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ON THE ROLE OF BOUNDARY CONDITIONS FOR CIP STABILIZATION OF HIGHER ORDER FINITE ELEMENTS*

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Abstract. We investigate the Continuous Interior Penalty (CIP) stabilization method for higher order finite elements applied to a convection diffusion equation with a small diffusion parameter ε. Performing numerical experiments, it turns out that strongly imposed Dirichlet boundary conditions lead to relatively bad numerical solutions. However, if the Dirichlet boundary conditions are imposed on the inflow part of the boundary in a weak sense and additionally on the whole boundary in an ε-weighted weak sense due to Nitsche then one obtains reasonable numerical results. In many cases, this holds even in the limit case where the parameter of the CIP stabilization is zero, i.e., where the standard Galerkin discretization is applied. We present an analysis which explains this effect.

Key words. diffusion-convection-reaction equation, finite elements, Nitsche type boundary conditions, error estimates

AMS subject classifications. 65N15, 65N30, 65N50

1. Introduction. We consider as a model problem the convection diffusion reaction equation and we are interested in stable and accurate discretizations for the case of a small diffusion parameter ε. Such discretizations can be also transferred to other problems like, for instance, the Navier–Stokes equations in the case of high Reynolds numbers. It is known that, unless the exact solution has no layers, the standard Galerkin discretization leads to un-feasible numerical solutions with non-physical oscillations. Therefore, a stabilization is necessary. Well established stabilization methods are the Streamline Upwind Petrov-Galerkin (SUPG) method and classical upwind techniques. Among these types, the SUPG method is capable also for higher order finite elements. Nevertheless, it has some undesirable features [7]. To overcome these problems, new types of stabilization methods have been developed in recent years, see [2, 3, 7, 13].

One popular method is the *Continuous Interior Penalty (CIP) stabilization* [4, 6, 7, 9, 10]. The idea is to add a penalty term which contains the jump of the gradient of the discrete solution across the inter-element faces. Thus, oscillations in the discrete solution are prohibited by a penalty which has a stabilizing effect. Moreover, this discretization is consistent since for a smooth exact solution the penalty term is zero. The method works also in the context of hp-FEM [6]. The additional costs of a CIP stabilization consist in a larger amount of memory and computational work due to a larger number of matrix entries in the stiffness matrix. However, the portion of the additional costs can be reduced essentially by taking suitable basis functions as we will show in a forthcoming paper.

In this paper, we present numerical experiments which show that strongly imposed Dirichlet boundary conditions within the CIP stabilization method lead to relatively bad numerical solutions. However, if the Dirichlet boundary conditions are imposed on the inflow part of the boundary in a weak sense and additionally on the whole boundary in an ε-weighted weak sense due to Nitsche [14] then one obtains reasonable numerical results. In many cases, this holds even in the limit case where the parameter of the CIP stabilization is zero, i.e., where the standard Galerkin discretization is applied. Also for the Navier-Stokes equations, it has been shown in [1] that weakly imposed Dirichlet boundary conditions are effective and superior to strongly imposed conditions. The aim of this paper is to explain the positive effect

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of a weak imposition of Dirichlet boundary conditions by a theoretical analysis. We prove that the discretization error is small uniformly with respect to ε in a subregion Ω_0 of the domain Ω which does not contain any layer. A different technique for proving local estimates of the discretization error can be found in the recent paper [8].

Another goal of the paper is to prove a global error estimate which justifies the use of locally adapted meshes. In the literature on CIP methods [4, 6, 7, 9]. it has been shown that, for small ε, the order of the discretization error in the L^2-norm is $\mathcal{O}(h^{k+1/2})$ where k is the polynomial degree of the finite element space. However, in all the mentioned papers, the final estimate is not completely in a locally adaptive fashion where a power of the local mesh size h_K of an element K is multiplied with a Sobolev norm of the exact solution on K. In order to overcome this problem, we have proved as a first step the "nearly" optimal order $\mathcal{O}(h^k)$ but in a completely adaptive fashion. The corresponding estimate gives a reasonable error bound if the local mesh size h_K is very small for all elements K located in the boundary layer where the local norm $|u|_{k+1,K}$ is large. Thus, the usage of locally adapted meshes is justified by our locally adaptive estimate.

