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REGULARIZATION PROPERTIES OF TIKHONOV REGULARIZATION WITH SPARSITY CONSTRAINTS*

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Abstract. In this paper, we investigate the regularization properties of Tikhonov regularization with a sparsity (or Besov) penalty for the inversion of nonlinear operator equations. We propose an a posteriori parameter choice rule that ensures convergence in the used norm as the data error goes to zero. We show that the method of surrogate functionals will at least reconstruct a critical point of the Tikhonov functional. Finally, we present some numerical results for a nonlinear Hammerstein equation.

Key words. inverse problems, sparsity

AMS subject classifications. 65J15, 65J20, 65J22

1. Introduction.

This paper is concerned with the stable computation of a solution of a nonlinear operator equation,

$$T(x) = y, \tag{1.1}$$

from noisy data y^δ and known data error bound $\|y^\delta - y\| \le \delta$. The operator T is assumed to be ill-posed, i.e., the solution of (1.1) does not depend continuously on the data. Thus, in the presence of noise, we have to use stabilizing algorithms for the inversion. These so-called regularization methods have been studied for a long time. For a good review, we refer to [12] and [19]. For nonlinear problems, the focus has been in particular on Tikhonov regularization with quadratic Hilbert space penalty [11, 22, 29, 30] and iterative methods such as the Landweber method [15, 21], the Gauss-Newton method [1, 3, 16], Newton's method [7, 14, 17] and the Levenberg-Marquardt method [13]. Although it seems at a first glance that Tikhonov regularization works under less restrictive conditions on the operator than most of the iterative methods, it still has the main drawback that a minimizer of the Tikhonov functional has to be computed, which requires an appropriate optimization method. These methods, however, will work under certain conditions on the operator only; see, e.g., [23, 24, 25]. All the methods mentioned have in common that they work in a Hilbert space setting only. Although in many applications a choice of the Hilbert space L_2 seems reasonable for the image space, this is not a priori the case for the domain of the operator T: In many applications, the operators T, $T'(x)$ and $T'(x)^*$ are represented by smoothing integral operators. Most of the standard regularization methods, e.g., Tikhonov regularization or Landweber iteration, will reconstruct an approximation in $R(T'(x)^*)$. If we assume $T'(x)^* : L_2(D) \to H^s(D)$ for all x, $s > 1/2$, $D \subset \mathbb{R}$, then the reconstructions will always be at least continuous, even if the solution of (1.1) has jumps. This well-known effect of over-smoothing of, e.g., Tikhonov regularization with quadratic Hilbert space penalty term, results in a good recovery of smooth functions, but fails if the solution has discontinuities or is spatially inhomogeneous; see [2] for an extensive discussion of these topics. A way to overcome these difficulties is by using suitable bases for the reconstruction. For instance, wavelet bases provide good localization in time and space, which makes them suitable for the reconstruction of functions with spatially inhomogeneous smoothness properties. Closely related to wavelet bases are Besov spaces, $B^s_{p,q}$, as the Besov norm of a function can be expressed equivalently via its coefficients with respect to a suitable wavelet basis; see also Section 2. The parameter s characterizes the

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smoothness of the function which is measured with respect to the norm in L_p. It turns out that functions that are smooth in principle but have few non-smooth parts, e.g., jumps, can still belong to a Besov space with high smoothness, although they might not belong to any Sobolev space H^s with $s > 1/2$; see, e.g., [6] for an example or [28] for general embedding results. Thus, when trying to reconstruct a solution of an operator equation that is mostly smooth, it is promising to stabilize the reconstruction by requiring it to belong to a suitable Besov space instead of a Sobolev space. This can be done by using a Besov norm as penalty term in Tikhonov regularization, i.e., we obtain an approximation to a solution by minimizing the functional

$$J_\alpha(x) = \|y^\delta - T(x)\|_{L_2}^2 + 2\alpha\|x\|_{B^s_{p,q}}^{(p)} . \tag{1.2}$$

The exponent (p) of the Besov norm indicates that we are considering both the penalties $\|x\|_{B^s_{p,q}}$ and $\|x\|^p_{B^s_{p,q}}$. If $p = q$, then the Besov norm of a function can be evaluated as a sum of its weighted wavelet coefficients $|\langle x, \Phi_\lambda\rangle|^p$, $\lambda \in \Lambda$, where $\{\Phi_\lambda\}_{\lambda\in\Lambda}$ is a suitable orthonormal wavelet basis and Λ denotes an index set. For fixed weights and $1 \le p < 2$ this means that we are putting a weaker penalty on functions with few but large coefficients, and a stronger penalty on functions with many but small coefficients. This effect is sharpened if p decreases to 1 and leads to the well known effect that a penalty with Besov norm and $p = q < 2$ will promote a sparse reconstruction with respect to the underlying (wavelet) basis. Thus, these types of penalties are also useful if it is known that the desired solution has a sparse representation in a given wavelet basis. This will often be the case if we want to reconstruct a compactly supported function with reasonably small support and use also a wavelet basis with compact support. For instance, in [27] we presented a reconstruction of a typical activity function in Single Photon Emission Computed Tomography, which had more than 90% zeros in the wavelet expansion. This sparsity pattern in the solution can only be recovered when using a sparsity penalty.

Sparsity penalties have been used by Donoho to achieve superresolution in imaging [8], and denoising of functions by wavelet thresholding is now a standard procedure in signal processing [9]. The idea of denoising a signal by using Tikhonov regularization (with identity as operator) and a Besov penalty was first emphasized by Chambolle et al. [5]. This approach was extended by Lorenz [18], who considered the denoising problem with penalties $\|\cdot\|^p_{B^s_{p,p}}$ and $\|\cdot\|_{B^s_{p,p}}$. Closely connected to the paper at hand is the article by Daubechies et al. [6]. They were the first to investigate (1.2) with a $linear$ operator T and penalty $\|\cdot\|^p_{B^s_{p,p}}$. The minimization of (1.2) requires the solution of a system of coupled nonlinear equations, which is rather cumbersome. Therefore they replaced the original functional by a sequence of so called *surrogate functionals* that are much easier to minimize, and showed that the associated sequence of minimizers converges toward the minimizer of (1.2). As the approximation quality of the minimizer x^δ_α depends on the noise as well as on the chosen regularization parameter α, they proposed a parameter choice rule $\alpha = \alpha(\delta)$ that ensures $x^\delta_{\alpha(\delta)} \to x^\dagger$ with $Tx^* = y$ as $\delta \to 0$.

Based on [6], we investigated in [27] the use of the functional (1.2) with $nonlinear$ operator T and one-homogeneous penalty, i.e., a penalty functional fulfilling $\Psi(a\cdot \boldsymbol{x}) = |a|\Psi(\boldsymbol{x})$ for $a \in \mathbb{R}$. In particular, we were using the functional

$$\Psi_{p,\beta}(\boldsymbol{x}) := \left(\sum_{\lambda\in\Lambda} \beta_\lambda |x_\lambda|^p\right)^{1/p} ,$$

where $\boldsymbol{x} = (x_\lambda)_{\lambda\in\Lambda}$ and $x_\lambda = \langle x, \Phi_\lambda\rangle$ denote the coefficients of a function x with respect to a given basis or frame, and β_λ is a sequence of weights bounded from below. This approach

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covers in particular the penalties $\|\cdot\|_{B^s_{p,p}}$ (but *not* $\|\cdot\|^p_{B^s_{p,p}}$). As in [6], we also used an iterative algorithm based on surrogate functionals for the minimization of the Tikhonov functional with the sparsity penalty, and proved the convergence of the iterates at least to a critical point. Due to the nonlinearity of T, the convergence results are weaker than in the linear case. For the penalty $\|\cdot\|_{B^1_{1,1}}$ we also provided an a priori parameter rule that ensures a convergence of the regularized solutions $x^\delta_{\alpha(\delta)}$ to a solution $x^\dagger$ of the equation with respect to the underlying norm. The minimization of (1.2) with a nonlinear operator was also considered by Bredies et al. [4]. They used a conditional gradient method, which turns out to be closely related to the surrogate functional method.

The one-homogenous penalty term $\Psi_{p,\beta}(\boldsymbol{x})$ was chosen because it usually penalizes nonsparse reconstructions stronger than the penalty $\Psi_{p,\beta}(\boldsymbol{x})^p$, and thus a sparser reconstruction can be expected. However, the use of the penalty $\Psi_{p,\beta}(\boldsymbol{x})$ has a severe drawback. The iterative minimization of (1.2) with a general one-homogeneous sparsity constraint by the surrogate functional approach requires the evaluation of a projection $P_{\mathcal{C}}$ on a convex set $\mathcal{C}$ within each iteration step [27]. The set $\mathcal{C}$ depends on the chosen penalty term. Unfortunately, in case of $\|\cdot\|_{B^s_{p,p}}$, the evaluation of the projection is, so far, explicitly known only for $p \in \{1, 2, \infty\}$. Although this includes the important case of an $B^1_{1,1}$ penalty, it also means that the method cannot be used for $1 < p < 2$ in practice. In this paper, we want to overcome this difficulty by abandoning the one-homogeneity of the penalty term, and consider the penalty $\Psi_{p,\beta}(\boldsymbol{x})^p$ instead. One hope is that the minimizers for Tikhonov regularization with penalty term $\Psi_{p,\beta}(\boldsymbol{x})^p$ are not that far away from the minimizers of Tikhonov regularization with penalty $\Psi_{p,\beta}(\boldsymbol{x})$, and that they have similar sparsity patterns. The main advantage of the new penalty is that the iteration for minimizing the associated surrogate functional can be evaluated for any $1 \le p \le 2$ and thus we are able to compute all the associated minimizers. This is important in particular if the solution is known to belong to $B^s_{p,p}$ with $1 < p < 2$, in which case it is advisable to choose the associated Besov norm as penalty. Also, numerical tests have shown that the computational effort for the reconstruction of a minimizer of the Tikhonov functional depends in particular on the chosen parameter p in the Besov norm. For $p = 1$, we observed a very slow convergence speed, which increases with p. Choosing a penalty with $1 < p < 2$ might thus allow us to trade degree of sparseness of the reconstruction for convergence speed.

