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ERROR ESTIMATE IN THE SINC COLLOCATION METHOD FOR VOLTERRA-FREDHOLM INTEGRAL EQUATIONS BASED ON DE TRANSFORMATION*

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Dedicated to Prof. K. Maleknejad on the occasion of his 62th birthday

Abstract. We present a method and experimental results for approximate solution of nonlinear Volterra-Fredholm integral equations by double exponential (DE) transformation based on the sinc collocation method. It is well known that by applying DE transformation the rate of convergence $O(\exp(-cN/\log N))$ is attained, where N is a parameter representing the number of terms of the sinc expansion. The purpose of this paper is to develop the work carried out in 2005 by Muhammad et al. [J. Comput. Appl. Math., 177 (2005), pp. 269–286], for the numerical solution of two dimensional nonlinear Volterra-Fredholm integral equations. We design a numerical scheme for these equations based on the sinc collocation method incorporated with the DE transformation. A new error estimation by truncation is also obtained which is shown to have an exponential order of convergence as in Muhammad et al. (op. cit.). Finally, the reliability and efficiency of the proposed scheme are demonstrated by some numerical experiments.

Key words. integral equation, sinc collocation method, double exponential transformation, Volterra-Fredholm integral equation, error estimation.

AMS subject classifications. 65D32, 45G10.

1. Introduction. In this paper, we consider the sinc collocation method for the numerical solution of nonlinear Volterra-Fredholm integral equations of the form

$$u(x,t) = g(x,t) + \int_a^t \int_\Omega F(x,t,\xi,\tau,u(\xi,\tau))d\xi d\tau, \quad (x,t) \in [a,T] \times \Omega, \tag{1.1}$$

where $u(x,t)$ is an unknown function, $g(x,t)$ and $F(x,t,\xi,\tau,u(\xi,\tau))$ are given analytical real-valued functions defined respectively on $D := [a,T] \times \mathbb{R}$ and $S \times \mathbb{R}$ (where $S = \{(x,t,\xi,\tau), \quad a \le \tau \le t \le T; \quad (x,\xi) \in \Omega \times \Omega\}$), with $F(x,t,\xi,\tau,u)$ nonlinear in u and Ω is a closed subset of $\mathbb{R}^n$. It is assumed that the given functions are such that (1.1) possesses a unique solution $u(x,t) \in C(D)$. Existence and uniqueness results for (1.1) may be found in [3, 8, 18, 26]; see also [7] and [10] for linear case.

For ease of exposition, we will restrict our discussion to the case where $\Omega := [a,b] \subseteq \mathbb{R}$. It will become clear that the following analysis can be extended to regions $\Omega := [a,b] \times [c,d] \subseteq \mathbb{R}^2$ and $\Omega \subseteq \mathbb{R}^3$. However, the problem of how to treat (1.1) numerically for general regions $\Omega \subseteq \mathbb{R}^n$, $(n \ge 2)$ remains to be solved.

Let the analytical integral operator K be defined as

$$(Ku)(x,t) = \int_a^t \int_\Omega F(x,t,\xi,\tau,u(\xi,\tau))d\xi d\tau,$$

so that equation (1.1) becomes:

$$u = g + Ku.$$

Equations of this type arise in the theory of nonlinear parabolic boundary value problems, the mathematical modeling of the spatio-temporal development of an epidemic, and

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various physical, mechanical, and biological problems; see [3, 26] for details. During the last 15 years, significant progress has been made in numerical analysis of linear and nonlinear versions of (1.1). For the linear case, some projection methods for numerical treatment of (1.1) are given in [4, 8, 10]. A Nyström type method is given by Hacia in [8]. Kauthen [10] studied continuous-time collocation and time-discretization collocation methods, and he analyzed their global discrete convergence properties and local and global superconvergence properties. The results of Kauthen have been extended to nonlinear Volterra-Fredholm equations by Brunner [1]. Guoqinag [5] considered the particular Trapezoidal Nyström method for (1.1) and gave the asymptotic error expansion. Also, Hadizadeh et al. [12] introduced the Adomian decomposition method for (1.1) and gave a bound for the solution. Usually, the convergence rates of these methods are of polynomial order with respect to N, where N represents the number of terms of the expansion or the number of points of the quadratures.

On the other hand, in the last three decades a variety of numerical methods based on the sinc approximation have been developed. Stenger [19, 20] and Lund and Bowers [11] provide overviews of the methods based on the sinc function for solving ODEs, PDEs and IEs; see [14, 19, 24, 27, 28]. The main purpose of the present research is to consider the Double Exponential transformation in the sinc collocation method for Volterra-Fredholm integral equations. The Double Exponential formula (DE), which is a quadrature formula based on double exponential transformation was first proposed by Takahasi and Mori [25] in 1974 for one dimensional numerical integration and is now recognized to be one of the most efficient quadrature formulas [13, 15]. The use of the DE transformation technique in the sinc method yields highly efficient numerical methods for interpolating, quadrature, approximation of transformations, differential and partial differential equations and integral equations. Here, we focus on using the DE transformation based on the sinc collocation method for approximate solution of nonlinear Volterra-Fredholm integral equations.