This paper is organized as follows. In Section 2, the model problem is described and notations are introduced. The two CIP stabilization methods with strong and weak imposition of the Dirichlet boundary conditions are presented in Section 3. Then, we show some numerical experiments in Section 4 which indicate that the weakly imposed boundary conditions yield much better results. We explain this effect by a theoretical analysis in Section 5. Finally, in Section 6, we investigate the numerical order of convergence for a test problem and compare it with the theoretical results.

2. Preliminaries and notations.

2.1. Model problem. Let $\Omega \subset \mathbb{R}^d$, $1 \le d \le 3$, be a bounded domain with a polygonal or polyhedral boundary. As a model problem we consider the following convection diffusion reaction problem for the unknown function $u : \Omega \to \mathbb{R}$:

$$\begin{aligned} -\varepsilon\triangle u + (\beta\cdot\nabla)u + \sigma u &= f \qquad \text{in} \quad \Omega, \\ u &= 0 \qquad \text{on} \quad \partial\Omega, \end{aligned} \tag{2.1}$$

where ε is a small positive parameter, $\beta : \Omega \to \mathbb{R}^d$ a convection field, $\sigma : \Omega \to \mathbb{R}$ a positive reaction coefficient and $f : \Omega \to \mathbb{R}$ a source term. For simplicity of the presentation, we consider only homogeneous Dirichlet boundary conditions but the results can be generalized to the case of non-homogeneous case. We assume that the functions β, σ and f are sufficiently smooth. To guarantee existence and uniqueness of the solution, we assume that there exists a constant σ_0 such that

$$\sigma(x) - \frac{1}{2}\operatorname{div}\beta(x) \ge \sigma_0 > 0 \qquad \forall x \in \Omega. \tag{2.2}$$

2.2. Notations. For a domain $G \subset \Omega$, let $|\cdot|_{m,p,G}$ and $\|\cdot\|_{m,p,G}$ denote the usual semi-norm and norm in the Sobolev spaces $W^{m,p}(G)$ and $(W^{m,p}(G))^d$, respectively. For the Hilbert spaces $H^m(G)$ and $(H^m(G))^d$, we omit the index p and denote the semi-norm and norm by $|\cdot|_{m,G}$ and $\|\cdot\|_{m,G}$. The inner product in $L^2(G)$ and $(L^2(G))^d$ will be denoted by $(\cdot,\cdot)_G$, whereas for a sufficiently smooth $(d-1)$-dimensional face E of G, the inner product in $L^2(E)$ is denoted by $\langle\cdot,\cdot\rangle_E$. By $\mathbb{P}_k(D)$ we denote the space of all polynomials on the domain $D \subset \mathbb{R}^n$, $1 \le n \le d$, with total degree less than or equal to k and by $\mathbb{Q}_k(D)$ the space of those polynomials where the maximum power in each coordinate is less than or equal to k. For a set $G \subset \mathbb{R}^d$, we denote by $\operatorname{int}(G)$ and $\overline{G}$ the interior and closure of G, respectively. If E is a $(d-1)$-dimensional manifold we denote by $|E|$ the $(d-1)$-measure of E.