This paper is organized as follows. In Section 2 we will summarize some basic facts on frames, wavelets and their connection to Besov spaces and we will introduce some assumptions on the nonlinear operator T necessary for the analysis of the proposed algorithms. Since the choice of the regularization parameter is important for the accuracy of the reconstructions, we will propose a parameter choice rule $\alpha = \alpha(\delta)$ in Section 3 that guarantees $x^\delta_{\alpha(\delta)} \to x^\dagger$ as $\delta \to 0$ for both penalties $\Psi_{p,\beta}(\boldsymbol{x}),\ \Psi_{p,\beta}(\boldsymbol{x})^p$, which includes the Besov penalties $\|\cdot\|_{B^s_{p,p}}$ and $\|\cdot\|^p_{B^s_{p,p}}$ for all $1 \le p \le 2$ and $s \ge 0$. To our knowledge, such a result is currently only available for $p = s = 1$. Since the convergence properties of the surrogate functional algorithm with penalty $\Psi_{p,\beta}(\boldsymbol{x})^p$ and nonlinear operator have not yet been analyzed, we will provide convergence results in Section 4. In particular, it will be shown that the iterates of the algorithm converge at least to a critical point of the functional. Finally, in Section 5 we present a simple numerical example in order to illustrate our theoretical results.

2. Wavelets, Besov spaces, and general assumptions on T.

Since our penalty terms work on sequences, a given function x has to be transformed into a sequence $\boldsymbol{x}$ that represents the function with respect to a given basis or frame. In this paper, we will use wavelet expansions, mainly due to their connections to Besov spaces. For a given wavelet system $\{\varphi, \psi\}$, every function $x \in L_2$ can be expanded as

$$x = \sum_{k\in\mathbb{Z}} \langle x, \varphi_{0k}\rangle \varphi_{0k} + \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}} \langle x, \psi_{jk}\rangle \psi_{jk} =: \sum_{\lambda\in\Lambda} \langle x, \Phi_\lambda\rangle \Phi_\lambda \ ,$$

where $\varphi_{0k}(t) = \varphi(t-k)$ and $\psi_{jk}(t) = 2^{j/2}\psi(2^j t - k)$ and Λ is an appropriately chosen index set.The operator

$$F : X \to \ell_2 \ , \qquad\qquad Fx = \boldsymbol{x} = \{\langle x, \Phi_\lambda\rangle\}_{\lambda\in\Lambda}$$

and its adjoint

$$F^*\boldsymbol{x} : \ell_2 \to X \ , \qquad\qquad F^*\boldsymbol{x} = \sum_{\lambda\in\Lambda} x_\lambda \Phi_\lambda$$

allow us to switch between functions and coefficients. As for all frames and bases we have an estimate

$$AI \le F^*F \le BI \tag{2.1}$$

with some constants $0 < A \le B$. Particularly important for us is the connection between wavelets and Besov norms $\|\cdot\|_{B^s_{p,p}}$: for a sufficiently smooth wavelet, we have

$$\|x\|^p_{B^s_{p,p}} \equiv \sum_{k\in\mathbb{Z}} |\langle x, \varphi_{0k}\rangle|^p + \sum_{j=0}^{\infty} 2^{j\sigma p} \sum_{k\in\mathbb{Z}} |\langle x, \psi_{jk}\rangle|^p \tag{2.2}$$

with $\sigma = s + d(1/2 - 1/p)$ in $\mathbb{R}^d$. We remark that for functions with compact support the double sum reduces to a simple one, since $\langle x, \psi_{jk}\rangle = 0$ for $|k|$ large enough. The norm in $B^s_{p,p}$ can therefore easily be expressed by the functional

$$\Psi_{p,\beta}(\boldsymbol{x}) := \left(\sum_{j=1}^{\infty} \beta_j |x_j|^p\right)^{1/p}$$

with

$$\beta_j \ge 1 \text{ for all } j \tag{2.3}$$

and x_j chosen so that $\Psi_{p,\beta}(\boldsymbol{x}) = \|x\|_{B^s_{p,p}}$.

If $\beta_j = 1$ holds for all j, then we will skip the subscript β and denote the functional by Ψ_p only. Clearly, as weighted ℓ_p-norms, the functionals $\Psi_{p,\beta}$ are weakly lower semi-continuous.

We will investigate the Tikhonov functionals

$$J_\alpha(\boldsymbol{x}) = \|y^\delta - T(F^*\boldsymbol{x})\|^2 + 2\alpha\Psi_{p,\beta}(\boldsymbol{x}) \ , \tag{2.4}$$

and

$$J_\alpha(\boldsymbol{x}) = \|y^\delta - T(F^*\boldsymbol{x})\|^2 + 2\alpha\Psi_{p,\beta}(\boldsymbol{x})^p \ . \tag{2.5}$$

As we are considering nonlinear operator equations, we cannot expect to get analytical results concerning regularization or optimization without additional conditions on the operator. In particular, we require the nonlinear operator T to meet the following conditions:

$$\boldsymbol{x}_k \overset{w}{\to} \boldsymbol{x} \Longrightarrow T(F^*\boldsymbol{x}_k) \to T(F^*\boldsymbol{x}) \ \ , \tag{2.6}$$

$$T'(F^*\boldsymbol{x}_k)^* z \to T'(F^*\boldsymbol{x})^* z \ \ , \text{ for all } z \ \ , \tag{2.7}$$

$$\|T'(x) - T'(x')\| \le L\|x - x'\| \ \ . \tag{2.8}$$

Lipschitz continuity of the derivative, i.e., (2.8), is a standard assumption for nonlinear inverse problems. It was already used for the analysis of Tikhonov regularization in a Hilbert space setting [11]. Combining (2.8) and (2.1) we get for the associated sequences $\boldsymbol{x} = Fx$, $\tilde{\boldsymbol{x}} = F\tilde{x}$,

$$\|T'(F^*\boldsymbol{x}) - T'(F^*\tilde{\boldsymbol{x}})\| \leq L\|F^*\boldsymbol{x} - F^*\tilde{\boldsymbol{x}}\| = L\|F^*F\boldsymbol{x} - F^*F\tilde{\boldsymbol{x}}\| \overset{(2.1)}{\leq} LB^{1/2}\|\boldsymbol{x} - \tilde{\boldsymbol{x}}\| . \tag{2.9}$$

If the operator $T : X \to Y$ does not fulfill (2.6), (2.7), then these properties hold in many cases if the setting is changed, e.g., by assuming more regularity of the solution of our equation. Assuming that there exists a function space X^s, and a compact embedding operator $i^s : X^s \to X$, we can consider $\tilde{T} = T \circ i^s : X^s \longrightarrow Y$. Lipschitz regularity of the derivative will be preserved, and one can show that the operator $\tilde{T}$ meets the conditions (2.6)-(2.9); see also [22, Sections 2.1 and 3]. In practical applications, X^s might be a Sobolev space H^{s+t} on a bounded domain with $s > 0$, and $X = H^t$.

3. A regularization result. The quality of reconstruction of regularization methods depends on a proper choice of the regularization parameter α in dependance of the data error. In this section we develop a parameter choice rule that guarantees convergence (with respect to the chosen penalty) of the regularized solution $\boldsymbol{x}_\alpha^\delta$ to the solution $\boldsymbol{x}^\dagger$ of equation (1.1) as $\delta \to 0$. We will propose an a priori parameter choice rule, that is, the regularization parameter can be obtained in dependence of the noise level only. The following two auxiliary result will be needed for the proof of Theorems 3.3 and 3.4:

LEMMA 3.1. *Let $1 \leq p \leq 2$. Then we have for a sequence $\boldsymbol{x} \in \ell_2$*

$$\|\boldsymbol{x}\|_{\ell_2} \leq \kappa_p \Psi_{p,\beta}(\boldsymbol{x}) . \tag{3.1}$$

with some constant $\kappa_p > 0$.

Proof. For the space of sequences we have $\ell_p \subset \ell_q$ for $1 \leq p \leq q < \infty$ and $\|\boldsymbol{x}\|_{\ell_q} \leq \kappa_p \|\boldsymbol{x}\|_{\ell_p}$. Consequently we have for $1 \leq p \leq 2$ and $\beta_\lambda \geq 1$

$$\|\boldsymbol{x}\|_{\ell_2} \leq \kappa_p \|\boldsymbol{x}\|_{\ell_p} = \kappa_p \left(\sum_{\lambda \in \Lambda} |\boldsymbol{x}_\lambda|^p \right)^{1/p} \overset{(2.3)}{\leq} \kappa_p \left(\sum_{\lambda \in \Lambda} \beta_\lambda |\boldsymbol{x}_\lambda|^p \right)^{1/p} = \kappa_p \Psi_{p,\beta}(\boldsymbol{x}). \quad \square$$

LEMMA 3.2. *Let $1 \leq p \leq 2$ and $a, b \in \mathbb{R}$ with $a > 0$ and $a + b \geq 0$. Then we have*

$$(a + b)^p \leq C_p(a^p + \operatorname{sgn}(b)|b|^p) . \tag{3.2}$$

Proof. Let us first consider the case $a, b \geq 0$. If $a \leq b$, then $(a + b)^p \leq 2^p b^p \leq 2^p(b^p + a^p)$. The same argument applies for $0 \leq b \leq a$, and thus $(a + b)^p \leq 2^p(a^p + b^p)$. Now let us assume $a > 0$, $b \leq 0$. Since $a + b \geq 0$ we obtain $a \geq |b|$. It follows that

$$\begin{aligned}
(a + b)^p &= (a + \operatorname{sgn}(b)|b|)^p = a^p \left(1 - \underbrace{\frac{|b|}{a}}_{\leq 1} \right)^p \\
&\overset{p \geq 1}{\leq} a^p \left(1 - \frac{|b|}{a} \right) \\
&\leq a^p \left(1 - \left(\frac{|b|}{a} \right)^p \right) \\
&= a^p - |b|^p = a^p + \operatorname{sgn}(b)|b|^p . \quad \square
\end{aligned}$$

For the following regularization result, we will extend the functionals (2.4), (2.5) in order to find a reconstruction closest to an a priori guess $\bar{\boldsymbol{x}}$ for the solution, which is done by replacing the penalty $\Psi_{p,\beta}(\boldsymbol{x})$ by $\Psi_{p,\beta}(\boldsymbol{x}-\bar{\boldsymbol{x}})$:

THEOREM 3.3. *Let $y^\delta \in Y$ satisfy $\|y^\delta - y\| \le \delta$ and let $\alpha(\delta)$ be chosen such that $\alpha(\delta) \to 0$ and $\delta^2/\alpha(\delta) \to 0$ as $\delta \to 0$. Assume there exists a solution of $T(F^*\boldsymbol{x}) = y$ with finite value $\Psi_{p,\beta}$, and the a priori guess $\bar{\boldsymbol{x}}$ is chosen with $\Psi_{p,\beta}(\bar{\boldsymbol{x}}) < \infty$. Then every sequence $\{\boldsymbol{x}_{\alpha_k}^{\delta_k}\}$ of minimizers of the functional $J_\alpha(\boldsymbol{x})$ with penalty $\Psi_{p,\beta}(\boldsymbol{x}-\bar{\boldsymbol{x}})$, defined in (2.4) where $\delta_k \to 0$ and $\alpha_k = \alpha(\delta_k)$ has a convergent subsequence. The limit of every convergent subsequence is a solution of $T(F^*\boldsymbol{x}) = y$ with minimal distance to $\bar{\boldsymbol{x}}$ with respect to $\Psi_{p,\beta}$. If, in addition, the solution $\boldsymbol{x}^\dagger$ with minimal distance to $\bar{\boldsymbol{x}}$ is unique, then we have*

$$\lim_{\delta\to 0} \boldsymbol{x}_{\alpha(\delta)}^\delta = \boldsymbol{x}^\dagger \ .$$

Proof. Let α_k and δ_k be as above , and $\boldsymbol{x}^\dagger$ be a solution of $T(F^*\boldsymbol{x})$ with minimal value of $\Psi_{p,\beta}(\boldsymbol{x}-\bar{\boldsymbol{x}})$. Since $\boldsymbol{x}_{\alpha_k}^{\delta_k}$ is a minimizer of J_{α_k}, we have

$$\|T(F^*\boldsymbol{x}_{\alpha_k}^{\delta_k}) - y^{\delta_k}\|^2 + 2\alpha_k\Psi_{p,\beta}(\boldsymbol{x}_{\alpha_k}^{\delta_k} - \bar{\boldsymbol{x}}) \le \delta_k^2 + 2\alpha_k\Psi_{p,\beta}(\boldsymbol{x}^\dagger - \bar{\boldsymbol{x}}) \ .$$

Hence $\|T(F^*\boldsymbol{x}_{\alpha_k}^{\delta_k}) - y^{\delta_k}\|^2 \le \delta_k^2 + 2\alpha_k\Psi_{p,\beta}(\boldsymbol{x}^\dagger - \bar{\boldsymbol{x}})$ and thus

$$\lim_{k\to\infty} T(F^*\boldsymbol{x}_{\alpha_k}^{\delta_k}) = y \ . \tag{3.3}$$

Moreover, we have $\Psi_{p,\beta}(\boldsymbol{x}_{\alpha_k}^{\delta_k} - \bar{\boldsymbol{x}}) \le \delta_k^2/\alpha_k(\delta_k) + \Psi_{p,\beta}(\boldsymbol{x}^\dagger - \bar{\boldsymbol{x}})$, which yields

$$\frac{1}{\kappa_p}\limsup_k \|\boldsymbol{x}_{\alpha_k}^{\delta_k} - \bar{\boldsymbol{x}}\|_{\ell_2} \overset{(3.1)}{\le} \limsup_k \Psi_{p,\beta}(\boldsymbol{x}_{\alpha_k}^{\delta_k} - \bar{\boldsymbol{x}}) \le \Psi_{p,\beta}(\boldsymbol{x}^\dagger - \bar{\boldsymbol{x}}) \ . \tag{3.4}$$

Thus, $\|\boldsymbol{x}_{\alpha_k}^{\delta_k}\|_{\ell_2}$ is bounded, and the sequence has a weakly convergent subsequence (in ℓ_2), again denoted by $\{\boldsymbol{x}_{\alpha_k}^{\delta_k}\}$,

$$\boldsymbol{x}_{\alpha_k}^{\delta_k} \rightharpoonup \boldsymbol{x}^* \ .$$

In particular, since T satisfies (2.6),

$$y \overset{(3.3)}{=} \lim_{k\to\infty} T(F^*\boldsymbol{x}_{\alpha_k}^{\delta_k}) = T(F^*\boldsymbol{x}^*) \ ,$$

and thus $\boldsymbol{x}^*$ is a solution of $T(F^*\boldsymbol{x}) = y$. Since $\Psi_{p,\beta}$ is weakly lower semi-continuous, we derive

$$\Psi_{p,\beta}(\boldsymbol{x}^* - \bar{\boldsymbol{x}}) \le \limsup \Psi_{p,\beta}(\boldsymbol{x}_{\alpha_k}^{\delta_k} - \bar{\boldsymbol{x}}) \overset{(3.4)}{\le} \Psi_{p,\beta}(\boldsymbol{x}^\dagger - \bar{\boldsymbol{x}}) \le \Psi_{p,\beta}(\boldsymbol{x}^* - \bar{\boldsymbol{x}}) \ . \tag{3.5}$$

The last inequality follows from the fact that $\boldsymbol{x}^\dagger$ is a solution with minimal value of $\Psi_{p,\beta}(\cdot)$. As a consequence, $\Psi_{p,\beta}(\boldsymbol{x}^* - \bar{\boldsymbol{x}}) = \Psi_{p,\beta}(\boldsymbol{x}^\dagger - \bar{\boldsymbol{x}})$, and $\boldsymbol{x}^*$ is also a solution with minimal $\Psi_{p,\beta}$-value.

Now we want to show $\Psi_{p,\beta}(\boldsymbol{x}_{\alpha_k}^{\delta_k} - \boldsymbol{x}^*) \to 0$ as $k \to \infty$. In order to do that, we need to rewrite the absolute value of a real number. Defining

$$\varphi(x,h) = \begin{cases} -\operatorname{sgn}(x)\cdot h & \text{if } 0 \neq \operatorname{sgn}(x) = \operatorname{sgn}(h) \text{ and } |x| > |h| \\ (\operatorname{sgn}(x)\cdot h - 2|x|) & \text{if } 0 \neq \operatorname{sgn}(x) = \operatorname{sgn}(h) \text{ and } |x| \le |h| \\ |h| & \text{if } 0 \neq \operatorname{sgn}(x) = -\operatorname{sgn}(h) \\ |h| & \text{if } x = 0\,. \end{cases} \tag{3.6}$$

We obtain

$$|x-h| = |x| + \varphi(x,h)\,. \tag{3.7}$$

Clearly, we have $|x| \geq 0$ and $|x| + \varphi(x,h) \geq 0$, and thus we get by (3.2)

$$(|x| + \varphi(x,h))^p \le C_p\left(|x|^p + \operatorname{sgn}(\varphi)|\varphi(x,h)|^p\right)\,. \tag{3.8}$$

Setting $x = (\boldsymbol{x}_{\alpha_k}^{\delta_k})_j$, $h = (\boldsymbol{x}^*)_j$ yields

$$\begin{aligned}
\Psi_{p,\beta}(\boldsymbol{x}_{\alpha_k}^{\delta_k} - \boldsymbol{x}^*)^p &= \sum_j \beta_j |(\boldsymbol{x}_{\alpha_k}^{\delta_k})_j - \boldsymbol{x}_j^*|^p \\
&\overset{(3.7)}{=} \sum_j \beta_j \left(|(\boldsymbol{x}_{\alpha_k}^{\delta_k})_j| + \varphi((\boldsymbol{x}_{\alpha_k}^{\delta_k})_j, \boldsymbol{x}_j^*)\right)^p \\
&\overset{(3.8)}{\le} C_p\left(\sum_j \beta_j |(\boldsymbol{x}_{\alpha_k}^{\delta_k})_j|^p + \sum_j \beta_j \operatorname{sgn}(\varphi)|\varphi((\boldsymbol{x}_{\alpha_k}^{\delta_k})_j, \boldsymbol{x}_j^*)|^p\right) \\
&= C_p\left(\Psi_{p,\beta}(\boldsymbol{x}_{\alpha_k}^{\delta_k})^p + \sum_j \beta_j \operatorname{sgn}(\varphi)|\varphi((\boldsymbol{x}_{\alpha_k}^{\delta_k})_j, \boldsymbol{x}_j^*)|^p\right) \\
&\overset{(3.5)}{\le} C_p\left(\Psi_{p,\beta}(\boldsymbol{x}^*) + \sum_j \beta_j \operatorname{sgn}(\varphi)|\varphi((\boldsymbol{x}_{\alpha_k}^{\delta_k})_j, \boldsymbol{x}_j^*)|^p\right)\,.
\end{aligned}$$

By the definition of $\varphi(x,h)$ in (3.6) follows

$$|\varphi(x,h)| = \begin{cases} |-\operatorname{sgn}(x)\cdot h| \le |h| & \text{if } 0 \neq \operatorname{sgn}(x) = \operatorname{sgn}(h) \text{ and } |x| > |h| \\ |\operatorname{sgn}(x)\cdot h - 2|x|| \le 3|h| & \text{if } 0 \neq \operatorname{sgn}(x) = \operatorname{sgn}(h) \text{ and } |x| \le |h| \\ |h| & \text{if } 0 \neq \operatorname{sgn}(x) = -\operatorname{sgn}(h) \\ |h| & \text{if } x = 0\,. \end{cases}$$

Therefore, we obtain

$$|\varphi(\boldsymbol{x}_j^*, \boldsymbol{x}_{\alpha_k\,j}^{\delta_k})|^p \le 3^p |\boldsymbol{x}_j^*|^p$$

and

$$\sum_j \beta_j |\varphi((\boldsymbol{x}_{\alpha_k}^{\delta_k})_j, \boldsymbol{x}_j^*)| \le 3^p \sum_j \beta_j |\boldsymbol{x}_j^*|^p = 3^p \Psi_{p,\beta}(\boldsymbol{x}^*)^p\,.$$

Thus $\sum_j 3^p \beta_j |\boldsymbol{x}_j^*|^p$ dominates $\sum_j \beta_j |\varphi((\boldsymbol{x}_{\alpha_k}^{\delta_k})_j, \boldsymbol{x}_j^*)|^p$, and we can interchange limit and sum,

$$\lim_{k\to\infty} \sum_j \beta_j \operatorname{sgn}(\varphi)|\varphi((\boldsymbol{x}_{\alpha_k}^{\delta_k})_j, \boldsymbol{x}_j^*)|^p = \sum_j \lim_{k\to\infty} \beta_j \operatorname{sgn}(\varphi)|\varphi((\boldsymbol{x}_{\alpha_k}^{\delta_k})_j, \boldsymbol{x}_j^*)|^p\,.$$

