $\breve{g} = \sum_{i=0}^{N} \breve{g}_i$. As in the linear case (see [29, p. 150]), $\breve{g}$ can be computed without knowing the exact solution u^*, since

$$\breve{g} = \sum_{i=0}^{N} \breve{g}_i = \sum_{i=0}^{N} Q_i(u^*) = \sum_{i=0}^{N} R_i^T \widetilde{A}_{\tau,i} R_i B(u^*) = \mathcal{M}^{-1} B(u^*) = \mathcal{M}^{-1} \hat{g}.$$

Using the matrix form of the nonlinear operator Q and the definition of $\breve{g}$, it is straightforward to prove that the nonlinear system $B(u) = \hat{g}$ is equivalent to the nonlinear system $Q(u) = \breve{g}$. In other words, we use the symmetric positive definite part as a preconditioner for the original nonlinear system. This idea has already been used by Cai and Xu [34] for nonsymmetric or indefinite problems. We can then define the Additive Schwarz-Richardson (ASR) algorithm for Problem (3.1).

ASR algorithm: given initial guesses u^0, $r^0 = Q(u^0) - \breve{g}$ and a stopping criterion, iterate for $k = 0, 1, \ldots$ until convergence

$$\begin{array}{ll} \text{solve the preconditioned system:} & \mathcal{M} r^k = B(u^k) - \hat{g}, \\ \text{update the solution:} & u^{k+1} = u^k - \lambda r^k, \end{array} \tag{3.9}$$

where λ is a properly chosen parameter; see Theorem 4.7.

We prove in the next section that the ASR iterations converge if λ is chosen properly, and its convergence rate depends on the parameters $C_0, \epsilon_{ij}, \omega$ defined previously.

4. An abstract convergence result. In order to prove our main result, Theorem 4.7, we need a few technical results established in the next few lemmas. We start by recalling a lemma due to Zhang [36], that plays an important role in proving the equivalence between $||\cdot||_{a_\tau}$ and $||\cdot||_{\mathcal{M}}$.

LEMMA 4.1. *Let $\widetilde{P}_i$ be the projection-like operator from V onto V_i defined by $\tilde{a}_{\tau,i}(\widetilde{P}_i u, v_i) = a_\tau(u, R_i^T v_i),\ v_i \in V_i$ and $P_{ad} = \sum_{i=0}^{N} R_i^T \widetilde{P}_i = \mathcal{M}^{-1} A_\tau$. Then*

$$a_\tau(P_{ad}^{-1} u, u) = \min_{\substack{u_i \in V_i \\ u = \sum R_i^T u_i}} \sum_{i=0}^{N} \tilde{a}_{\tau,i}(u_i, u_i). \tag{4.1}$$

LEMMA 4.2. *The $\mathcal{M}-$norm and $a_\tau-$norm are equivalent, i.e.,*

$$\frac{1}{C_0^2}||u||_{\mathcal{M}}^2 \leq ||u||_{a_\tau}^2 \leq 2\omega(1+\rho(\mathcal{E}))||u||_{\mathcal{M}}^2,$$

where C_0, ω and $\rho(\mathcal{E})$ are the constants defined in the three assumptions of the abstract Schwarz theory, inequalities (3.3)–(3.5).

Proof. a) Lower bound. Using Lemma 4.1 and the stable decomposition assumption (3.3), we have

$$\begin{aligned} ||u||_{\mathcal{M}}^2 &= u^T \mathcal{M} u = u^T A_\tau A_\tau^{-1} \mathcal{M} u = a_\tau(A_\tau^{-1}\mathcal{M}u, u) \\ &= a_\tau(P_{ad}^{-1} u, u) \leq \sum_{i=0}^{N} \tilde{a}_{\tau,i}(u_i, u_i) \leq C_0^2 ||u||_{a_\tau}^2 . \end{aligned}$$

b) Upper bound. Let $\bar{u} = \sum_{i=1}^{N} R_i^T u_i$. From the strengthened Cauchy-Schwarz inequality (3.4) and the local stability assumption (3.5), it follows that

$$\begin{aligned} a_\tau\left(\bar{u}, \bar{u}\right) &= \sum_{i,j=1}^{N} a_\tau(R_i^T u_i, R_j^T u_j) \leq \sum_{i,j=1}^{N} \epsilon_{ij} a_\tau(R_i^T u_i, R_i^T u_i)^{1/2} a_\tau(R_j^T u_j, R_j^T u_j)^{1/2} \\ &\leq \rho(\mathcal{E}) \sum_{i=1}^{N} a_\tau(R_i^T u_i, R_i^T u_i) \leq \rho(\mathcal{E})\omega \sum_{i=1}^{N} \tilde{a}_{\tau,i}(u_i, u_i). \end{aligned}$$

Each element $u \in V_h$ can be written as $u = R_0^T u_0 + \bar{u}$. Using the bilinearity of a_τ, the local stability assumption and the last bound, we have

$$\begin{aligned} a_\tau(u,u) &= a_\tau\left(R_0^T u_0 + \bar{u}, R_0^T u_0 + \bar{u}\right) = a_\tau(R_0^T u_0, R_0^T u_0) + a_\tau(\bar{u}, \bar{u}) + 2a_\tau(R_0^T u_0, \bar{u}) \\ &\leq \omega \tilde{a}_{\tau,0}(u_0, u_0) + \rho(\mathcal{E})\omega \sum_{i=1}^{N} \tilde{a}_{\tau i}(u_i, u_i) + 2a_\tau(R_0^T u_0, \overline{u}). \end{aligned}$$

Since $2a_\tau(R_0^T u_0, \overline{u}) \leq 2a_\tau(R_0^T u_0, R_0^T u_0)^{1/2} a_\tau(\overline{u}, \overline{u})^{1/2} \leq a_\tau(R_0^T u_0, R_0^T u_0) + a_\tau(\overline{u}, \overline{u})$, we obtain

$$a_\tau(u,u) \leq 2\omega(1+\rho(\mathcal{E})) \sum_{i=0}^{N} \tilde{a}_{\tau,i}(u_i, u_i),\ \forall u_i \in V_i, u = \sum_{i=0}^{N} R_i^T u_i.$$