Recently, Muhammad and Mori [15] established a method of indefinite numerical integration based on DE transformation incorporated into sinc expansion of the integrand which gives results with high efficiency. Muhammad et al. [14] successfully applied the sinc collocation method based on the DE transformation to the numerical solution of one dimensional linear integral equations and found that the error bound is $O(\exp(-cN/\log N)),\ c > 0$. We note that in the standard setup of the sinc numerical methods the errors are known to be $O(\exp(-c\sqrt{N}))$ with some $c > 0$, where N is the number of nodes or bases used in methods. According to Burchard and Höllig [2], this convergence rate for analytic functions is optimal and much faster than that of polynomial order. However, as we pointed out, the errors in the sinc numerical methods based on DE transformation are $O(\exp(-cN/\log N))$ with some $c > 0$, which is also meaningful both theoretically and practically. It has also been found that these error bounds are the best possible in a certain mathematical sense.

The purpose of the present research is to develop the work carried out by Muhammad et al. [14] for the numerical solution of two dimensional nonlinear Volterra-Fredholm integral equations using the DE transformation based on the sinc collocation method. We shall provide an alternative strategy with respect to [14] to show that the convergence rate is of exponential order. We also demonstrate the reliability and efficiency of the proposed scheme by some numerical experiments.

The layout of the paper is as follows: In Section 2, we give basic definitions, assumptions and preliminaries of the sinc approximations and related topics. In Section 3, a sinc collocation method based on DE transformation is considered for numerical solution of nonlinear Volterra-Fredholm integral equations. In Section 4, we provide the order of convergence of the scheme using a new approach, and, finally, Section 5 contains the details of our numerical implementation and some experimental results.

2. Basic definitions and preliminaries. The goal of this section is to recall the notations and definitions of sinc functions from [14, 17], which state some known results and derive useful formulas that are important for other sections.

DEFINITION 2.1. *The exponential integrals* $Ei(a,z)$ *are defined for* $Re(z)>0$ *by*

$$Ei(a,z)=\int_1^{\infty} e^{-sz} z^a dz.$$

This classical definition is extended by analytic continuation to the entire complex plane using, $Ei(a,z)=z^{a-1}\Gamma(1-a,z)$, with the exception of the point 0 in the case of $Ei(1,z)$.

Now, let f be a function defined on $\mathbb{R}$, and $h>0$ is a step size, then the *Whittaker Cardinal* is defined by the series

$$C(f,h)(x)=\sum_{j=-\infty}^{\infty} f(jh)S(j,h)(x),$$

whenever this series converges, and

$$S(j,h)(x)=\frac{\sin[\pi(x-jh)/h]}{\pi(x-jh)/h},$$

is known as jth sinc function.

Throughout this paper, let $d>0$, and $\mathcal{D}_d$ denote the region $\{z=x+iy, \quad |y|<d\}$ in the complex plan $\mathbb{C}$ and ϕ the conformal map of a simply connected domain $\mathcal{D}$ in the complex domain onto $\mathcal{D}_d$, such that $\phi(a)=-\infty, \phi(b)=\infty$, where a,b are boundary points of $\mathcal{D}$. Let ψ denote the inverse map of ϕ, and let the arc Γ, with end points $a,b(a,b\in\Gamma)$, be given by $\Gamma=\psi(-\infty,\infty)$. For $h>0$, let the points x_k on Γ be given by $x_k=\psi(kh), k\in\mathbb{Z}$.

Moreover, let $H^1(\mathcal{D}_d)$ be the family of all functions g is analytic in $\mathcal{D}_d$, such that

$$\mathcal{N}_1(g,\mathcal{D}_d)=\lim_{\epsilon\to 0}\int_{\partial\mathcal{D}_{d(\epsilon)}}|g(t)||dt|<\infty,$$
$$\mathcal{D}_{d(\epsilon)}=\left\{t\in\mathbb{C}, \quad |\mathrm{Re}\, t|<\tfrac{1}{\epsilon}, \quad |\mathrm{Im}\, t|<d(1-\epsilon)\right\}.$$

We recall the following definitions from [14], that will become instrumental in establishing our useful formulas.

DEFINITION 2.2. *A function* g *is said to decay double exponentially, if there exist constants* α *and* C*, such that*

$$|g(t)|\le C\exp(-\alpha\exp|t|), \quad t\in(-\infty,\infty).$$

Equivalently, a function g *is said to decay double exponentially with respect to the conformal map* ϕ*, if there exist positive constants* α *and* C *such that*

$$|g(\phi(t))\phi'(t)|\le C\exp(-\alpha\exp|t|), \quad t\in(-\infty,\infty).$$

Here, we suppose that $K_\phi^\alpha(\mathcal{D}_d)$ denotes the family of functions g where $g(\phi(t))\phi'(t)$ belongs to $H^1(\mathcal{D}_d)$ and decays double exponentially with respect to ϕ. If f belongs to $K_\phi^\alpha(\mathcal{D}_d)$ with respect to ϕ, then we have the following formulas for definite and indefinite integrals based on DE transformation which is given and fully discussed in [14, 15, 25]:

$$\int_a^b f(x)dx=h\sum_{i=-N}^{N} f\left(\phi(ih)\right)\phi'(ih)+O\left(\exp(-\frac{2\pi dN}{\log(2\pi dN/\alpha)})\right),$$

and

$$\int_a^s f(x)dx = h\sum_{i=-N}^{N} f(\phi(ih))\phi'(ih)\left(\frac{1}{2}+\frac{1}{\pi}\,\mathrm{Si}\,(\frac{\pi\phi^{-1}(s)}{h}-i\pi)\right)$$
$$+\,O\left(\frac{\log N}{N}\exp(-\frac{\pi dN}{\log(\pi dN/\alpha)})\right),$$

, where $\mathrm{Si}(t)$ is the Sine integral defined by

$$\mathrm{Si}(t)=\int_0^t \frac{\sin\tau}{\tau}d\tau,$$

and we truncate the infinite summations at $j = \pm N$ in such a way that N and the mesh size h satisfies $h = \frac{1}{N}\log(2\pi dN/\alpha)$ or $\frac{1}{N}\log(\pi dN/\alpha)$, for definite or indefinite integrals, respectively.

Now, let $\phi_j : \mathcal{D}_j \to \mathcal{D}_d$ be conformal maps of simply connected domains $\mathcal{D}_j$ in the complex domain onto $\mathcal{D}_d$ and $\psi_j = {\phi_j}^{-1}$, $(j = 1, 2)$. If $f(x, y)$ is double exponentially with respect to conformal maps ϕ_1 and ϕ_2, respectively, then we can expand $f(x, y)$ in terms of the following sinc cardinal series,

$$f\left(\phi_1(t),\phi_2(s)\right)\phi_1'(t)\phi_2'(s)=\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} f(\phi_1(ih),\phi_2(jl))\phi_1'(ih)\phi_2'(jl)$$
$$\times\,\mathrm{sinc}\,(\frac{t}{h}-i)\,\mathrm{sinc}\,(\frac{s}{l}-j)+e(t,s,h,l,d),$$

where $e(t, s, h, l, d)$ is the remainder term which follows from applying the sinc cardinal series for the function $f(x, y)$ which depends on variables s, t and mesh sizes h and l correspounding to variables x, y and the parameter d as the width of region $\mathcal{D}_d$. By taking the integral of the above expression with respect to x and y, respectively, we have

$$\int_a^x\int_c^d f(x,y)dxdy=\int_{-\infty}^{\phi_1^{-1}(x)}\int_{-\infty}^{\infty} f(\phi_1(t),\phi_2(s))\phi_1'(t)\phi_2'(s)dtds.$$

Carrying out term-wise integration, it follows that

$$\int_a^x\int_c^d f(x,y)dxdy = \tag{2.1}$$
$$hl\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} f(\phi_1(ih),\phi_2(jl))\phi_1'(ih)\phi_2'(jl)(\frac{1}{2}+\frac{1}{\pi}\,\mathrm{Si}\,(\frac{\pi\phi_1^{-1}(x)}{l}-j\pi))$$
$$+O\left(\exp(\frac{-2\pi d}{h})\right)+O\left(l\exp(\frac{-\pi d}{l})\right).$$

3. The numerical analysis of the method. In this section, we derive formulas for numerical solution of nonlinear Volterra-Fredholm integral equations of mixed type (1.1). Without loss of generality and owing to the large variety of kernels and nonlinearities that occur in practice, we assume that in (1.1)

$$F(x,t,\xi,\tau,0)=0,$$
$$F(x,t,\xi,\tau,u(\xi,\tau))=k(x,t,\xi,\tau)H(\xi,\tau,u(\xi,\tau)).$$

This situation arises naturally in many applied problems, e.g., some concrete models in the area of reaction diffusion equations, physical and biological problems; see [1, 3, 18]. Note that for ease of exposition here we will restrict our discussion to the case where Ω is an interval, e.g., $\Omega := [b,c] \subseteq \mathbb{R}$. It will become clear that the following analysis can be extended to regions $\Omega \subseteq \mathbb{R}^2$ and $\Omega \subseteq \mathbb{R}^3$. However, the problem of how to treat (1.1) numerically for general regions $\Omega \subseteq \mathbb{R}^n$, $(n \geq 2)$ will become much more complicated and remains to be solved.

So, the main equation converts to the equation

$$u(x,t) = g(x,t) + \int_a^t \int_b^c k(x,t,\xi,\tau) H(\xi,\tau,u(\xi,\tau)) d\xi d\tau, \tag{3.1}$$

where $u(x,t)$ is an unknown function, $g(x,t)$, $k(x,t,\xi,\tau)$ and $H(\xi,\tau,u(\xi,\tau))$ are given analytic real-valued functions defined respectively on regions $D := [b,c] \times [a,T]$, $S := \{(x,t,\xi,\tau) : b \leq x,\ \xi \leq c;\ a \leq \tau \leq t \leq T\}$ and $D \times \mathbb{R}$ with $H(\xi,\tau,u)$ nonlinear in u and are such that (3.1) possesses a unique solution $u(x,t) \in C(D)$. Furthermore, we assume that the nonlinear function H is bounded which satisfies the generalized Lipschitz condition with constant L.