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The domain Ω is decomposed into elements $K \in \mathcal{T}_h$ which are assumed to be open quadrilaterals or hexahedra. We denote by h_K the diameter of the element $K \in \mathcal{T}_h$ and by $h := max_{K\in\mathcal{T}_h} h_K$ the mesh-size. In the following, we introduce some notation to describe the element faces of the mesh. For an element $K \in \mathcal{T}_h$, we denote by $\mathcal{E}(K)$ the set of all $(d-1)$-dimensional faces of K. Let $\mathcal{E}_h := \bigcup_{K\in\mathcal{T}_h} \mathcal{E}(K)$ be the set of all element faces of the grid $\mathcal{T}_h$. We split $\mathcal{E}_h$ as $\mathcal{E}_h = \mathcal{E}_h^0 \cup \mathcal{E}_h^b$ where $\mathcal{E}_h^0$ denotes the set of all inner faces and $\mathcal{E}_h^b$ the set of all faces located at the boundary of Ω. For each inner face $E \in \mathcal{E}_h^0$, there exist exactly two different elements denoted by $K(E)$ and $K'(E)$ such that E is one of their faces. For the boundary faces $E \in \mathcal{E}_h^b$, there is only one element denoted by $K(E)$ which has E as one of its faces and we set formally $K'(E) := \emptyset$. For an element $K \in \mathcal{T}_h$ and a face $E \in \mathcal{E}(K)$, we denote by n_E^K the unit normal on E pointing outward with respect to K. We assign to each face $E \in \mathcal{E}_h$ the unit normal $n_E := n_E^{K(E)}$ and the mesh size $h_E := h_{K(E)}$. For an element-wise given function v and an inner face $E \in \mathcal{E}_h^0$, we denote by $[v]_E$ the jump of v across E defined as

$$[v_h]_E(x) := v_h\big|_{K(E)}(x) - v_h\big|_{K'(E)}(x) \qquad \forall x \in E.$$

Throughout this paper, C will denote a generic constant which may have different values at different places. All these constants occurring inside of any estimate will be independent of ε and the local and global mesh parameters h_K and h.

3. Two discretizations with CIP stabilization. The weak formulation of problem (2.1) reads: *Find* $u \in H_0^1(\Omega)$ *such that*

$$a(u,v) = (f,v)_\Omega \qquad \forall\, v \in H_0^1(\Omega), \tag{3.1}$$

where

$$a(u,v) := \varepsilon(\nabla u, \nabla v)_\Omega + (\beta\cdot\nabla u, v)_\Omega + (\sigma u, v)_\Omega.$$

Let $k \geq 1$ denote a fixed polynomial degree. Then, we define the finite element space V_h as

$$V_h := \{v \in H^1(\Omega) :\, v\big|_K \circ F_K \in \mathbb{Q}_k(\widehat{K}) \quad \forall K \in \mathcal{T}_h\}, \tag{3.2}$$

where $\widehat{K} = (-1,1)^d$ denotes the reference element and $F_K : \widehat{K} \to K$ the multilinear reference mapping of the element $K \in \mathcal{T}_h$.

In the following, we describe two discretizations of problem (3.1) which differ in the treatment of the boundary conditions.

3.1. Strong Dirichlet boundary conditions. Here, the Dirichlet boundary conditions are contained in the finite element space in a strong sense, i.e., we will work with the space

$$V_h^D := V_h \cap H_0^1(\Omega)$$

and the discrete problem reads : *Find* $u_h^D \in V_h^D$ *such that*

$$a(u_h^D, v_h) + s_h(u_h^D, v_h) = (f, v_h)_\Omega \qquad \forall\, v_h \in V_h^D. \tag{3.3}$$

The bilinear form $s_h(\cdot,\cdot)$ acts as a stabilization term and is defined as

$$s_h(w,v) := \sum_{E\in\mathcal{E}_h^0} \gamma h_E^2 \left\langle [\nabla w]_E, [\nabla v]_E \right\rangle_E \qquad \forall\, w, v \in H^2(\Omega) + V_h,$$

where γ is a non-negative stabilization parameter which is independent of h and ε. The case $\gamma = 0$ corresponds to the standard Galerkin discretization of (3.1).

The stabilization method (3.3) is called *continuous interior penalty* (CIP) method since it penalizes if the solution u_h^D has large jumps in the gradient across the element faces. In particular, functions which have oscillations on a relatively large subdomain will get a large penalty. Conversely, if the gradient of a function w is smooth enough, as for instance for a smooth solution u of (3.1), then there is no penalty, i.e.,

$$s_h(w, v_h) = 0 \qquad \forall w \in H^2(\Omega),\ v_h \in V_h. \tag{3.4}$$

This implies the consistency of the discretization (3.3).

