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ON THE EQUIVALENCE OF PRIMAL AND DUAL SUBSTRUCTURING PRECONDITIONERS*

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Abstract. After a short historical review, we present four popular substructuring methods: FETI-1, BDD, FETI-DP, BDDC, and derive the primal versions to the two FETI methods, called P-FETI-1 and P-FETI-DP, as proposed by Fragakis and Papadrakakis. The formulation of the BDDC method shows that it is the same as P-FETI-DP and the same as a preconditioner introduced by Cros. We prove the equality of eigenvalues of a particular case of the FETI-1 method and of the BDD method by applying a recent abstract result by Fragakis.

Key words. domain decomposition methods, iterative substructuring, finite element tearing and interconnecting, balancing domain decomposition, BDD, BDDC, FETI, FETI-DP, P-FETI-DP.

AMS subject classifications. 65N55, 65M55, 65Y05.

1. Introduction. Substructuring methods are among the most popular and widely used methods for the solution of systems of linear algebraic equations obtained by finite element discretization of second order elliptic problems. This paper provides a review of recent results on the equivalence of several substructuring methods in a common framework, complemented by some details not published previously.

In Section 2 we give a brief review of the history of these methods. After introducing the basic concepts of substructuring in Section 3, we formulate the dual methods, FETI-1 and FETI-DP in Section 4, and derive their primal versions, P-FETI-1 and P-FETI-DP, originally introduced in [20]. However the derivation was omitted in [20]. Next, in Section 5, we formulate the primal methods, BDD and BDDC . Finally, we study connections between the methods in Section 6. We revisit our recent proof that the P-FETI-DP is in fact the same method as the BDDC [35] and the preconditioner by Cros [8]. Next, we translate some of the abstract ideas from [19, 20] into a framework usual in the domain decomposition literature. We recall from [20] that for a certain variant of FETI-1, the P-FETI-1 method is the same algorithm as BDD. Then we derive a recent abstract result by Fragakis [19] in this special case to show that the eigenvalues of BDD and that particular version of FETI-1 are the same. It is notable that this is the variant of FETI-1 devised to deal with difficult, heterogeneous problems [1].

2. Historical remarks. In this section, we provide a short overview of iterative substructuring, also known as non-overlapping domain decomposition. Rather than attempting a complete unbiased survey, our review centers on works connected to the BDD and FETI theory by the second author and collaborators.

Consider a second order, selfadjoint, positive definite elliptic problem, such as the Laplace equation or linearized elasticity, discretized by finite elements with characteristic element size h. Given sufficient boundary conditions, the global stiffness matrix is nonsingular, and

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its condition number grows as $O(h^{-2})$ for $h \to 0$. However, if the domain is divided into substructures consisting of disjoint unions of elements and the interior degrees of freedom of each substructure are eliminated, the resulting matrix on the boundary degrees of freedom has a condition number that grows only as $O(H^{-1}h^{-1})$, where $H \gg h$ is the characteristic size of the substructure. This fact has been known early on; see Keyes and Gropp [22] and, for a recent rigorous treatment, Brenner [4]. The elimination of the interior degrees of freedom is also called *static condensation*, and the resulting reduced matrix is called the *Schur complement*. Because of the significant decrease of the condition number, one can substantially accelerate iterative methods by investing some work up front in the Cholesky decomposition of the stiffness matrix on the interior degrees of freedom, and then just run back substitution in each iteration. The finite element matrix is assembled separately in each substructure. This process is called *subassembly*. The elimination of the interior degrees of freedom in each substructure can be done independently, which is important for parallel computing: each substructure can be assigned to an independent processor. The substructures are then treated as large elements, with the Schur complements playing the role of the local stiffness matrices of the substructures; see [22, 43] for more details.

The process just described is the background of *primal iterative substructuring methods*. Here, the condition that the values of degrees of freedom common to several substructures coincide is enforced strongly, by using a single variable to represent them. The improvement of the condition number from $O(h^{-2})$ to $O(H^{-1}h^{-1})$, straightforward implementation, and the potential for parallel computing explain the early popularity of iterative substructuring methods [22]. However, further preconditioning is needed. Perhaps the most basic preconditioner for the reduced problem is a diagonal one. Preconditioning of a matrix by its diagonal helps to take out the dependence on scaling and variation of coefficients and grid sizes. But the diagonal of the Schur complement is expensive to obtain. It is usually better to avoid computing the Schur complement explicitly and only use multiplication by the reduced substructure matrices, which can be implemented by solving a *Dirichlet* problem on each substructure. Probing methods (Chan and Mathew [6]) use such matrix-vector multiplication to estimate the diagonal entries of the Schur complement.

In *dual iterative substructuring methods*, also called FETI methods, the condition that the values of degrees of freedom common to several substructures coincide is enforced weakly, by Lagrange multipliers. The original degrees of freedom are then eliminated, resulting in a system for the Lagrange multipliers, with the system operator consisting essentially of an assembly of the inverses of the Schur complements. Multiplication by the inverses of the Schur complements can be implemented by solving a *Neumann* problem on each substructure. The assembly process is modified to ensure that the Neumann problems are consistent, giving rise to a natural coarse problem. The system for the Lagrange multipliers is solved again iteratively. This is the essence of the FETI method by Farhat and Roux [18], later called FETI-1. The condition number of the FETI-1 method with diagonal preconditioning grows as $O(h^{-1})$ and is bounded independently of the number of substructures; see Farhat, Mandel, and Roux [17]. For a small number of substructures, the distribution of the eigenvalues of the iteration operator is clustered at zero, resulting in superconvergence of conjugate gradients; however, for more than a handful of substructures, the superconvergence is lost and the speed of convergence is as predicted by the $O(h^{-1})$ growth of the condition number [17].

For large problems and large number of substructures, *asymptotically optimal preconditioners* are needed. These preconditioners result typically in condition number bounds of the form $O(\log^{\alpha}(1 + H/h))$ (the number 1 is there only to avoid the value $\log 1 = 0$). In particular, the condition number is bounded independently of the number of substructures and the bounds grow only slowly with the substructure size. Such preconditioners require a

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coarse problem, and *local preconditioning* that inverts approximately (but well enough) the diagonal submatrices associated with segments of the interfaces between the subtructures or the substructure matrices themselves. The role of the local preconditioning is to slow down the growth of the condition number as $h \to 0$, while the role of the coarse problem is to provide global exchange of information in order to bound the condition number independently of the number of substructures. Many such asymptotically optimal primal methods were designed in the 1980s and 1990s; e.g., Bramble, Pasciak, and Schatz [2, 3], Dryja [11], Dryja, Smith, and Widlund [13], Dryja and Widlund [14], and Widlund [46]. However, those algorithms require additional assumptions and information that may not be readily available from finite element software, such as an explicit assumption that the substructures form a coarse triangulation and that one can build coarse linear functions from its vertices.