Since $\boldsymbol{x}_{\alpha_k}^{\delta_k} \xrightarrow{\ell_2} \boldsymbol{x}^*$, we have in particular $(\boldsymbol{x}_{\alpha_k}^{\delta_k})_j \to (\boldsymbol{x}^*)_j$ for $k \to \infty$, and thus $(\boldsymbol{x}_{\alpha_k}^{\delta_k})_j \to \boldsymbol{x}_j^*$ for $k \to \infty$. Now assume that $\boldsymbol{x}_j^* \neq 0$. Then there exists k_0 such that $(\boldsymbol{x}_{\alpha_k}^{\delta_k})_j \neq 0$ and $\operatorname{sgn}((\boldsymbol{x}_{\alpha_k}^{\delta_k})_j) = \operatorname{sgn}(\boldsymbol{x}_j^*)$ for all $k \geq k_0$. According to the definition (3.6) of φ, we have for $k \geq k_0$

$$\varphi((\boldsymbol{x}_{\alpha_k}^{\delta_k})_j, \boldsymbol{x}_j^*) = \begin{cases} -\operatorname{sgn}((\boldsymbol{x}_{\alpha_k}^{\delta_k})_j) \cdot \boldsymbol{x}_j^* = -|\boldsymbol{x}_j^*| \\ \\ \operatorname{sgn}((\boldsymbol{x}_{\alpha_k}^{\delta_k})_j) \cdot \boldsymbol{x}_j^* - 2|(\boldsymbol{x}_{\alpha_k}^{\delta_k})_j| = |\boldsymbol{x}_j^*| - 2|(\boldsymbol{x}_{\alpha_k}^{\delta_k})_j| \end{cases}$$

which gives

$$\lim_{k\to\infty} \varphi((\boldsymbol{x}_{\alpha_k}^{\delta_k})_j, \boldsymbol{x}_j^*) = -|\boldsymbol{x}_j^*| \,,$$

and

$$\lim_{k\to\infty} \operatorname{sgn}(\varphi)|\varphi((\boldsymbol{x}_{\alpha_k}^{\delta_k})_j, \boldsymbol{x}_j^*)|^p = -|\boldsymbol{x}_j^*|^p \,.$$

We wish to remark that we do not have to consider the case $\boldsymbol{x}_j^* = 0$, because $\varphi((\boldsymbol{x}_{\alpha_k}^{\delta_k})_j, \boldsymbol{x}_j^*) = 0$ holds then anyway, and we can exclude these indices in all sums. Consequently,

$$\begin{aligned} 0 &\leq \lim_{k\to\infty} \Psi_{p,\beta}(\boldsymbol{x}_{\alpha_k}^{\delta_k} - \boldsymbol{x}^*) \leq C_p \left(\Psi_{p,\beta}(\boldsymbol{x}^*) + \lim_{k\to\infty} \sum_j \beta_j \operatorname{sgn}(\varphi)|\varphi((\boldsymbol{x}_{\alpha_k}^{\delta_k})_j, \boldsymbol{x}_j^*)|^p \right) \\ &= C_p \left(\Psi_{p,\beta}(\boldsymbol{x}^*) + \sum_j \lim_{k\to\infty} \beta_j \operatorname{sgn}(\varphi)|\varphi((\boldsymbol{x}_{\alpha_k}^{\delta_k})_j, \boldsymbol{x}_j^*)|^p \right) \\ &= C_p \left(\Psi_{p,\beta}(\boldsymbol{x}^*) - \sum_j \beta_j |\boldsymbol{x}_j^*|^p \right) = 0 \,, \end{aligned}$$

which proves $\boldsymbol{x}_{\alpha_k}^{\delta_k} \to \boldsymbol{x}^*$ with respect to $\Psi_{p,\beta}$ and also with respect to ℓ_2. If $\boldsymbol{x}^*$ is unique, our assertion about the convergence of $\boldsymbol{x}_{\alpha(\delta)}^{\delta}$ follows by the convergence principles from the fact that every sequence has a convergent subsequence with the same limit $\boldsymbol{x}^\dagger$. ☐

The same techniques as above also can be applied in case of the penalty $\Psi_{p,\beta}(\boldsymbol{x} - \bar{\boldsymbol{x}})^p$:

THEOREM 3.4. *Let $y^\delta \in Y$ with $\|y^\delta - y\| \leq \delta$ and let $\alpha(\delta)$ be chosen with $\alpha(\delta) \to 0$ and $\delta^2/\alpha(\delta) \to 0$ as $\delta \to 0$. Assume there exists a solution of $T(F^*\boldsymbol{x}) = y$ with finite value of $\Psi_{p,\beta}$, and the a priori guess $\bar{\boldsymbol{x}}$ fulfills $\Psi_{p,\beta}(\bar{\boldsymbol{x}}) < \infty$ as well. Then every sequence $\{\boldsymbol{x}_{\alpha_k}^{\delta_k}\}$ of minimizers of the functional $J_\alpha(\boldsymbol{x})$ with penalty $\Psi_{p,\beta}(\boldsymbol{x} - \bar{\boldsymbol{x}})^p$, defined in (2.5) where $\delta_k \to 0$ and $\alpha_k = \alpha(\delta_k)$ has a convergent subsequence. The limit of every convergent subsequence is a solution of $T(F^*\boldsymbol{x}) = y$ with minimal value of $\Psi_{p,\beta}(\boldsymbol{x} - \bar{\boldsymbol{x}})$. If, in additition, the solution $\boldsymbol{x}^\dagger$ with minimal $\Psi_{p,\beta}(\boldsymbol{x} - \bar{\boldsymbol{x}})$ is unique, then we have*

$$\lim_{\delta\to 0} \boldsymbol{x}_{\alpha(\delta)}^{\delta} = \boldsymbol{x}^\dagger \,.$$

Proof. The proof is almost the same as above:

Let α_k and δ_k be as above , and $\boldsymbol{x}^\dagger$ a solution of $T(F^*\boldsymbol{x})$ with minimal value of $\Psi_{p,\beta}(\boldsymbol{x} - \bar{\boldsymbol{x}})$. Since $\boldsymbol{x}_{\alpha_k}^{\delta_k}$ is a minimizer of J_{α_k}, we have

$$\|T(F^*\boldsymbol{x}_{\alpha_k}^{\delta_k}) - y^{\delta_k}\|^2 + 2\alpha_k \Psi_{p,\beta}(\boldsymbol{x}_{\alpha_k}^{\delta_k} - \bar{\boldsymbol{x}})^p \leq \delta_k^2 + 2\alpha_k \Psi_{p,\beta}(\boldsymbol{x}^\dagger - \bar{\boldsymbol{x}})^p \,.$$

Hence we have $\|T(F^*\boldsymbol{x}_{\alpha_k}^{\delta_k}) - y^{\delta_k}\|^2 \le \delta_k^2 + 2\alpha_k \Psi_{p,\beta}(\boldsymbol{x}^\dagger - \bar{\boldsymbol{x}})^p$ and thus

$$\lim_{k\to\infty} T(F^*\boldsymbol{x}_{\alpha_k}^{\delta_k}) = y\ .$$

Moreover, we have $\Psi_{p,\beta}(\boldsymbol{x}_{\alpha_k}^{\delta_k} - \bar{\boldsymbol{x}}) \le \delta_k^2/\alpha_k(\delta_k) + \Psi_{p,\beta}(\boldsymbol{x}^\dagger - \bar{\boldsymbol{x}})^p$, which yields

$$\frac{1}{\kappa_p}\limsup_k \|\boldsymbol{x}_{\alpha_k}^{\delta_k} - \bar{\boldsymbol{x}}\|_{\ell_2}^p \overset{(3.1)}{\le} \limsup_k \Psi_{p,\beta}(\boldsymbol{x}_{\alpha_k}^{\delta_k} - \bar{\boldsymbol{x}})^p \le \Psi_{p,\beta}(\boldsymbol{x}^\dagger - \bar{\boldsymbol{x}})^p\ .$$

Thus, $\|\boldsymbol{x}_{\alpha_k}^{\delta_k}\|_{\ell_2}$ is bounded, and the sequence has a weakly convergent subsequence (in ℓ_2), again denoted by $\{\boldsymbol{x}_{\alpha_k}^{\delta_k}\}$, with

$$\boldsymbol{x}_{\alpha_k}^{\delta_k} \rightharpoonup \boldsymbol{x}^*\ .$$

In particular, as T satisfies (2.6),

$$y \overset{(3.3)}{=} \lim_{k\to\infty} T(F^*\boldsymbol{x}_{\alpha_k}^{\delta_k}) = T(F^*\boldsymbol{x}^*)\ ,$$

and thus $\boldsymbol{x}^*$ is a solution of $T(F^*\boldsymbol{x}) = y$. By assumption, $\Psi_{p,\beta}$ is weakly semi continuous, and thus we derive

$$\Psi_{p,\beta}(\boldsymbol{x}^* - \bar{\boldsymbol{x}})^p \le \limsup \Psi_{p,\beta}(\boldsymbol{x}_{\alpha_k}^{\delta_k} - \bar{\boldsymbol{x}})^p \overset{(3.4)}{\le} \Psi_{p,\beta}(\boldsymbol{x}^\dagger - \bar{\boldsymbol{x}})^p \le \Psi_{p,\beta}(\boldsymbol{x}^* - \bar{\boldsymbol{x}})^p\ .$$

The last inequality follows from the fact that $\boldsymbol{x}^\dagger$ is a solution with minimal value of $\Psi_{p,\beta}(\boldsymbol{x} - \bar{\boldsymbol{x}})$. As a consequence, $\Psi_{p,\beta}(\boldsymbol{x}^* - \bar{\boldsymbol{x}}) = \Psi_{p,\beta}(\boldsymbol{x}^\dagger - \bar{\boldsymbol{x}})$, and $\boldsymbol{x}^*$ is also a solution with minimal distance to $\bar{\boldsymbol{x}}$ with respect to $\Psi_{p,\beta}$.