3.2. Nitsche type boundary conditions. In the second discretization of (3.1), the Dirichlet boundary conditions are imposed in a weak sense due to Nitsche [14]. Using the finite element space $V_h^N := V_h$, the discrete problem reads: *Find* $u_h^N \in V_h^N$ *such that*

$$a(u_h^N, v_h) + s_h(u_h^N, v_h) + n_h(u_h^N, v_h) = (f, v_h)_\Omega \qquad \forall\, v_h \in V_h^N, \tag{3.5}$$

where the bilinear form $n_h(\cdot,\cdot)$ is defined as

$$\begin{aligned} n_h(w, v) :=& \varepsilon \sum_{E\in\mathcal{E}_h^b} \left\{ -\left\langle \frac{\partial w}{\partial n_E}, v \right\rangle_E - \left\langle w, \frac{\partial v}{\partial n_E} \right\rangle_E + \frac{\gamma_N}{h_E} \langle\, w, v\,\rangle_E \right\} \\ &+ \sum_{E\in\mathcal{E}_h^-} \langle\, |\beta\cdot n_E| w, v\,\rangle_E \end{aligned} \tag{3.6}$$

for all $w, v \in V_h + H^2(\Omega)$. The parameter γ_N is independent of ε and h_E and has to be chosen sufficiently large; see Sections 4 and 5. The set $\mathcal{E}_h^-$ consists of all boundary faces E that are contained in the so-called *inflow boundary part* Γ_-, i.e.,

$$\mathcal{E}_h^- := \{E \in \mathcal{E}_h^b : E \subset \Gamma_-\} \quad \text{with} \quad \Gamma_- := \{x \in \partial\Omega :\ \beta(x)\cdot n(x) < 0\},$$

where n denotes the unit normal on $\partial\Omega$ pointing outward with respect to Ω. Let us assume that Γ_- is completely represented by the faces $E \subset \mathcal{E}_h^-$, i.e., $\overline{\Gamma_-} = \bigcup_{E\subset\mathcal{E}_h^-} E$. In the 1D-case, a face E corresponds to a nodal point x_E and the boundary integral is defined as $\langle\, u, v\,\rangle_E := u(x_E)v(x_E)$.

Note that the first sum in (3.6) has been proposed by Nitsche [14] for the Laplace equation. Whereas the first two terms in this sum are chosen due to consistency and symmetry of the method the third term penalizes the case if the solution u_h^N of (3.5) does not satisfy the Dirichlet boundary condition $u_h^N = 0$. However, we have to take into account that there is the factor ε in front of the sum which means that for small values $\varepsilon \ll h_E$ the penalty is small. This implies that for the solution u_h^N of (3.5), the fulfillment of the boundary condition is neglected as long as ε is small compared to the local mesh sizes h_E of element faces at the boundary $\partial\Omega$. If we would have (by some appropriate adaptive refinement strategy) a fine mesh along the boundary, the penalty would take effect yielding a good fulfillment of the boundary condition. This makes sense along the boundary part $\partial\Omega\setminus\Gamma_-$ since in general the solution would have boundary layers there. However, at the inflow part Γ_- of the boundary, there are no boundary layers in general such that a mesh refinement would make no sense. Therefore, the second sum is added in (3.6) which penalizes (also for small ε) the case if the solution would violate the Dirichlet boundary condition at Γ_-. This term has already been proposed in the literature for problems with $\varepsilon = 0$, see for instance the textbook of Johnson [12, Sect. 9.5].

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4. Numerical comparison of strong and weak Dirichlet conditions. In this section, we present some numerical experiments that compare the strong imposition of the Dirichlet boundary conditions in discretization (3.3) with the weak imposition in discretization (3.5). In order to simplify the graphical output of the exact and numerical solution, we restrict ourselves to a one-dimensional test problem which contains the typical difficulty of a convection dominated convection diffusion problem, namely a small ε and the presence of a boundary layer. We consider the problem

$$
\begin{aligned}
L_\varepsilon u := -\varepsilon u'' + u' + u &= f \quad \text{in} \quad \Omega = (0,1), \\
u &= 0 \quad \text{on} \quad \partial\Omega,
\end{aligned} \tag{4.1}
$$

where f is chosen such that the exact solution is:

$$
u(x) = \underbrace{e^x + x - 1 - (e-1)x}_{r(x)} - \underbrace{\frac{e^{(x-1)/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}}}_{s(x)} .
$$