Practitioners desire methods that work algebraically with arbitrary substructures, even if a theory may be available only in special cases (first results on extending the theory to quite arbitrary substructures are given in Dohrmann, Klawonn, and Widlund [10] and Klawonn, Rheinbach, and Widlund [23]). They also prefer methods formulated in terms of the substructure matrices only, with minimal additional information. In addition, the methods should be robust with respect to various irregularities of the problem. Two such methods have emerged in early 1990s: the Finite Element Tearing and Interconnecting (FETI) method by Farhat and Roux [18], and the Balancing Domain Decomposition (BDD) by Mandel [31]. Essentially, the FETI method (with the Dirichlet preconditioner) preconditions the assembly of the inverses of the Schur complements by an assembly of the Schur complements, and the BDD method preconditions assembly of Schur complements by an assembly of the inverses, with a suitable coarse problem added. Of course, the assembly weights and other details play an essential role.

The BDD method added a coarse problem to the local Neumann-Neumann preconditioner by DeRoeck and Le Tallec [41], which consisted of the assembly (with weights) of pseudoinverses of the local matrices of the substructure. Assembling the inverses of the local matrices is an idea similar to the Element-by-Element (EBE) method by Hughes et al. [21]. The method was called Neumann-Neumann because the preconditioner requires solution of Neumann problems on all substructures, in contrast to an earlier Neumann-Dirichlet method, which, for a problem with two substructures, required the solution of a Neumann problem on one and a Dirichlet problem on the other [46]. The coarse problem in BDD was constructed from the natural nullspace of the problem (constant for the Laplace equation, rigid body motions for elasticity) and solving the coarse problem guaranteed consistency of local problems in the preconditioner. The coarse correction was then imposed variationally, just as the coarse correction in multigrid methods. The $O(\log^2(1 + H/h))$ bound was then proved [31].

In the FETI method, solving the local problems on the substructures to eliminate the original degrees of freedom has likewise required working in the complement of the nullspace of the substructure matrices, which gave a rise to a natural coarse problem. Since the operator employs inverse of the Schur complement (solving a Neumann problem) an optimal preconditioner employs multiplication by the Schur complement (solving a Dirichlet problem), hence the preconditioner was called the Dirichlet preconditioner. The $O(\log^3(1 + H/h))$ bound was proved by Mandel and Tezaur [36], and $O(\log^2(1 + H/h))$ for a certain variant of the method by Tezaur [44]; see also Klawonn and Widlund [25] for further discussion.

Because the interface to the BDD and FETI method required only the multiplication by the substructure Schur complements, solving systems with the substructure Schur complements, and information about the substructure nullspace, the methods got quite popular and widely used. In Cowsar, Mandel, and Wheeler [7], the multiplications were implemented as solution of mixed problems on substructures. However, neither the BDD nor the FETI

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method worked well for 4th order problems (plate bending). The reason was essentially that both methods involve “tearing” a vector of degrees of freedom reduced to the interface, and, for 4th order problems, the “torn” function has energy that grows as negative power of h, unlike for 2nd order problems, where the energy grows only as a positive power of $\log 1/h$. The solution was to prevent the “tearing” by fixing the function at the substructure corners; then only its derivative along the interface gets “torn”, which has energy again only of the order $\log 1/h$. Preventing such “tearing” can be generally accomplished by increasing the coarse space, since the method runs in the complement to the coarse space. For the BDD method, this was relatively straightforward, because the algebra of the BDD method allows arbitrary enlargement of the coarse space. The coarse space that does the trick contains additional functions with spikes at corners, defined by fixing the value at the corner and minimizing the energy. With this improvement, $O(\log^2(1 + H/h))$ condition number bound was proved and fast convergence was recovered for 4th order problems (Le Tallec, Mandel, and Vidrascu [28, 29]). In the FETI method, unfortunately, the algebra requires that the coarse space is made of exactly the nullspace of the substructure matrices, so a simple enlargement of the coarse space is not possible. Therefore, a version of FETI, called FETI-2, was developed by Mandel, Tezaur, and Farhat [38], with a second correction by coarse functions concentrated at corners, wrapped around the original FETI method variationally much like BDD, and the $O(\log^3(1 + H/h))$ bound was proved again. However, the BDD and FETI methods with the modifications for 4th order problems were rather unwieldy (especially FETI-2), and, consequently, not as widely used.

The breakthrough came with the Finite Element Tearing and Interconnecting - Dual, Primal (FETI-DP) method by Farhat et al. [15], which enforced the continuity of the degrees of freedom on a substructure corner as in the primal method by representing them by one common variable, while the remaining continuity conditions between the substructures are enforced by Lagrange multipliers. The primal variables are again eliminated and the iterations run on the Lagrange multipliers. The elimination process can be organized as solution of sparse system and it gives rise to a natural coarse problem, associated with substructure corners. In 2D, the FETI-DP method was proved to have condition number bounded as $O(\log^2(1 + H/h))$ both for 2nd order and 4th order problems by Mandel and Tezaur [37]. However, the method does not converge as well in 3D and averages over edges or faces of substructures need to be added as coarse variables for fast convergence (Klawonn, Widlund, and Dryja [27], Farhat, Lesoinne, and Pierson [16]), and the $O(\log^2(1 + H/h))$ bound can then be proved [27].

The Balancing Domain Decomposition by Constraints (BDDC) was developed as a primal alternative the FETI-DP method by Dohrmann [9]. The BDDC method imposes the equality of coarse degrees of freedom on corners and of averages by constraints. In the case of only corner constraints, the coarse basis functions are the same as in the BDD method for 4th order problems from [28, 29]. The bound $O(\log^2(1 + H/h))$ for BDDC was first proved by Mandel and Dohrmann [33]. The BDDC and the FETI-DP are currently the most advanced versions of the BDD and FETI families of methods.

The convergence properties of the BDDC and FETI-DP methods were quite similar, yet it came as a surprise when Mandel, Dohrmann, and Tezaur [34] proved that the spectra of their preconditioned operators are in fact identical, once all the components are same. This result came at the end of a long chain of ties discovered between BDD and FETI type method. Algebraic relations between FETI and BDD methods were pointed out by Rixen et al. [40], Klawonn and Widlund [25], and Fragakis and Papadrakakis [20]. An important common bound on the condition number of both the FETI and the BDD method in terms of a single inequality was given by Klawonn and Widlund [25]. Fragakis and Papadrakakis [20], who

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derived certain primal versions of FETI and FETI-DP preconditioners (called P-FETI-1 and P-FETI-DP), have also observed that the eigenvalues of BDD and a certain version of FETI are identical along with the proof that the primal version of this particular FETI algorithm gives a method same as BDD. The proof of equality of eigenvalues of BDD and FETI was given just recently in more abstract framework by Fragakis [19]. Mandel, Dohrmann, and Tezaur [34] have proved that the eigenvalues of BDDC and FETI-DP are identical and they have obtained a simplified and fully algebraic version (i.e., with no undetermined constants) of a common condition number estimate for BDDC and FETI-DP, similar to the estimate by Klawonn and Widlund [25] for BDD and FETI. Simpler proofs of the equality of eigenvalues of BDDC and FETI-DP were obtained by Li and Widlund [30], and by Brenner and Sung [5], who also gave an example when BDDC has an eigenvalue equal to one but FETI-DP does not. A primal variant of P-FETI-DP was proposed by Cros [8], giving a conjecture that BDDC and P-FETI-DP is in fact the same method, which was first shown on a somehow more abstract level in our recent work [35].

