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APPROXIMATION OF THE MINIMAL GERŠGORIN SET OF A SQUARE COMPLEX MATRIX*

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Abstract. In this paper, we address the problem of finding a numerical approximation to the minimal Geršgorin set, $\Gamma^{\mathcal{R}}(A)$, of an irreducible matrix A in $\mathbb{C}^{n,n}$. In particular, boundary points of $\Gamma^{\mathcal{R}}(A)$ are related to a well-known result of Olga Taussky.

Key words. eigenvalue localization, Geršgorin theorem, minimal Geršgorin set.

AMS subject classifications. 15A18, 65F15

1. Introduction.

Given an irreducible matrix $A = [a_{i,j}]$ in $\mathbb{C}^{n,n}$, its i-th Geršgorin disk is defined, with $N := \{1, 2, \ldots, n\}$, by

$$\Gamma_i(A) := \{z \in \mathbb{C} : |z - a_{i,i}| \leq r_i(A) := \sum_{j \in N \setminus \{i\}} |a_{i,j}|\} \quad (i \in N), \tag{1.1}$$

and the union of all these disks, denoted by

$$\Gamma(A) := \bigcup_{i \in N} \Gamma_i(A), \tag{1.2}$$

is called the *Geršgorin set* for A. A well-known result of Geršgorin [2] gives us that $\Gamma(A)$ contains the spectrum, $\sigma(A)$, of A, i.e.,

$$\sigma(A) := \{\lambda \in \mathbb{C} : \det(\lambda I - A) = 0\} \subseteq \Gamma(A). \tag{1.3}$$

Continuing, for any $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T > \mathbf{0}$ in $\mathbb{R}^n$, i.e., $x_i > 0$ for all $i \in N$, let $X := \operatorname{diag}[x_1, x_2, \ldots, x_n]$ denote the associated nonsingular diagonal matrix. Then, $X^{-1}AX$ has the same eigenvalues as A. Thus, with the Geršgorin disks for $X^{-1}AX$ now given by

$$\Gamma_i^{r^{\mathbf{x}}}(A) := \{z \in \mathbb{C} : |z - a_{i,i}| \leq r_i^{\mathbf{x}}(A) := \sum_{j \in N \setminus \{i\}} \frac{|a_{i,j}| x_j}{x_i}\} \quad (i \in N), \tag{1.4}$$

and with the associated Geršgorin set,

$$\Gamma^{r^{\mathbf{x}}}(A) := \bigcup_{i \in N} \Gamma_i^{r^{\mathbf{x}}}(A), \tag{1.5}$$

then

$$\sigma(A) \subseteq \Gamma^{r^{\mathbf{x}}}(A), \quad \text{for any } \mathbf{x} > \mathbf{0} \text{ in } \mathbb{R}^n. \tag{1.6}$$

The inclusion of (1.6) is also a well-known result of Geršgorin [2]. Clearly, the following intersection,

$$\Gamma^{\mathcal{R}}(A) := \bigcap_{\mathbf{x} > \mathbf{0} \text{ in } \mathbb{R}^n} \Gamma^{r^{\mathbf{x}}}(A), \tag{1.7}$$

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called the *minimal Geršgorin set* in [4, 6], is always a subset of $\Gamma^{r^{\mathbf{x}}}(A)$, for *any* $\mathbf{x} > \mathbf{0}$ in $\mathbb{R}^n$, thereby giving the sharpest inclusion set for $\sigma(A)$, with respect to *all* positive diagonal similarity transforms $X^{-1}AX$ of A.

This sharpness can also be expressed in the following way; cf. [6, Theorem 4.5]. With

$$\hat{\Omega}(A) := \{B = [b_{i,j}] \in \mathbb{C}^{n,n} : b_{i,i} = a_{i,i} \text{ and } |b_{i,j}| \leq |a_{i,j}| \text{ for } i \neq j \ (i, j \in N)\}, \tag{1.8}$$

then

$$\sigma\big(\hat{\Omega}(A)\big) := \bigcup_{B \in \hat{\Omega}(A)} \sigma(B) = \Gamma^{\mathcal{R}}(A), \tag{1.9}$$

i.e., each point of $\Gamma^{\mathcal{R}}(A)$ is an eigenvalue of *some* matrix B in $\hat{\Omega}(A)$.

