Electronic Transactions on Numerical Analysis.
Volume 31, pp. 49-67, 2008.

ISSN 1068-9613.

ETNA
Kent State University
etna@mcs.kent.edu

A TECHNIQUE FOR COMPUTING MINORS OF BINARY HADAMARD MATRICES AND APPLICATION TO THE GROWTH PROBLEM*

CHRISTOS KRAVVARITIS† AND MARILENA MITROULI†

Abstract. A technique to compute all the possible minors of order $n - j$ of binary Hadamard matrices with entries $(0, 1)$ is introduced. The method exploits the properties of such matrices S and also the symmetry and special block structure appearing when one forms the matrix $D^T D$, where D is a submatrix of S. Theoretically, the method works for every pair of values n and j and provides general analytical formulae. The whole process can be standardized and implemented as a computer algorithm. The usefulness of such a method is justified by the application to the growth problem. This study gives also more insight into some structural properties of these matrices and leads to the formulation of the growth conjecture for binary Hadamard matrices.

Key words. Binary Hadamard matrices, determinant calculus, symbolic computations, Gaussian elimination, growth problem.

AMS subject classifications. 15A15, 05B20, 65F40, 65F05, 65G50.

1. Introduction.

1.1. Orthogonal matrices and minors. An *orthogonal matrix* Q of order n satisfies $QQ^T = Q^TQ = I_n$. By this definition, orthogonal matrices have determinant ± 1, the inverse of an orthogonal matrix is its transpose, the product of two orthogonal matrices is an orthogonal matrix and they yield the property that the Euclidean matrix norm is unitarily invariant. Therefore it can be proved that they have some important numerical properties; e.g., the product of any matrix with an orthogonal matrix is always stable (in the sense that it gives only a small and acceptable error) and orthogonally similar matrices have the same eigenvalues.

An interesting generalization of orthogonal matrices is the concept of generalized normalized orthogonal matrices, as described by the following definition.

DEFINITION 1.1. *A matrix $A = (a_{ij})$ is called* normalized *if* $\max_{i,j} |a_{ij}| = 1$. *A normalized $n \times n$ matrix A is called* normalized orthogonal *if $AA^T = A^TA = c(A)I_n$, for some constant $c(A)$, and* generalized normalized orthogonal *if $AA^T = A^TA = c(A)(I_n + J_n)$, where J_n denotes the matrix of order n whose entries are all ones.*

A similar definition was given in [4]. These matrices are generalized to within a row scaling, i.e., the product of such a matrix with its transpose gives a multiple either of the identity matrix I_n (e.g., Hadamard and weighing matrices) or of the similarly special structured matrix $I_n + J_n$; e.g., binary Hadamard matrices.

The purpose of this paper is to study the properties of generalized normalized orthogonal matrices and, in particular, the computation of minors of binary Hadamard matrices, having entries $(0, 1)$. In general, it is difficult to obtain *analytical formulae* for minors of various orders for a given arbitrary matrix. A very interesting result to compute *numerically* all principal minors of a matrix, yielding an $O(2^n)$ algorithm, was presented in [10]. The derivation of analytical formulae for minors of the orthogonal matrices discussed in this work is possible due to their special structure and properties.

A *Hadamard matrix* H of order n is a matrix with elements ± 1 satisfying the orthogonality relation $HH^T = H^TH = nI_n$. It can be proved [4, 11] that if H is a Hadamard

* Received October 25, 2007. Accepted June 30, 2008. Published online on December 4, 2008. Recommended by Reinhard Nabben.

†Department of Mathematics, University of Athens, Panepistemiopolis, 15784 Athens, Greece ({ckrav,mmitroul}@math.uoa.gr). This research was financially supported by ΠΕΝΕΔ 03ΕΔ 740 of the Greek General Secretariat for Research and Technology.

ETNA
Kent State University
etna@mcs.kent.edu

matrix of order n then $n = 1, 2$ or $n \equiv 0 \,(\mathrm{mod}\, 4)$. However, there is still a conjecture that remains to be settled, on whether Hadamard matrices exist for every n being a multiple of 4. For more details on Hadamard matrices the reader can consult [8, 14, 22, 25].

TABLE 1.1
Values of minors for Hadamard matrices of general order n

minor	values of minors
$n-1$	$n^{n/2-1}$
$n-2$	0, $2n^{n/2-2}$
$n-3$	0, $4n^{n/2-3}$
$n-4$	0, $8n^{n/2-4}$, $16n^{n/2-4}$

The first known results concerning minors of Hadamard matrices were obtained in [23] for the $n-1$, $n-2$, and $n-3$, minors. In [17] all the possible $n-4$ minors of Hadamard matrices were calculated theoretically by a method that led to a numerical algorithm.

TABLE 1.2
Values of minors for Hadamard matrices of orders 12 and 16

minor	values of minors
$n-5$	0, $16n^{n/2-5}$, $32n^{n/2-5}$, $48n^{n/2-5}$
$n-6$	0, $32n^{n/2-6}$, $64n^{n/2-6}$, $96n^{n/2-6}$, $128n^{n/2-6}$, $160n^{n/2-6}$
$n-7$	0, $64n^{n/2-7}$, $128n^{n/2-7}$, $192n^{n/2-7}$, $256n^{n/2-7}$, $320n^{n/2-7}$, $384n^{n/2-7}$, $448n^{n/2-7}$, $512n^{n/2-7}$, $576n^{n/2-7}$

The general results for the $n-j$ minors, $j = 1, \ldots, 4$, of Hadamard matrices are summarized in Table 1.1. The values for $j = 5, 6, 7$ have been proved only for $n = 12$ and 16 [20], due to computational difficulties of the existing methods, and are given in Table 1.2. It can be seen that all the possible values of the $n-j$ minors, $j = 1, \ldots, 7$, follow a specific pattern. This observation constitutes the following open conjecture for the possible values of minors of Hadamard matrices.

CONJECTURE 1.2 (**Conjecture for minors of Hadamard matrices**). *All possible minors of dimension $(n-j) \times (n-j)$, $j \geq 1$, of Hadamard matrices are either 0 or*

$$p \cdot n^{(n/2)-j}, \qquad \textit{for } \; p = 2^{j-1}, 2 \cdot 2^{j-1}, 3 \cdot 2^{j-1}, \ldots, s \cdot 2^{j-1},$$

where

$$s \cdot 2^{j-1} = \max\{\det(A) | A \in \mathbb{R}^{j \times j}, \textit{with entries } \pm 1\}$$

and the value 0 is excluded from the case $j = 1$.

The maximum determinant values for ± 1 matrices are given in Table 1.3. The study of the above conjecture is expected to lead to useful results concerning the possible values of determinants of ± 1 matrices, which are not exactly specified even for relatively small orders ($n = 8$). The relevant known results are given in [4, 19].

A *binary Hadamard matrix* (called also *S-matrix*) is an $n \times n$ $(0, 1)$ matrix formed by taking an $(n+1) \times (n+1)$ Hadamard matrix in which the entries in the first row and column are $+1$, changing $+1$ to 0, and -1 to $+1$, and deleting the first row and column. Therefore, $n \equiv 3 \,(\mathrm{mod}\, 4)$. A binary Hadamard matrix satisfies $SS^T = S^T S = \frac{1}{4}(n+1)(I_n + J_n)$

ETNA
Kent State University
etna@mcs.kent.edu

TABLE 1.3
Maximum determinants of ± 1 *matrices*

n	1	2	3	4	5	6	7
max. det	1	2	4	16	48	160	576

and $SJ_n = J_nS = \frac{1}{2}(n+1)J_n$. Further information on binary Hadamard matrices, their applications and related problems can be found in [8, 12, 22, 25] and in the references therein.

REMARK 1.3. It is important to emphasize that the present work deals only with the specific binary Hadamard matrices that are obtained from the cores of Hadamard matrices according to the construction described above. These binary Hadamard matrices are actually the incidence matrices of symmetric balanced incomplete block designs (SBIBDs) with parameters $(4t-1, 2t, t)$ [5, 8]. Indeed, if H is a Hadamard matrix of order $4t$, then it can be written in the form

$$H = \begin{bmatrix} 1 & e^T \\ e & A \end{bmatrix},$$

where $e^T = (1, 1, \ldots, 1)$ is the $1 \times (4t-1)$ vector with elements 1. Then the matrix $C = \frac{1}{2}(J_{4t-1} - A)$ is the incidence matrix of an SBIBD with parameters $(4t-1, 2t, t)$. An SBIBD with parameters $(4t-1, 2t, t)$ is the complement of an SBIBD with parameters $(4t-1, 2t-1, t-1)$, the incidence matrix of which is constructed as $\frac{1}{2}(A + J_{4t-1})$. In [18] values of minors for various families of $(1, -1)$ incidence matrices of SBIBDs were computed and the growth problem was discussed for them.