It is interesting to note that the choice of assembly weights in the BDD preconditioner was known at the very start from the work of DeRoeck and Le Tallec [41] and before, while the choice of weights for FETI type method is much more complicated. A correct choice of weights is essential for the robustness of the methods with respect to scaling the matrix in each substructure by an arbitrary positive number (the "independence of the bounds on jumps in coefficients"). For the BDD method, such convergence bounds were proved by Mandel and Brezina [32], using a similar argument as in Sarkis [42] for Schwarz methods; see also Dryja, Sarkis, and Widlund [12]. For the FETI methods, a proper choice of weights was discovered only much later; see Rixen and Farhat [39], Farhat, Lesoinne and Pierson [16] for a special case, Klawonn and Widlund [25] for a more general case and convergence bounds, and a detailed discussion in Mandel, Dohrmann, and Tezaur [34].

3. Substructuring components for a model problem. We first show how the spaces and operators we will work with arise in the standard substructuring theory for a model problem obtained by a discretization of the second order elliptic problem. Consider a bounded domain $\Omega \subset \mathbb{R}^d$ decomposed into nonoverlapping subdomains (alternatively called substructures) denoted Ω_i, $i = 1, ..., N$, which form a conforming triangulation of the domain Ω. Each substructure is a union of a uniformly bounded number of Lagrangean $P1$ or $Q1$ finite elements, such that the nodes of the finite elements between substructures coincide. The boundary of Ω_i is denoted by $\partial\Omega_i$. The nodes contained in the intersection of at least two substructures are called boundary nodes. The union of all boundary nodes of all substructures is called the interface Γ, and Γ_i is the interface of substructure Ω_i. The space of vectors of local degrees of freedom on Γ_i is denoted by W_i and $W = W_1 \times \cdots \times W_N$. Let $S_i : W_i \to W_i$ be the Schur complement operator obtained by eliminating all interior degrees of freedom of Ω_i, i.e., those that do not belong to interface Γ_i. We assume that the matrices S_i are symmetric positive semidefinite and consider global vectors and matrices in the block form

$$w = \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix}, \quad w \in W, \quad S = \begin{bmatrix} S_1 & & \\ & \ddots & \\ & & S_N \end{bmatrix}. \tag{3.1}$$

The problem we wish to solve is the constrained minimization of energy,

$$\frac{1}{2}a(u,u) - \langle r, u\rangle \to \min, \qquad \text{subject to } u \in \widehat{W}, \tag{3.2}$$

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where $\widehat{W} \subset W$ is the space of all vectors of degrees of freedom on the substructures that coincide on the interfaces, and the bilinear form

$$a(u,v) = \langle Su, v \rangle, \quad \forall u, v \in W,$$

is assumed to be positive definite on $\widehat{W}$. In the variational form, problem (3.2) can be written as

$$u \in \widehat{W} : a(u,v) = \langle r, v \rangle, \quad \forall v \in \widehat{W}. \tag{3.3}$$

The global Schur complement $\widehat{S} : \widehat{W} \mapsto \widehat{W}'$ associated with a is defined by

$$a(u,v) = \langle \widehat{S}u, v \rangle, \quad \forall u, v \in \widehat{W}.$$

Defining R as the natural embedding of the space $\widehat{W}$ into the space W, i.e.,

$$R : \widehat{W} \to W, \quad R : u \in \widehat{W} \longmapsto u \in W, \tag{3.4}$$

we can write (3.3) equivalently as the system of linear algebraic equations

$$\widehat{S}u = r, \qquad \text{where } \widehat{S} = R^T S R. \tag{3.5}$$

The BDDC and FETI-DP, as the two-level preconditioners, are characterized by the selection of certain *coarse degrees of freedom*, such as values at the corners and averages over edges or faces of substructures; for their general definition see, e.g., [26]. So, we define $\widetilde{W} \subset W$ as the subspace of all functions such that the values of any coarse degrees of freedom have a common value for all relevant substructures and vanish on $\partial\Omega$, and such that

$$\widehat{W} \subset \widetilde{W} \subset W.$$

The space $\widetilde{W}$ has to be selected in the design of the preconditioner so that the bilinear form $a(\cdot,\cdot)$ is positive definite on $\widetilde{W}$. The operator $\widetilde{S} : \widetilde{W} \to \widetilde{W}'$ associated with a is defined by

$$a(u,v) = \langle \widetilde{S}u, v \rangle, \quad \forall u, v \in \widetilde{W}.$$

REMARK 3.1. The idea to restrict the bilinear form $a(\cdot,\cdot)$ from the space W into the subspace $\widetilde{W}$ is closely related to the concept of subassembly, as employed in [30].

In formulation of dual methods from the FETI family, we introduce the matrix

$$B = [B_1, \ldots, B_N],$$

which enforces the continuity across substructure interfaces and it is defined as follows: each row B corresponds to a degree of freedom common to a pair of substructures i and j. The entries of the row are zero except for one $+1$ in the block i and one -1 in the block j, so that the condition

$$Bu = 0 \Longleftrightarrow u \in \widehat{W},$$

and using (3.4), clearly

$$BR = 0. \tag{3.6}$$

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An important ingredient of substructuring methods is the averaging operator $E : W \to \widehat{W}$ defined as

$$E = R^T D_P,$$

where $D_P : W \to W$ is a given weight matrix such that the decomposition of unity property holds,

$$ER = I. \tag{3.7}$$

In terms of substructuring, E is an averaging operator that maps the substructure local degrees of freedom to global degrees of freedom.

The last ingredient is the matrix B_D constructed from B as

$$B_D = [D_{D1}B_1, \ldots, D_{DN}B_N],$$

where the matrices D_{Di} are determined from D_P; see [27, 34] for details.

Finally, we shall assume (cf. [34, equation (10)]) that

$$B_D^T B + RE = I, \tag{3.8}$$

which easily implies $EB_D^T B = E(I - RE) = E - ERE = 0$, and so

$$B^T B_D E^T = 0. \tag{3.9}$$

4. P-FETI family of methods. We review the FETI-1 and FETI-DP preconditioners followed in each case by a formulation of their primal versions denoted as P-FETI-1 and P-FETI-DP, respectively.