Unlike the Geršgorin set $\Gamma(A)$ of (1.2) or $\Gamma^{r^{\mathbf{x}}}(A)$ of (1.5), the minimal Geršgorin set $\Gamma^{\mathcal{R}}(A)$ of (1.7) is not in general easy to determine numerically. The aim of this paper is to find a *reasonable approximation* of $\Gamma^{\mathcal{R}}(A)$, with a finite number of calculations, which contains $\Gamma^{\mathcal{R}}(A)$, and for which a limited number of boundary points of this approximation are actual boundary points of $\Gamma^{\mathcal{R}}(A)$. The determination of these latter boundary points are then related to a famous sharpening, by Olga Taussky [3], of the Geršgorin set of (1.2).

2. Background. Given an irreducible matrix $A = [a_{i,j}]$ in $\mathbb{C}^{n,n}$, its associated irreducible matrix $Q(z) = [q_{i,j}(z)]$, in $\mathbb{R}^{n,n}$, is defined by

$$q_{i,i}(z) := -|z - a_{i,i}|, \text{ and } q_{i,j}(z) := |a_{i,j}|, \text{ for } i \neq j \ (i, j \in N). \tag{2.1}$$

If

$$\mu(z) := \max_{i \in N} |z - a_{i,i}|, \tag{2.2}$$

then the matrix $B(z) := [b_{i,j}(z)] \in \mathbb{R}^{n,n}$, defined by

$$b_{i,i}(z) := \mu(z) - |z - a_{i,i}|, \text{ and } b_{i,j}(z) := |a_{i,j}|, \ i \neq j \ (i, j \in N), \tag{2.3}$$

satisfies

$$B(z) = Q(z) + \mu(z) I_n, \tag{2.4}$$

where $B(z)$ is a nonnegative irreducible matrix in $\mathbb{R}^{n,n}$. Then, from the Perron-Frobenius theory of nonnegative matrices, the matrix $B(z)$ possesses a positive real eigenvalue, $\rho(B(z))$, called the *Perron root* of $B(z)$, which is characterized as follows. For any $\mathbf{x} > \mathbf{0}$ in $\mathbb{R}^{n,n}$, either

$$\min_{i \in N}\{\big(B(z)\mathbf{x}\big)_i / x_i\} < \rho(B(z)) < \max_{i \in N}\{\big(B(z)\mathbf{x}\big)_i / x_i\}, \tag{2.5}$$

or

$$B(z)\mathbf{x} = \rho(B(z))\mathbf{x}. \tag{2.6}$$

Thus, if we set

$$\nu(z) := \rho(B(z)) - \mu(z) \ \ (\text{all } \ z \in \mathbb{C}), \tag{2.7}$$

then $\nu(z)$ is a real-valued function, defined for all $z \in \mathbb{C}$. Moreover, from (2.5) and (2.6), for any $\mathbf{x} > \mathbf{0}$ in $\mathbb{R}^n$ and any $z \in \mathbb{C}$, either

$$\min_{i \in N}\{\big(Q(z)\mathbf{x}\big)_i / x_i\} < \nu(z) < \max_{i \in N}\{\big(Q(z)\mathbf{x}\big)_i / x_i\}, \tag{2.8}$$

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or

$$Q(z)\mathbf{x} = \nu(z)\mathbf{x}, \tag{2.9}$$

the last equation giving us that $\nu(z)$ is an eigenvalue of $Q(z)$.

The following connection of the function $\nu(z)$ of (2.7) to the minimal Geršgorin set, $\Gamma^{\mathcal{R}}(A)$, comes from [4] and [6]:

$$z \in \Gamma^{\mathcal{R}}(A) \text{ if and only if } \nu(z) \geq 0, \tag{2.10}$$

and

$$\text{if } z \in \partial\Gamma^{\mathcal{R}}(A), \text{ then } \nu(z) = 0. \tag{2.11}$$