Applying (2.1) to the function k in (3.1) gives

$$\int_a^t \int_b^c k(x,t,\xi,\tau) H(\xi,\tau,u(\xi,\tau)) d\xi d\tau \approx$$
$$hl \sum_{i=-N}^{N} \sum_{j=-M}^{M} k(x,t,\varepsilon_i,\zeta_j)\phi_1'(ih)\phi_2'(jl)\left(\frac{1}{2} + \frac{1}{\pi}\,\mathrm{Si}\,(\frac{\pi\phi_1^{-1}(x)}{l} - j\pi)\right) H(\varepsilon_i,\zeta_j,u_{ij}),$$

where u_{ij} denotes an approximate value of $u(xi,t_j)$ and

$$\begin{array}{lll}
\xi & = \phi_1(\eta) = \frac{T-a}{2}\tanh(\frac{\pi}{2}\sinh\eta) + \frac{T+a}{2}, & \\
\tau & = \phi_1(\theta) = \frac{T-a}{2}\tanh(\frac{\pi}{2}\sinh\theta) + \frac{T+a}{2}, & \\
x & = \phi_1(\xi), & -\infty \leq \xi \leq \infty, \\
t & = \phi_2(\tau), & -\infty \leq \tau \leq \phi_2^{-1}(x), \\
\varepsilon_i & = \phi_1(ih), & i = -N, \ldots, N, \\
\zeta_j & = \phi_2(jl), & j = -M, \ldots, M.
\end{array}$$

Note that the points $x_i = \phi_1(ih), t_j = \phi_2(jl), i = -N, \ldots, N, j = -M, \ldots, M$, are the sinc points. If we replace the second term on the right hand side of (3.1), with above relation we have

$$u(x,t) - hl \sum_{i=-N}^{N} \sum_{j=-M}^{M} k(x,t,\varepsilon_i,\zeta_j)\phi_1'(ih)\phi_2'(jl)$$
$$\times \left(\frac{1}{2} + \frac{1}{\pi}\,\mathrm{Si}\,(\frac{\pi\phi_1^{-1}(x)}{l} - j\pi)\right) H(\varepsilon_i,\zeta_j,u_{ij}) \approx g(x,y).$$

The dimension of this nonlinear system is $(2N+1)(2M+1)$ and there are $(2N+1)(2M+1)$ unknowns $u_{ij}, i = -N, ..., N, j = -M, ..., M$ to be determined, so we can apply the collocation sinc points $x_k = \phi_1(kh),\ \ t_p = \phi_2(pl),\ \ k = -N, ..., N$, $p = -M, ..., M$. Following [14, p. 274], note that due to this choice of points, the first term, $\frac{\pi\phi_1^{-1}(x)}{l}$ in argument of Si, reduces to $\frac{\pi\phi_1^{-1}(x_k)}{l} = \pi k$.

Substituting x_k into x and t_p into t in above expression and collocating it, we have the following system of $(2N+1)(2M+1)$ nonlinear equations with $(2N+1)(2M+1)$ unknowns u_{ij},

$$\begin{aligned}
&u_{nm} - hl \sum_{i=-N}^{N} \sum_{j=-M}^{M} k(x_n, t_m, \xi_i, \tau_j) \phi_1'(ih) \phi_2'(jl) \\
&\quad \times \left(\frac{1}{2} + \frac{1}{\pi} Si\left(\frac{\pi \phi_1^{-1}(x)}{l} - j\pi\right) \right) H(\varepsilon_i, \zeta_j, u_{ij}) = g(x_m, t_n), \\
&n = -N, ..., N; \;\; m = -M, ..., M.
\end{aligned}$$

Solving the above nonlinear system of equations, we obtain u_{ij} which is an approximate solution of $u(x_i, t_j)$ at sinc points x_i, t_j. However, here due to analytic behavior and non-singularity of the kernels and functions of the equation, we may be obtain the approximate solution of the nonlinear system using any mathematical software, e.g., Maple®; see test problems in Section 5. Furthermore, in order to get an approximate value of $u(x, t)$ at an arbitrary points x, t, we may have to approximate the integral by suitable quadrature. If this is done by interpolatory quadrature formulas with abscissas given by the collocation points, we have

$$u_{NM}(x, t) = g(x, t) + hl \sum_{i=-N}^{N} \sum_{j=-M}^{M} k(x, t, \xi_i, \tau_j) \phi_1'(ih) \phi_2'(jl)\, \eta_{l,j}\, H(\xi_i, \tau_j, u_{ij}), \tag{3.2}$$

where

$$\eta_{l,j} = \left(\frac{1}{2} + \frac{1}{\pi} Si\left(\frac{\pi \phi_1^{-1}(x)}{l} - j\pi\right)\right).$$