4.1. P-FETI-1. In the case of the FETI-1 method, the problem (3.2) is formulated as minimization of total subdomain energy subject to the continuity condition

$$\frac{1}{2}a(w, w) - \langle f, w\rangle \to \min, \quad \text{subject to } w \in W, \quad Bw = 0, \tag{4.1}$$

which is equivalent to a saddle point system: find $(w, \lambda) \in W \times \Lambda$ such that

$$\begin{array}{lcl} Sw + B^T\lambda & = & f, \\ Bw & = & 0. \end{array} \tag{4.2}$$

First, note that S is invertible on $\operatorname{null} B$ and λ is unique up to a component in $\operatorname{null} B^T$, so Λ is selected to be $\operatorname{range} B$. Let Z be matrix with linearly independent columns, such that

$$\operatorname{range} Z = \operatorname{null} S. \tag{4.3}$$

Since S is semi-definite, it must hold for the first equation to be solvable that

$$f - B^T\lambda \in \operatorname{range} S = (\operatorname{null} S)^\perp = (\operatorname{range} Z)^\perp = \operatorname{null} Z^T,$$

so, equivalently, we require that

$$Z^T(f - B^T\lambda) = 0. \tag{4.4}$$

Eliminating w from the first equation of (4.2) as

$$w = S^+(f - B^T\lambda) + Za, \tag{4.5}$$

substituting in the second equation of (4.2) and rewriting (4.4), we get

$$\begin{aligned} BS^+B^T\lambda - BZa &= BS^+f, \\ -Z^TB^T\lambda &= -Z^Tf. \end{aligned}$$

Denoting $G = BZ$ and $F = BS^+B^T$ this system becomes

$$\begin{aligned} F\lambda - Ga &= BS^+f, \\ -G^T\lambda &= -Z^Tf. \end{aligned} \tag{4.6}$$

Multiplying the first equation by $(G^TQG)^{-1}G^TQ$, where Q is some symmetric and positive definite scaling matrix, we can compute a as

$$a = (G^TQG)^{-1}G^TQ(F\lambda - BS^+f). \tag{4.7}$$

The first equation in (4.6) thus becomes

$$F\lambda - G(G^TQG)^{-1}G^TQ(F\lambda - BS^+f) = BS^+f. \tag{4.8}$$

Introducing

$$P = I - QG(G^TQG)^{-1}G^T,$$

as the Q-orthogonal projection onto $\operatorname{null} G^T$, we get that (4.8) corresponds to the first equation in (4.6) multiplied by P^T. So, the system (4.6) can be written in the decoupled form as

$$\begin{aligned} P^TF\lambda &= P^TBS^+f, \\ G^T\lambda &= Z^Tf. \end{aligned}$$

The initial value of λ is chosen to satisfy the second equation in (4.6), so

$$\lambda_0 = QG(G^TQG)^{-1}Z^Tf. \tag{4.9}$$

Substituting λ_0 into (4.7) gives initial value of a as

$$a_0 = (G^TQG)^{-1}G^TQ(F\lambda_0 - BS^+f). \tag{4.10}$$

Since we are looking for $\lambda \in \operatorname{null} G^T$, the FETI-1 method is a preconditioned conjugate gradient method applied to the system

$$P^TFP\lambda = P^TBS^+f. \tag{4.11}$$

In the primal version of the FETI-1 preconditioner, the assembled and averaged solution u is obtained from (4.5), using equations (4.10) and (4.9), as

$$\begin{aligned} u &= Ew \\ &= E\left[S^+(f - B^T\lambda_0) + Za_0\right] \\ &= E\left[S^+(f - B^T\lambda_0) + Z(G^TQG)^{-1}G^TQ(F\lambda_0 - BS^+f)\right] \\ &= E\left[S^+(f - B^T\lambda_0) + Z(G^TQG)^{-1}G^TQ(BS^+B^T\lambda_0 - BS^+f)\right] \\ &= E\left[I - Z(G^TQG)^{-1}G^TQB\right]S^+(f - B^T\lambda_0) \\ &= E\left[(I - Z(G^TQG)^{-1}G^TQB)S^+(I - B^TQG(G^TQG)^{-1}Z^T)\right]E^Tr \\ &= EH^TS^+HE^Tr \\ &= M_{\text{P-FETI}}r, \end{aligned}$$

where we have denoted by

$$H = I - B^T QG(G^T QG)^{-1} Z^T, \tag{4.12}$$

and so

$$M_{\text{P-FETI}} = EH^T S^+ HE^T, \tag{4.13}$$

is the associated primal preconditioner P-FETI-1, same as [20, equation (79)].

4.2. P-FETI-DP. In the case of the FETI-DP, the problem (3.2) is formulated as minimization of total subdomain energy subject to the continuity condition

$$\frac{1}{2}a(w,w) - \langle f, w\rangle \to \min, \quad \text{subject to } w \in \widetilde{W}, \quad Bw = 0. \tag{4.14}$$

Compared to the formulation of FETI-1 in (4.1), we have now used the subspace $\widetilde{W} \subset W$ such that the operator $\widetilde{S}$ associated with $a(\cdot,\cdot)$ on the space $\widetilde{W}$ is positive-definite. In this case, (4.14) is equivalent to setting up a saddle point system: find $(w,\lambda) \in \widetilde{W} \times \Lambda$ such that

$$\begin{array}{lcl} \widetilde{S}w + B^T\lambda & = & f, \\ Bw & = & 0. \end{array} \tag{4.15}$$

Since $\widetilde{S}$ is invertible on $\widetilde{W}$, solving for w from the first and substituting into the second equation of (4.15), we get

$$B\widetilde{S}^{-1}B^T\lambda = B\widetilde{S}^{-1}f,$$

which is the dual system to be solved by preconditioned conjugate gradients, with the Dirichlet preconditioner defined by

$$M_{\text{FETI-DP}} = B_D \widetilde{S} B_D^T.$$

Next, we will derive the P-FETI-DP preconditioner using the original paper by Farhat et. al. [15] in order to verify the P-FETI-DP algorithm given in [20, equation (90)] for the corner constraints. We split the global vector of degrees of freedom u into the vector of global coarse degrees of freedom, denoted by u_c, and the vector of remaining degrees of freedom, denoted by u_r. We note that we could perform a change of basis (cf. [24, 26, 30]) to make all primal constraint (such as averages over edges or faces) explicit, i.e., each coarse degrees of freedom would correspond to an explicit degree of freedom in the vector u_c. Thus, we decompose the space $\widetilde{W}$ as (cf. [34, Remark 5])

$$\widetilde{W} = \widetilde{W}_c \oplus \widetilde{W}_r, \tag{4.16}$$

where the space $\widetilde{W}_c$ consists of functions that are continuous across interfaces, have a nonzero value at one coarse degree of freedom at a time, and zero at other coarse degrees of freedom, and the space $\widetilde{W}_r$ consists of functions with coarse degrees of freedom equal to zero. The solution splits into the solution of the global coarse problem in the space $\widetilde{W}_c$ and the solution of independent subdomain problems on the space $\widetilde{W}_r$.

Let $R_c^{(i)}$ be a map of global coarse variables to its subdomain component, i.e.,

$$R_c^{(i)} u_c = u_c^{(i)}, \qquad R_c = \begin{bmatrix} R_c^{(1)} \\ \vdots \\ R_c^{(N)} \end{bmatrix},$$

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let B_r be an operator enforcing the interface continuity of u_r by

$$B_r u_r = 0, \qquad B_r = \begin{bmatrix} B_r^{(1)} & \dots & B_r^{(N)} \end{bmatrix},$$

and let the mappings E_r^T and E_c^T distribute the primal residual r to the subdomain forces and to the global coarse problem right-hand side, respectively.