It remains to estimate $\Psi_{p,\beta}(\boldsymbol{x}_{\alpha_k}^{\delta_k} - \boldsymbol{x}^*)^p$, which can be done exactly as in the previous proof. □

4. Computation of a minimizer for the Tikhonov functional with penalty $\Psi_{p,\beta}^p(\boldsymbol{x})$. We want to use surrogate functionals for the minimization of (2.5). They were first investigated in Daubechies et al. [6] for the minimization of (2.5) with a linear operator. A first convergence result of the algorithm with nonlinear operator and quadratic Hilbert space penalty was presented in [26], and the minimization of (2.4) with one-homogeneous sparsity constraint was investigated in [27]. As mentioned earlier, the use of the penalty $\Psi_{p,\beta}(\boldsymbol{x})$ has the disadvantage that iteration process necessary for the computation of a minimizer of the surrogate functional can only be carried out explicitly for $p \in 1, 2, \infty$. As we will see, this is not the case if the penalty $\Psi_{p,\beta}(\boldsymbol{x})^p$ is used instead. What remains an open question is the convergence of the surrogate functional algorithm for minimizing (2.5) and nonlinear operator, which will be considered in this chapter. Although the results from the one-homogeneous case do not carry over directly, it turns out that some of the proofs will be almost the same. In these cases, we will only refer to the corresponding proofs in [27].

We want to compute a minimizer of the functional

$$J_\alpha(\boldsymbol{x}) = \|y^\delta - T(F^*\boldsymbol{x})\|^2 + 2\alpha\Psi_{p,\beta}(\boldsymbol{x})^p. \tag{4.1}$$

The minimizer (or at least a critical point of the functional) will be computed by the method of surrogate functionals, that is we consider

$$J_\alpha^s(\boldsymbol{x},\boldsymbol{a}) = \|y^\delta - T(F^*\boldsymbol{x})\|^2 + 2\alpha\Psi_{p,\beta}(\boldsymbol{x})^p + C\|\boldsymbol{x}-\boldsymbol{a}\|^2 - \|T(F^*\boldsymbol{x}) - T(F^*\boldsymbol{a})\|^2 \tag{4.2}$$

and create a sequence of iterates $\{\boldsymbol{x}_k\}$ by

$$\boldsymbol{x}_{k+1} = \arg\min_{\boldsymbol{x}} J_\alpha^s(\boldsymbol{x}, \boldsymbol{x}_k) .$$

In the following we will show that the sequence $\boldsymbol{x}_k$ converges towards a critical point of the functional. Since (4.2) contains a negative term, it is not clear whether the functional $\bar{J}_\alpha^s(\boldsymbol{x}, \boldsymbol{a})$ is bounded from below. It turns out that this can be ensured by a proper choice of the constant C in (4.2):

PROPOSITION 4.1. *For given $\alpha > 0$ and $\boldsymbol{x}_0 \in \ell_2$, we define a ball K_r by*

$$K_r := \{\boldsymbol{x} \in \ell_2 : \Psi_{p,\beta}(\boldsymbol{x})^p \le r\}$$

with radius

$$r := \frac{\|y^\delta - T(F^*\boldsymbol{x}_0)\|_Y^2 + 2\alpha\Psi_{p,\beta}(\boldsymbol{x}_0)^p}{2\alpha}.$$

Let the constant C be chosen as

$$C := 2B \max\left\{ \left(\sup_{\boldsymbol{x}\in K_r} \|T'(F^*\boldsymbol{x})\| \right)^2, L\sqrt{\|y^\delta - T(F^*\boldsymbol{x}_0)\|^2 + 2\alpha\Psi_{p,\beta}(\boldsymbol{x}_0)^p} \right\}, \tag{4.3}$$

where L is the Lipschitz constant of the Fréchet derivative of T and B denotes the upper frame constant in (2.1). Then the functionals $J_\alpha^s(\boldsymbol{x}, \boldsymbol{x}_k)$ are bounded from below for all $\boldsymbol{x} \in \ell_2$ and all $k \in \mathbb{N}$ and have thus minimizers. For the minimizer $\boldsymbol{x}_{k+1}$ of $J_\alpha^s(\boldsymbol{x}, \boldsymbol{x}_k)$ holds $\boldsymbol{x}_{k+1} \in K_r$. Moreover, the sequences $\{J_\alpha(\boldsymbol{x}_k)\}_{k=0,1,2,\ldots}$ and $\{J_\alpha^s(\boldsymbol{x}_{k+1}, \boldsymbol{x}_k)\}_{k=0,1,2,\ldots}$ are non-increasing, and

$$2BL\|y^\delta - T(F^*\boldsymbol{x}_k)\| \le C \tag{4.4}$$

holds.

The proof is the same as in Lemma 1, Proposition 2 and Corollary 3 in [27], one only has to replace $\Psi(\mathbf{L}\boldsymbol{x})$ by $\Psi_{p,\beta}(\boldsymbol{x})^p$.

PROPOSITION 4.2. *Let $1 < p \le 2$. Then the necessary condition for a minimizer of (4.2) is given by the equation*

$$0 = -FT'(F^*\boldsymbol{x})^*(y^\delta - T(F^*\boldsymbol{a})) - C(\boldsymbol{a} - \boldsymbol{x}) + \alpha p D(\boldsymbol{x}) , \tag{4.5}$$

where

$$(D(\boldsymbol{x}))_j = \beta_j \operatorname{sgn}(x_j)|x_j|^{p-1} .$$

Proof. By standard Hilbert space methods we obtain for $h \in \ell_2$

$$\begin{aligned} J_\alpha^s(\boldsymbol{x}+\boldsymbol{h}, \boldsymbol{a}) &= \|y^\delta - T(F^*\boldsymbol{x})\|^2 + C\|\boldsymbol{x} - \boldsymbol{a}\|^2 - \|T(F^*\boldsymbol{x}) - T(F^*\boldsymbol{a})\|^2 \\ &- 2\langle FT'(F^*\boldsymbol{x})^*(y^\delta - T(F^*\boldsymbol{a})) + C(\boldsymbol{a} - \boldsymbol{x}), \boldsymbol{h}\rangle_{\ell_2} + 2\alpha(\Psi_{p,\beta}(\boldsymbol{x}+\boldsymbol{h})^p - \Psi_{p,\beta}(\boldsymbol{x}))^p + O(\|\boldsymbol{h}\|^2) . \end{aligned}$$

Morover, we get

$$\begin{aligned} \Psi_{p,\beta}(\boldsymbol{x}+\boldsymbol{h})^p &= \sum \alpha_j |x_j + h_j|^p = \sum \beta_j |x_j|^p + p \sum \beta_j \operatorname{sgn}(x_j)|x_j|^{p-1} h_j + O(\|\boldsymbol{h}\|^2) \\ &= \Psi_{p,\beta}(\boldsymbol{x})^p + p\langle D(\boldsymbol{x}), \boldsymbol{h}\rangle_{\ell_2} + O(\|\boldsymbol{h}\|^2) . \end{aligned}$$

Altogether, we have shown

$$J_\alpha^s(\boldsymbol{x}+\boldsymbol{h},\boldsymbol{a})-J_\alpha^s(\boldsymbol{x},\boldsymbol{a}) = -2\langle FT'(F^*\boldsymbol{x})^*(y^\delta - T(F^*\boldsymbol{a}))+C(\boldsymbol{a}-\boldsymbol{x})-\alpha pD(\boldsymbol{x}),\boldsymbol{h}\rangle_{\ell_2}+O(\|\boldsymbol{h}\|^2)\ ,$$

which proves the assertion. ◻

We wish to remark that a similiar expression for the necessary condition was derived in [6] in case of a linear operator.

Setting

$$M(\boldsymbol{x},\boldsymbol{a}) = \frac{1}{C}FT'(F^*\boldsymbol{x})(y^\delta - T(F^*a)) + a\ ,$$

equation (4.5) can be rewritten as

$$\underbrace{(I+\frac{\alpha}{C}pD)\boldsymbol{x}}_{=:\Phi(x)} = M(\boldsymbol{x},\boldsymbol{a})\ ,$$

and, in order to compute a critical point of (4.2), it remains to solve the fixed point equation

$$\boldsymbol{x} = \Phi^{-1}(M(\boldsymbol{x},\boldsymbol{a})) =: U(\boldsymbol{x},\boldsymbol{a})\ . \tag{4.6}$$

Since the operator D is defined on the coefficients, so are Φ and Φ^{-1}, i.e., with $\boldsymbol{x}=(x_j)$, we have to solve the equation

$$x_j + \frac{\alpha}{C}p\beta_j|x_j|^{p-1}\operatorname{sgn}(x_j) = M(\boldsymbol{x},\boldsymbol{a})_j$$

for every j. This equation always obtains a unique solution, as the left hand side is strictly monotonically increasing. We would like to find a solution of the fixed point equation (4.6) by a classical fixed point iteration, i.e. we want to compute the new iterate as $\boldsymbol{x}_{k+1} = \lim_{l\to\infty}\boldsymbol{x}_{k,l}$, $\boldsymbol{x}_{k,l+1} = U(\boldsymbol{x}_{k,l},\boldsymbol{x}_k)$. Thus we have to show that U is a contraction:

LEMMA 4.3. *The operator U is Lipschitz continuous,*

$$\|U(\boldsymbol{x},\boldsymbol{a}) - U(\tilde{\boldsymbol{x}},\boldsymbol{a})\| \le q\|\boldsymbol{x}-\tilde{\boldsymbol{x}}\|$$

with constant

$$q = \frac{BL}{C}\sqrt{J_\alpha(\boldsymbol{a})}\ .$$

Proof. The operator Φ^{-1} is non - expansive, e.g., we have $\|\Phi^{-1}(\boldsymbol{x}) - \Phi^{-1}(\tilde{\boldsymbol{x}})\| \le \|\boldsymbol{x}-\tilde{\boldsymbol{x}}\|$; see [6]. We observe

$$\begin{aligned}
\|U(\boldsymbol{x},\boldsymbol{a}) - U(\tilde{\boldsymbol{x}},\boldsymbol{a})\| &\le \|M(\boldsymbol{x},\boldsymbol{a}) - M(\tilde{\boldsymbol{x}},\boldsymbol{a})\| \\
&\le \frac{B^{1/2}}{C}\|FT'(F^*\boldsymbol{x}) - FT'(F^*\tilde{\boldsymbol{x}})\|\|y^\delta - T(F^*\boldsymbol{a})\| \\
&\overset{(2.9)}{\le} \frac{BL}{C}\|\boldsymbol{x}-\tilde{\boldsymbol{x}}\|\sqrt{\|y^\delta - T(F^*\boldsymbol{a})\|^2 + \alpha\Psi_p(\boldsymbol{a})^p}. \quad ◻
\end{aligned}$$