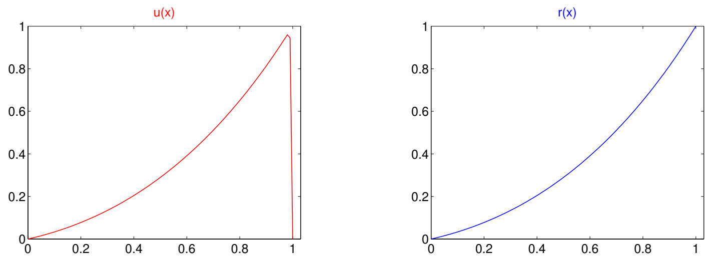

FIG. 4.1. *Exact solution $u(x)$ for $\varepsilon = 10^{-3}$ and solution $r(x)$ of the reduced problem (4.2).*

The inflow part of the boundary is $\Gamma_- = \{0\}$ and the boundary layer is located at the boundary point $x = 1$. The exact solution can be decomposed as $u(x) = r(x) - s(x)$, see Figure 4.1, where r is the solution of the so-called *reduced problem*

$$
\begin{aligned}
L_0 r := \beta \cdot \nabla r + \sigma r &= f \quad \text{in} \quad \Omega, \\
r &= 0 \quad \text{on} \quad \Gamma_-,
\end{aligned} \tag{4.2}
$$

and s is the boundary layer correction. Note that s is small almost everywhere except in a very small subregion of $x = 1$. In particular, it holds

$$
\|s\|_{0,\Omega} = \|u - r\|_{0,\Omega} \le C\varepsilon^{1/2}.
$$

Therefore, in the case of small ε, the discrete solution u_h should also be a good L^2-approximation of r, i.e., the finite element space V_h^D or V_h^N of discretization (3.3) or (3.5), respectively, should be capable for a good approximation of r. Since, in general, r is not zero at $\partial\Omega \setminus \Gamma_-$, the space V_h^D is not a good candidate whereas V_h^N is well suited to approximate r. From this point of view, we would expect that discretization (3.5) yields better results than (3.3). This will be confirmed by the following numerical experiments.

For the computations, we have used the value $\varepsilon = 10^{-3}$ and an equidistant mesh with $h = 1/20$. In Figure 4.2, we show for each discretization the exact solution u of (4.1) (dashed line) and the corresponding discrete solution $u_h = u_h^D$ of problem (3.3) or $u_h = u_h^N$ of problem (3.5) (solid line) with the polynomial degree $k = 1$ or $k = 3$, respectively. In