EXAMPLE 1.4. The following are three binary Hadamard matrices of small order:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

1.2. Notation and preliminary remarks. Whenever a determinant or minor is mentioned in this work, we mean its magnitude, i.e., the absolute value. The symbols I_n and J_n stand for the identity matrix and the matrix of order n whose entries are all one, respectively. Whenever information on the dimension is not essential, the indices are omitted. We denote by $A(j)$ the absolute value of the determinant of the $j \times j$ principal submatrix in the upper left corner of the matrix A, i.e., $A(j)$ is the magnitude of the $j \times j$ leading principal minor of A. An $m \times n$ matrix having all its entries equal to $x \in \mathbb{R}$ will be denoted by $x_{m \times n}$. If x consists of more than one terms (e.g., $x = k-1$), then parentheses are used around x to avoid confusion.

The notation $(\kappa - \lambda)I + \lambda J$ will be frequently used as a compact notation for a matrix

ETNA
Kent State University
etna@mcs.kent.edu

of the form

$$\begin{bmatrix} \kappa & \lambda & \cdots & \lambda \\ \lambda & \kappa & \cdots & \lambda \\ \vdots & & \ddots & \\ \lambda & \lambda & \cdots & \kappa \end{bmatrix}.$$

If it is essential to specify the order n of such a matrix, then it will be denoted by $X_{n\times n} \equiv (\kappa-\lambda)I_n + \lambda J_n$, for instance,

$$X_{2\times 2} = \begin{bmatrix} \kappa & \lambda \\ \lambda & \kappa \end{bmatrix}.$$

Let $\mathbf{y}_{\beta+1}$ be the column vectors containing the binary representation of each integer $\beta = 2^j - 1$. Define the $j \times 1$ vectors $\mathbf{u}_k = \mathbf{y}_{2^j-k+1},\ k = 1,\ldots,2^j$. We write U_j for all the matrices with j rows and the appropriate number of columns, in which $\mathbf{u}_k$ occurs u_k times. Thus,

$$U_j = \begin{array}{ccccc} \overbrace{1...1}^{u_1} & \overbrace{1...1}^{u_2} & \ldots & \overbrace{0...0}^{u_{2^j-1}} & \overbrace{0...0}^{u_{2^j}} \\ 1...1 & 1...1 & \ldots & 0...0 & 0...0 \\ . & . & \ldots & . & . \\ . & . & \ldots & . & . \\ 1...1 & 1...1 & \ldots & 0...0 & 0...0 \\ 1...1 & 0...0 & \ldots & 1...1 & 0...0 \end{array} = \begin{array}{ccccc} u_1 & u_2 & \ldots & u_{2^{j-1}-1} & u_{2^{j-1}} \\ 1 & 1 & \ldots & 0 & 0 \\ 1 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 1 & 0 \end{array}.$$

The matrix U_j is important in this study because it depicts a general form for j arbitrary rows of a binary Hadamard matrix.

EXAMPLE 1.5.

$$U_3 = \begin{array}{cccccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{array}$$

$$U_4 = \begin{array}{cccccccccccccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 & u_9 & u_{10} & u_{11} & u_{12} & u_{13} & u_{14} & u_{15} & u_{16} \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{array}$$

We provide some useful formulae for two specially structured matrices, which will be used repeatedly throughout the paper.

LEMMA 1.6. *Let $A = (\kappa-\lambda)I_v + \lambda J_v$, where κ and λ are integers. Then,*

$$\det A = [\kappa + (v-1)\lambda](\kappa-\lambda)^{v-1} \tag{1.1}$$

and, for $\kappa \neq \lambda, -(v-1)\lambda$, A is nonsingular with

$$A^{-1} = \frac{1}{\kappa^2 + (v-2)\kappa\lambda - (v-1)\lambda^2}\{[\kappa + (v-1)\lambda]I_v - \lambda J_v\}. \tag{1.2}$$

ETNA
Kent State University
etna@mcs.kent.edu

Equation (1.2) is a special case of the Sherman-Morrison formula [3, p. 239], which computes the inverse of a rank-one correction of a nonsingular matrix B as

$$(B - uv^T)^{-1} = B^{-1} + \frac{B^{-1}uv^TB^{-1}}{1 - v^TB^{-1}u},$$

where u, v are vectors and $v^TB^{-1}u \neq 1$. Indeed, we obtain (1.2) letting $B = (\kappa - \lambda)I_v$, $u = -\lambda[1\,1\ \ldots\ 1]^T$ and $v = [1\,1\ \ldots\ 1]^T$.

LEMMA 1.7 (Schur determinant formula [15, p. 21]). *Let* $B = \left[\begin{smallmatrix} B_1 & B_2 \\ B_3 & B_4 \end{smallmatrix}\right]$. *If* B_1 *is nonsingular, then*

$$\det B = \det B_1 \cdot \det(B_4 - B_3B_1^{-1}B_2). \tag{1.3}$$

If B_4 *is nonsingular, then*

$$\det B = \det B_4 \cdot \det(B_1 - B_2B_4^{-1}B_3). \tag{1.4}$$

The paper is organized as follows. In Section 2, the strategy to compute all the possible $n - j$ minors of binary normalized orthogonal matrices is outlined. In Section 3, we provide some useful lemmas and results concerning minors of binary Hadamard matrices as well as an algorithm suited for this purpose. In Section 4, the growth problem is described in general, the growth conjecture for binary Hadamard matrices is formulated and information about pivot patterns of binary Hadamard matrices is given. Finally, Section 5 summarizes the results of this work and highlights further improvements and possible extensions.

2. The numerical technique for the evaluation of minors. In this section, we present the technique we propose to calculate all the possible $(n - j) \times (n - j)$ minors of binary normalized orthogonal matrices of order n in a general context. However, it can be better understood through the theoretical proofs and the algorithm of Section 3.

In order to compute all the possible $(n - j) \times (n - j)$ minors of a binary Hadamard matrix S, we write it in the form

$$S = \begin{bmatrix} M & U_j \\ V_j^T & D \end{bmatrix},$$

where M and D are square matrices of orders j and $n - j$, respectively. The purpose is to compute, for every possible upper left $j \times j$ submatrix M, the values of the determinant of D, which is actually the required minor. The submatrices U_j and V_j^T contain all possible $j \times (n-j)$ columns and $(n-j) \times j$ rows, respectively, which can appear in S, so they must be created very carefully according to the properties of S. Although, with appropriate row and/or column interchanges and, if necessary, with multiplications by -1, it is possible to achieve $U_j = V_j$ for some binary Hadamard matrices, in general it is not. It is very important that the same columns are clustered together in U_j, as in Section 1.2. In this manner, the computation is simplified by the block form, and the derivation of analytical formulae is possible.

From the order of S, the inner products of its first j rows and, if necessary, the total number of ones and zeros in the first j rows of S, we set up and solve a system whose unknowns are the numbers of columns of U_j. If an exact solution can be found (i.e., there are no parameters in the solution), the method provides a general formula. If some parameters exist in the solution, we must determine upper bounds (depending on n) that give all feasible values for the parameters. Thus, in this case the result will not be general, but dependent on n.

Afterwards, the matrix D^TD (or equivalently DD^T) is evaluated taking into account that $S^TS = SS^T = \frac{1}{4}(n+1)(I_n + J_n)$, and the result is written appropriately in block form. The sizes of the blocks are known since they correspond to the solution of the system of equations described above. Finally, we aim at deriving $\det D^TD$ by successive applications of formula (1.3) or (1.4), with the help of (1.1) and (1.2).

Due to the properties of S, all diagonal blocks of D^TD are of the form $(a-b)I + bJ$ and each of the remaining blocks consists only of a constant. Also, D^TD is always symmetric. These properties are always taken into account during the computations, so that every matrix multiplication, inversion and determinant evaluation is not performed explicitly, but in an efficient manner with the help of (1.1)–(1.4).