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Taking the minimum over the all decompositions of u and using Lemma 4.1, we conclude

$$a_\tau(u,u) \le 2\omega(1+\rho(\mathcal{E})) \min_{\substack{u_i \in V_i \\ u=\sum R_i^T u_i}} \sum_i \tilde{a}_{\tau,i}(u_i,u_i) = 2\omega(1+\rho(\mathcal{E}))a_\tau(P_{ad}^{-1}u,u)$$

$$= 2\omega(1+\rho(\mathcal{E}))||u||^2_{\mathcal{M}}. \quad \square$$

LEMMA 4.3. *There exists a constant* $\delta_0 = \frac{1}{C_0^2} > 0$*, such that*

$$(Q(u+z)-Q(u),z)_{\mathcal{M}} \ge \delta_0 ||z||^2_{\mathcal{M}},\ \forall u,z \in V_h.$$

Proof. The definition of $Q(u)$, the linearity of b in the second argument, the strict monotonicity of $\tilde{b}$, and the lower bound of Lemma 4.2 imply

$$\begin{aligned}
(Q(u+z)-Q(u),z)_{\mathcal{M}} &= z^T\mathcal{M}(Q(u+z)-Q(u)) \\
&= z^T\mathcal{M}\mathcal{M}^{-1}(B(u+z)-B(u)) \\
&= z^T(B(u+z)-B(u)) \\
&= \sum_{j=1}^n z_j(b_j(u+z)-b_j(u)) \\
&= \sum_{j=1}^n z_j\left[(u+z,\phi_j)+\tau b(u+z,\phi_j)-(u,\phi_j)-\tau b(u,\phi_j)\right] \\
&= (z,\sum_{j=1}^n z_j\phi_j)+\tau\left[b(u+z,\sum_{j=1}^n z_j\phi_j)-b(u,\sum_{j=1}^n z_j\phi_j)\right] \\
&= (z,z)+\tau\left[b(u+z,z)-b(u,z)\right] \\
&= ||z||^2_{a_\tau}+\tau\left[\tilde{b}(u+z,z)-\tilde{b}(u,z)\right] \ge ||z||^2_{a_\tau} \\
&\ge \frac{1}{C_0^2}||z||^2_{\mathcal{M}}.
\end{aligned}$$

Using a particular decomposition of z, we are able to give another proof of this lemma. Since P_{ad} is invertible (see [33, Lemma 2.5]), we can decompose $z \in V_h$ as $z = \sum_j R_j^T z_j$, where

$$z_j = \widetilde{P}_j P_{\text{ad}}^{-1} z,$$

$$a_\tau(z,z) = a_\tau(u+z,z) - a_\tau(u,z) = \sum_j \left[a_\tau(u+z,R_j^Tz_j) - a_\tau(u,R_j^Tz_j)\right]$$

$$= \sum_j \left[\tilde{a}_{\tau j}(\widetilde{Q}_j(u+z),z_j) - \tilde{a}_{\tau j}(\widetilde{Q}_j(u),z_j)\right] - \sum_j \tau\left[\tilde{b}(u+z,R_j^Tz_j) - \tilde{b}(u,R_j^Tz_j)\right]$$

$$= \sum_j \left[\tilde{a}_{\tau j}(\widetilde{Q}_j(u+z),z_j) - \tilde{a}_{\tau j}(\widetilde{Q}_j(u),z_j)\right] - \tau\left[\tilde{b}(u+z,z) - \tilde{b}(u,z)\right].$$

Using the monotonicity of $\tilde{b}$, we can write

$$a_\tau(z,z) \le \sum_j \tilde{a}_{\tau j}(\widetilde{Q}_j(u+z) - \tilde{Q}_j(u),z_j).$$

By the definition of $z_j,\ \widetilde{P}_j = \widetilde{A}_{\tau j}^{-1} R_j A_\tau,\ P_{\rm ad} = \mathcal{M}^{-1} A_\tau$, and we have

$$\begin{aligned}
||z||^2_{a_\tau} &\leq \sum_j \tilde{a}_{\tau j}(\widetilde{Q}_j(u+z) - \widetilde{Q}_j(u), z_j) = \sum_j z_j^T \widetilde{A}_{\tau j}\left[\widetilde{Q}_j(u+z) - \widetilde{Q}_j(u)\right] \\
&= \sum_j z^T \left(P_{\rm ad}^{-1}\right)^T \widetilde{P}_j^T \widetilde{A}_{\tau j} \widetilde{A}_{\tau j}^{-1} R_j \left[B(u+z) - B(u)\right] \\
&= z^T \left(P_{\rm ad}^{-1}\right)^T \left(\sum_j \widetilde{P}_j^T R_j\right) \left[B(u+z) - B(u)\right] \\
&= z^T \left(P_{\rm ad}^{-1}\right)^T \left(\sum_j A_\tau^T R_j^T \left(\widetilde{A}_{\tau j}^{-1}\right)^T R_j\right) \left[B(u+z) - B(u)\right] \\
&= z^T \left(P_{\rm ad}^{-1}\right)^T A_\tau \mathcal{M}^{-1} \left[B(u+z) - B(u)\right] = z^T (A_\tau^{-1}\mathcal{M})^T A_\tau \mathcal{M}^{-1} \left[B(u+z) - B(u)\right] \\
&= z^T \mathcal{M}^T (A_\tau^{-1})^T A_\tau \mathcal{M}^{-1} \left[B(u+z) - B(u)\right] = z^T \mathcal{M} \left[Q(u+z) - Q(u)\right] \\
&= \left(z, Q(u+z) - Q(u)\right)_{\mathcal{M}}.
\end{aligned}$$

We conclude the proof by using Lemma 4.2. ☐

LEMMA 4.4. *For $i = 0, 1, \ldots, N$, we have that*

a) $$||Q_i(u+z) - Q_i(u)||^2_{a_\tau} \leq \omega \Big[a_\tau(z, Q_i(u+z) - Q_i(u)) + \tau\tilde{b}(u+z, Q_i(u+z) - Q_i(u)) - \tau\tilde{b}(u, Q_i(u+z) - Q_i(u))\Big];$$

b) $||Q_i(u+z) - Q_i(u)||_{a_\tau} \leq \omega(1 + c\tilde{L})||z||_{a_\tau}$.