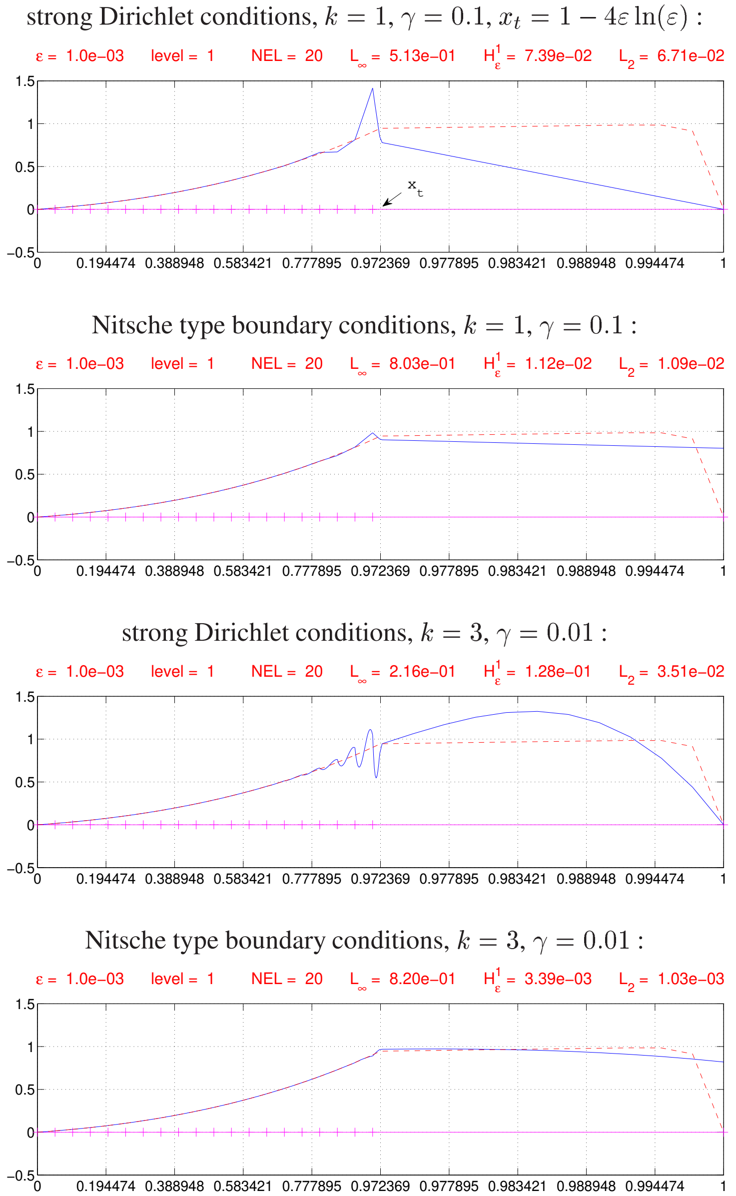

FIG. 4.2. *Comparison between strong and Nitsche type Dirichlet boundary conditions for $\varepsilon = 10^{-3}$; the exact solution u is plotted by a dashed line and the numerical solution u_h by a solid line.*

the discretization (3.5), we have used the parameter $\gamma_N = 10$. The boundary integrals in the discretization are defined for the 1D-case as $\langle\, u, v \,\rangle_E := u(x_E)v(x_E)$, where x_E denotes the point associated with the element face E of the grid. In order to see what happens in the small boundary layer region, we have stretched in all pictures the layer interval $(x_t, 1)$ to the interval $(0.5, 1)$ where the *transition point* x_t is defined by $x_t := 1 - 4\varepsilon|\ln(\varepsilon)|$. The remaining interval $(0, x_t)$ is compressed to the interval $(0, 0.5)$. In the pictures, we have also presented some discretely computed error norms measured on the non-layer region $\Omega_0 := (0, 0.95)$. By L_∞ we denote the error norm $||u - u_h||_{0,\infty,\Omega_0}$, by H^1_ε the term $\varepsilon^{1/2}|u - u_h|_{1,2,\Omega_0}$ and by L_2

the error $\|u - u_h\|_{0,2,\Omega_0}$. We clearly see that, for both polynomial degrees $k = 1$ and $k = 3$, the discretization (3.5) with Nitsche type boundary conditions gives much better results than discretization (3.3) with strong imposition of the Dirichlet boundary conditions. This holds not only in the non-layer region Ω_0 but also in the region $(0.95, 1)$ containing the boundary layer.

5. Analysis for Nitsche type boundary conditions. For a subdomain $G \subset \Omega$, let us define the following semi-norm

$$\|\!|v|\!\|_G := \varepsilon^{1/2}|v|_{1,G} + \|v\|_{0,G} + \|\,|\beta\cdot n|^{1/2}v\,\|_{0,\Gamma_-} + s_h(v,v)^{1/2}. \tag{5.1}$$

Then, we can prove the following local estimates of the discretization error.