The fixed point iteration $\boldsymbol{x}_{k,l+1} = U(\boldsymbol{x}_{k,l}, \boldsymbol{x}_k)$ converges only if the operator $U(\cdot, \boldsymbol{x}_k)$ forms a contraction for all $k \in \mathbb{N}$. This will be shown in the following

PROPOSITION 4.4. *The fixed point map $U(\boldsymbol{x}, \boldsymbol{x}_k)$ for solving the fixed point equation (4.6) is a contraction for all $k = 0, 1, 2, \dots$ and all $\alpha \geq 0$.*

Proof. By the definition of C in (4.3) and Lemma 4.3 (setting $\boldsymbol{a} = \boldsymbol{x}_0$), we deduce that $U(\boldsymbol{x}, \boldsymbol{x}_0)$ is a contraction with

$$q = \frac{BL}{C}\sqrt{J_\alpha(\boldsymbol{x}_0)} \leq \frac{1}{2} < 1.$$

Proposition 4.1 states that the sequence of functional values $J_\alpha(\boldsymbol{x}_k)$ is nonincreasing, and we complete the proof by

$$\begin{aligned}\|U(\boldsymbol{x}, \boldsymbol{x}_k) - U(\tilde{\boldsymbol{x}}, \boldsymbol{x}_k)\|_{\ell_2} &\leq \frac{BL}{C}\sqrt{J_\alpha(\boldsymbol{x}_k)}\|\boldsymbol{x} - \tilde{\boldsymbol{x}}\|_{\ell_2} \\ &\leq \dots \leq \frac{BL}{C}\sqrt{J_\alpha(\boldsymbol{x}_0)}\|\boldsymbol{x} - \tilde{\boldsymbol{x}}\|_{\ell_2} \leq \frac{1}{2}\|\boldsymbol{x} - \tilde{\boldsymbol{x}}\|_{\ell_2}. \quad \square\end{aligned}$$

The reconstructed fixed point of $U(\boldsymbol{x}, \boldsymbol{x}_k)$ is certainly a critical point of the surrogate functional, but it remains to show that it is also a global minimizer of $J_\alpha^s(\boldsymbol{x}, \boldsymbol{x}_k)$:

PROPOSITION 4.5. *The necessary equation (4.6) for a minimum of the functional $J_\alpha^s(\boldsymbol{x}, \boldsymbol{x}_k)$ has a unique fixed point, and the fixed point iteration converges towards the minimizer.*

Proof. To verify this assertion, we have to investigate the Taylor expansion of J_α^s more closely. By Taylor's expansion for T and the Lipschitz-continuity of T' we get

$$T(F^*\boldsymbol{x} + F^*\boldsymbol{h}) = T(F^*\boldsymbol{x}) + T'(F^*\boldsymbol{x})F^*\boldsymbol{h} + R(F^*\boldsymbol{x}, F^*\boldsymbol{h}) \tag{4.7}$$

with

$$\|R(F^*\boldsymbol{x}, F^*\boldsymbol{h})\|_Y \leq \frac{B^{1/2}L}{2}\|\boldsymbol{h}\|_{\ell_2}^2 .$$

We have

$$J_\alpha^s(\boldsymbol{x} + \boldsymbol{h}, \boldsymbol{x}_k) - J_\alpha^s(\boldsymbol{x}, \boldsymbol{x}_k) = \partial J_\alpha^s(\boldsymbol{x}, \boldsymbol{x}_k)\boldsymbol{h} + r(\boldsymbol{x}, \boldsymbol{x}_k, \boldsymbol{h})$$

and

$$\Psi_{p,\beta}^p(\boldsymbol{x} + \boldsymbol{h}) - \Psi_{p,\beta}^p(\boldsymbol{x}) = \partial\Psi_{p,\beta}^p(\boldsymbol{x})\boldsymbol{h} + r_1(\boldsymbol{x}, \boldsymbol{h}) .$$

$\Psi_{p,\beta}^p$ is convex, and for convex functionals

$$r_1(\boldsymbol{x}, \boldsymbol{h}) \geq 0 \tag{4.8}$$

holds. Using (4.7) we get

$$\begin{aligned}J_\alpha^s(\boldsymbol{x} + \boldsymbol{h}, \boldsymbol{x}_k) - J_\alpha^s(\boldsymbol{x}, \boldsymbol{x}_k) = \partial J_\alpha^s(\boldsymbol{x}, \boldsymbol{x}_k)\boldsymbol{h} &+ C\|\boldsymbol{h}\|^2 - 2\langle y^\delta - T(F^*\boldsymbol{x}), R(F^*\boldsymbol{x}, F^*\boldsymbol{h})\rangle \\ &+2\alpha \cdot r_1(\boldsymbol{x}, \boldsymbol{h}) \\ \geq \partial J_\alpha^s(\boldsymbol{x}, \boldsymbol{x}_k)\boldsymbol{h} &+ \left(C - BL\|y^\delta - T(F^*\boldsymbol{x})\|\right)\|\boldsymbol{h}\|^2 + 2\alpha \cdot r_1(\boldsymbol{x}, \boldsymbol{h}) .\end{aligned}$$

Now, $\boldsymbol{x}_{k+1}$ is at least a critical point of the functional, and thus we have

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$$J_\alpha^s(\boldsymbol{x}_{k+1}+\boldsymbol{h},\boldsymbol{x}_k) - J_\alpha^s(\boldsymbol{x}_{k+1},\boldsymbol{x}_k) \geq \left(C - BL\|y^\delta - T(F^*\boldsymbol{x})\|\right)\|\boldsymbol{h}\|^2 + 2\alpha \cdot r_1(\boldsymbol{x},\boldsymbol{h}) \overset{(4.4),(4.8)}{\geq} \frac{C}{2}\|\boldsymbol{h}\|^2$$

for all $\boldsymbol{h}$, and thus $\boldsymbol{x}_{k+1}$ is the only global minimizer of $J_\alpha^s(\boldsymbol{x},\boldsymbol{x}_k)$. □

The Tikhonov functional with nonlinear operator is usually not globally convex. As a consequence, the functional might have several, even local, minimizer. This is not the case for the surrogate functional:

PROPOSITION 4.6. *Let T be a twice differentiable operator. Then the functional $J_\alpha^s(\boldsymbol{x},\boldsymbol{x}_{k+1})$ is strictly convex.*

The above result ensures the existence of a unique minimizer of the surrogate functional. The proof of this Proposition is exactly the same as (the somewhat long) proof of Proposition 12 in [27], since the proof works on the surrogate functional without the penalty term only.

In a final step we would like to investigate the convergence properties of the sequence of iterates $\{\boldsymbol{x}_k\}_{k\in\mathbb{N}}$.

PROPOSITION 4.7. *The sequence of iterates $\{\boldsymbol{x}_k\}_{k\in\mathbb{N}}$ has a convergent subsequence, and every convergent subsequence converges towards a critical point of the Tikhonov functional (4.1). If there exists at least one isolated limit of one subsequence, then the whole sequence converges towards a critical point of the functional.*

This Proposition is a summary of the results given in Section 5 of [27], and most arguments carry over directly. We will therefore only give a short sketch of the proof. First, it is shown that there exists a weak convergent subsequence of $\{\boldsymbol{x}_k\}$, which follows from the fact that $\boldsymbol{x}_k \in K_r(\boldsymbol{x}_0)$ for all k. In a next step, it is shown that the limit of every convergent subsequence fulfills the necessary condition for a minimizer of the Tikhonov functional. The proofs are only using the fact that the penalty term is a convex and weakly lower semi-continuous functional with existent sub-differential, and apply thus also to $\Psi_{p,\beta}^p$, see the proofs of Lemma 13-17 in [27]. In order to prove that the sequence converges also with respect to the ℓ_2-norm, we have to use the structure of the penalty $\Psi_{p,\beta}^p$. For a weakly convergent subsequence $\boldsymbol{x}_{k_l} \rightharpoonup \boldsymbol{x}_\alpha^\star$ it is shown that $\|\boldsymbol{x}_{k_l}\| \to \|\boldsymbol{x}_\alpha^\star\|$ holds, and thus the subsequence converges strongly, see the proof of Theorem 18 in [27], which can be carried over with small and obvious changes. The convergence of the whole sequence in case of an isolated minimizer can be taken from [26].

To summarize the results of this section, we have shown that the iteration $\boldsymbol{x}_{k+1} = \arg\min_{\boldsymbol{x}} J_\alpha^s(\boldsymbol{x},\boldsymbol{x}_k)$ converges at least to a critical minimizer of the Tikhonov functional (4.1), and the minimizer of $J_\alpha^s(\boldsymbol{x},\boldsymbol{x}_k)$ can be obtained as the limit of the fixed point iteration $\boldsymbol{x}_{k,l+1} = U(\boldsymbol{x}_{k,l},\boldsymbol{x})$.

4.1. A remark on the inversion of Φ. In order to carry out the above given fixed point iteration, we have to invert the mapping Φ for every set of coefficients. Setting

$$f(t_j) = t_j + \beta_j \frac{\alpha}{C} p|t_j|^{p-1}\operatorname{sgn}(t_j)$$

we have to consider the problem of finding a solution of the equation $f(t_j) - z_j = 0$ with given $z_j, j \in \mathbb{N}$. For simplicity, we will drop the index j in the following. However, we should keep in mind that not only the data z, but also the weight β changes with every index. It has already been mentioned that this simple one-dimensional problem admits a unique solution. Applying Newton's method seems to be a good idea for solving the equation. Unfortunately, it turns out that Newton's method quite often fails for $1 < p < 2$, in particular for $0 \leq z \leq 1$.