4. Error analysis. In this section, we derive the error estimation for two-dimensional nonlinear integral equations of mixed type. In order to estimate the error, we used the supremum norm error between the numerical approximation u_{ij} and the exact solution $u(x_i, t_j)$ at the sinc grid points, which is determined and reported as

$$\|u\| = \sup_{ij} |u_{ij} - u(x_i, t_j)|.$$

Let $u_{NM}(x, y)$ and $u(x, y)$ be the approximate and exact solution of (3.1), respectively, given by (3.2). Also, suppose that $k(., ., s, .)$, $\;k(., ., ., t)$ belong to $K^{\alpha}_{\phi_1}(\mathcal{D}_d)$ and $K^{\alpha}_{\phi_2}(\mathcal{D}_d)$ uniformly for all s, $\;t$, and that $u(\phi_1(t), \phi_2(s))$ is analytic and bounded on $\mathcal{D}_d$.

First, we prove a lemma giving an upper bound for a DE transformation.

LEMMA 4.1. *The function ϕ_1 which is defined as a DE transformation satisfies the following relation:*

$$\sum_{|i|>N} |\phi_1'(ih)| \le \frac{2\exp(-\alpha \exp(hN))}{\alpha h \exp(hN)},$$

where α is a constant, N is a parameter representing the number of terms of the sinc expansion and $h = \frac{1}{N}\log(2\pi d N/\alpha)$.

Proof. It is not difficult to show that for a sufficiently small values $\epsilon > 0$

$$|\phi_1'(t)| = \frac{(b-a)}{2} \frac{\frac{\pi}{2}\cosh t}{\cosh^2(\frac{\pi}{2}\sinh t)} = O\left(\exp(-\frac{\pi(1-\epsilon)}{2}\exp t)\right).$$

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Let $\alpha = \frac{\pi(1-\epsilon)}{2}$, so we have

$$\begin{aligned}
\sum_{|i|>N} |\phi_1'(ih)| &\leq 2 \sum_{i=N+1}^{\infty} |\phi_1'(ih)|, \\
&\leq 2 \int_N^{\infty} \exp(-\alpha \exp(hx)) dx, \\
&\leq \frac{2}{\alpha h \exp(hN)} \int_N^{\infty} \alpha h \exp(hx) \exp(-\alpha \exp(hx)) dx, \\
&= \frac{2 \exp(-\alpha \exp(hN))}{\alpha h \exp(hN)}.
\end{aligned}$$

Note that, for evaluating (2.1) we need to truncate the infinite summations at some $i = \pm N$ and $j = \pm M$. As we have pointed out, the discretization error due to the replacement of the integral with the first infinite summation is approximately equal to $\exp(-2\pi d/N)$, while the truncation error due to the truncation of the first infinite summation from the Definition 2.2 is approximately equal to $\exp(-\alpha \exp Nh)$. This leads to $h = \frac{1}{N} \log(2\pi dN/\alpha)$. $\square$

As a consequence of this lemma, we have the following corollary.

COROLLARY 4.2. *Let $l = \frac{1}{M} \log(\pi dM/\alpha)$, where α is a constant and M is a parameter representing the number of terms of the sinc expansion. Then the function ϕ_2 which is defined as a DE transformation satisfies the following relation:*

$$\sum_{|j|>M} |\phi_2'(jl)| \leq \frac{\exp(-\alpha \exp(lM))}{\alpha l \exp(lM)}.$$

Proof. We refrain from going into details. For evaluating the second infinite summation in (2.1) at some $j = \pm M$, we must truncate the summation the same as in the first summation. Using a similar procedure as outlined in Lemma 4.1, we obtain the result. $\square$

From the mentioned lemma and corollary, we can estimate the error with a different approach with to respect to [14] and obtain the exponential order of convergence in error estimation. The result is summarized as follows.

THEOREM 4.3. *Consider the Volterra-Fredholm integral equation (3.1) and suppose that the hypothesis for the kernel k and functions H and u at the beginning of the previous section holds. Then the sinc collocation approximation error $u(x,t)$ based on DE transformation satisfies*

$$\|u - u_{NM}\| = O\left(\frac{\exp(-cN)}{\log(N)} + \frac{\exp(-c'M)}{\log(M)} \right),$$

where the constants c and c' are independent of N and M.

Proof. Let us set

$$\|u(x,y) - u_{NM}(x,y)\| =$$

$$\| \int_a^x \int_b^c k(x,y,s,t) H(s,t,u(s,t)) ds dt - hl \sum_{i=-N}^{N} \sum_{j=-M}^{M} k_{x,y,\phi_1,\phi_2}\, \eta_{l,j}\, H(\phi_1(ih), \phi_2(jl), u_{ij}) \|$$

$$\leq A + B + C + D,$$

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where

$$A = \| \sum_{i=-\infty}^{\infty} \sum_{|j|<M} k_{x,y,\phi_1,\phi_2}\ \eta_{l,j} \|,$$

$$B = \| \sum_{|i|<N} \sum_{j=-\infty}^{\infty} k_{x,y,\phi_1,\phi_2}\ \eta_{l,j} \|,$$

$$C = \| \sum_{|i|<N} \sum_{|j|<M} k_{x,y,\phi_1,\phi_2}\ \eta_{l,j} - H(\phi_1(ih), \phi_2(jl), u_{ij}) \|,$$

$$D = \| \sum_{|i|>N} \sum_{|j|>M} k_{x,y,\phi_1,\phi_2}\ \eta_{l,j} \|,$$

and

$$k_{x,y,\phi_1,\phi_2} = k(x, y, \phi_1(ih), \phi_2(jl))\phi_1'(ih)\phi_2'(jl)\ H(\phi_1(ih), \phi_2(jl), u(\phi_1(ih), \phi_2(jl))),$$

with $\eta_{l,j}$ defined in (3.2).