It is important to emphasize that the proposed method calculates all the possible $(n-j)\times(n-j)$ minors. The selected rows, which are written as the first j rows of A, do not necessarily appear there, but can be located anywhere in the matrix. They can be moved to the first rows with appropriate row and/or column interchanges. We write them at the top only for the sake of better presentation and without any loss of generality. The fact that we examine *all the possible upper left $j \times j$ submatrices* guarantees that with this technique we calculate *all the possible $(n-j)\times(n-j)$ minors of A* and that we don't miss any of their values.

It is also important to stress that the method can be very easily modified to work for Hadamard matrices as well. We believe it is sensible to present it through the example of binary Hadamard matrices, which represents a comprehensive but not extensive range of calculations and yields new results.

The method can be implemented symbolically in a computer algebra system, like Maple, which guarantees the accuracy of the results avoiding roundoff errors, and preserves analytical formulae.

3. Main results. In this section, we prove theoretical formulae for the $n-j$ minors, $j = 1,2,3,4$, of binary Hadamard matrices. We explain why the proof of results for $j > 2$ is more complicated and why it can be carried out with this technique only for specific, fixed values of n. First we give the following, almost straightforward, lemma. Throughout this section, we set $k := \frac{1}{4}(n+1)$.

LEMMA 3.1. *The determinant of a binary Hadamard matrix S of order n is*

$$2^{-n}(n+1)^{\frac{n+1}{2}}.$$

Proof. From the definition of S we have

$$SS^T = k(I_n + J_n) = k \begin{bmatrix} 2 & 1 & \cdots \\ 1 & 2 & \\ \vdots & & \ddots \end{bmatrix} = k[(2-1)I_n + J_n].$$

Equation (1.1) gives

$$\det SS^T = k^n[2 + (n-1)] = k^n(n+1) = \frac{(n+1)^{n+1}}{4^n}.$$

Since $\det SS^T = (\det S)^2$, we have

$$|\det S| = 2^{-n}(n+1)^{\frac{n+1}{2}},$$

which proves the result. □

REMARK 3.2. Lemma 3.1 can also be proved by using the explicit connection between a Hadamard matrix H of order $n+1$ and a binary Hadamard matrix S of order n, namely

$$H - J_{n+1} = \begin{bmatrix} 0 & 0 \\ 0 & -2S \end{bmatrix}.$$

To this end, we may use the invertibility of any $n \times n$ submatrix of H, since all such minors are nonzero (cf. Table 1.1); this idea does not seem to be applicable to derive more results on minors of S, since invertibility cannot be guaranteed for submatrices of H of smaller order.

Before proceeding to further computations, we need the following result, which gives the number of ones and zeros in every row and column of a binary Hadamard matrix.

LEMMA 3.3. *Every row and every column of a binary Hadamard matrix S of order n has $\frac{n+1}{2}$ ones and $\frac{n-1}{2}$ zeros.*

Proof. The result is derived from the property $SJ_n = J_nS = \frac{1}{2}(n+1)J_n$, which shows that the sum of the entries of every row and column of a binary Hadamard matrix is $\frac{n+1}{2}$, considering that the entries of the matrix are only (0,1). □

PROPOSITION 3.4. *Let S be a binary Hadamard matrix of order n. Then all the possible $(n-1) \times (n-1)$ minors of S are $2^{1-n}(n+1)^{\frac{n-1}{2}}$.*

Proof. Since S is a binary Hadamard matrix of order n, it can be written in one of the following two forms:

$$S = \left[\begin{array}{c|c} 1 & \overbrace{1\ \ldots\ 1}^{(n-1)/2}\quad \overbrace{0\ \ldots\ 0}^{(n-1)/2} \\ \hline 1 & \\ \vdots & \\ 1 & \mathrm{A} \\ 0 & \\ \vdots & \\ 0 & \end{array}\right] \quad \text{or} \quad \left[\begin{array}{c|c} 0 & \overbrace{1\ \ldots\ 1}^{(n+1)/2}\quad \overbrace{0\ \ldots\ 0}^{(n-3)/2} \\ \hline 1 & \\ \vdots & \\ 1 & \mathrm{A}' \\ 0 & \\ \vdots & \\ 0 & \end{array}\right],$$

where the first columns contain the appropriate number of ones and zeros below the horizontal line, so that they have $\frac{n+1}{2}$ ones and $\frac{n-1}{2}$ zeros.

From the definition of S, $S^TS = k(I_n + J_n)$, it follows that the $(n-1) \times (n-1)$ matrix A^TA has the form

$$A^TA = \begin{bmatrix} A_1 & A_2 \\ A_2 & A_3 \end{bmatrix},$$

where

$$\begin{aligned} A_1 &= [2k-1-(k-1)]I_{\frac{n-1}{2}} + (k-1)J_{\frac{n-1}{2}}, \\ A_2 &= kJ_{\frac{n-1}{2}}, \\ A_3 &= (2k-k)I_{\frac{n-1}{2}} + kJ_{\frac{n-1}{2}}. \end{aligned}$$

From (1.4) we have

$$\det A^TA = \det A_3 \cdot \det(A_1 - A_2A_3^{-1}A_2), \tag{3.1}$$

while equation (1.1) gives

$$\det A_3 = 2^{-n}(n+1)^{\frac{n+1}{2}}. \tag{3.2}$$

ETNA
Kent State University
etna@mcs.kent.edu

With the help of (1.2) we obtain

$$A_3^{-1} = \frac{1}{k(n+1)} \left\{ [(n-1)+2] I_{\frac{n-1}{2}} - 2J_{\frac{n-1}{2}} \right\},$$

and careful calculations give

$$A_1 - A_2 A_3^{-1} A_2 = \frac{1}{4} \left\{ [(n-1)+2] I_{\frac{n-1}{2}} - 2J_{\frac{n-1}{2}} \right\}.$$

Equation (1.1) yields

$$\det(A_1 - A_2 A_3^{-1} A_2) = 2^{2-n}(n+1)^{\frac{n-3}{2}} \tag{3.3}$$

and substituting (3.2) and (3.3) in (3.1) we obtain $\det A^T A = 2^{2-2n}(n+1)^{n-1}$, so that

$$|\det A| = 2^{1-n}(n+1)^{\frac{n-1}{2}}.$$

It is now obvious that we obtain the same value for $\det A$, independently from the possible position of A inside S, if S is compelled to be in the first possible form. Working similarly for the second possible form of S yields the same result for $\det A'$. Hence, we conclude that all the possible $(n-1) \times (n-1)$ minors of S are of magnitude $2^{1-n}(n+1)^{\frac{n-1}{2}}$. ☐

PROPOSITION 3.5. *Let S be a binary Hadamard matrix of order n, $n > 2$. Then all the possible $(n-2) \times (n-2)$ minors of S are 0 or $2^{3-n}(n+1)^{\frac{n-3}{2}}$.*

Proof. There are 2^4=16 possible cases for the upper left 2×2 corner of S, since the possible entries are 0 and 1. The proof is illustrated for one matrix, as the other cases can be handled in a similar fashion. Since S is an $n \times n$ binary Hadamard matrix, we suppose that it can be written in the following form:

$$S = \left[\begin{array}{cc|cccc}
1 & 1 & \overbrace{1 \ \dots \ 1}^{u} & \overbrace{1 \ \dots \ 1}^{v} & \overbrace{0 \ \dots \ 0}^{x} & \overbrace{0 \ \dots \ 0}^{y} \\
1 & 0 & 1 \ \dots \ 1 & 0 \ \dots \ 0 & 1 \ \dots \ 1 & 0 \ \dots \ 0 \\
\hline
1 & 1 & & & & \\
\vdots & \vdots & & & & \\
1 & 1 & & & & \\
1 & 0 & & & & \\
\vdots & \vdots & & & & \\
1 & 0 & & \mathrm{B} & & \\
0 & 1 & & & & \\
\vdots & \vdots & & & & \\
0 & 1 & & & & \\
0 & 0 & & & & \\
\vdots & \vdots & & & & \\
0 & 0 & & & &
\end{array} \right].$$

From the order of the matrix S, the inner product of its first two rows and the total number of zeros in the first two rows (according to Lemma 3.3), we obtain the following system of four equations

$$\left\{ \begin{array}{l}
u + v + x + y = n - 2 \\
1 + u = \frac{n+1}{4} \\
x + y = \frac{n-1}{2} \\
1 + v + y = \frac{n-1}{2}
\end{array} \right.$$