It is also known (cf. [6], Theorem 4.6), from the assumption that A is irreducible, that

$$\nu(a_{i,i}) > 0, \text{ for all } i \in N. \tag{2.12}$$

Further, given any real number θ with $0 \leq \theta < 2\pi$, it is known (cf. [6], Theorem 4.6) that there is a largest number $\hat{\varrho}_i(\theta) > 0$ such that

$$\nu(a_{i,i} + \hat{\varrho}_i(\theta)e^{i\theta}) = 0, \text{ and } \nu(a_{i,i} + te^{i\theta}) \geq 0, \text{ for all } 0 \leq t < \hat{\varrho}_i(\theta), \tag{2.13}$$

so that the entire complex interval $[a_{i,i} + te^{i\theta}]_{t=0}^{\hat{\varrho}_i(\theta)}$ lies in $\Gamma^{\mathcal{R}}(A)$. This implies that the set

$$\bigcup_{\theta=0}^{2\pi} [a_{i,i} + te^{i\theta}]_{t=0}^{\hat{\varrho}_i(\theta)} \tag{2.14}$$

is a *star-shaped* subset of $\Gamma^{\mathcal{R}}(A)$, for each $i \in N$, with

$$\nu(a_{i,i} + \hat{\varrho}_i(\theta)e^{i\theta}) \in \partial\Gamma^{\mathcal{R}}(A). \tag{2.15}$$

The results of (2.14) and (2.15) will be used below.

Next, we recall the famous result of Olga Taussky [3], on a sharpening of the Geršgorin Circle Theorem: Let $A = [a_{i,j}]$ in $\mathbb{C}^{n,n}$ be irreducible. If $\lambda \in \sigma(A)$ is such that $\lambda \notin int\ \Gamma_i(A)$ for each $i \in N$, i.e., $|\lambda - a_{i,i}| \geq r_i(A)$ for each $i \in N$, then

$$|\lambda - a_{i,i}| = r_i(A), \text{ for each } i \in N, \tag{2.16}$$

i.e., *each* Geršgorin circle $\{z \in \mathbb{C} : |z - a_{i,i}| = r_i(A)\}$ passes through λ.

To complete this section, we include the following:

$$\text{If } \nu(z) = 0, \text{ then } \det Q(z) = 0. \tag{2.17}$$

This follows directly from (2.9), since $\nu(z)$ is an eigenvalue of $Q(z)$. Finally, from [6, Exercise 7, p. 108], we also have that

$$\text{for any } z \text{ and } z' \text{ in } \mathbb{C}, \ |\nu(z) - \nu(z')| \leq |z - z'|, \tag{2.18}$$

so that $\nu(z)$ is *uniformly continuous* in $\mathbb{C}$. This also will be used below.

3. Numerical procedure for approximating $\Gamma^{\mathcal{R}}(A)$. With the given irreducible matrix $A = [a_{ij}]$ in $\mathbb{C}^{n,n}$, choose any j in N, and set $z = a_{j,j}$. Next, we assume that the nonnegative irreducible matrix $B(a_{j,j})$, which has at least one zero diagonal entry from (2.3), is a *primitive matrix*; cf. of [5, Section 2.2]. (We note that this is certainly the case if some diagonal entry of $B(a_{j,j})$ is positive. More generally, if $B(a_{j,j})$ is not primitive (i.e., $B(a_{j,j})$ is cyclic of some index $p \geq 2$), then any simple shift of $B(a_{j,j})$ into $B(a_{j,j}) + \varepsilon I_n$ *is* primitive for each $\varepsilon > 0$.)

With $B(a_{j,j})$ primitive, then, starting with an $\mathbf{x}^{(0)} > \mathbf{0}$ in $\mathbb{R}^n$, the power method gives convergent upper and lower estimates for $\rho(B(a_{j,j}))$, i.e., if $\mathbf{x}^{(m)} := B^m(a_{j,j})\mathbf{x}^{(0)}$ for all $m \geq 1$, then with $\mathbf{x}^{(m)} := [x_1^{(m)}, x_2^{(m)}, ..., x_n^{(m)}]^T$, we have

$$\underline{\lambda_m} := \min_{i \in N}\{\frac{x_i^{(m+1)}}{x_i^{(m)}}\} \leq \rho(B(a_{j,j})) \leq \max_{i \in N}\{\frac{x_i^{(m+1)}}{x_i^{(m)}}\} =: \overline{\lambda_m} \tag{3.1}$$

for all $m \geq 1$, with

$$\lim_{m \to \infty} \underline{\lambda_m} = \rho(B(a_{j,j})) = \lim_{m \to \infty} \overline{\lambda_m}. \tag{3.2}$$

In this way, using (2.4), (2.7), and (2.9), convergent upper and lower estimates of $\nu(a_{j,j})$ can be numerically obtained. (These estimations of $\nu(a_{j,j})$ do not need great accuracy for graphing purposes, as the example in Section 4 shows).