Now, for each term of the above expression we can find an upper bound. Due to the conditions on the kernel and transformation, we have for each i

$$\sum_{|j|<M} \|k_{x,y,\phi_1,\phi_2}\eta_{l,j}\| < L_1.$$

Using Lemma 4.1, for the first term of the decomposition we get

$$A = L_1 \sum_{i=-\infty}^{\infty} |\phi_1'(ih)| < L_1 \frac{2\exp(-\alpha \exp(hN))}{\alpha h \exp(hN)},$$

and similarly, according to what we have done, the upper bound for the second term is

$$B = L_2 \sum_{j=-\infty}^{\infty} |\phi_2'(jl)| < L_2 \frac{\exp(-\alpha \exp(lM))}{\alpha l \exp(lM)}.$$

For the third term by definition of reported norm and assumptions of the function H, it follows that

$$C = \sum_{|i|<N} \sum_{|j|<M} \|k_{x,y,\phi_1,\phi_2}\ \eta_{l,j} - H(\phi_1(ih), \phi_2(jl), u_{ij})\| < (2N+1)L_3\ L\ \|u\|,$$

where for each i

$$L_3 = \sum_{|j|<M} \|k_{x,y,\phi_1,\phi_2}\ \eta_{l,j}\| / \|H\|.$$

Finally, for the last term, using the assumptions on the kernel and the definition of double exponential transformation which is given in the beginning of Section 2, we have

$$\|k(x, y, \phi_1(ih), \phi_2(jl))\phi_1'(ih)\| < c\ \exp(-\alpha \exp(ih)),$$
$$\|k(x, y, \phi_1(ih), \phi_2(jl))\phi_2'(jl)\| < c\ \exp(-\alpha \exp(jl)).$$

So, we can write

$$\begin{aligned} D &= \sum_{|i|>N} \sum_{|j|>M} \|k(x, y, \phi_1(ih), \phi_2(jl))\phi_1'(ih)\phi_2'(jl)\| \\ &< \int_{R_+} \int_{R_+} c' \exp(-\alpha \exp(hx)) \exp(-\alpha \exp(ly)) dx dy \\ &< c' \int_{R_+} \exp(-\alpha \exp(hx)) dx \int_{R_+} \exp(-\alpha \exp(ly)) dy \\ &< c' Ei(1, \alpha e^{hx}) Ei(1, \alpha e^{ly}), \end{aligned}$$

where $c' = \frac{c^2}{hl} L_4,\ L_4 = \sup_{ij} |H(\phi_1(ih), \phi_2(lj), u(\phi_1(jh), \phi_2(il))\ \eta_{h,j}|$.

Finally, by considering the errors occuring above, we have

$$\|u(x,y) - u_{NM}(x,y)\| \le L_1 \left[\frac{2\exp(-\alpha\exp(hN))}{\alpha h \exp(hN)}\right] + L_2 \left[\frac{\exp(-\alpha\exp(lM))}{\alpha l \exp(lM)}\right] + (2N+1) L_3\ L\ \|u\| + c' Ei(1, \alpha e^{hx}) Ei(1, \alpha e^{ly}).$$

Observing now that $hN = \log(2\pi dN/\alpha)$ and $lM = \log(\pi dM/\alpha)$, we obtain the estimate stated in the theorem. ☐

5. Numerical experiments. In this section, the theoretical results of the previous sections are illustrated by means of some numerical examples. We consider the following two test problems.

EXAMPLE 5.1. Consider the following nonlinear Volterra-Fredholm integral equation,

$$u(x,y) = f(x,y) + \int_0^x \int_0^1 G(x,y,s,t) \sin u(s,t) ds dt, \qquad (x,y) \in [0,1] \times [0,1],$$

where $G(x,y,s,t) = xy(s+t)$ and $f(x,y) = -y(x^2 - \sin x - x \sin x)$. The exact solution of this problem is $u(x,y) = xy$.

EXAMPLE 5.2. (From [1] and [5].)

$$u(x,y) = f(x,y) + \int_0^y \int_0^1 G(x,y,s,t)(1 - e^{-u(s,t)}) ds dt, \quad (x,y) \in [0,1] \times [0,1],$$

where $G(x,y,s,t) = \frac{x(1-s^2)}{(1+y)(1+t^2)}$ and $f(x,y) = -\log\left(1 + \frac{xy}{1+y^2}\right) + \frac{xy^2}{8(1+y)(1+y^2)}$, with the exact solution $u(x,y) = -\log\left(1 + \frac{xy}{1+y^2}\right)$.