ETNA
Kent State University
etna@mcs.kent.edu

which has the exact solution

$$(u, v, x, y) = \frac{1}{4}(n-3, n-3, n+1, n-3).$$

According to the properties of an $n \times n$ binary Hadamard matrix, the $(n-2) \times (n-2)$ matrix $B^T B$ has the form

$$B^T B = \left[\begin{array}{c|ccc} B_{1_{u\times u}} & (k-1)_{u\times v} & (k-1)_{u\times x} & k_{u\times y} \\ \hline (k-1)_{v\times u} & B_{2_{v\times v}} & k_{v\times x} & k_{v\times y} \\ (k-1)_{x\times u} & k_{x\times v} & B_{2_{x\times x}} & k_{x\times y} \\ k_{y\times u} & k_{y\times v} & k_{y\times x} & B_{3_{y\times y}} \end{array}\right] \equiv \left[\begin{array}{cc} B_{1_{u\times u}} & F \\ F^T & G \end{array}\right],$$

where

$$\begin{aligned} B_1 &= [(2k-2) - (k-2)]I + (k-2)J, \\ B_2 &= [(2k-1) - (k-1)]I + (k-1)J, \\ B_3 &= (2k-k)I + kJ. \end{aligned}$$

Thus, according to (1.3),

$$\det B^T B = \det B_{1_{u\times u}} \cdot \det(G - F^T B_{1_{u\times u}}^{-1} F). \tag{3.4}$$

From (1.1) we have

$$\det B_{1_{u\times u}} = \frac{\sqrt{2}}{2}(n^2 - 6n + 25)\left(\frac{n+1}{4}\right)^{\frac{n}{4}}(n+1)^{-\frac{7}{4}} \tag{3.5}$$

and from (1.2) $B_{1_{u\times u}}^{-1} = (k_1 - \lambda_1)I_u + \lambda_1 J_u$, where

$$k_1 = \frac{4(n^2 - 10n + 53)}{(n-1)(n^2 - 6n + 25)} \quad \text{and} \quad \lambda_1 = \frac{16(n-7)}{(n-1)(n^2 - 6n + 25)}.$$

Hence,

$$G - F^T B_{1_{u\times u}}^{-1} F = \left[\begin{array}{cc} K_{1_{v\times v}} & N_2 \\ N_2^T & N_1 \end{array}\right],$$

where the blocks $K_{1_{v\times v}}$, N_1, and N_2 are calculated. For the sake of brevity and clarity of presentation, we refrain from stating explicitly the form of all intermediate matrices analytically; however they are available from the authors on request.

From (1.3) we have

$$\det(G - F^T B_{1_{u\times u}}^{-1} F) = \det K_{1_{v\times v}} \cdot \det(N_1 - N_2^T K_{1_{v\times v}}^{-1} N_2). \tag{3.6}$$

From this point on, the idea of the proof is to apply consecutively formula (1.3) for the block matrices appearing, and carry out the calculations with the help of (1.1) and (1.2).

Proceeding similarly as above, we calculate $\det K_{1_{v\times v}}$ and $\det(N_1 - N_2^T K_{v\times v}^{-1} N_2)$ making use of (1.1), (1.2), and (1.3). We have

$$\det K_{1_{v\times v}} = \frac{2\sqrt{2}(n^3 - n^2 - 5n + 61)}{n^2 - 6n + 25}\left(\frac{n+1}{4}\right)^{\frac{n}{4}}(n+1)^{-\frac{7}{4}} \tag{3.7}$$

and

$$N_1 - N_2^T K_{1_{v\times v}}^{-1} N_2 = \begin{bmatrix} P_{1x\times x} & Q_1 \\ Q_1^T & P_{2y\times y} \end{bmatrix},$$

where the blocks $P_{1x\times x}$, Q_1, and $P_{2y\times y}$ are obtained as described.

According to (1.3),

$$\det(N_1 - N_2^T K_{1_{v\times v}}^{-1} N_2) = \det P_{1x\times x}\ \det(P_{2y\times y} - Q_1^T P_{1x\times x}{}^{-1} Q_1). \tag{3.8}$$

From (1.1) and (1.2) we have

$$\det P_{1x\times x} = \frac{2\sqrt{2}(n+1)^{\frac{1}{4}}(n^2-2n+13)(\frac{n+1}{4})^{\frac{n}{4}}}{n^3-n^2-5n+61} \tag{3.9}$$

and

$$P_{2y\times y} - Q_1^T P_{1x\times x}{}^{-1} Q_1 = R_{3y\times y}, \tag{3.10}$$

where $R_{3y\times y} = (k_2 - \lambda_2)I_y + \lambda_2 J_y$, $k_2 = \frac{n^3-5n^2+19n+25}{4(n^2-2n+13)}$ and $\lambda_2 = -\frac{n^2-2n-3}{n^2-2n+13}$.

Equation (1.1) gives

$$\det R_{3y\times y} = \frac{8\sqrt{2}(n+1)^{\frac{1}{4}}(\frac{n+1}{4})^{\frac{1}{4}}}{n^2-2n+13}. \tag{3.11}$$

Finally, from (3.4)–(3.11) we have

$$\begin{aligned}\det B^T B &= \det E_{u\times u}\ \det K_{1_{v\times v}}\ \det P_{1x\times x}\ \det R_{3y\times y} \\ &= \frac{64(\frac{n+1}{4})^n}{(n+1)^3} = 4^{3-n}(n+1)^{n-3}.\end{aligned}$$

Hence, $|\det B| = 2^{3-n}(n+1)^{\frac{n-3}{2}}$. Similarly, we handle all the possible remaining cases for the upper left hand corner and obtain the same result and the value 0. □

The following two lemmas specify the possible number of columns of a binary Hadamard matrix if only few rows of it are considered. They are useful to carry out proofs like that of Proposition 3.5 for $(n-j)\times(n-j)$ minors, $j > 2$. In such cases, the linear systems occurring from the properties of binary Hadamard matrices contain parameters and cannot be solved exactly. Lemma 3.7 can be used to establish bounds for the parameters in the solutions of the systems, which actually represent columns of a binary Hadamard matrix, if the first j rows are considered separately. Hence, there exist constraints on the number of columns of a binary Hadamard matrix, which moreover limit the calculations of the proposed technique. Since the upper bounds for the parameters are dependant on the order n, we cannot provide general results in these cases, but only for specific values of n.

LEMMA 3.6. *Let S be a binary Hadamard matrix of order n, $n > 2$. Then for every triple of rows of S the number of columns which are*

(a) $(1,1,1)^T$ *or* $(0,0,0)^T$ *is* $(n-3)/4$,
(b) $(1,1,0)^T$ *or* $(0,0,1)^T$ *is* $(n+1)/4$,
(c) $(1,0,1)^T$ *or* $(0,1,0)^T$ *is* $(n+1)/4$,
(d) $(1,0,0)^T$ *or* $(0,1,1)^T$ *is* $(n+1)/4$.

Proof. Three rows of the binary Hadamard matrix S can be written as

$$\overbrace{\begin{matrix}1&\dots&1\\1&\dots&1\\1&\dots&1\end{matrix}}^{u_1}\quad\overbrace{\begin{matrix}1&\dots&1\\1&\dots&1\\0&\dots&0\end{matrix}}^{u_2}\quad\overbrace{\begin{matrix}1&\dots&1\\0&\dots&0\\1&\dots&1\end{matrix}}^{u_3}\quad\overbrace{\begin{matrix}1&\dots&1\\0&\dots&0\\0&\dots&0\end{matrix}}^{u_4}\quad\overbrace{\begin{matrix}0&\dots&0\\1&\dots&1\\1&\dots&1\end{matrix}}^{u_5}\quad\overbrace{\begin{matrix}0&\dots&0\\1&\dots&1\\0&\dots&0\end{matrix}}^{u_6}\quad\overbrace{\begin{matrix}0&\dots&0\\0&\dots&0\\1&\dots&1\end{matrix}}^{u_7}\quad\overbrace{\begin{matrix}0&\dots&0\\0&\dots&0\\0&\dots&0\end{matrix}}^{u_8}.$$