The equations of equilibrium can now be written (cf. [15, equation (9)–(10)]) as

$$\begin{array}{llllll} S_{rr}^{(i)} w_r^{(i)} & + S_{rc}^{(i)} R_c^{(i)} w_c & + B_r^{(i)T} \lambda & = & f_r^{(i)}, \\ \sum_{i=1}^{N} R_c^{(i)T} S_{rc}^{(i)T} w_r^{(i)} & + \sum_{i=1}^{N} R_c^{(i)T} S_{cc}^{(i)} R_c^{(i)} w_c & & = & f_c, \\ \sum_{i=1}^{N} B_r^{(i)} w_r^{(i)} & & & = & 0, \end{array}$$

where the first equation corresponds to independent subdomain problems, second corresponds to the global coarse problem and the third enforces the continuity of local problems. This system can be re-written as

$$\begin{bmatrix} S_{rr} & S_{rc} R_c & B_r^T \\ (S_{rc} R_c)^T & \widetilde{S}_{cc} & 0 \\ B_r & 0 & 0 \end{bmatrix} \begin{bmatrix} u_r \\ u_c \\ \lambda \end{bmatrix} = \begin{bmatrix} f_r \\ f_c \\ 0 \end{bmatrix}, \tag{4.17}$$

where $f_r = E_r^T r$, $f_c = E_c^T r$, and the blocks are defined as

$$\widetilde{S}_{cc} = \sum_{i=1}^{N} R_c^{(i)T} S_{cc}^{(i)} R_c^{(i)}, \quad S_{rr} = \begin{bmatrix} S_{rr}^{(1)} & & \\ & \ddots & \\ & & S_{rr}^{(N)} \end{bmatrix}, \quad S_{rc} R_c = \begin{bmatrix} S_{rc}^{(1)} R_c^{(1)} \\ \vdots \\ S_{rc}^{(N)} R_c^{(N)} \end{bmatrix}.$$

REMARK 4.1. Note that the system (4.17) is just the expanded system (4.15).

Expressing u_r from the first equation in (4.17), we get

$$u_r = S_{rr}^{-1} \left(f_r - S_{rc} R_c u_c - B_r^T \lambda \right).$$

Substituting for u_r into the second equation in (4.17) gives

$$\widetilde{S}_{cc}^{*} u_c - (S_{rc} R_c)^T S_{rr}^{-1} B_r^T \lambda = f_c - (S_{rc} R_c)^T S_{rr}^{-1} f_r,$$

where $\widetilde{S}_{cc}^{*} = \widetilde{S}_{cc} - R_c^T S_{rc}^T S_{rr}^{-1} S_{rc} R_c$. Inverting $\widetilde{S}_{cc}^{*}$, we get that

$$u_c = \widetilde{S}_{cc}^{*\,-1} \left[f_c - (S_{rc} R_c)^T S_{rr}^{-1} f_r + (S_{rc} R_c)^T S_{rr}^{-1} B_r^T \lambda \right].$$

After initialization with $\lambda = 0$, not mentioned in [19, 20], but which can be used (cf. [45,

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Section 6.4]), the assembled and averaged solution is

$$
\begin{aligned}
u &= E_r u_r + E_c u_c \\
&= E_r S_{rr}^{-1}\left\{ f_r - S_{rc} R_c \widetilde{S}_{cc}^{*\,-1} \left(f_c - (S_{rc}R_c)^T S_{rr}^{-1} f_r\right)\right\} \\
&\quad + E_c \widetilde{S}_{cc}^{*\,-1}\left(f_c - (S_{rc}R_c)^T S_{rr}^{-1} f_r\right) \\
&= E_r S_{rr}^{-1} f_r - E_r S_{rr}^{-1} S_{rc} R_c \widetilde{S}_{cc}^{*\,-1} f_c \\
&\quad + E_r S_{rr}^{-1} S_{rc} R_c \widetilde{S}_{cc}^{*\,-1} (S_{rc}R_c)^T S_{rr}^{-1} f_r \\
&\quad + E_c \widetilde{S}_{cc}^{*\,-1} f_c - E_c \widetilde{S}_{cc}^{*\,-1} (S_{rc}R_c)^T S_{rr}^{-1} f_r \\
&= E_r S_{rr}^{-1} f_r \\
&\quad + (E_c - E_r S_{rr}^{-1} S_{rc} R_c) \widetilde{S}_{cc}^{*\,-1} \left(f_c - (S_{rc}R_c)^T S_{rr}^{-1} f_r\right) \\
&= M_{\text{P-FETI-DP}}\, r,
\end{aligned}
$$

where

$$
\begin{aligned}
M_{\text{P-FETI-DP}} &= E_r S_{rr}^{-1} E_r^T \\
&\quad + (E_c - E_r S_{rr}^{-1} S_{rc} R_c) \widetilde{S}_{cc}^{*\,-1} (E_c^T - R_c^T S_{rc}^T S_{rr}^{-1} E_r^T)
\end{aligned} \tag{4.18}
$$

is the associated preconditioner P-FETI-DP, as in [20, equation (90)].

5. BDD family of methods. We recall two primal preconditioners from the Balancing Domain Decomposition (BDD) family by Mandel in [31], namely the original BDD and Balancing Domain Decomposition by Constraints (BDDC) introduced by Dohrmann [9].

5.1. BDD. The BDD is a Neumann-Neumann algorithm (cf. [14]) with a simple coarse grid correction, introduced by Mandel [31]. The name of the preconditioner comes from an idea to *balance* the residual. We say that $v \in \widehat{W}$ is balanced if

$$Z^T E^T v = 0.$$

Let us denote the "balancing" operator as

$$C = EZ, \tag{5.1}$$

so the columns of C are equal to the weighted sum of traces of the subdomain zero energy modes. Next, let us denote by $S_C\widehat{S}$ the $\widehat{S}$-*orthogonal* projection onto the range of C, so that

$$S_C = C\left(C^T \widehat{S} C\right)^{-1} C^T,$$

and by P_C the complementary projection to $S_C\widehat{S}$, defined as

$$P_C = I - S_C\widehat{S}. \tag{5.2}$$

The BDD preconditioner [31, Lemma 3.1] can be written in our settings as

$$
\begin{aligned}
M_{\text{BDD}} &= \left[(I - S_C\widehat{S}) E S^+ E^T \widehat{S}(I - S_C\widehat{S}) + S_C\widehat{S}\right]\widehat{S}^{-1} \\
&= \left[(I - S_C\widehat{S}) E S^+ E^T (\widehat{S}\widehat{S}^{-1} - \widehat{S} S_C \widehat{S}\widehat{S}^{-1}) + S_C\widehat{S}\widehat{S}^{-1}\right] \\
&= P_C E S^+ E^T P_C^T + S_C,
\end{aligned} \tag{5.3}
$$

where S_C serves as the coarse grid correction; see [31, 32] and [20] for details.