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Even if a relatively good starting value is available, Newton's method breaks down: If, for example, we have $p = 1.1$, $z = -0.1224$, $\beta_j \frac{\alpha}{C} p = 2$ and $t_0 = -0.0048$, then the iteration oscillates after a few iterations with $t_{2k} = -1.7749$ and $t_{2k+1} = 1.5937$. The reason for this behaviour is that the derivative of $f(t)$, for $1 < p < 2$, is singular at zero and approaches 1 for larger values of t. In our example, the correct value would have been $t = -7.3676e - 13$, which is close to the singularity point of the derivative and thus explaines why the method fails. Also, a transformation to a fixed point equation $t = \beta \frac{\alpha}{C} p|t|^{p-1} \operatorname{sgn}(t) - z$ does not help, as the belonging fixed point iteration fails. However, a method that always works is the method of bisection:

LEMMA 4.8. *Let $a < b$ be chosen such that the unique solution t_* of $f(t) = z$ fulfils $a \le t_* \le b$. Then the method of bisection converges towards t_*. If t_k denotes the sequence of iterates, then we have*

$$|t_k - t_*| \le 2^{-k}|b - a| \; .$$

For the proof we only have to use the fact that $f(t)$ is a strictly increasing function with $\lim_{t\to-\infty} f(t) = -\infty$, $\lim_{t\to\infty} f(t) = \infty$; the convergence rate holds always for bisection. In order to minimize the numerical effort of the method, it is crucial to choose $a,\ b$ as close to t_* as possible. Let us start with a remark on the function f:

LEMMA 4.9. *The function f obeys the properties*

$$\begin{aligned} f(-t) &= -f(t) \\ t \ge 0 &\Leftrightarrow z \ge 0 \\ z \ge 0 &\Rightarrow t_* \in [0, z] \; , \end{aligned}$$

where t_* denotes the solution of $f(t) = z$.

As a consequence, it is sufficient to consider the case $t,\ z \ge 0$. In order to compute the values a, b needed in Lemma 4.9, we proceed as follows: Firstly, we have to find strictly increasing functions $\tilde{f}, \bar{f}$ with

$$\bar{f}(t) \le f(t) \le \tilde{f}(t) \; .$$

In a next step, we compute the unique solutions $\bar{t}, \tilde{t}$ of the equations $\bar{f}(t) = z$, $\tilde{f}(t) = z$. Because of $\bar{f}(\bar{t}) = z \le f(\bar{t})$, $\bar{t}$ is an upper bound for t_*, and with the same argument, $\tilde{t}$ is a lower bound, i.e., we can choose in Lemma 4.8 $a = \tilde{t}$, $b = \bar{t}$. Let us first consider

The case $\mathbf{0 \le t \le 1}$.

This is true if $1 + \beta \frac{\alpha}{C} p \le z$ holds. Setting $q = \beta \frac{\alpha}{C} p$, we have in this case

$$t(1+q) \le t + q|t|^{p-1} \le |t|^{p-1}(1+q) \; ,$$

but we can even find a better estimate: For $p - 1 \le 1/2$ we have

$$0 \le t \le |t|^{2(p-1)} \le |t|^{p-1} \; ,$$

and thus

$$f(t) = t + q|t|^{p-1} \le \underbrace{|t|^{2(p-1)} + q|t|^{p-1}}_{=:\tilde{f}(t)} \le |t|^{p-1}(1+q) \; .$$

One easily sees that the equation $\tilde{f}(t) = z$ has the solution

$$\tilde{t} = \left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + z} \right)^{1/(p-1)} \; . \tag{4.9}$$

In case $(p-1) > 1/2$ we use the estimate

$$t + q|t|^{p-1} \le |t|^{p-1}(1+q) \ ,$$

and thus

$$\tilde{t} = \left(\frac{z}{1+q}\right)^{1/(p-1)} . \tag{4.10}$$

Let us now derive an upper bound for t_*. We start again with the case $p-1 \le \frac{1}{2}$. Then it follows

$$t + qt \le \underbrace{t + q|t|^{1/2}}_{=:\bar{f}(t)} \le t + qt^{p-1} \ .$$

The solution of $\bar{f}(t) = z$ is easily derived as

$$\bar{t} = \left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + z}\right)^2 . \tag{4.11}$$

In the case $p-1 > 1/2$, we write

$$t + qt^{p-1} = t(1 + qt^{p-2})$$

and replace $p(t) = t^{p-2}$ in $[0,1]$ by its linear Taylor approximation around 1, given by

$$g(t) = (p-2)t + (3-p) \ .$$

As $p(t)$ is a convex and strictly increasing on $(0,1]$, we have $1 \le g(t) \le p(t)$ on $[0,1]$, and therefore

$$t + qt \le \underbrace{t(1 + q \cdot g(t))}_{=:\bar{f}(t)} \le t + qt^{p-1} \ .$$

The solution of $\bar{f}(t) = z$ is easily derived as

$$\bar{t} = \frac{1+q(3-p)}{2q(2-p)} - \sqrt{\left(\frac{1+q(3-p)}{2q(2-p)}\right)^2 - \frac{z}{q(2-p)}} \ . \tag{4.12}$$

The case $t > 1$.

In principle, the bounds for this case can be achieved simultaneously as above, the functions $\tilde{f}(t)$, $\bar{f}(t)$, as well as $\tilde{t}$, $\bar{t}$, exchange their roles. We get

$$t + q \cdot t^{p-1} \le \begin{cases} t + q \cdot t^{1/2} & \text{for } p-1 \le 1/2 \\ t + q \cdot t & \text{for } p-1 > 1/2 \end{cases} ,$$

and $\tilde{t}$ is given by (4.12) if $p-1 \le 1/2$ and by (4.10) if $p-1 > 1/2$.

To bound $f(t)$ from below, we use for $p-1 \le 1/2$ the bound

$$\bar{f}(t) = t^{2(p-1)} + qt^{p-1} \le t + qt^{p-1} \ ,$$

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and thus $\bar{t}$ is given by (4.9). The case $p-1 \geq 1/2$ is a bit more complicated: We use the estimate

$$t(1+q\cdot t^{p-2}) \geq t(1-q\cdot g(t)) = \bar{f}(t)\ ,$$

where $g(t)=mt+n$ denotes the linear Taylor-approximation of the function $p(t)=t^{p-2}$ at point $t=z$, and m,n given by

$$\begin{aligned} m &= (p-2)\cdot z^{p-3} \\ n &= z^{p-2}(3-p)\ , \end{aligned}$$

and the solution of $\bar{f}(t)=z$ is given by

$$\bar{t} = -\frac{1+qn}{2qm} + \sqrt{\left(\frac{1+qn}{2qm}\right)^2 + \frac{z}{qm}}\ . \tag{4.13}$$

We summarize the derived bounds in Table 4.1. Using bisection with these upper and lower bounds gives a fast converging algorithm for the computation of a solution of the equation $f(t_i)=z_i$ for each i.

TABLE 4.1
Collection of the formulas for the computation of $\bar{t}$, $\tilde{t}$. The numbers in brackets refer to the right hand side of the belonging references.

	$t \leq 1$		$t > 1$	
	$p-1 \leq 1/2$	$p-1 > 1/2$	$p-1 \leq 1/2$	$p-1 > 1/2$
$\tilde{t} =$	(4.9)	(4.10)	(4.12)	(4.10)
$\bar{t} =$	(4.11)	(4.12)	(4.9)	(4.13)

5. A numerical example. In this section we want to confirm our analytical results. In particular, we demonstrate that the parameter choice rule proposed in Theorem 3.4 yields a convergent regularization scheme, and that we are able to minimize (2.5) by our surrogate functional approach. Moreover, we confirm properties of our regularization approach that are already known in case of a linear operator but have to be verified for nonlinear operators also, e.g., that sparse reconstructions are obtained and that the degree of sparseness depends on the chosen parameter p.

We consider a simple nonlinear Hammerstein equation $T(x)=y$, where the operator $T: H^u([0,1]) \to L_2([0,1])$, $u \geq 0$ is given by

$$T(x)(s) = \int_0^s x^2(t)\,dt \qquad 0 \leq s \leq 1.$$

This type of operator equation has been used frequently for numerical test computations [10, 20]. Due to its structure, the numerical errors in the evaluation of T and its derivative can be easily controlled. This is important in order to observe convergence rates numerically: Since the reconstruction always suffers from additional numerical errors, we have have to secure that these errors also tend to zero as the data error tends to zero. This is often difficult in more realistic examples, e.g., Single Photon Emission Computed Tomography (SPECT), for which some numerical results with $B^1_{1,1}(\Omega)$ penalty, $\Omega \subset \mathbb{R}^2$ were shown in [27].

The Fréchet derivative of T and its adjoint are easily computed by

$$T'(x)h(s) = 2\int_0^s x(t)h(t)\,dt$$

$$T'(x)^*g = i_u^*\left(2\cdot x(\cdot)\int_\cdot^1 g(s)\,ds\right) ,$$

where i_u^* denotes the adjoint of the embedding operator $i_u : H^u \to L_2$. Please note that i_u is a compact operator. As T is also well defined and continuous on L_2, we have on H^u $T(x) = T(i_u x)$, and T will transform weak convergent sequences in H^u to convergent sequences in L_2. By the arguments given below (2.9), the operator T therefore meets the conditions (2.6) , (2.7) if the operator T' is Lipschitz continuous, which can be seen by

$$\begin{aligned}\|T'(x)h - T'(\tilde{x})h\|^2 &= \int_0^1\left(\int_0^s (x(t)-\tilde{x}(t))h(t)\,dt\right)^2 ds\\ &\le \int_0^1 \|x-\tilde{x}\|^2\|h\|^2\,ds = \|x-\tilde{x}\|^2\|h\|^2 .\end{aligned}$$

Since we are going to use a wavelet basis in our computations, we have for the frame bounds $A = B = 1$, and thus (2.9) is also fulfilled.