From the order of the matrix S, the inner product of its first three rows and the total number of ones and zeros in the first three rows (according to Lemma 3.3) we get the following system of ten equations

$$\begin{cases} u_1+u_2+u_3+u_4+u_5+u_6+u_7+u_8=n\\ u_1+u_2=\frac{n+1}{4}\\ u_1+u_3=\frac{n+1}{4}\\ u_1+u_5=\frac{n+1}{4}\\ u_1+u_2+u_3+u_4=\frac{n+1}{2}\\ u_5+u_6+u_7+u_8=\frac{n-1}{2}\\ u_1+u_2+u_5+u_6=\frac{n+1}{2}\\ u_3+u_4+u_7+u_8=\frac{n-1}{2}\\ u_1+u_3+u_5+u_7=\frac{n+1}{2}\\ u_2+u_4+u_6+u_8=\frac{n-1}{2} \end{cases}$$

which has the solution

$$\begin{aligned} &u_1=\tfrac{n-3}{4}-u_8\\ &u_2=u_8+1\\ &u_3=u_8+1\\ &u_4=\tfrac{n-3}{4}-u_8\\ &u_5=u_8+1\\ &u_6=\tfrac{n-3}{4}-u_8\\ &u_7=\tfrac{n-3}{4}-u_8\\ &u_8=u_8. \end{aligned}$$

From the solution we see that

$$\begin{aligned} &u_1+u_8=\tfrac{n-3}{4},\\ &u_2+u_7=u_3+u_6=u_4+u_5=\tfrac{n+1}{4}, \end{aligned}$$

and the result follows immediately. □

LEMMA 3.7. *Let S be a binary Hadamard matrix of order n, $n>2$. For all the 2^j possible columns $\mathbf{u}_1,\dots,\mathbf{u}_{2^j}$ of S (or U_j) included in the first j rows, $j\geq 3$, it holds*

$$\begin{cases} 0\leq u_i\leq \dfrac{n-3}{4}, & i\in\left\{1,\dots,\frac{1}{8}\cdot 2^j\right\}\cup\left\{\frac{7}{8}\cdot 2^j+1,\dots,2^j\right\}\\ 0\leq u_i\leq \dfrac{n+1}{4}, & \textit{otherwise.} \end{cases}$$

Proof. If we consider separately the first three rows from the first j rows mentioned in the statement of the lemma, we observe for the 2^j possible columns $\mathbf{u}_i$ of U_j, $i=1,\dots,2^j$, that

$$\begin{array}{rcccccl}
\mathbf{u}_1(1:3) & = & \ldots & = & \mathbf{u}_{\frac{1}{8}2^j}(1:3) & = & (1,1,1)^T\\
\mathbf{u}_{\frac{1}{8}2^j+1}(1:3) & = & \ldots & = & \mathbf{u}_{\frac{2}{8}2^j}(1:3) & = & (1,1,0)^T\\
\mathbf{u}_{\frac{2}{8}2^j+1}(1:3) & = & \ldots & = & \mathbf{u}_{\frac{3}{8}2^j}(1:3) & = & (1,0,1)^T\\
\mathbf{u}_{\frac{3}{8}2^j+1}(1:3) & = & \ldots & = & \mathbf{u}_{\frac{4}{8}2^j}(1:3) & = & (1,0,0)^T\\
\mathbf{u}_{\frac{4}{8}2^j+1}(1:3) & = & \ldots & = & \mathbf{u}_{\frac{5}{8}2^j}(1:3) & = & (0,1,1)^T\\
\mathbf{u}_{\frac{5}{8}2^j+1}(1:3) & = & \ldots & = & \mathbf{u}_{\frac{6}{8}2^j}(1:3) & = & (0,1,0)^T\\
\mathbf{u}_{\frac{6}{8}2^j+1}(1:3) & = & \ldots & = & \mathbf{u}_{\frac{7}{8}2^j}(1:3) & = & (0,0,1)^T\\
\mathbf{u}_{\frac{7}{8}2^j+1}(1:3) & = & \ldots & = & \mathbf{u}_{\frac{8}{8}2^j}(1:3) & = & (0,0,0)^T
\end{array}$$

where $\mathbf{u}_i(1:3)$ denotes the first three entries (using Matlab notation) of the column $\mathbf{u}_i$ of U_j. This observation arises easily from a combinatorial counting and can also be verified on the matrices U_3 and U_4 given in Section 1.2.

From Lemma 3.6 we conclude

$$\begin{array}{rcccccccccccl}
u_1 & + & \ldots & + & u_{\frac{1}{8}2^j} & + & u_{\frac{7}{8}2^j+1} & + & \ldots & + & u_{\frac{8}{8}2^j} & = & \frac{n-3}{4}\\
u_{\frac{1}{8}2^j+1} & + & \ldots & + & u_{\frac{2}{8}2^j} & + & u_{\frac{6}{8}2^j+1} & + & \ldots & + & u_{\frac{7}{8}2^j} & = & \frac{n+1}{4}\\
u_{\frac{2}{8}2^j+1} & + & \ldots & + & u_{\frac{3}{8}2^j} & + & u_{\frac{5}{8}2^j+1} & + & \ldots & + & u_{\frac{6}{8}2^j} & = & \frac{n+1}{4}\\
u_{\frac{3}{8}2^j+1} & + & \ldots & + & u_{\frac{4}{8}2^j} & + & u_{\frac{4}{8}2^j+1} & + & \ldots & + & u_{\frac{5}{8}2^j} & = & \frac{n+1}{4}.
\end{array}$$

The result follows immediately from these relations, by taking into account that $u_i \geq 0$, as they denote numbers of columns. ☐

In addition, another difficulty in calculating $(n-j)\times(n-j)$ minors, $j>2$, arises from the fact that we have to examine 2^{j^2} possible upper left corners, which are singled out from the general form of j rows of a binary Hadamard matrix, denoted by U_j. This observation, in combination with the fact that the computations in the proofs of Propositions 3.4 and 3.5 follow a standard procedure based on the successive applications of formula (1.3) or (1.4), led us to develop this technique from an algorithmic point of view. Thus, we constructed the following *Minors Algorithm*, with the intent of calculating all the possible $(n-j)\times(n-j)$ minors, $j \geq 1$, of binary Hadamard matrices.

The implementation on a computer algebra package takes as input every possible upper left submatrix. For each one of them, the necessary computations are performed symbolically. The Minors Algorithm can be applied theoretically for every value of n and j. Lemma 3.7 is used for finding the possible values of the parameters in the solutions of the linear systems appearing at step 2. The symbol V_j stands for all the possible columns with entries 0 and 1, like U_j; we choose a different notation in order to show that the matrices U_j and V_j are not necessarily the same.

The Minors Algorithm

Input: All possible $j\times j$ matrices M, which can exist in the upper left corner of an $n\times n$ binary Hadamard matrix $S=\begin{bmatrix} M & U_j\\ V_j^T & D\end{bmatrix}$.

Output: Absolute values of all the possible $(n-j)\times(n-j)$ minors of S.

FOR EVERY matrix M

Step 1: *FORM* the system of $1+\binom{j}{2}+2j$ equations and 2^j unknowns u_i that results from counting the columns, the inner products of every two distinct rows of the matrix $[M\ \ U_j]$ and the total number of ones and zeros in every row of $[M\ \ U_j]$.

Step 2: *SOLVE* the system for all u_i.

ETNA
Kent State University
etna@mcs.kent.edu

FOR all the parameters attaining the values $0, \ldots, \frac{n-3}{4}$ or $\frac{n+1}{4}$

Step 3: *IF* $u_i \geq 0$ and u_i is an integer, $i = 1, \ldots, 2^j$

$$D^T D \equiv \begin{bmatrix} E_1 & F_1 \\ F_1^T & G_1 \end{bmatrix}$$

$$G_1 - F_1^T E_1^{-1} F_1 \equiv \begin{bmatrix} E_2 & F_2 \\ F_2^T & G_2 \end{bmatrix}$$

ELSE there are no acceptable solutions

END IF

Step 4: *FOR* $k = 2, \ldots, 2^j - 2$

$$G_k - F_k^T E_k^{-1} F_k \equiv \begin{bmatrix} E_{k+1} & F_{k+1} \\ F_{k+1}^T & G_{k+1} \end{bmatrix}$$

END

Step 5: $E_{2^j} = G_{2^j-1} - F_{2^j-1}^T E_{2^j-1}^{-1} F_{2^j-1}$

Step 6: $\det D^T D := \prod_{i=1}^{2^j} \det E_i, \; |\det D| = \sqrt{\det D^T D}$