Proof. a) Using the definition of Q_i, the local stability assumption, the definition of $\widetilde{Q}_i$, the linearity of a_τ, and the Lipschitz continuity of $\tilde{b}$, we have,

$$\begin{aligned}
&||Q_i(u+z) - Q_i(u)||^2_{a_\tau} \\
&\qquad = a_\tau(Q_i(u+z) - Q_i(u), Q_i(u+z) - Q_i(u)) \\
&\qquad = a_\tau(R_i^T(\widetilde{Q}_i(u+z) - \widetilde{Q}_i(u)), R_i^T(\widetilde{Q}_i(u+z) - \widetilde{Q}_i(u))) \\
&\qquad \leq \omega \tilde{a}_\tau(\widetilde{Q}_i(u+z) - \widetilde{Q}_i(u), \widetilde{Q}_i(u+z) - \widetilde{Q}_i(u)) \\
&\qquad = \omega \Big[(u+z, R_i^T(\widetilde{Q}_i(u+z) - \widetilde{Q}_i(u))) + \tau b(u+z, R_i^T(\widetilde{Q}_i(u+z) - \widetilde{Q}_i(u))) \\
&\qquad\qquad - (u, R_i^T(\widetilde{Q}_i(u+z) - \widetilde{Q}_i(u))) - \tau b(u, R_i^T(\widetilde{Q}_i(u+z) - \widetilde{Q}_i(u)))\Big] \\
&\qquad = \omega \left[(z, Q_i(u+z) - Q_i(u)) + \tau a(z, Q_i(u+z) - Q_i(u)) + \right. \\
&\qquad\qquad \left. \tau\tilde{b}(u+z, Q_i(u+z) - Q_i(u)) - \tau\tilde{b}(u, Q_i(u+z) - Q_i(u))\right].
\end{aligned}$$

b) By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&||Q_i(u+z) - Q_i(u)||^2_{a_\tau} \\
&\qquad \leq \omega \Big[||z||_{a_\tau} ||Q_i(u+z) - Q_i(u)||_{a_\tau} + \tau\tilde{L}||z|| \cdot ||Q_i(u+z) - Q_i(u)||\Big].
\end{aligned}$$

The coercivity of the bilinear form $a(\cdot,\cdot)$ implies that $\tau||z||^2 \le c\tau a(z,z) \le c a_\tau(z,z)$ and consequently we have $||Q_i(u+z)-Q_i(u)||^2_{a_\tau} \le \omega(1+c\tilde{L})||z||_{a_\tau}||Q_i(u+z)-Q_i(u)||_{a_\tau}$. □

LEMMA 4.5. *There exists a constant* $\breve{\delta}_1^{\,2} = 2\omega^2(\rho(\mathcal{E})^2+1)(1+c\tilde{L})^2 > 0$, *such that*

$$||Q(u+z)-Q(u)||_{a_\tau} \le \breve{\delta}_1||z||_{a_\tau},\ \forall u,v \in V_h.$$

Proof. Define $\bar{Q}(u) = \sum\limits_{i=1}^{N} Q_i(u)$, as in the proof of the upper bound of Lemma 4.2. The bilinearity of a_τ, the definition of $Q_i(u)$, and the strengthened Cauchy-Schwarz inequality, imply that

$$||\bar{Q}(u+z)-\bar{Q}(u)||^2_{a_\tau} \le \rho(\mathcal{E})\sum_{i=1}^{N} a_\tau(Q_i(u+z)-Q_i(u),Q_i(u+z)-Q_i(u)).$$

From Lemma 4.4, we obtain

$$\begin{aligned}
&||\ \bar{Q}(u+z)-\bar{Q}(u)||^2_{a_\tau}\\
&\le \omega\rho(\mathcal{E})\sum_{i=1}^{N}[a_\tau(z,Q_i(u+z)-Q_i(u))+\\
&\qquad\qquad + \tau\tilde{b}(u+z,Q_i(u+z)-Q_i(u)) - \tau\tilde{b}(u,Q_i(u+z)-Q_i(u))]\\
&\le \rho(\mathcal{E})\omega\left[a_\tau(z,\bar{Q}(u+z)-\bar{Q}(u)) + \tau\tilde{b}(u+z,\bar{Q}(u+z)-\bar{Q}(u)) - \tau\tilde{b}(u,\bar{Q}(u+z)-\bar{Q}(u))\right]\\
&\le \rho(\mathcal{E})\omega(1+c\tilde{L})||z||_{a_\tau}||\bar{Q}(u+z)-\bar{Q}(u)||_{a_\tau}.
\end{aligned}$$

Hence,

$$||\bar{Q}(u+z)-\bar{Q}(u)||_{a_\tau} \le \rho(\mathcal{E})\omega(1+c\tilde{L})||z||_{a_\tau}. \tag{4.2}$$

Since $Q(u) = Q_0(u)+\bar{Q}(u)$, relation (4.2) and Lemma 4.4 give us

$$\begin{aligned}
&||\ Q(u+z)-Q(u)||^2_{a_\tau} \le 2||Q_0(u+z)-Q_0(u)||^2_{a_\tau} + 2||\bar{Q}(u+z)-\bar{Q}(u)||^2_{a_\tau}\\
&\le 2[\omega(1+\tau\tilde{L})]^2||z||^2_{a_\tau} + 2[\rho(\mathcal{E})\omega(1+\tau\tilde{L})]^2||z||^2_{a_\tau} = 2\omega^2(1+\tau\tilde{L})^2(1+\rho(\mathcal{E})^2)||z||^2_{a_\tau}.\ \square
\end{aligned}$$

LEMMA 4.6. *There exists a constant* $\delta_1 = C_0^2\breve{\delta}_1^{\,2}2\omega(1+\rho(\mathcal{E}))$, *such that*

$$||Q(u+z)-Q(u)||^2_{\mathcal{M}} \le \delta_1||z||^2_{\mathcal{M}}.$$

Proof. This bound is a direct consequence of Lemma 4.2 and Lemma 4.5:

$$\begin{aligned}
||Q(u+z)-Q(u)||^2_{\mathcal{M}} &\le C_0^2||Q(u+z)-Q(u)||^2_{a_\tau}\\
&\le C_0^2\breve{\delta}_1^{\,2}||z||^2_{a_\tau}\\
&\le C_0^2\breve{\delta}_1^{\,2}2\omega(1+\rho(\mathcal{E}))||z||^2_{\mathcal{M}}.\ \square
\end{aligned}$$

Using these lemmas, we are now in a position to prove the main convergence result for our ASR method.