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5.2. BDDC. Following a similar path as Li and Widlund [30], we will assume that each constraint can be represented by an explicit degree of freedom and that we can decompose the space $\widetilde{W}$ as in (4.16). We note that the original BDDC in [9, 33] is mathematically equivalent, but algorithmically it treats the corner coarse degrees of freedom and edge in the definition of $\widetilde{W}$ in different ways. The BDDC is the method of preconditioned conjugate gradients for the assembled system (3.5) with the preconditioner M_{BDDC} defined by (cf. [30, equation (27)])

$$M_{\text{BDDC}} = T_{\text{sub}} + T_0,$$

where $T_{\text{sub}} = E_r S_{rr}^{-1} E_r^T$ is the subdomain correction obtained by solving independent problems on subdomains, and $T_0 = E\Psi(\Psi^T S \Psi)^{-1} \Psi^T E^T$ is the coarse grid correction. Here Ψ are the coarse basis functions defined by energy minimization,

$$\operatorname{tr} \Psi^T S \Psi \to \min .$$

Since we assume that each constraint corresponds to an explicit degree of freedom, the coarse basis functions Ψ can be easily determined via the analogy to the discrete harmonic functions, discussed, e.g., in [45, Section 4.4]; the functions Ψ are equal to 1 in the coarse degrees of freedom and have energy minimal extension with respect to the remaining degrees of freedom u_r, so they are precisely given as

$$\Psi = \begin{bmatrix} R_c \\ -S_{rr}^{-1} S_{rc} R_c \end{bmatrix}.$$

Then, we can compute

$$\begin{aligned}
\Psi^T S \Psi &= \begin{bmatrix} R_c^T & -R_c^T S_{rc}^T S_{rr}^{-1} \end{bmatrix} \begin{bmatrix} S_{cc} & S_{rc}^T \\ S_{rc} & S_{rr} \end{bmatrix} \begin{bmatrix} R_c \\ -S_{rr}^{-1} S_{rc} R_c \end{bmatrix} \\
&= R_c^T S_{cc} R_c - R_c^T S_{rc}^T S_{rr}^{-1} S_{rc} R_c \\
&= \widetilde{S}_{cc} - R_c^T S_{rc}^T S_{rr}^{-1} S_{rc} R_c = \widetilde{S}_{cc}^*,
\end{aligned}$$

followed by

$$\begin{aligned}
&E\Psi[\Psi^T S \Psi]^{-1} \Psi^T E^T \\
&= E \begin{bmatrix} R_c \\ -S_{rr}^{-1} S_{rc} R_c \end{bmatrix} {\widetilde{S}_{cc}^*}^{-1} \begin{bmatrix} R_c^T & -R_c^T S_{rc}^T S_{rr}^{-1} \end{bmatrix} E^T \\
&= (E_c - E_r S_{rr}^{-1} S_{rc} R_c) {\widetilde{S}_{cc}^*}^{-1} (E_c^T - R_c^T S_{rc}^T S_{rr}^{-1} E_r^T).
\end{aligned}$$

So, the BDDC preconditioner takes the form

$$\begin{aligned}
M_{\text{BDDC}} &= E_r S_{rr}^{-1} E_r^T + \\
&\quad + (E_c - E_r S_{rr}^{-1} S_{rc} R_c) {\widetilde{S}_{cc}^*}^{-1} (E_c^T - R_c^T S_{rc}^T S_{rr}^{-1} E_r^T).
\end{aligned} \tag{5.4}$$

6. Connections of the preconditioners. We review from [20, Section 8] that a certain version of P-FETI-1 gives exactly the same algorithm as BDD. Next, we state the equivalence of P-FETI-DP and BDDC preconditioners. Finally, we translate the abstract proof relating the spectra of primal and dual preconditioners [19, Theorem 4] in the case of FETI-1 and BDD.

THEOREM 6.1 ([20, Section 8]). *If Q is chosen to be the Dirichlet preconditioner, the P-FETI-1 and the BDD preconditioners are the same.*

Proof. We will show that the P-FETI-1 in (4.13) with $Q = B_D S B_D^T$ is the same as the BDD in (5.3). So, similarly as in [20, pp. 3819–3820], from (4.12) we get

$$\begin{aligned} H &= I - B^T Q G (G^T Q G)^{-1} Z^T \\ &= I - B^T B_D S B_D^T B Z (Z^T B^T B_D S B_D^T B Z)^{-1} Z^T \\ &= I - A_R (Z^T A_R)^{-1} Z^T, \end{aligned}$$

where

$$A_R = B^T B_D S B_D^T B Z.$$

Using (3.8), definitions of C in (5.1), $\widehat{S}$ in (3.5), and because $SZ = 0$ by (4.3),

$$\begin{aligned} A_R &= (I - E^T R^T) S (I - RE) Z \\ &= SZ - SREZ - E^T R^T SZ + E^T R^T SREZ \\ &= SZ - SRC - E^T R^T SZ + E^T \widehat{S} C \\ &= (E^T \widehat{S} - SR) C, \end{aligned}$$

and, similarly,

$$Z^T A_R = Z^T (E^T \widehat{S} - SR) C = C^T \widehat{S} C - Z^T SREZ = C^T \widehat{S} C.$$

Using the two previous results, (5.2), and symmetries of $\widehat{S}$ and S_c, we get

$$\begin{aligned} H E^T &= \left(I - A_R (Z^T A_R)^{-1} Z^T \right) E^T \\ &= E^T - A_R (Z^T A_R)^{-1} Z^T E^T \\ &= E^T - (E^T \widehat{S} - SR) C (C^T \widehat{S} C)^{-1} C^T \\ &= E^T - (E^T \widehat{S} - SR) S_C \\ &= E^T - E^T \widehat{S} S_C + SRS_C \\ &= E^T (I - \widehat{S} S_C) + SRS_C \\ &= E^T P_C^T + SRS_C. \end{aligned}$$

Next, the matrix S_C satisfies the relation

$$\begin{aligned} S_C R^T S S^+ SRS_C = S_C R^T SRS_C &= S_C \widehat{S} S_C \\ &= C (C^T \widehat{S} C)^{-1} C^T \widehat{S} C (C^T \widehat{S} C)^{-1} C^T \\ &= C (C^T \widehat{S} C)^{-1} C^T = S_C. \end{aligned}$$

Since, by definition $P_C C = 0$, using (3.7) we get for some Y that

$$\begin{aligned} P_C E S^+ SRS_C &= P_C E (I + ZY) RS_C \\ &= P_C ERS_C + P_C EZYRS_C \\ &= P_C S_C + P_C CYRS_C \\ &= P_C S_C \\ &= (I - S_C \widehat{S}) S_C \\ &= S_C - S_C = 0, \end{aligned}$$

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and the same is true for the transpose, so $S_C R^T S S^+ E^T P_C^T = 0$.