END {for all the parameters }

END {for every matrix M }

PROPOSITION 3.8. *Let S be a binary Hadamard matrix of order $n = 11$. Then all possible $(n-3) \times (n-3)$ minors of the matrix S are 0 or $2^{5-n}(n+1)^{\frac{n-5}{2}}$.*

Proof. The idea is similar to the proof of Proposition 3.5. S is written in the form

$$S = \begin{bmatrix} M & U_3 \\ V_3^T & D \end{bmatrix}$$

and all the $2^9 = 512$ possible 3×3 upper left corners M are taken as input for the Minors Algorithm. The familiar properties of S lead to a linear system which has the same left hand side as the system in the proof of Lemma 3.6, but different right hand sides, according to the upper left hand corner selected for M. The solutions $u_1, \ldots, u_8$, representing the numbers of columns of S are expressed in terms of the parameter u_8, which is allowed to take the values $0, 1, 2$, according to Lemma 3.7. For the acceptable solutions having positive integer components, the rest of the procedure of the Minors Algorithm is carried out in order to specify the determinant of D for each M. For instance, at Step 3 of the algorithm the matrix $D^T D$ has the form

$$D^T D = \begin{bmatrix} E_1 & k_2 & k_2 & k_1 & k_2 & k_1 & k_1 & k \\ & F & k_1 & k_1 & k_1 & k_1 & k & k \\ & & F & k_1 & k_1 & k & k_1 & k \\ & & & G & k & k & k & k \\ & & & & F & k_1 & k_1 & k \\ & & & & & G & k & k \\ & & & & & & G & k \\ & & & & & & & H \end{bmatrix},$$

where $k = \frac{n+1}{4}$, $k_1 = k - 1$, $k_2 = k - 2$, $E_1 = kI_{u_1} + (k-3)J_{u_1}$, $F = kI + (k-2)J$, $G = kI + (k-1)J$ and $H = kI + kJ$. The diagonal blocks $E_1, F, F, \ldots, G, H$, are of known orders $u_1 \times u_1$, $u_2 \times u_2, \ldots, u_8 \times u_8$. The elements k, k_1, k_2, represent blocks of appropriate sizes (according to the notation introduced in Section 1.2), but the subscripts are omitted for a more compact presentation. From now on, the sequence of matrices E_k, F_k, G_k, $k = 2, \ldots, 8$, results from Steps 4 and 5 of the Minors Algorithm. For the sake of brevity we do not describe all the matrices explicitly. ☐

ETNA
Kent State University
etna@mcs.kent.edu

By applying the Minors Algorithm for $n = 11$ and $j = 4$, the following result is derived.

PROPOSITION 3.9. *Let S be a binary Hadamard matrix of order $n = 11$. Then all possible $(n-4) \times (n-4)$ minors of S are 0, $2^{7-n}(n+1)^{\frac{n-7}{2}}$ or $2^{8-n}(n+1)^{\frac{n-7}{2}}$.*

4. Application to the growth problem.

4.1. Description of the problem. The backward error analysis for Gaussian elimination (GE) on a matrix $A = (a_{ij}^{(0)})$ is traditionally expressed in terms of the *growth factor*

$$g(n, A) = \frac{\max_{i,j,k} |a_{ij}^{(k)}|}{\max_{i,j} |a_{ij}^{(0)}|},$$

which involves all the elements $a_{ij}^{(k)}$, $k = 0, 1, 2, \ldots, n-1$, that occur during the elimination [3, 13, 24]. Matrices with the property that no row and column exchanges are needed during GE with complete pivoting are called *completely pivoted* (CP) or feasible. For a CP matrix A we have

$$g(n, A) = \frac{\max\{p_1, p_2, \ldots, p_n\}}{|a_{11}^{(0)}|}, \tag{4.1}$$

where $p_1, p_2, \ldots, p_n$ are the pivots of A. According to known theorems [3, 24], it is clear that the stability of GE depends on the growth factor. If $g(n, A)$ is of order 1, the elimination process is stable. If $g(n, A)$ is larger, we must expect instability. The study of the values assumed by $g(n, A)$, and the specification of pivot patterns, are referred to as *the growth problem*.

In [2] Cryer conjectured that "for real matrices $g(n, A) \leq n$, with equality if and only if A is a Hadamard matrix". This conjecture became one of the most famous open problems in Numerical Analysis and has been investigated since then by many mathematicians. The inequality was finally shown to be false in [9], however its second part is still an open problem.

Since binary Hadamard matrices are connected with Hadamard matrices, it is sensible to apply GE with complete pivoting on equivalent (i.e., obtained by row and/or column interchanges) binary Hadamard matrices of various orders and write down their pivot patterns and growth factors.

Tables 4.1 and 4.2 show some pivot patterns which appear if GE with complete pivoting is applied, experimentally, to 200000 equivalent binary Hadamard matrices for each order $n = 15$, 19,23, 31, and 39. The last column gives the total number of pivot patterns that appeared in the experiments. Especially for $n = 15$, the binary Hadamard matrices are obtained separately from Hadamard matrices of order 16, which are classified in five equivalence classes I,II,...,V, see [25]. It is interesting to mention that the total numbers of pivot patterns observed experimentally for Hadamard matrices of order 16 from the equivalence classes I,II,...,V, are 9,15,10,12,12, respectively. On the contrary, for $n > 15$ the total numbers of pivot patterns of binary Hadamard matrices are significantly fewer than the ones obtained from the corresponding Hadamard matrices; see [16]. The $(n-3)$th pivot, i.e., the fourth counting from the last pivot, is always $\frac{n+1}{8}$, except for binary Hadamard matrices of order 15 obtained from the I-equivalence class of Hadamard matrices of order 16. Finally, it is interesting to compare the pivot patterns of Hadamard and binary Hadamard matrices. We observe (cf. [16]) that with the exception of the first pivot, which is in both cases always 1, the pivots of binary Hadamard matrices are about one half of the pivots of the corresponding Hadamard matrices, and consequently the growth factors are halved, as well. This fact points out the significance for Gaussian elimination of inserting the entry 0 in binary Hadamard matrices instead of the entry -1 of Hadamard matrices.

TABLE 4.1
Pivot patterns of binary Hadamard matrices of order $n = 15$

class	pivot pattern	number
I	$(1, 1, 2, 1, \frac{4}{3}, 1, 2, 1, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$	12
	$(1, 1, 2, 1, \frac{4}{3}, 2, 3, 1, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$	
	$(1, 1, 2, \frac{3}{2}, \frac{4}{3}, 1, 2, 2, 2, 2, 4, 4, 4, 4, 8)$	
II	$(1, 1, 2, 1, \frac{5}{3}, \frac{6}{5}, 2, 1, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$	15
	$(1, 1, 2, 1, \frac{5}{3}, \frac{6}{5}, 2, \frac{4}{3}, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$	
	$(1, 1, 2, 1, \frac{5}{3}, \frac{6}{5}, 2, \frac{8}{5}, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$	
III	$(1, 1, 2, 1, \frac{4}{3}, \frac{9}{5}, 2, 1, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$	18
	$(1, 1, 2, 1, \frac{5}{3}, \frac{9}{5}, 2, 1, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$	
	$(1, 1, 2, 1, \frac{5}{3}, \frac{9}{5}, 2, \frac{4}{3}, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$	
IV/V	$(1, 1, 2, 1, \frac{5}{3}, \frac{9}{5}, 2, 1, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$	16
	$(1, 1, 2, 1, \frac{5}{3}, \frac{9}{5}, 2, 2, 2, 2, \frac{12}{5}, \frac{8}{3}, 2, 4, 4, 8)$	
	$(1, 1, 2, 1, \frac{5}{3}, \frac{9}{5}, 2, 2, \frac{20}{9}, \frac{12}{5}, \frac{8}{3}, 2, 4, 4, 8)$	

These numerical experiments, in combination with the theoretical results of Section 3, lead to the following new conjecture.

CONJECTURE 4.1 (**Growth conjecture for binary Hadamard matrices**). *Let S be a binary Hadamard matrix of order n. Reduce S by GE with complete pivoting. Then, for large enough n,*

(i) $g(n, S) = \frac{n+1}{2}$;

(ii) Every pivot before the last has magnitude at most $\frac{n+1}{2}$;

(iii) The three last pivots are (in backward order) $\frac{n+1}{2}$, $\frac{n+1}{4}$, $\frac{n+1}{4}$;

(iv) The $(n-3)$th pivot can be $\frac{n+1}{8}$ *or* $\frac{n+1}{4}$;

(v) The first three pivots are equal to $1,\ 2,\ 2$*; the fourth can take the values* 1 *or* $3/2$.