THEOREM 5.1 (Local estimate). *Let V_h^N be the space V_h defined in (3.2) with the polynomial degree $k \geq 1$ and $u_h = u_h^N$ the solution of the discrete problem (3.5). Assume (2.2), $\gamma \geq 0$, $r \in H^{k+1}(\Omega)$ for the solution r of the reduced problem (4.2) and $u \in H^2(\Omega)$ for the solution u of (2.1). Furthermore, let $\Omega_0 \subset \Omega$ be a subdomain excluding all boundary layers in the sense that there exists a constant C independent of ε such that*

$$\varepsilon^{1/2}|u - r|_{1,\Omega_0} + \|u - r\|_{0,\Omega_0} \leq C\varepsilon. \tag{5.2}$$

Then, for sufficiently large γ_N, $\varepsilon \leq h$ and $m := 2 - d/2$, it holds with $\underline{h} := \min_{K\in\mathcal{T}_h} h_K$

$$\|\!|u - u_h|\!\|_{\Omega_0} \leq C\big(\frac{\varepsilon}{\underline{h}^m} + h^k\big)\,\|r\|_{k+1,\Omega}. \tag{5.3}$$

Moreover, if in addition it holds that

$$|u - r|_{1,\Omega_0} \leq C\varepsilon \tag{5.4}$$

with a constant C independent of ε, then we have

$$|u - u_h|_{1,\Omega_{0h}} \leq C\underline{h}_0^{-1}\big(\frac{\varepsilon}{\underline{h}^m} + h^k\big)\,\|r\|_{k+1,\Omega}, \tag{5.5}$$

where $\underline{h}_0 := \min_{K\in\mathcal{T}_h(\Omega_0)} h_K$ with $\mathcal{T}_h(\Omega_0) := \{K \in \mathcal{T}_h : K \subset \Omega_0\}$, and $\Omega_{0h} := \mathrm{int}\big(\bigcup_{K\in\mathcal{T}_h(\Omega_0)}\overline{K}\big)$.

Proof. Step 1. At first we prove the estimate

$$a_h(r - u, v_h) \leq C\frac{\varepsilon}{\underline{h}^m}\|r\|_{2,\Omega}\|\!|v_h|\!\|_\Omega \qquad \forall v_h \in V_h^N, \tag{5.6}$$

where $a_h(\cdot,\cdot)$ is defined as

$$a_h(w, v_h) := a(w, v_h) + s_h(w, v_h) + n_h(w, v_h), \quad \forall w \in H^2(\Omega) + V_h^N, \quad \forall v_h \in V_h^N. \tag{5.7}$$

Setting $w := r - u$ and using (2.1), (4.2) and (3.4), we get

$$\begin{aligned} a_h(w, v_h) &= \varepsilon(\nabla r, \nabla v_h)_\Omega + \varepsilon \sum_{E\in\mathcal{E}_h^b}\Big\{-\Big\langle \frac{\partial r}{\partial n_E}, v_h\Big\rangle_E - \Big\langle r, \frac{\partial v_h}{\partial n_E}\Big\rangle_E \\ &\qquad + \frac{\gamma_N}{h_E}\langle r, v_h\rangle_E\Big\} \\ &= -\varepsilon(\triangle r, v_h)_\Omega + \varepsilon \sum_{E\in\mathcal{E}_h^b\setminus\mathcal{E}_h^-}\Big\{-\Big\langle r, \frac{\partial v_h}{\partial n_E}\Big\rangle_E + \frac{\gamma_N}{h_E}\langle r, v_h\rangle_E\Big\}. \end{aligned}$$

The first term is estimated as

$$-\varepsilon(\triangle r, v_h)_\Omega \le C\varepsilon \|r\|_{2,\Omega} \|v_h\|_\Omega .$$

By standard arguments (transformation to the reference element $\widehat{K}$, trace theorem on $\widehat{K}$, transformation back and inverse inequalities) we can prove for all boundary faces $E \in \mathcal{E}_h^b$ the estimates

$$\|\frac{\partial v_h}{\partial n_E}\|_{0,E} \le C_1 h_E^{-1/2} |v_h|_{1,K(E)} \le C h_E^{-3/2} \|v_h\|_{0,K(E)} \qquad \forall v_h \in V_h^N, \tag{5.8}$$

and

$$\|v_h\|_{0,E} \le C h_E^{-1/2} \|v_h\|_{0,K(E)} \qquad \forall v_h \in V_h^N. \tag{5.9}$$

From the continuous embedding $H^2(\Omega) \hookrightarrow C(\overline{\Omega})$, we get

$$\|r\|_{0,E} \le C|E|^{1/2} \|r\|_{2,\Omega} \le C h_E^{(d-1)/2} \|r\|_{2,\Omega} \qquad \forall E \in \mathcal{E}_h^b.$$