Using these results, the P-FETI-1 preconditioner from (4.13) becomes

$$\begin{aligned} M_{\text{P-FETI}} &= EH^T S^+ HE^T \\ &= (S_C R^T S + P_C E) S^+ (E^T P_C^T + SRS_C) \\ &= S_C R^T S S^+ E^T P_C^T + S_C R^T S S^+ SRS_C \\ &\quad + P_C E S^+ E^T P_C^T + P_C E S^+ SRS_C \\ &= P_C E S^+ E^T P_C^T + S_C, \end{aligned} \tag{6.1}$$

and we see that (6.1) is the same as the definition of BDD in (5.3). ◻

THEOREM 6.2. *The P-FETI-DP and the BDDC preconditioners are the same.*

Proof. The claim follows directly comparing the definitions of both preconditioners, P-FETI-DP in equation (4.18) and BDDC in equation (5.4). ◻

COROLLARY 6.3. *Comparing the preconditioner proposed by Cros [8, equation (4.8)] with the definitions (4.18) and (5.4), it follows that this preconditioner can be interpreted as either P-FETI-DP or BDDC.*

In the remaining, we will show the equality of eigenvalues of BDD and FETI-1, being Q the Dirichlet preconditioner.

LEMMA 6.4. *The two preconditioned operators can be written as*

$$\begin{aligned} M_{\mathit{FETI}} \mathcal{F} &= (B_D S B_D^T)(B \widetilde{S}^+ B^T), \\ M_{\mathit{BDD}} \widehat{S} &= (E \widetilde{S}^+ E^T)(R^T S R), \end{aligned}$$

where

$$\widetilde{S}^+ = H^T S^+ H.$$

Proof. First, $M_{\text{FETI}} = B_D S B_D^T$, which is the Dirichlet preconditioner. From (4.11), using the definition of H by (4.12), we get

$$\begin{aligned} \mathcal{F} &= P^T F P \\ &= P^T B S^+ B^T P \\ &= \left(I - G(G^T Q G)^{-1} G^T Q^T\right) B S^+ B^T \left(I - QG(G^T Q G)^{-1} G^T\right) \\ &= \left(B - BZ(G^T Q G)^{-1} G^T Q^T B\right) S^+ \left(B^T - B^T Q G (G^T Q G)^{-1} Z^T B^T\right) \\ &= B\left(I - Z(G^T Q G)^{-1} G^T Q B\right) S^+ \left(I - B^T Q G (G^T Q G)^{-1} Z^T\right) B^T \\ &= B H^T S^+ H B^T = B \widetilde{S}^+ B^T. \end{aligned}$$

Next, $\widehat{S}$ is defined by (3.5). By Theorem 6.1, we can use (4.13) for M_{BDD} to get

$$M_{\text{BDD}} = EH^T S^+ H E^T = E \widetilde{S}^+ E^T. \qquad ◻$$

Before proceeding to the main result, we need to prove two technical Lemmas relating the operators S and $\widetilde{S}^+$. The first Lemma establishes [19, Assumptions (13) and (22)] as well as [19, Lemma 3] for FETI-1 and BDD.

LEMMA 6.5. *The operators S, $\widetilde{S}^+$ defined by (3.1) and Theorem 6.4, respectively, satisfy*

$$\widetilde{S}^+ S R = R, \tag{6.2}$$

$$\widetilde{S}^+ S \widetilde{S}^+ = \widetilde{S}^+. \tag{6.3}$$

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Moreover, the following relations are valid

$$B\widetilde{S}^{+}SR = 0, \tag{6.4}$$
$$\widetilde{S}^{+}B^{T}B_{D}S\widetilde{S}^{+}E^{T} = 0. \tag{6.5}$$

Proof. First, from (4.3) and symmetry of S it follows that

$$\begin{aligned} HS &= \left(I - B^{T}QG(G^{T}QG)^{-1}Z^{T}\right)S \\ &= S - B^{T}QG(G^{T}QG)^{-1}Z^{T}S = S. \end{aligned}$$

Using $H^{T} = I - Z(G^{T}QG)^{-1}G^{T}QB$, we get

$$\begin{aligned} H^{T}S^{+}S &= H^{T}(I + ZY) = H^{T} + H^{T}ZY \\ &= H^{T} + \left[I - Z(G^{T}QG)^{-1}G^{T}QB\right]ZY \\ &= H^{T} + ZY - Z(G^{T}QG)^{-1}G^{T}QGY \\ &= H^{T} + ZY - ZY = H^{T}, \end{aligned}$$

so

$$\widetilde{S}^{+}S = H^{T}S^{+}HS = H^{T}S^{+}S = H^{T}.$$

Finally, from the previous equation and (3.6), we get (6.2) as

$$\widetilde{S}^{+}SR = H^{T}R = \left(I - Z(G^{T}QG)^{-1}G^{T}QB\right)R = R,$$

and, since H^{T} is a projection, we immediately get also (6.3) as

$$\widetilde{S}^{+}S\widetilde{S}^{+} = H^{T}\widetilde{S}^{+} = H^{T}H^{T}S^{+}H = \widetilde{S}^{+}.$$

Next, noting (3.6), (6.4) follows directly from (6.2).

Using (6.2)–(6.3) and (3.8)–(3.9), we get (6.5) as

$$\begin{aligned} \widetilde{S}^{+}B^{T}B_{D}S\widetilde{S}^{+}E^{T} &= \widetilde{S}^{+}(I - E^{T}R^{T})S\widetilde{S}^{+}E^{T} \\ &= \widetilde{S}^{+}S\widetilde{S}^{+}E^{T} - \widetilde{S}^{+}E^{T}R^{T}S\widetilde{S}^{+}E^{T} \\ &= \widetilde{S}^{+}E^{T} - \widetilde{S}^{+}E^{T}R^{T}E^{T} \\ &= \widetilde{S}^{+}(I - E^{T}R^{T})E^{T} \\ &= \widetilde{S}^{+}B^{T}B_{D}E^{T} = 0. \qquad \square \end{aligned}$$

The next Lemma is a particular version of [19, Theorem 4] for FETI-1 and BDD.