Next, assume, for convenience, that $\nu(a_{j,j}) > 0$ is accurately known, and select any real θ, with $0 \leq \theta < 2\pi$. The numerical goal now is to estimate the largest $\hat{\varrho}_j(\theta)$, with sufficient accuracy, where, from (2.2),

$$\nu(a_{j,j} + \hat{\varrho}_j(\theta)e^{i\theta}) = 0, \text{ with } \nu(a_{j,j} + (\hat{\varrho}_j(\theta) + \varepsilon)e^{i\theta}) < 0 \tag{3.3}$$

for all sufficiently small $\varepsilon > 0$. By definition, we then have that

$$a_{j,j} + \hat{\varrho}_j(\theta)e^{i\theta} \text{ is a boundary point of } \Gamma^{\mathcal{R}}(A). \tag{3.4}$$

This means, from the min-max conditions (2.8)-(2.9), that there is an $\mathbf{x} > \mathbf{0}$, in $\mathbb{R}^n$, such that (cf. (2.9))

$$Q(a_{j,j} + \hat{\varrho}_j(\theta)e^{i\theta})\mathbf{x} = \mathbf{0}, \text{ where } \mathbf{x} = [x_1, x_2, ..., x_n]^T > \mathbf{0}. \tag{3.5}$$

Equivalently, on calling $a_{j,j} + \hat{\varrho}_j(\theta)e^{i\theta} =: z_j(\theta)$, we can express (3.5), using the definition of (2.1), as

$$|z_j(\theta) - a_{i,i}| = \sum_{k \in N \backslash \{i\}} |a_{i,k}| x_k / x_i, \quad (\text{all } i \in N), \tag{3.6}$$

which can be interpreted, from (2.16), simply as Olga Taussky's boundary result. What is perhaps more interesting is that it is geometrically *unnecessary* now to determine the components of the vector $\mathbf{x} > \mathbf{0}$ in $\mathbb{R}^n$, for which (3.6) is valid. This follows since knowing the boundary point $z_j(\theta)$ of $\Gamma^{\mathcal{R}}(A)$, and knowing each of the centers, $\{a_{i,i}\}_{i \in N}$, of the associated n Geršgorin disks, then all the circles of (3.6) can be directly drawn, without knowing the components of the vector $\mathbf{x}$.

We return to the numerical estimation of $\hat{\varrho}_j(\theta)$, which satisfies (3.3)-(3.5). Setting $z := a_{j,j}$ and $z' := a_{j,j} + \hat{\varrho}_j(\theta)e^{i\theta}$, we know from (2.18) that

$$\hat{\varrho}_j(\theta) \geq \nu(a_{j,j}) > 0. \tag{3.7}$$

Consider then the number $\nu(a_{j,j} + \nu(a_{j,j})e^{i\theta})$. If this number is positive, then increase the number $\nu(a_{j,j})$ to $\nu(a_{j,j}) + \Delta$, $\Delta > 0$, until $\nu(a_{j,j} + (\nu(a_{j,j}) + \Delta)e^{i\theta})$ is negative, and apply a bisection search to the interval $[\nu(a_{j,j}), \nu(a_{j,j}) + \Delta]$ to determine $\hat{\varrho}_j(\theta)$, satisfying (3.3). (Again, as in the estimation of $\nu(a_{j,j})$, estimates of $\hat{\varrho}_j(\theta)$ do not need great accuracy for graphing purposes.) We remark that a similar bisection search, on z, can be directly applied to

$$\det Q\big(\nu(a_{j,j} + \hat{\varrho}_j(\theta)e^{i\theta})\big) = 0, \tag{3.8}$$

as a consequence of (2.11) and (2.15), but this requires, however, the evaluation of an $n \times n$ determinant.