In order to compute $T'(x)^*$ we have to evaluate the operator i_u^*. Its definition depends on the chosen norm and inner product on H^u. We are going to use the wavelet based definition of the norm given in (2.2). As $H^u = B^u_{2,2}$, we have

$$\|x\|^2_{H^u} = \sum_{k\in\mathbb{Z}} |\langle x,\varphi_{0k}\rangle|^2 + \sum_{j=0}^\infty 2^{2ju}\sum_{k\in\mathbb{Z}}|\langle x,\psi_{jk}\rangle|^2 ,$$

and the corresponding inner product is given by

$$\langle x,y\rangle_{H^u} = \sum_{k\in\mathbb{Z}}\langle x,\varphi_{0k}\rangle\langle y,\varphi_{0k}\rangle + \sum_{j=0}^\infty 2^{2ju}\sum_{k\in\mathbb{Z}}\langle x,\psi_{jk}\rangle\langle y,\psi_{jk}\rangle .$$

The operator i_u^* is defined via the equation $\langle i_u x, y\rangle_{L_2} = \langle x, i_u^* y\rangle_{H^u}$. We have

$$\begin{aligned}\langle i_u x,y\rangle_{L_2} &= \sum_{k\in\mathbb{Z}}\langle x,\varphi_{0k}\rangle\langle y,\varphi_{0k}\rangle + \sum_{j=0}^\infty\sum_{k\in\mathbb{Z}}\langle x,\psi_{jk}\rangle\langle y,\psi_{jk}\rangle\\ &= \sum_{k\in\mathbb{Z}}\langle x,\varphi_{0k}\rangle\langle y,\varphi_{0k}\rangle + \sum_{j=0}^\infty 2^{2ju}\sum_{k\in\mathbb{Z}}\langle x,\psi_{jk}\rangle\frac{\langle y,\psi_{jk}\rangle}{2^{2ju}} = \langle x, i_u^*y\rangle_{H^u} ,\end{aligned}$$

with

$$i_u^* y = \sum_{k\in\mathbb{Z}}\langle y,\varphi_{0k}\rangle\varphi_{0k} + \sum_{j=0}^\infty\sum_{k\in\mathbb{Z}}\frac{\langle y,\psi_{jk}\rangle}{2^{2ju}}\psi_{jk} .$$

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The adjoint of the embedding operator damps the detail coefficients in the wavelet expansion for each level differently. If $u > 0$ is chosen very small, then the influence of i_u^* (at least in the numerical implementation) is almost not detectable. In our numerical example, we want to impose as little additional smoothness to the solutions as possible , which is secured by choosing the parameter u small enough. For the experiments, we were setting $u = 10^{-6}$. Thus, we are close to the L_2 setting.

The constant C in the definition of the surrogate functional plays an important role. Its accurate value, given in (4.3), is difficult to evaluate. If C is chosen too large, then experiments have shown that the convergence of the iterates toward the minimizer of the original functional is slowed down. However, if it is chosen to small, no convergence can be achieved. In fact, we observed that a to small C almost immediately leads to strongly divergent iterates. Therefore we chose the largest constant that still produced convergent iterates, which was the case for $C = 2$.

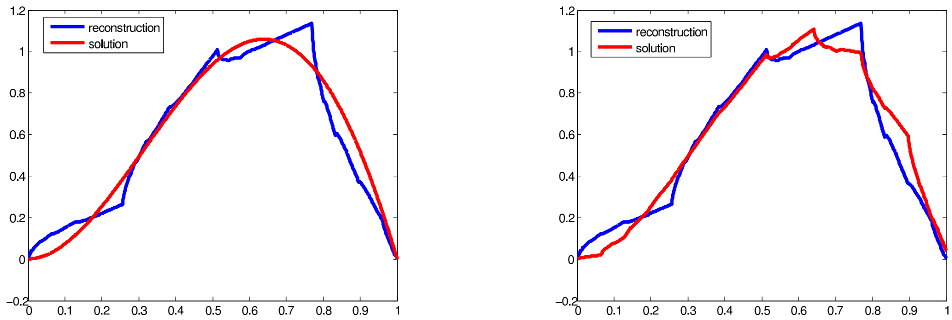

FIG. 5.1. *Reconstructions from y^δ_{full} (left) and y^δ_{sparse} (right). The data noise was approx. 1%.*

For our test, we chose one solution to be

$$x^\dagger_{full}(t) = 8t^2 - 8t^3 - 2t^4 + 2t^5 \ .$$

Performing a wavelet transform of $x^\dagger_{full}$ (we used the discret wavelet transform, the function was given at 1000 points equispaced in $[0, 1]$), it becomes apparent that the function has no sparse representation at least with respect to the Daubechies wavelets db2-db4. Thus, for comparison, we computed a sparse version $x^\dagger_{sparse}$ of $x^\dagger_{full}$ by keeping only the first 15 coefficients (i.e., the approximation coefficients and the detail coefficients on the first level). For both functions, the associated data y_{full}, y_{sp} was computed and contaminated with noise. Afterwards, a reconstruction with the Besov penalty $\| \cdot \|^{1.2}_{B^1_{1.2,1.2}}$ was performed. The regularization parameter was linked to the data noise by $\alpha = 10 \cdot \delta$.

Figure 5.1 shows the reconstruction results for both sets of data. If we compare the reconstructions for both cases, we realize that they are almost the same. Even in the nonsparse solution case, the sparsity constraint is so strong that it surpresses small coefficients in the reconstruction even when they are there, and reconstruct only a few large coefficients. This underlines the well known observation that sparsity constraints work best if the solution is known to be sparse. On the other hand, if there is a lot of noise in the data, we are anyway able to reconstruct large coefficients only, and the influence of the noise might be considerably reduced by the sparsity constraint. We wish to add that the reconstruction quality depends heavily on the regularization parameter α, and a different parameter choice rule might result in much better reconstructions even for a nonsparse solution. However, to our knowledge no other parameter choice rule for sparsity constraints has been investigated so far.

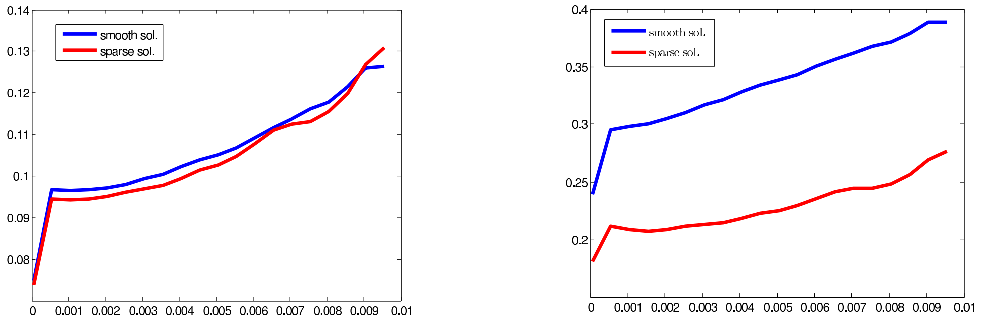

FIG. 5.2. *Reconstruction quality (ordinate) of the sparse and nonsparse solution in dependence of the data error (abscissa) with respect to* L_2 *(left) and* $B^1_{1.2,1.2}$ *(right).*

In a next test, we observed the convergence speed of the method. We measured the distance between the true solutions $x^\dagger_{full}$, $x^\dagger_{sp}$ and the belonging reconstructions first with respect to L_2 and then with respect to $B^1_{1.2,1.2}$. Figure 5.2 shows that the reconstruction quality in L_2 is the same in both cases. It resembles the well known fact that the value of the L_2-norm of a function is often well approximated by using only few wavelet coefficients. This seems not to be the case for the $B^1_{1.2,1.2}$ norm, where we see a clear difference in the reconstruction quality. It is not surprising that the sparse function is better reconstructed than the nonsparse. However, as also the nonsparse solution has a finite value of the $B^1_{1.2,1.2}$-norm, we do expect a convergence to the true solution in both cases, which is confirmed by our numerical test.

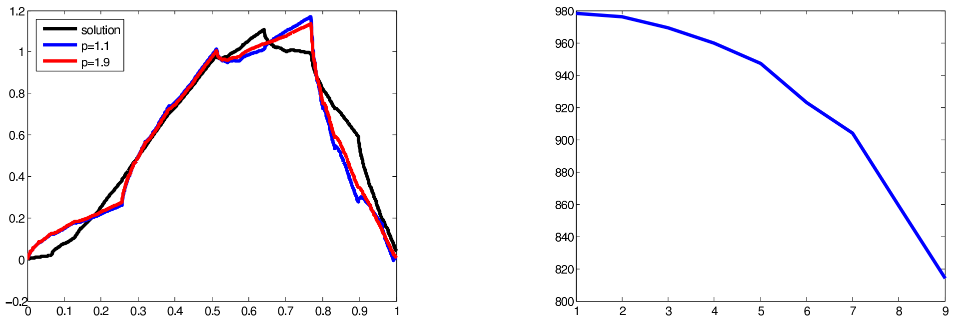

FIG. 5.3. *Left: Solution and reconstruction for* $p = 1.1$, $p = 1.9$. *Right: number of zero coefficients (ordinate) in the reconstruction with* $p \in [1.1, 1.9]$ *(abscissa). The solution had 1000 zero coefficients.*

Next, we performed reconstructions with a $B^1_{p,p}$-penalty for different values of $1 < p$ and fixed regularization parameter. The reconstructions differ slightly; see Figure 5.3, (left picture) . In particular, we find that the number of zero coefficients decreases with increasing p, as can be seen in Figure 5.3 (right picture). This confirms our expectation that a smaller p yields a more sparse reconstruction. It would therefore be interesting to investigate also the case $p < 1$. This is, however, a difficult task, since the belonging sparsity constraint is not even convex. In a final test we compare the distribution of the small wavelet coefficients for reconstructions with $L_2 = B^0_{2,2}$ and $B^1_{1.1,1.1}$ penalties, see the histogram plot in Figure 5.4. Both distributions have a large peak around zero, indicating that most coefficients are close to zero. However, the width of the peaks shows also that we have more moderately small coefficients for the L_2 reconstruction, which confirms again the ability of the chosen $B^1_{1.1,1.1}$ constraint to provide a sparse reconstruction.

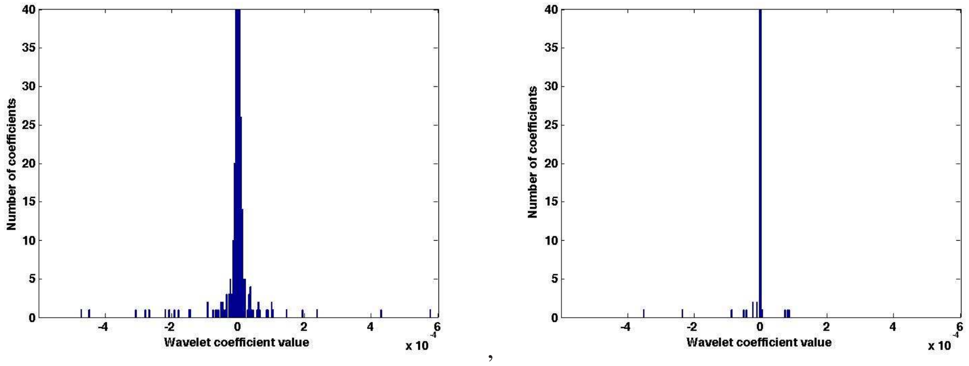

FIG. 5.4. *Histogram plot of the wavelet coefficient distribution around zero. Left: Reconstruction with L_2 penalty. Right: Reconstruction with $B^1_{1.1,1.1}$ penalty.*

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