THEOREM 4.7. *If* $0 < \lambda < 2\delta_0/\delta_1^2$, *then the* **ASR** *method for Problem (3.8) converges, i.e.,*

$$||u^k - u^*||^2_{\mathcal{M}} \leq P(\lambda)^k ||u^0 - u^*||^2_{\mathcal{M}},$$

where $P(\lambda) = 1 - 2\lambda\delta_0 + \lambda^2\delta_1^2$ *and* $\delta_0,\ \delta_1$ *are the constants defined in Lemma 4.3 and Lemma 4.6, respectively.*

Proof. Let $e^k = u^k - u^*$ and $r^k = Q(u^k) - Q(u^*)$ be the error and the residual at step k. Then

$$e^{k+1} = u^{k+1} - u^* = u^k - \lambda r^k - u^* = e^k - \lambda r^k,$$

and

$$\begin{aligned} ||e^{k+1}||^2_{\mathcal{M}} &= (e^{k+1}, e^{k+1})_{\mathcal{M}}(e^k - \lambda r^k, e^k - \lambda r^k)_{\mathcal{M}} \\ &= ||e^k||^2_{\mathcal{M}} - 2\lambda(e^k, r^k)_{\mathcal{M}} + \lambda^2||r^k||^2_{\mathcal{M}}. \end{aligned}$$

Lemma 4.3 with $u = u^*$ and $z = u^k - u^*$ yields

$$\begin{aligned} -(e^k, r^k)_{\mathcal{M}} &= -(u^k - u^*, Q(u^k) - Q(u^*))_{\mathcal{M}} \\ &= -(u^k - u^*, Q(u^k - u^* + u^*) - Q(u^*))_{\mathcal{M}} \\ &\leq -\delta_0||e^k||^2_{\mathcal{M}}. \end{aligned}$$

Using Lemma 4.6 with $u = u^*$ and $z = u^k - u^*$, we obtain

$$||r_k||^2_{\mathcal{M}} = ||Q(u^k) - Q(u^*)||^2_{\mathcal{M}} = ||Q(u^k - u^* + u^*) - Q(u^*)||^2_{\mathcal{M}} \leq \delta_1^2||e^k||^2_{\mathcal{M}}.$$

Therefore

$$||e^{k+1}||^2_{\mathcal{M}} \leq (1 - 2\lambda\delta_0 + \lambda^2\delta_1^2)||e^k||^2_{\mathcal{M}} = P(\lambda)||e^k||^2_{\mathcal{M}}.$$

If we choose $0 < \lambda < \frac{2\delta_0}{\delta_1^2}$, then $P(\lambda) < 1$, and we obtain the convergence of the ASR method. □

REMARK 4.8. $P(\lambda)$ attains its minimum at $\lambda_{min} = \frac{\delta_0}{\delta_1^2}$ and $P(\lambda_{min}) = 1 - \frac{\delta_0^2}{\delta_1^2} < 1$.

REMARK 4.9. If we drop the coarse space V_0 and define $Q(u) = \sum_{i=1}^{N} Q_i(u)$, the ASR algorithm is convergent. It is easy to see that in this case $\delta_0 = \frac{1}{C_0^2}$ and $\delta_1 = C_0\rho(\mathcal{E})\omega(1+c\tilde{L})$.

REMARK 4.10. The ASR performance depends on the choice of the parameter λ in the Richardson iteration. This parameter can be chosen:
i) theoretically, by approximately minimizing the quadratic function $P(\lambda)$ in Theorem 4.7 if the constants δ_0 and δ_1 can be estimated;
ii) numerically, by running a few tests cases (as we have done in Section 6, Figure 6.1) and selecting a value which gives an approximate minimum of the ASR iteration count;
iii) automatically, by using one of the step-length strategies available in the literature of numerical optimization, such as the one described in [26]; see the results reported in Section 6, Table 6.2.
We remark that the numerical results of Figure 6.1 show that the ASR iteration count is not very sensitive to the choice of λ near the minimum, but only near the endpoints of the convergence interval.

5. Convergence estimates for parabolic problem. Our additive Schwarz preconditioner is build as in the linear case. We partition the domain Ω into shape regular nonoverlapping subdomains $\Omega_i,\ 1 \le i \le N$, of characteristic diameter H, defining a shape-regular coarse mesh $\mathcal{T}_H$. Each subdomain Ω_i is extended to a larger one, Ω_i', by adding the elements of the fine mesh $\mathcal{T}_h$ within a distance δ_i from its boundary. We assume that the partition $\{\Omega_i'\}$ satisfies the finite covering assumption (see, e.g., [33]), and we denote by $\delta = \max_i \delta_i$ the overlap size. Using the above decomposition, a one-level method is defined by the local spaces $V_i = \{v \in H_0^1(\Omega_i')|\ v|_T \text{ is linear},\ \forall T \in T_{h,i}\},\ 1 \le i \le N$, and the local bilinear forms $\tilde{a}_{\tau,i}(u_i,v_i) = a_\tau(R_i^T u_i, R_i^T v_i),\ \forall u_i,\ v_i \in V_i$, with interpolation operators $R_i^T : V_i \longrightarrow V,\ 1 \le i \le N$. We then build a two-level algorithm by defining the coarse finite element space $V_0 = \{v \in H_0^1(\Omega)|\ v \text{ is continuous and } v|_T \text{ is linear},\ \forall T \in T_H\}$ and the operator R_0^T, which interpolates the coarse functions onto the fine mesh.

We consider the variational parabolic problem: given $u(t_0,x) = u_0(x)$ and right-hand side G, find $u(t) \in H^1(\Omega)$, such that $\forall t \in (t_0,T)$,

$$\left(\frac{\partial u}{\partial t}, v\right) + a(u,v) + (F(u),v) = (G,v), \tag{5.1}$$

where

$$a(u,v) = \int_\Omega \sum_{i,j=1}^{d} a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\, dx,$$

with $a_{ij} \in C^1(\Omega)$, such that $a_{ij}(x) = a_{ji}(x),\ \forall x \in \Omega \subset \mathbf{R}^d$ for all $i,j = 1,\dots,d$ and F a monotone nonlinear function.