4.2. Pivot patterns of binary Hadamard matrices. The object of this section is to demonstrate the unique pivot pattern of a CP binary Hadamard matrix of order 11, in other words to show that every equivalent binary Hadamard matrix can have only this pivot pattern if GE with complete pivoting is applied to it or, equivalently, if GE is performed on a CP binary Hadamard matrix of order 11.

A naive search performing all possible row and/or column interchanges in order to find all possible binary Hadamard matrices would require $(11!)^2 \approx 10^{15}$ trials. Such computer search would not be completed in reasonable time. In addition, the pivot pattern of each one of these matrices should be computed. Another obstacle when dealing with orders greater than 11 is the fact that the pivot pattern is not invariant under equivalence row and/or column interchanges, i.e., it is possible that equivalent matrices can have different pivot patterns; cf. Table 4.1. Hence, in order to derive results about pivot patterns one cannot work with a representative matrix of an equivalence class of binary Hadamard matrices, e.g., the set of equivalent matrices with same determinant. Thus, one has to take all the possible matrices

ETNA
Kent State University
etna@mcs.kent.edu

TABLE 4.2
Pivot patterns of binary Hadamard matrices of orders $n = 19, 23, 31, 39$

n	pivot pattern	number
19	$(1, 1, 2, \frac{3}{2}, \frac{5}{3}, \frac{9}{5}, 2, \ldots, \frac{5}{2}, \frac{5}{2}, \frac{10}{3}, \frac{5}{2}, 5, 5, 10)$	187
	$(1, 1, 2, \frac{3}{2}, \frac{5}{3}, \frac{9}{5}, \frac{9}{4}, \ldots, \frac{25}{9}, 3, 5, \frac{5}{2}, 5, 5, 10)$	
	$(1, 1, 2, \frac{3}{2}, \frac{5}{3}, \frac{9}{5}, \frac{5}{2}, \ldots, \frac{25}{8}, \frac{15}{4}, 5, \frac{5}{2}, 5, 5, 10)$	
23	$(1, 1, 2, 1, \frac{5}{3}, \frac{8}{5}, 2, \ldots, 3, 3, 4, 3, 6, 6, 12)$	228
	$(1, 1, 2, \frac{3}{2}, \frac{5}{2}, \frac{9}{5}, 3, \ldots, \frac{10}{3}, \frac{18}{5}, 6, 3, 6, 6, 12)$	
	$(1, 1, 2, \frac{3}{2}, 2, 2, 4, \ldots, \frac{15}{4}, \frac{9}{2}, 6, 3, 6, 6, 12)$	
31	$(1, 1, 2, 1, \frac{5}{3}, \frac{8}{5}, 2, \ldots, 4, 4, \frac{16}{3}, 4, 8, 8, 16)$	595
	$(1, 1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{5}{2}, \ldots, 4, 4, 8, 4, 8, 8, 16)$	
	$(1, 1, 2, \frac{3}{2}, 2, 2, 3, \ldots, 4, 4, 8, 4, 8, 8, 16)$	
39	$(1, 1, 2, 1, \frac{5}{3}, \frac{9}{5}, 2, \ldots, 5, 5, \frac{20}{3}, 5, 10, 10, 20)$	10000
	$(1, 1, 2, \frac{3}{2}, \frac{5}{3}, \frac{9}{5}, 2, \ldots, 5, 5, \frac{20}{3}, 5, 10, 10, 20)$	
	$(1, 1, 2, \frac{3}{2}, 2, 2, 3, \ldots, \frac{25}{4}, \frac{15}{2}, 10, 5, 10, 10, 20)$	

into account. We show how the results of Section 3 can be used to calculate pivots from the end of the pivot structure in order to save significant computational time.

First, we give two useful properties for CP matrices.

LEMMA 4.2 ([2], [7, p. 26], [21]). *Let A be a CP matrix.*

(i) *The magnitude of the pivots appearing after application of GE algorithm on A is given by*

$$p_j = \frac{A(j)}{A(j-1)}, \quad j = 1, 2, \ldots, n, \quad A(0) = 1. \tag{4.2}$$

(ii) *The maximum $j \times j$ leading principal minor of A, when the first $j-1$ rows and columns are fixed, is $A(j)$.*

THEOREM 4.3. *If GE is applied to a CP binary Hadamard matrix of order n, $n > 2$, the last two pivots are (in backward order) $\frac{n+1}{2}$ and $\frac{n+1}{4}$.*

Proof. From Lemma 3.1 we have $\det S \equiv S(n) = 2^{-n}(n+1)^{\frac{n+1}{2}}$. Propositions 3.4 and 3.5, in combination with Lemma 4.2 (ii), imply that for a CP binary Hadamard matrix it holds $S(n-1) = 2^{1-n}(n+1)^{\frac{n-1}{2}}$ and $S(n-2) = 2^{3-n}(n+1)^{\frac{n-3}{2}}$. By substituting these values in relation (4.2) and taking into account Lemma 3.1, we obtain

$$p_n = \frac{S(n)}{S(n-1)} = \frac{2^{-n}(n+1)^{\frac{n+1}{2}}}{2^{1-n}(n+1)^{\frac{n-1}{2}}} = \frac{n+1}{2}$$

and

$$p_{n-1} = \frac{S(n-1)}{S(n-2)} = \frac{2^{1-n}(n+1)^{\frac{n-1}{2}}}{2^{3-n}(n+1)^{\frac{n-3}{2}}} = \frac{n+1}{4}. \qquad \square$$

ETNA
Kent State University
etna@mcs.kent.edu

Considering the interpretation of a binary Hadamard matrix as SBIBD $(4t-1, 2t, t)$ we can state that the two last pivots (in backward order) are $2t$ and t. It is interesting to observe that the respective values for the complementary SBIBD $(4t-1, 2t-1, t-1)$ are $2t$ and $2t$ [18].

In the remainder of this paper we apply the results of Section 3 to calculate $(n-j) \times (n-j)$, $j > 2$, minors for $n = 11$ fixed.

PROPOSITION 4.4. *If GE is applied to a CP binary Hadamard matrix of order 11, the third and fourth pivot from the end are* 3 *and* $\frac{3}{2}$*, respectively.*

Proof. Propositions 3.8 and 3.9, in combination with Lemma 4.2 (ii), yield that for a CP binary Hadamard matrix S of order $n = 11$ it holds $S(n-3) = 2^{5-n}(n+1)^{\frac{n-5}{2}} = 27$ and $S(n-4) = 2^{8-n}(n+1)^{\frac{n-7}{2}} = 18$. By substituting these values and the general value for $S(n-2)$ in relation (4.2), we obtain

$$p_{n-2} = \frac{S(n-2)}{S(n-3)} = \frac{81}{27} = 3 \quad \text{and} \quad p_{n-3} = \frac{S(n-3)}{S(n-4)} = \frac{27}{18} = \frac{3}{2}. \qquad \square$$

Next, we illustrate how the values of the minors $S(j)$, $j = 1, \ldots, 6$, for a CP binary Hadamard matrix of order 11 can be specified, so that relation (4.2) can be used for the computation of the first six pivots. First we give the following useful result.

LEMMA 4.5. *The maximum absolute value of the determinant of all* $n \times n$ *matrices with elements 0 and 1 is given in the following table for* $n = 1, \ldots, 6$

n	1	2	3	4	5	6
max. det	1	1	2	3	5	9

More information and results on determinants of $(0, 1)$ matrices can be found in [1, 6, 11, 26].

PROPOSITION 4.6. *Let* S *be a CP binary Hadamard matrix of order 11. Then* $S(1) = 1$*,* $S(2) = 1$*,* $S(3) = 2$*,* $S(4) = 3$*,* $S(5) = 5$*, and* $S(6) = 9$*.*

Proof. Consider S as the third matrix of the Example 1.4, which is a binary Hadamard matrix of order 11. We observe that $S(1) = 1$, $S(2) = 1$, $S(3) = 2$, $S(4) = 3$, $S(5) = 5$, and $S(6) = 9$. These values of the minors $S(j)$, $j = 1, \ldots, 6$, are the maximum values for $j \times j$, $j = 1, \ldots, 6$, matrices with elements 0 and 1, as it can be verified by Lemma 4.5.