LEMMA 6.6. *The following identities are valid:*

$$\begin{aligned} T_{D}(M_{FETI}\mathcal{F}) &= (M_{BDD}\widehat{S})T_{D}, \qquad & T_{D} &= E\widetilde{S}^{+}B^{T}, \\ T_{P}(M_{BDD}\widehat{S}) &= (M_{FETI}\mathcal{F})T_{P}, \qquad & T_{P} &= (M_{FETI}\mathcal{F})B_{D}SR. \end{aligned}$$

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Proof. Using the transpose of (6.5) and (6.4), we derive the first identity as

$$\begin{aligned}
T_D(M_{\text{FETI}}\mathcal{F}) &= E\widetilde{S}^+B^TB_DSB_D^TB\widetilde{S}^+B^T \\
&= E\widetilde{S}^+(I-E^TR^T)S(I-RE)\widetilde{S}^+B^T \\
&= E\widetilde{S}^+S(I-RE)\widetilde{S}^+B^T - E\widetilde{S}^+E^TR^TS\widetilde{S}^+B^T \\
&\quad + E\widetilde{S}^+E^TR^TSRE\widetilde{S}^+B^T \\
&= E\widetilde{S}^+SB_D^TB\widetilde{S}^+B^T - E\widetilde{S}^+E^TR^TS\widetilde{S}^+B^T \\
&\quad + (E\widetilde{S}^+E^T)(R^TSR)T_D \\
&= (M_{\text{BDD}}\widehat{S})T_D.
\end{aligned}$$

Similarly, using (6.5) and (6.4), we derive the second identity as

$$\begin{aligned}
T_P(M_{\text{BDD}}\widehat{S}) &= (M_{\text{FETI}}\mathcal{F})B_DSRE\widetilde{S}^+E^TR^TSR \\
&= (M_{\text{FETI}}\mathcal{F})B_DS(I-B_D^TB)\widetilde{S}^+(I-B^TB_D)SR \\
&= (M_{\text{FETI}}\mathcal{F})B_DS\widetilde{S}^+(I-B^TB_D)SR \\
&\quad - (M_{\text{FETI}}\mathcal{F})B_DSB_D^TB\widetilde{S}^+SR \\
&\quad + (M_{\text{FETI}}\mathcal{F})B_DSB_D^TB\widetilde{S}^+B^TB_DSR \\
&= M_{\text{FETI}}B\widetilde{S}^+B^TB_DS\widetilde{S}^+E^TR^TSR \\
&\quad - (M_{\text{FETI}}\mathcal{F})B_DSB_D^TB\widetilde{S}^+SR \\
&\quad + (M_{\text{FETI}}\mathcal{F})(B_DSB_D^T)(B\widetilde{S}^+B^T)B_DSR \\
&= (M_{\text{FETI}}\mathcal{F})(M_{\text{FETI}}\mathcal{F})B_DSR. \\
&= (M_{\text{FETI}}\mathcal{F})T_P. \qquad \square
\end{aligned}$$

THEOREM 6.7. *Under the assumption of Lemma 6.6, the spectra of the preconditioned operators $M_{BDD}\widehat{S}$ and $M_{FETI\text{-}1}\mathcal{F}$ satisfy the relation*

$$\sigma(M_{BDD}\widehat{S}) \setminus \{1\} = \sigma(M_{FETI\text{-}1}\mathcal{F}) \setminus \{0,1\}.$$

Moreover, the multiplicity of any common eigenvalue $\lambda \neq 0,1$ is identical for the two preconditioned operators.

Proof. Let u_D be a (nonzero) eigenvector of the preconditioned FETI-1 operator corresponding to the eigenvalue λ_D. Then, by Lemma 6.6, we have

$$T_D(M_{\text{FETI-1}}\mathcal{F})u_D = (M_{\text{BDD}}\widehat{S})T_Du_D,$$

so T_Du_D is an eigenvector of the preconditioned BDD operator corresponding to the eigenvalue λ_D, provided that $T_Du_D \neq 0$. So, we assume that $T_Du_D = 0$, but then it is also true that

$$\begin{aligned}
0 = B_DSR(T_Du_D) &= B_DSRE\widetilde{S}^+B^Tu_D \\
&= B_DS(I-B_D^TB)\widetilde{S}^+B^Tu_D = B_DS\widetilde{S}^+B^Tu_D - B_DSB_D^TB\widetilde{S}^+B^Tu_D \\
&= B_DS\widetilde{S}^+B^Tu_D - (M_{\text{FETI}}\mathcal{F})u_D = B_DS\widetilde{S}^+B^Tu_D - \lambda_Du_D,
\end{aligned}$$

so that

$$B_DS\widetilde{S}^+B^Tu_D = \lambda_Du_D.$$

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Note that, by (6.2) and (3.6), we get

$$\begin{aligned}(B_D S\widetilde{S}^{+}B^T)^2 &= B_D S\widetilde{S}^{+}B^T B_D S\widetilde{S}^{+}B^T \\ &= B_D S\widetilde{S}^{+}(I - E^T R^T)S\widetilde{S}^{+}B^T \\ &= B_D S\widetilde{S}^{+}S\widetilde{S}^{+}B^T - B_D S\widetilde{S}^{+}E^T R^T S\widetilde{S}^{+}B^T \\ &= B_D S\widetilde{S}^{+}B^T - B_D S\widetilde{S}^{+}E^T R^T B^T \\ &= B_D S\widetilde{S}^{+}B^T,\end{aligned}$$

so $B_D S\widetilde{S}^{+}B^T$ is a projection, and therefore $\lambda_D = 0, 1$.

Next, Let u_P be a (nonzero) eigenvector of the preconditioned BDD operator corresponding to the eigenvalue λ_P. Then, by Lemma 6.6, we have

$$T_P(M_{\text{BDD}}\widehat{S}) = (M_{\text{FETI}}\mathcal{F})T_P,$$

so $T_P u_P$ is an eigenvector of the preconditioned FETI-1 operator corresponding to the eigenvalue λ_P, provided that $T_P u_P \neq 0$. So, we assume that $T_P u_P = 0$, but then, using (6.2) and (3.7), we also get

$$\begin{aligned}0 &= T_D(T_P u_P) = T_D(M_{\text{FETI}}\mathcal{F})B_D S R u_P \\ &= (M_{\text{BDD}}\widehat{S})T_D B_D S R u_P = (M_{\text{BDD}}\widehat{S})E\widetilde{S}^{+}B^T B_D S R u_P \\ &= M_{\text{BDD}}\widehat{S}E\widetilde{S}^{+}(I - E^T R^T)S R u_P \\ &= M_{\text{BDD}}\widehat{S}E\widetilde{S}^{+}S R u_P - M_{\text{BDD}}\widehat{S}E\widetilde{S}^{+}E^T R^T S R u_P \\ &= M_{\text{BDD}}\widehat{S}u_P - M_{\text{BDD}}\widehat{S}E\widetilde{S}^{+}E^T R^T S R u_P \\ &= M_{\text{BDD}}\widehat{S}u_P - (M_{\text{BDD}}\widehat{S})^2 u_P,\end{aligned}$$

which is the same as

$$\lambda_P u_P - \lambda_P^2 u_P = \lambda_P(1 - \lambda_P)u_P = 0,$$

and therefore $\lambda_P = 0, 1$.

Finally, let $\lambda \neq 0, 1$ be an eigenvalue of the operator $M_{\text{BDD}}\widehat{S}$ with multiplicity m. From the previous arguments, the eigenspace corresponding to λ is mapped by the operator T_P into an eigenspace of $M_{\text{FETI-1}}\mathcal{F}$ and since this mapping is one-to-one, the multiplicity of λ corresponding to $M_{\text{FETI-1}}\mathcal{F}$ is $n \geq m$. By the same argument, we can prove the opposite inequality and the conclusion follows. □

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