To summarize, given an irreducible matrix $A = [a_{i,j}]$ in $\mathbb{C}^{n,n}$, our procedure for approximating its minimal Geršgorin set, $\Gamma^{\mathcal{R}}(A)$, is to first determine, with reasonable accuracy, the positive numbers $\{\nu(a_{j,j})\}_{j\in N}$, and then, again with reasonable accuracy, to determine a few boundary points $\{\omega_k\}_{k=1}^m$ of $\Gamma^{\mathcal{R}}(A)$. For each such boundary point ω_k of $\Gamma^{\mathcal{R}}(A)$, there is an associated Geršgorin set, consisting of the union of the n Geršgorin disks, namely,

$$\Gamma^{\omega_k}(A) := \bigcup_{i\in N} \{z \in \mathbb{C} : |z - a_{i,i}| \leq |\omega_k - a_{i,i}|\}, \tag{3.9}$$

and their intersection,

$$\bigcap_{k=1}^{m} \Gamma^{\omega_k}(A), \tag{3.10}$$

gives an approximation to $\Gamma^{\mathcal{R}}(A)$, for which $\Gamma^{\mathcal{R}}(A)$ is a *subset*, and for which m points, of the boundary of $\bigcap_{k=1}^m \Gamma^{\omega_k}(A)$, are *boundary points* of $\Gamma^{\mathcal{R}}(A)$.

4. An easy example. Consider the irreducible 3×3 matrix

$$C = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \tag{4.1}$$

whose minimal Geršgorin set, $\Gamma^{\mathcal{R}}(C)$, is shown with the inner blue boundary in Figure 4.1. (This minimal Geršgorin set, $\Gamma^{\mathcal{R}}(C)$, also appears as the set with boundary (1) (2) (3) of [6, Figure 4.4].) For the vector $\mathbf{x}_0 = [1, 1, 1]^T \in \mathbb{R}^3$, the associated Geršgorin set $\Gamma^{r^{\mathbf{x}_0}}(C)$, turns out to be simply

$$\Gamma^{r^{\mathbf{x}_0}}(C) = \{z \in \mathbb{C} : |z - 2| \leq 2\}. \tag{4.2}$$

The boundary of this set is the (outer) *black circle* in Figure 4.1.

Next, starting with the diagonal entry, $z = 2$, of the matrix C, we estimate $\nu(2)$, which is positive from (2.12). As $\mu(2) = 1$ from (2.2), the associated nonnegative irreducible matrix $B(2)$ from (2.3) is

$$B(2) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

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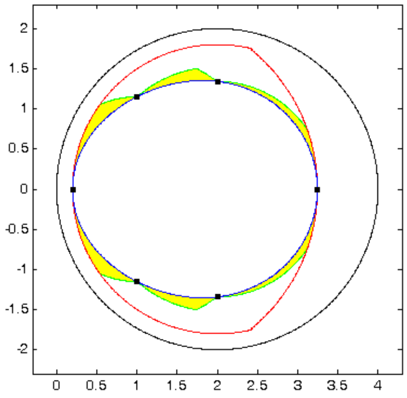

FIG. 4.1.

and a few power method iterations (see (3.1)-(3.2)), starting with $\mathbf{x}_0 = [2, 1, 2]^T$, gives that $\rho(B(2)) \doteq 2.2$. More precisely[1], $\rho(B(2)) = 2.24697$, so that from (2.7) we have $\nu(2) = 1.24697$.

Next, we search on the ray $2 + t$, with $t \geq 0$, for the largest value $\hat{t}$ for which $\nu(2 + \hat{t}) = 0$ and $\nu(2 + t) \geq 0$ for all $0 \leq t \leq \hat{t}$. Using the inequality of (2.18), it follows that $\hat{t} \geq \nu(2) = 1.24697$. However, in this particular case, it happens that $\hat{t} = 1.24697$, so that $z_1 = 3.24697$ is such that $\nu(z_1) = 0$, with $z_1 \in \partial\Gamma^{\mathcal{R}}(C)$. Similarly, on considering the diagonal entry $1 = c_{2,2}$, we approximate $\nu(1)$, which turns out to be $\nu(1) = 0.80194$, and then searching on the ray $1 - t$, $t \geq 0$, we similarly obtain $\nu(z_2) = 0$ with $z_2 = 0.19806$, and with $z_2 \in \partial\Gamma^{\mathcal{R}}(C)$. Calling $\Gamma^{r_1^{\mathbf{x}}}(C)$ and $\Gamma^{r_2^{\mathbf{x}}}(C)$ the associated Geršgorin sets, then the intersection of the three sets, $\bigcap_{j=0}^{2} \Gamma^{r_j^{\mathbf{x}}}(C)$, is shown in Figure 4.1 with the *red boundary*, where the boundary of the minimal Geršgorin set, $\Gamma^{\mathcal{R}}(C)$, is shown in *blue*.