We note that a binary Hadamard matrix of order 11 is unique under equivalence, since it is derived from a Hadamard matrix of order 12 which is unique under equivalence [25]. Therefore, the matrices with maximum determinants exist in every binary Hadamard matrix of order 11, since they have been proved to exist in one. If the matrix is CP, the matrices with maximum determinants must appear in the upper left corner, according to Lemma 4.2 (ii), and this completes the proof. $\square$

PROPOSITION 4.7. *Let* S *be a CP binary Hadamard matrix of order 11. Then the first six pivots of* S *are 1, 1, 2,* $\frac{3}{2}$*,* $\frac{5}{3}$*, and* $\frac{9}{5}$*.*

Proof. The pivot values in the statement are obtained by substituting appropriately the results of Proposition 4.6 in formula (4.2). $\square$

PROPOSITION 4.8. *If GE with complete pivoting is performed on a binary Hadamard matrix of order 11 the pivot pattern is* $(1, 1, 2, \frac{3}{2}, \frac{5}{3}, \frac{9}{5}, 2, \frac{3}{2}, 3, 3, 6)$*.*

Proof. The first six pivots of a binary Hadamard matrix S are given in Proposition 4.7 and the last four pivots in Proposition 4.4 and Theorem 4.3. The seventh pivot can be found from the property that the determinant of the matrix equals the product of the pivots, i.e.,

$$\det S = \prod_{i=1}^{11} p_i \quad \Rightarrow \quad p_7 = \frac{\det S}{\prod_{i=1, i\neq 7}^{11} p_i} = 2. \qquad \square$$

THEOREM 4.9. *If GE with complete pivoting is performed on a binary Hadamard matrix of order 11, i.e., an SBIBD* $(11, 6, 3)$*, the growth factor is 6.*

Proof. Proposition 4.8 and equation (4.1) yield the required growth factor, taking into account that, according to Lemma 4.2 (ii), the element $a_{11}^{(0)}$ of a CP binary Hadamard matrix can be only 1. ☐

Following a similar, easier procedure, we can obtain the following pivot patterns as well.

PROPOSITION 4.10. *If GE with complete pivoting is performed on binary Hadamard matrices of orders 3 and 7, i.e., SBIBDs* $(3, 2, 1)$ *and* $(7, 4, 2)$*, the pivot patterns are* $(1, 1, 2)$ *and* $(1, 1, 2, 1, 2, 2, 4)$*, respectively.*

The importance of Theorem 4.9 and Proposition 4.10 is that they guarantee stability when solving linear systems with the respective matrices using GE with complete pivoting, since they have small growth factors (of order 1), and hence they do not allow the existence of significant roundoff errors.

5. Conclusions. We proposed a technique to calculate all the possible $(n - j) \times (n - j)$ minors of binary Hadamard matrices. This also reveals some properties regarding their structure. The theoretical idea leads to an algorithm that overcomes the difficulties arising from the laborious calculations done by hand. Theoretically, the algorithm works for every pair of values n and j. The usefulness of such a method is justified by its application to a problem of Numerical Linear Algebra, known as the growth problem. The results obtained, in combination with the extensive numerical experiments, lead to the formulation of the growth conjecture for binary Hadamard matrices.

An important open problem is whether a method can be found to prove general results independently of the presence of parameters in the solution of the linear systems appearing in the method. Furthermore, it is still not understood why the value $(n + 1)/4$ as $(n - 3)$th pivot (cf. Tables 4.1 and 4.2) appears only for binary Hadamard matrices of order 15 obtained from Hadamard matrices of order 16 belonging to the I-class of equivalence, and specifically only in one pivot pattern. Finally, the proof of the conjecture for minors of Hadamard matrices, probably in connection with possible values of determinants of ± 1 matrices, and the existence of binary Hadamard matrices for every $n \equiv 3 \pmod 4$ are open problems, too.

Acknowledgments. We would like to thank the referees for their valuable comments and suggestions that contributed to a significant improvement of the presentation of this paper.

REFERENCES

[1] J. BRENNER AND L. CUMMINGS, *The Hadamard Maximum Determinant Problem*, Amer. Math. Monthly, 79 (1972), pp. 626–630.
[2] C. W. CRYER, *Pivot size in Gaussian elimination*, Numer. Math., 12 (1968), pp. 335–345.
[3] B. N. DATTA, *Numerical Linear Algebra and Applications*, Brooks/Cole, Pacific Grove, 1995.
[4] J. DAY AND B. PETERSON, *Growth in Gaussian elimination*, Amer. Math. Monthly, 95 (1988), pp. 489–513.
[5] J. H. DINITZ AND D. R. STINSON, *Contemporary Design Theory: A Collection of Surveys*, John Wiley & Sons, New York, 1992.
[6] H. EHLICH, *Determinanten Abschätzungen für binäre Matrizen*, Math. Z., 83 (1964), pp. 123–132.
[7] F. R. GANTMACHER, *The Theory of Matrices, vol. 1.*, Chelsea, New York, 1959.
[8] A. V. GERAMITA AND J. SEBERRY, *Orthogonal Designs: Quadratic Forms and Hadamard Matrices*, Marcel Dekker, New York-Basel, 1979.
[9] N. GOULD, *On growth in Gaussian elimination with pivoting*, SIAM J. Matrix Anal. Appl., 12 (1991), pp. 354–361.
[10] K. GRIFFIN AND M. J. TSATSOMEROS, *Principal minors, Part I: A method for computing all the principal minors of a matrix*, Linear Algebra Appl., 419 (2006), pp. 107–124.
[11] J. HADAMARD, *Résolution d'une question relative aux déterminants*, Bull. Sci. Math., 17 (1893), pp. 30–31.
[12] M. HARWIT AND N. J. A. SLOANE, *Hadamard Transform Optics*, Academic Press, New York, 1979.
[13] N. J. HIGHAM, *Accuracy and Stability of Numerical Algorithms*, SIAM, Philadelphia, 2002.

ETNA
Kent State University
etna@mcs.kent.edu

[14] K. J. HORADAM, *Hadamard Matrices and their Applications*, Princeton University Press, Princeton, 2007.
[15] R. A. HORN AND C. R. JOHNSON, *Matrix Analysis*, Cambridge University Press, Cambridge, 1985.
[16] C. KOUKOUVINOS, E. LAPPAS, M. MITROULI AND J. SEBERRY, *On the complete pivoting conjecture for Hadamard matrices of small orders*, J. Res. Prac. Inf. Tech., 33 (2001), pp. 298–309.
[17] C. KOUKOUVINOS, M. MITROULI AND J. SEBERRY, *An algorithm to find formulae and values of minors of Hadamard matrices*, Linear Algebra Appl., 330 (2001), pp. 129–147.
[18] C. KOUKOUVINOS, M. MITROULI AND J. SEBERRY, *Values of minors of* $(1,-1)$ *incidence matrices of SBIBDs and their application to the growth problem*, Des. Codes Cryptogr., 23 (2001), pp. 267–282.
[19] C. KRAVVARITIS AND M. MITROULI, *Evaluation of minors associated to weighing matrices*, Linear Algebra Appl., 426 (2007), pp. 774–809.
[20] C. KRAVVARITIS AND M. MITROULI, *Computations for minors of Hadamard matrices*, Bull. Greek Math. Soc., 54 (2007), pp. 221–238.
[21] C. KRAVVARITIS, M. MITROULI AND J. SEBERRY, *On the pivot structure for the weighing matrix* $W(12,11)$, Linear Multilinear Algebra, 55 (2007), pp. 471–490.
[22] J. SEBERRY AND M. YAMADA, *Hadamard matrices, sequences and block designs*, in Contemporary Design Theory: A Collection of Surveys, J. H. Dinitz and D. J. Stinson, eds., John Wiley & Sons, New York, 1992, pp. 431–560.
[23] F. R. SHARPE, *The maximum value of a determinant*, Bull. Amer. Math. Soc., 14 (1907), pp. 121–123.
[24] L. N. TREFETHEN AND D. BAU, III, *Numerical Linear Algebra*, SIAM, Philadelphia, 1997.
[25] W. D. WALLIS, A. P. STREET, AND J. S. WALLIS, *Combinatorics: Room Squares, Sum-Free Sets, Hadamard Matrices*, Lectures Notes in Math., 292, Springer, New York, 1972.
[26] J. WILLIAMSON, *Determinants whose elements are 0 and 1*, Amer. Math. Monthly, 53 (1946), pp. 427–434.