Here, since $a = 0,\ T = 1,\ b = 0,\ c = 1$, we employed

$$x = \phi_1(t) = \tfrac{1}{2}\tanh(\tfrac{\pi}{2}\sinh t) + \tfrac{1}{2}, \qquad x_i = \phi_1(ih),$$

$$y = \phi_2(s) = \tfrac{1}{2}\tanh(\tfrac{\pi}{2}\sinh s) + \tfrac{1}{2}, \qquad y_j = \phi_2(jl).$$

Calculations were carried out in double precision arithmetic with Maple® software. In order to compare the efficiency of the procedure and observe the convergence behavior, we used $N = M = 3$ for the number of function evaluations and set the step size $h = l$ by $h = l = 2^{-k}, k = 2, 4, \ldots$.

The maximum absolute errors are given in Tables 5.1 and 5.2 for Examples 5.1 and 5.2, respectively, and Figures 5.2 and 5.4 show the behavior rate of convergence. Our numerical results confirm the results obtained by Brunner [1] and Guoqiang [5]. However, due to the fact that the convergence rate here is based on the double exponential transformation, it is much faster than that of polynomial order of any expansion scheme such as [1] and [5].

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TABLE 5.1
Max-Error for Example 5.1.

k	$h = l$	Max-Error
2	2.500000E-01	1.470588E-03
4	6.250000E-02	6.703078E-04
8	3.906250E-03	2.485672E-06
10	9.765625E-04	8.412087E-08
13	1.220703E-04	4.907670E-11
16	1.525879E-05	-7.672500E-13
17	7.629395E-06	-1.498800E-15

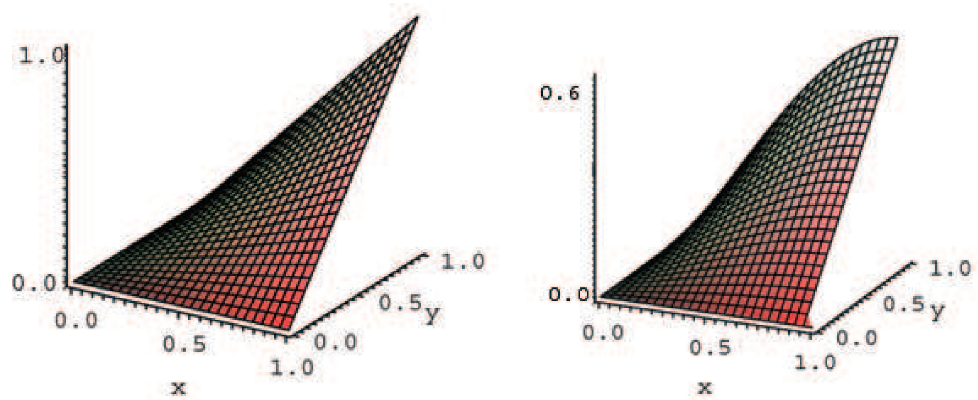

FIG. 5.1. *The exact (left) and approximate (right) solution of Example 5.1.*

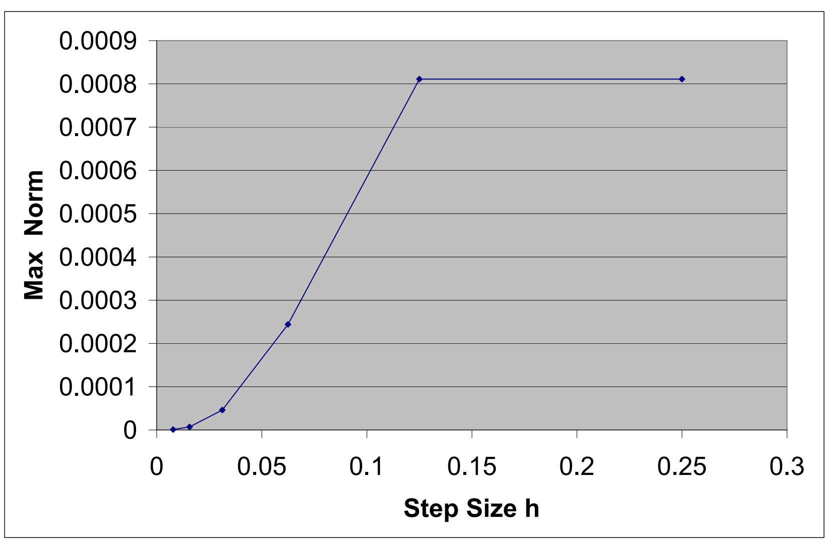

FIG. 5.2. *Rate of convergence of the scheme for Example 5.1.*

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TABLE 5.2
Max-Error for Example 5.2.

k	$h = l$	Max-Error
2	2.500000E-01	8.112627E-04
4	6.250000E-02	2.441878E-04
8	3.906250E-03	9.571530E-08
10	9.765625E-04	1.003290E-09
13	1.220703E-04	7.690000E-13
14	6.103516E-05	6.630000E-14
17	7.629395E-06	3.000000E-17