We see from Figure 4.1 that the set with the red boundary is a set in the complex plane which contains $\Gamma^{\mathcal{R}}(C)$ and has two real boundary points, shown as the black squares z_1 and z_2, in common with $\Gamma^{\mathcal{R}}(C)$. Continuing, knowing $\nu(a_{1,1} = a_{3,3} = 2) = 1.24697$ and $\nu(a_{2,2} = 1) = 0.80194$, we then look for four additional points of $\partial\Gamma^{\mathcal{R}}(C)$ which are found on the four rays: $2 \pm it$, $t \geq 0$, and $1 \pm it$, $t \geq 0$. This gives us the following four points $\{z_j\}_{j=3}^{6}$ of $\Gamma^{\mathcal{R}}(C)$:

$$z_3 = 1 + i(1.150963),\ z_4 = \overline{z_3},\ z_5 = 2 + i(1.34236),\ z_6 = \overline{z_5}.$$

The intersection now of the above associated six Geršgorin sets is shown in Figure 4.1 with the *green* boundary, which includes $\Gamma^{\mathcal{R}}(C)$ and has six boundary points in common with $\partial\Gamma^{\mathcal{R}}(C)$, shown as solid black squares. The region between the green boundary of $\Gamma^{\mathcal{R}}(C)$ and its blue boundary is colored in *yellow*, which can be seen as small "roofs" composed of segments of circular arcs.

The amount of numerical calculation to obtain a good approximation to $\Gamma^{\mathcal{R}}(C)$ is moderate. It is further evident that *better* approximations to $\Gamma^{\mathcal{R}}(C)$, having more points in common with $\partial\Gamma^{\mathcal{R}}(C)$, can be similarly constructed.

5. Comparisons with Brualdi sets. Given an irreducible matrix $A = [a_{ij}]$ in $\mathbb{C}^{n,n}$, $n \geq 2$, one can similarly associate with A a minimal Brauer set, $\mathcal{K}^{\mathcal{R}}(A)$, as well as a minimal

[1] All such numbers are truncated after five decimal digits.

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Brualdi set $\mathcal{B}^{\mathcal{R}}(A)$, as described in [6, Section 4.3]. However, it is known (see [6, Theorem 4.15]) that all of these sets are equal, i.e.,

$$\Gamma^{\mathcal{R}}(A) = \mathcal{K}^{\mathcal{R}}(A) = \mathcal{B}^{\mathcal{R}}(A), \tag{5.1}$$

but the approximation of, say, the minimal Brualdi set $\mathcal{B}^{\mathcal{R}}(A)$, would now differ from our approximations of the minimal Geršgorin set, $\Gamma^{\mathcal{R}}(A)$, described earlier in this paper. For matrices having a very *large* number of nonzero off-diagonal entries, it is *unlikely* (see [6, Section 2.3]) that a similar numerical approximation of the minimal Brualdi set, $\mathcal{B}^{\mathcal{R}}(A)$, which from (5.1) equals $\Gamma^{\mathcal{R}}(A)$, would be numerically *competitive* with our numerical approach of Section 3 for approximating $\Gamma^{\mathcal{R}}(A)$. But, in the case of the matrix C of (4.1), there are just two associated Brualdi cycles, $\gamma_1 = (13)$ and $\gamma_2 = (23)$, for this matrix C, so that the approximation of $\Gamma^{\mathcal{R}}(C)$, via Brualdi sets, in this case, is easy. In particular, for any $\mathbf{x} = [x_1, x_2, x_3]^T > 0$ in $\mathbb{R}^3$, its associated *Brualdi lemniscates* (cf. [6, eq. (4.78)]) are