The abstract convergence result of Theorem 4.7 now can be applied to get explicit bounds in terms of the discretization parameters. For simplicity, we consider the simplest case, where we use exact solvers (i.e., $\tilde{a}_{\tau,i}(u_i,v_i) = a_\tau(R_i^T u_i, R_i^T v_i), \forall u_i, v_i \in V_i$), so that $\omega = 1$ in the local stability assumption. We also assume that there are at most N^c nonzero elements in each row of $\mathcal{E} \le N^c$, so that $\rho(\mathcal{E}) \le N^c$ in the strengthened Cauchy-Schwarz inequality; see [33, Lemma 2.10]. Therefore, we need only to bound the constant C_0 in the stable decomposition assumption.

LEMMA 5.1. *The stable decomposition constant C_0 can be bounded by*

a) $C_0^2 \le C \max\left\{1 + \dfrac{H}{\delta},\ 1 + \dfrac{\tau}{H\delta}\right\}$, *in the one-level case,*

b) $C_0^2 \le C\left(1 + \dfrac{H}{\delta}\right)$, *in the two-level case,*

where C is a constant independent of $H,\ h$ and δ.

Proof. a) One-level case. Following the classical proof for the linear elliptic case (see, e.g., [33]), we define $u_i = R_i(I^h(\theta_i u)),\ i = 1,\dots,N$, where $\{\theta_i \in W^{1,\infty},\ 1 \le i \le N\}$ defines a partition of unity and I^h is the nodal piecewise linear interpolant on the fine mesh $\mathcal{T}_h$. By the approximations properties of I^h and the small overlap lemma (see [33], Lemma 3.10 and eqs.(3.20) - (3.23) or the original Lemma 3.1 in [16]), we obtain

$$|R_i^T u_i|^2_{H^1(\Omega)} \equiv |I^h(\theta_i u)|^2_{H^1(\Omega_i')} \le C\left(1 + \frac{H}{\delta}\right)|u|^2_{H^1(\Omega_i')} + \frac{1}{H_i\delta_i}||u||^2_{L^2(\Omega_i')}. \tag{5.2}$$

The equivalence between the $L^2(\Omega)-$norm and the discrete $L^2(\Omega)-$norm implies that

$$(R_i^T u_i, R_i^T u_i)_{L^2(\Omega)} \equiv ||I^h(\theta_i u)||^2_{L^2(\Omega_i')} \le C||u||^2_{L^2(\Omega_i')}. \tag{5.3}$$

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Using the coercivity of $a(\cdot,\cdot)$, (5.2), (5.3), and the finite covering assumption, we have

$$\begin{aligned}
\sum_{i=1}^{N} \tilde{a}_{\tau,i}(u_i,u_i) &= \sum_{i=1}^{N} a_\tau(R_i^T u_i, R_i^T u_i) = \sum_{i=1}^{N} \left[(R_i^T u_i, R_i^T u_i)_{L^2(\Omega)} + \tau a(R_i^T u_i, R_i^T u_i)\right] \\
&\leq C \sum_{i=1}^{N} \left[(R_i^T u_i, R_i^T u_i)_{L^2(\Omega)} + \tau |R_i^T u_i|^2_{H^1(\Omega)}\right] \\
&\leq C \sum_{i=1}^{N} \left\{ ||u||^2_{L^2(\Omega_i')} + \tau \left[\left(1+\frac{H}{\delta}\right) |u|^2_{H^1(\Omega_i')} + \frac{1}{H\delta} ||u||^2_{L^2(\Omega_i')} \right] \right\} \\
&\leq C \sum_{i=1}^{N} \left[\left(1+\frac{\tau}{H\delta}\right) ||u||^2_{L^2(\Omega_i')} + \tau \left(1+\frac{H}{\delta}\right) |u|^2_{H^1(\Omega_i')} \right] \\
&\leq C \left(1+\frac{\tau}{H\delta}\right) ||u||^2_{L^2(\Omega)} + \tau \left(1+\frac{H}{\delta}\right) |u|^2_{H^1(\Omega)}.
\end{aligned}$$

The coercivity of the bilinear form $a(\cdot,\cdot)$ implies that $|u|^2_{H^1(\Omega)} \leq C a(u,u)$ and therefore

$$\begin{aligned}
\sum_{i=1}^{N} \tilde{a}_{\tau,i}(u_i,u_i) &\leq C \left[\left(1+\frac{\tau}{H\delta}\right) ||u||^2_{L^2(\Omega)} + \tau \left(1+\frac{H}{\delta}\right) a(u,u) \right] \\
&\leq C \max \left\{ 1+\frac{H}{\delta}, 1+\frac{\tau}{H\delta} \right\} \left[||u||^2_{L^2(\Omega)} + \tau a(u,u) \right].
\end{aligned}$$

b) Two-level case. Suppose that the coarse mesh is quasi-uniform. Then the bound for C_0^2 can be obtained as in [33] by letting $I^H : L^2(\Omega) \longrightarrow V_0$ be the L^2-projection of u, $(I^H u, v)_{L^2(\Omega)} = (u, v)$, $\forall u \in L^2(\Omega)$, $v \in V_0$, and $u_0 = I^H u$, $w = u - I^h u_0$, $u_i = I^h(\theta_i w)$. □

REMARK 5.2. The bound a) of Lemma 5.1 implies that if the ratio $\tau/(H\delta)$ is small enough, then a constant upper bound $C_0^2 \leq C\left(1+\frac{H}{h}\right)$ still holds for the one-level ASR algorithm.

REMARK 5.3. In case of small overlap $\delta = h$, the estimates of Lemma 5.1 become

a) $C_0^2 \leq C \max\left\{1+\frac{H}{h},\ 1+\frac{\tau}{Hh}\right\}$, in the one-level case,

b) $C_0^2 \leq C\left(1+\frac{H}{h}\right)$, in the two-level case.