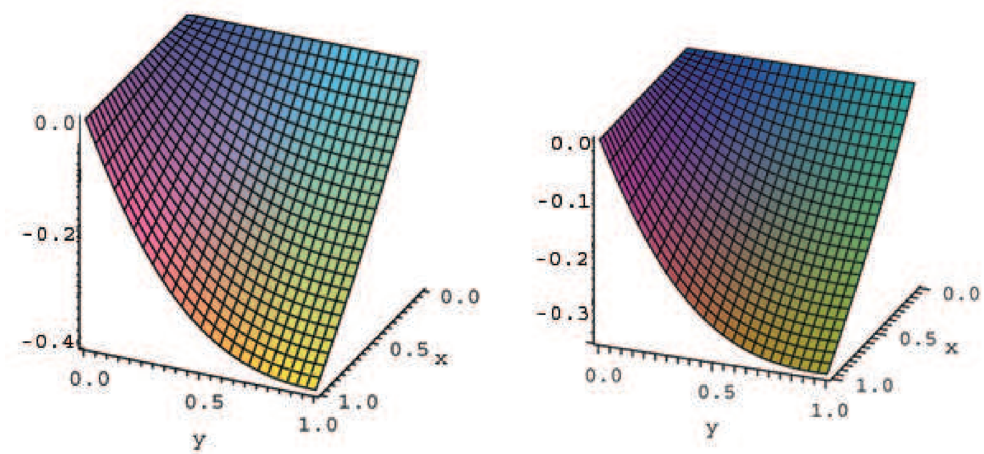

FIG. 5.3. *The exact (left) and approximate (right) solution of Example 5.2.*

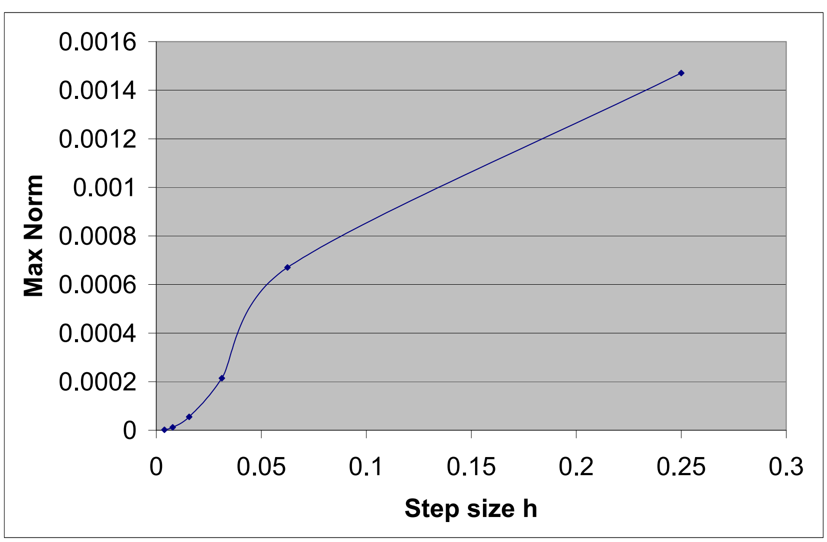

FIG. 5.4. *Rate of convergence of the scheme for Example 5.2.*

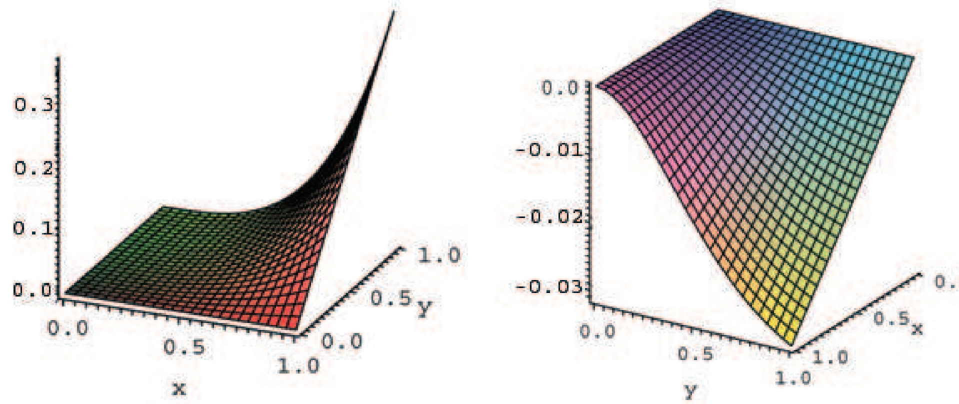

FIG. 5.5. *Error caused of Examples 5.1 (left) and 5.2 (right).*

6. Conclusion. In this paper, the sinc collocation method based on the double exponential transformation was applied to two dimensional nonlinear Voltera-Fredholm integral equations. The scheme and its transformation is a natural and promising strategy in the numerical solution of mixed type integral equations. We observe here that significant improvements have been obtained compared with the numerical results reported by others. We can improve the accuracy of the solution by selecting the appropriate parameters and the large values of N, M.

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