$$\mathcal{B}^{r^{\mathbf{x}}}_{\gamma_1}(C) = \{z \in \mathbb{C} : |z-2|^2 \le r_1^{\mathbf{x}}(C) \cdot r_2^{\mathbf{x}}(C) = (\frac{x_3}{x_1}) \cdot (\frac{x_1+x_2}{x_3}) = \frac{x_1+x_2}{x_1}\}, \tag{5.2}$$

and

$$\mathcal{B}^{r^{\mathbf{x}}}_{\gamma_2}(C) = \{z \in \mathbb{C} : |z-1||z-2| \le (\frac{x_3}{x_2}) \cdot (\frac{x_1+x_2}{x_3}) = \frac{x_1+x_2}{x_2}\}, \tag{5.3}$$

so that its associated Brualdi set is (cf. [6, eq. (2.40)])

$$\mathcal{B}^{r^{\mathbf{x}}}(C) = \mathcal{B}^{r^{\mathbf{x}}}_{\gamma_1}(C) \bigcup \mathcal{B}^{r^{\mathbf{x}}}_{\gamma_2}(C). \tag{5.4}$$

Now, knowing that $z_1 = 3.24697$ is a boundary point of $\Gamma^{\mathcal{R}}(C)$, we determine $\mathbf{x}_1 > 0$ and $\mathbf{x}_2 > 0$ so that $z_1 = 3.24697$ is a boundary point of $\mathcal{B}^{r^{\mathbf{x}_1}}(C)$. For this particular point $z_1 = 3.24697$, the associated Brualdi set, consisting of the union of two Brualdi lemniscate sets, is such that the boundary of *each* Brualdi lemniscate passes through z_1. (This is exactly the analog of Olga Taussky Theorem in the Geršgorin case; see [1] and [6, Theorem 2.8].) The union of these two Brualdi lemniscate sets can be verified to reduce to

$$\mathcal{B}^{r^{\mathbf{x}_1}}(C) = \{z \in \mathbb{C} : |z-1| \cdot |z-2| \le 2.80193\}.$$

Similarly, for the point $z_2 = 0.19806$, the associated Brualdi set has its two lemniscate sets passing through z_2, and the union of these two Brualdi lemniscate sets can be verified to reduce to the disk

$$\mathcal{B}^{r^{\mathbf{x}_2}}(C) = \{z \in \mathbb{C} : |z-2| \le 1.80193\}.$$

The boundary of the intersection $\mathcal{B}^{r^{\mathbf{x}_1}}_{\gamma_1}(C) \bigcap \mathcal{B}^{r^{\mathbf{x}_2}}_{\gamma_2}(C)$ is shown in Figure 5.1 with the *green* boundary. Also shown in Figure 5.1, with the *red* boundary, is the related Geršgorin set from Figure 4.1, which also has z_1 and z_2 as common points with the minimal Geršgorin set $\Gamma^{\mathcal{R}}(C)$.

From Figure 5.1, we see that $\mathcal{B}^{r^{\mathbf{x}_1}}_{\gamma_1}(C) \bigcap \mathcal{B}^{r^{\mathbf{x}_2}}_{\gamma_2}(C)$ is a *proper subset* of the related Geršgorin set, where the difference between these sets is shown in *yellow*. This is not unexpected, as it is known (cf. [6, eq. (4.80)]) that, for any matrix A in $\mathbb{C}^{n,n}$,

$$\mathcal{B}^{r^{\mathbf{x}}}(A) \subseteq \Gamma^{r^{\mathbf{x}}}(A), \quad \text{for any } \mathbf{x} > \mathbf{0} \text{ in } \mathbb{R}^n.$$

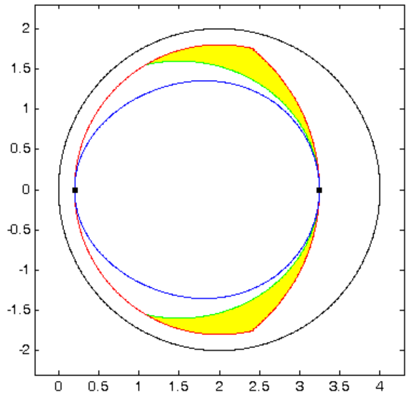

FIG. 5.1.

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