Therefore, in scaled speed-up tests with constant ratio H/h only the two-level ASR algorithm is scalable, since the term $\tau/(Hh)$ asymptotically dominates the one-level bound for any fixed value of τ. Nevertheless, for a moderate number of subdomains (i.e., for $1/H$ small enough), the one-level bound is dominated by the first term $1+\frac{H}{h}$, which yields a "temporary" scalability; see Figure 6.2.

REMARK 5.4. In case of generous overlap $\delta = CH$, the estimates of Lemma 5.1 agree with the estimates obtained by Cai [7, 8] for linear parabolic problems. In such case, Cai proved that if τ/H^2 is small enough, then the one-level Schwarz algorithm is scalable and the two-level algorithm satisfies an optimal constant bound. Lemma 5.1 extends these estimates to nonlinear parabolic problems and to the case of variable overlap.

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TABLE 6.1

Comparison of ASR and Linearly Implicit Euler methods (GMRES with AS preconditioner): scaled speed-up test with fixed subdomain size $H/h = 4$, small overlap size $\delta = h$, and increasing number of subdomains N (and nodes). iter = iteration counts, cpu = cpu times, err = relative errors with exact solution.

	ASR with $\lambda = 0.4$				Lin. Impl. Euler: GMRES with AS prec.						
	one-level		two-levels		one-level		one-level, rest = 20		two-levels		
N	iter	cpu	iter	cpu	iter	cpu	iter	cpu	iter	cpu	err
2×2	40	0.07	43	0.07	10	0.09	10	0.10	11	0.07	6.76e-3
4×4	70	0.57	36	0.29	17	0.15	17	0.20	16	0.15	1.67e-3
6×6	123	2.88	35	0.87	20	0.59	20	0.84	17	0.49	7.44e-4
8×8	197	11.20	36	2.15	23	1.76	20+5	2.93	16	1.26	4.18e-4
10×10	293	36.80	36	4.61	26	4.48	20+7	7.78	16	3.02	2.67e-4
12×12	410	99.12	36	9.07	29	11.31	20+9	18.54	16	6.63	1.86e-4
14×14	549	236.93	36	16.25	31	29.12	20+14	51.33	16	15.69	1.36e-4
16×16	709	511.79	36	27.08	34	77.29	20+16	128.71	16	37.42	1.04e-4
18×18	891	1013.27	36	43.18	36	162.50	20+18	188.71	16	76.17	8.26e-5

6. Numerical results. In this section, we report the results of MATLAB numerical experiments with the ASR method applied to the nonlinear parabolic problem (5.1) with the linear elliptic bilinear form and the nonlinear function

$$a(u, v) = \sum_{i}^{N} \int_{\Omega_i} \sigma_i \nabla u \nabla v dx, \qquad f(u) = 0.5u + u^3.$$

The elliptic coefficients σ_i are equal to 1 in Tables 6.2-6.4, while they are piecewise constant with jump discontinuities across subdomain boundaries in the last tests of Figures 6.3 and 6.4. The domain is the unit square Ω and the right-hand side g is chosen so that $u^*(t, x) = t \sin(\pi x) \sin(\pi y)$ is the exact solution when $\sigma_i = 1$. We consider $t_0 = 0$, $u_0(x) = 0$, and we compute the solution for $t = \tau = 0.01$. The iteration process is stopped when $||r_k||_{\mathcal{M}}/||r_0||_{\mathcal{M}} \leq 1e{-8}$, and we denote the relative error by $err = ||u - u^*||_{l^2(\Omega)}/||u^*||_{l^2(\Omega)}$.

Comparison between ASR and Linearly Implicit Euler methods. We start with a comparison between our ASR method (3.9) and the Linearly Implicit Euler method, consisting in applying to the original nonlinear problem (2.2) a single Newton step, see e.g. Deuflhard [14], Lang [18], and solve the resulting linear system by GMRES with the Additive Schwarz preconditioner $\mathcal{M}$ defined in (3.7). This method has the advantage of requiring only the solution of a linear problem per time step, but it requires the computation of the Jacobian of the nonlinear operator. We remark that the Jacobian is not needed in our ASR algorithm and it might even be practically uncomputable for some nonlinear problems, such as the monodomain and bidomain systems coupled with realistic ionic models; see Munteanu [21]. In our notations, the Linearly Implicit Euler method for the nonlinear system (3.1) $B(u^m) = \hat{g}$ at time step t_m becomes:

solve the Jacobian system: $\quad J_B^m s^m = \hat{g} - B(u^m)$
by GMRES with the Additive Schwarz preconditioner $\mathcal{M}$ in (3.7),
update the solution: $\quad u^{m+1} = u^m + s^m,$

where J_B^m is the Jacobian of the nonlinear operator B at u^m. Given our assumption (3.2) on the structure of B, this Jacobian matrix also can be written in terms of the nonlinear form $b(u, v)$ and the mass matrix M as $J_B^m = M + \tau J_b^m$, with J_b^m equal to the Jacobian of b at u^m.

Table 6.1 compares the two algorithms by reporting their iteration counts (iter) and MATLAB cpu times (cpu) on a PC (Acer Aspire 3680) in a scaled speed-up test, where the subdomain size $H/h = 4$ is kept fixed and the number of subdomains (hence nodes) is increased

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TABLE 6.2
ASR scaled speed-up test with fixed subdomain size $H/h = 4$, small overlap size $\delta = h$, and increasing number of subdomains N (and nodes): ASR with fixed $\lambda = 0.4$ and random RHS (second and third columns); ASR with step-lenght strategy for λ and exact solution (fourth and fifth columns). iter = iteration counts, err = relative errors with exact solution.

	ASR with $\lambda = 0.4$ random RHS		ASR with step-lenght strategy for λ exact solution RHS			
	one-level	two-levels	one-level		two-levels	
N	iter	iter	iter	err	iter	err
2×2	40	42	25	6.76e-3	24	6.76e-3
4×4	70	38	37	1.67e-3	25	1.67e-3
6×6	122	39	69	7.44e-4	24	7.44e-4
8×8	196	40	117	4.18e-4	24	4.18e-4
10×10	291	42	157	2.67e-4	27	2.67e-4
12×12	407	41	223	1.86e-4	22	1.86e-4
14×14	544	42	-	-	25	1.36e-4
16×16	703	43	-	-	23	1.04e-4
18×18	882	43	-	-	24	8.26e-5

from 2×2 to 18×18. For both algorithms the right-hand side is chosen so that the exact solution is u^* given above and we can determine the relative errors (err) reported in the last column of the table, that decrease proportionally to the mesh refinement (i.e., increasing subdomains) as expected. In agreement with the theory (see Remark 5.3), in the one-level case the number of iterations of both algorithms increases (much less for Linearly Implicit Euler, column 5), while in the two-level case the iteration counts remain bounded and both algorithms are scalable (with a better upper bound of 16 for Linearly Implicit Euler, column 6, than 36 for ASR, column 3). The cpu times behave accordingly, i.e., they increase strongly for the one-level algorithms and show a more moderate increase for the two-level algorithms. The most relevant comparison between the two-level algorithms shows that the ASR cpu times are initially slightly larger than the Linearly Implicit Euler cpu times, but as the problem size increases (for $N \geq 14 \times 14$) the ASR times equal and then definitely improve over the Lineary Implicit Euler times (27.08 v. 37.42 sec. for $N = 16 \times 16$ and 43.18 v. 76.17 sec. for $N = 18 \times 18$). These results indicate that ASR can be asymptotically more efficient than Linearly Implicit Euler as the problem size and number of subdomains increase. We remark that this is only a partial indication because of the serial implementation of the two algorithms, where the subdomain problems of the Additive Schwarz preconditioners are solved sequentially; it would be much more significant to compare the parallel cpu times for the two algorithms on modern distributed computing architectures (which is beyond the scope of this paper).

ASR scalability with random RHS and λ step-lenght strategy. Table 6.2 reports the results of a scaled speed-up test analogous to Table 6.1, but focuses only on the ASR algorithm with minimal overlap $\delta = h$. In the left part of the table (columns 2 and 3), the parameter λ is again fixed at 0.4 but the RHS is randomly distributed. In the right part of the table (column 4 and 5), λ is chosen by the step-length strategy of [26] (columns 4 and 5) and the RHS is again the one associated to the exact solution u^* given above. The results confirm that in the one-level case the number of ASR iterations (iter) increases, while in the two-level case this number remains bounded and the ASR algorithm is scalable. The step-length strategy for the selection of λ yields better iteration counts (around 22 - 27 iterations) than the fixed $\lambda = 0.4$ selection (around 36 iterations), but at the expense of a much larger cpu time (not shown). The results with a random right-hand side show the same scalability of the two-level ASR algorithm, only with slightly larger iteration counts (now with an upper bound of 43 iterations).

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TABLE 6.3
ASR standard speed-up test with fixed mesh size $h = 1/48$, small overlap size $\delta = h$, and increasing number of subdomains N (of decreasing size H/h). ASR with fixed $\lambda = 0.4$ (second and third columns); ASR with step-lenght strategy for λ (fourth and fifth columns). iter = iteration counts, err = relative errors with exact solution.

	$\lambda = 0.4$				λ: step-length strategy			
	one-level		two-levels		one-level		two-levels	
N	iter	err	iter	err	iter	err	iter	err
2×2	155	1.74e-4	70	1.74e-4	86	1.74e-4	43	1.74e-4
4×4	192	1.74e-4	58	1.74e-4	114	1.74e-4	32	1.74e-4
6×6	245	1.74e-4	47	1.74e-4	149	1.74e-4	27	1.74e-4
8×8	301	1.74e-4	41	1.74e-4	-	-	23	1.74e-4

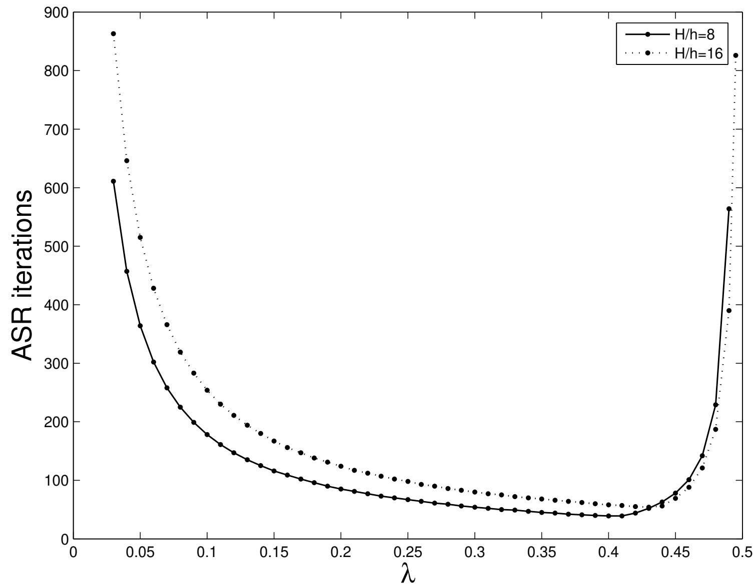

FIG. 6.1. *ASR iterations counts as a function of the parameter λ; $N = 2 \times 2$, overlap size $\delta = h$.*

ASR standard speed-up. We study the ASR performance in a standard speed-up test where the global problem size is fixed ($h = 1/48$) and the number of subdomains is increased from 2×2 to 8×8, hence decreasing the ratio H/h. The same quantities (iter and err) of Table 6.2 are reported in Table 6.3. As predicted by the theory, only in the two-level case, the ASR iteration counts improve as the subdomain size H decreases, since the term $\tau/(Hh)$ dominates the one-level bound in Remark 5.3b).

ASR dependence on λ. Figure 6.1 confirms the theoretical prediction of Theorem 4.7, showing the ASR iteration counts as a function of the parameter λ for $N = 2\times 2$ subdomains, overlap $\delta = h$, and two mesh sizes $h = 1/8$ (continuous line), $h = 1/16$ (dashed line). The explicit formula of Theorem 4.7 shows that the parabola $P(\lambda)$ attains its minimum inside a right interval of 0 and tends to 1 at its endpoints; correspondingly, the ASR convergence rate attains a minimum inside an interval $(0, \alpha),\ \alpha > 0$ and degenerates at the interval endpoints.

ASR dependence on δ. Table 6.4 shows that the ASR iteration counts improve with increasing overlap size δ, for both the one- and two-level ASR algorithms, in agreement with Lemma 5.1. In the two-level case, the improvement becomes irrelevant for overlap sizes

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TABLE 6.4
Effect of increasing the overlap size δ in one- and two-level ASR methods for fixed mesh size $h = 1/48$, number of subdomains $N = 2 \times 2$, $\lambda = 0.4$ iter = iteration counts, err = relative errors with exact solution.

overlap size δ	one-level iter	one-level err	two-levels iter	two-levels err
h	155	1.74e-4	70	1.74e-4
2h	82	1.74e-4	46	1.74e-4
3h	59	1.74e-4	37	1.74e-4
4h	49	1.74e-4	37	1.74e-4

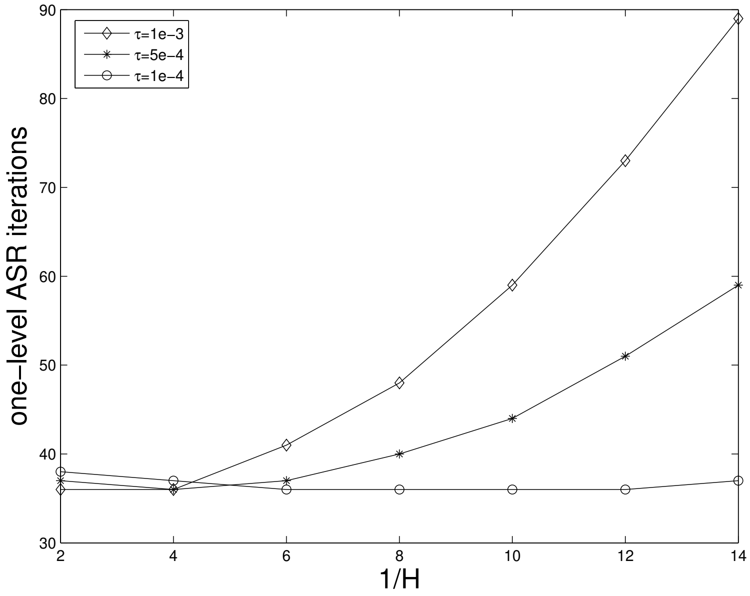

FIG. 6.2. *One-level ASR iterations counts for increasing number of subdomains $1/H$ and different time-step sizes τ, fixed ratio $H/h = 4$, $\delta = h$.*

larger than $3h$.

ASR scalability dependence on τ. Figure 6.2 reports a different validation of the one-level bound by reducing the time-step size τ and performing a scaled speed-up test as in Table 6.2 (H/h fixed and small overlap $\delta = h$). While we already know that for a given value of τ the one-level ASR algorithm is not scalable (Remark 5.3 and Table 6.2), we nevertheless expect a reduction of τ to give bounded iteration counts up to a critical number of subdomains N_τ ("temporary" scalability) since the first term dominates the maximum in Remark 5.3a), while for $N > N_\tau$ we expect increasing iteration counts since the second term dominates the maximum in Remark 5.3a). The results in Figure 6.2 show that this is indeed the case: for $\tau = 1e-3,\ N_\tau \sim 4 \times 4$, for $\tau = 5e-4,\ N_\tau \sim 6 \times 6$ and for $\tau = 1e-4,\ N_\tau \sim 12 \times 12$.

ASR performance for elliptic coefficients with jump discontinuities across subdomains. Finally, Figures 6.3 and 6.4 report the ASR results when the coefficients σ_i of the linear elliptic operator are piecewise constant and present jump discontinuities across subdomain boundaries. We considered a decomposition of Ω into 7×7 subdomains with $H/h = 4,\ \delta = h$; we chose a larger tolerance $tol = 1e-4$ in the stopping criterion in order to test the convergence (or lack thereof) of the unpreconditioned algorithm, up to a maximum

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1	1	1	1	1	1	1
1	1	1	1	1	1	1
1	1	σ_i	σ_i	σ_i	1	1
1	1	σ_i	σ_i	σ_i	1	1
1	1	σ_i	σ_i	σ_i	1	1
1	1	1	1	1	1	1
1	1	1	1	1	1	1

σ_i	one-level iter	two-levels iter	no prec iter
10^{-2}	63	17	-
10^{-1}	66	17	-
1	77	17	29818
10	76	17	-
10^2	72	17	-

FIG. 6.3. *Iteration counts of one- and two-level ASR method with $\lambda = 0.4$ for a test problem with discontinuous coefficient σ_i in the linear elliptic operator; the domain is decomposed into $N = 7 \times 7$ subdomains (left panel, with values of σ_i indicated), $H/h = 4$, $tol = 1e-4$*

0	-3	1	0	-1	2	0
-2	2	0	1	-3	3	0
1	0	-2	2	3	2	1
2	-3	2	0	2	1	-3
3	0	1	2	0	1	3
-3	1	0	3	-3	3	0
0	-1	1	2	-2	3	-3

one-level iter	two-levels iter
40	17

FIG. 6.4. *Iteration counts of one- and two-level ASR method with $\lambda = 0.4$ for a test problem with random discontinuous coefficient σ_i in the linear elliptic operator; the domain is decomposed into $N = 7 \times 7$ subdomains (left panel), with indicated the values of the exponents α_i in the coefficients $\sigma_i = 10^{\alpha_i}$, $H/h = 4$, $tol = 1e-4$*

number of iterations $maxit = 3 \cdot 10^4$. We first considered a configuration where inside the 3×3 central subdomains Ω_i (see Figure 6.3, left) the elliptic coefficients σ_i have varying values ranging from $1e-2$ to $1e+2$, while they are equal to 1 in the other surrounding subdomains. The table in Figure 6.3, right, show that the two-level ASR method is unaffected by the size of the discontinuity, the one-level is almost unaffected but with iteration counts more than four times larger, and the unpreconditioned Richardson iteration essentially does not converge within $maxit$ iterations. We also considered a second configuration where the coefficients $\sigma_i = 10^{\alpha_i}$ have random exponents α_i given in Figure 6.4, left, with a variation of six orders of magnitude. The two-level ASR iteration counts are the same as before (17), the one-level iteration counts are a bit better (40) and the unpreconditioned method again does not converge.

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