Summarizing all these estimates, we obtain (5.6).
Step 2. We show that there exists a constant $C_2 > 0$ independent of h and ε such that

$$C_2 \|v_h\|_\Omega^2 \le a_h(v_h, v_h) \qquad \forall v_h \in V_h^N. \tag{5.10}$$

Partial integration and (2.2) yield

$$\begin{aligned} a(v_h, v_h) &= \varepsilon |v_h|_{1,2,\Omega}^2 + (\sigma - \frac{1}{2}\operatorname{div}\beta, v_h^2)_\Omega + \frac{1}{2}\left\langle \beta\cdot n, v_h^2 \right\rangle_{\partial\Omega} \\ &\ge \varepsilon |v_h|_{1,2,\Omega}^2 + \sigma_0 \|v_h\|_{0,\Omega}^2 - \frac{1}{2}\||\beta\cdot n|^{1/2} v_h\|_{0,\Gamma_-}^2 . \end{aligned} \tag{5.11}$$

For $n_h(\cdot,\cdot)$ defined in (3.6), we get

$$n_h(v_h, v_h) = \sum_{E\in\mathcal{E}_h^b} \left\{ -2\varepsilon \left\langle \frac{\partial v_h}{\partial n_E}, v_h \right\rangle_E + \frac{\varepsilon\gamma_N}{h_E}\|v_h\|_{0,E}^2 \right\} + \||\beta\cdot n|^{1/2} v_h\|_{0,\Gamma_-}^2 .$$

Using (5.8), the first term on the right hand side can be estimated as

$$\begin{aligned} \left| \sum_{E\in\mathcal{E}_h^b} \left\langle \frac{\partial v_h}{\partial n_E}, v_h \right\rangle_E \right| &\le \sum_{E\in\mathcal{E}_h^b} C_1 h_E^{-1/2} |v_h|_{1,K(E)} \|v_h\|_{0,E} \\ &\le \sum_{E\in\mathcal{E}_h^b} \left\{ \frac{1}{4}|v_h|_{1,K(E)}^2 + \frac{C_1^2}{h_E}\|v_h\|_{0,E}^2 \right\} . \end{aligned}$$

If we choose γ_N large enough such that $\gamma_N \ge 4C_1^2$, we get

$$n_h(v_h, v_h) \ge -\frac{1}{2}\varepsilon |v_h|_{1,\Omega}^2 + \sum_{E\in\mathcal{E}_h^b} \frac{\varepsilon\gamma_N}{2h_E}\|v_h\|_{0,E}^2 + \||\beta\cdot n|^{1/2} v_h\|_{0,\Gamma_-}^2 .$$

Together with (5.11) and (5.7) this implies

$$\begin{aligned} a_h(v_h, v_h) &\ge \frac{1}{2}\varepsilon |v_h|_{1,\Omega}^2 + \sigma_0 \|v_h\|_{0,\Omega}^2 + \frac{1}{2}\||\beta\cdot n|^{1/2} v_h\|_{0,\Gamma_-}^2 \\ &\quad + \sum_{E\in\mathcal{E}_h^b} \frac{\varepsilon\gamma_N}{2h_E}\|v_h\|_{0,E}^2 + s_h(v_h, v_h) \\ &\ge C_2 \|v_h\|_\Omega^2 . \end{aligned} \tag{5.12}$$

Step 3. Let $i_h r \in V_h^N$ be the standard finite element interpolate of the reduced solution $r \in H^{k+1}(\Omega)$. We prove that

$$a_h(i_h r - r, v_h) \le C\left\{\varepsilon h^{k-1} + h^k\right\} |r|_{k+1,\Omega} \|\!|v_h|\!\|_\Omega \qquad \forall v_h \in V_h^N. \tag{5.13}$$

For the interpolation error $\eta := i_h r - r$, we have the well-known local estimates

$$\begin{aligned} |\eta|_{m,K} &\le Ch_K^{k+1-m}|r|_{k+1,K}, && m = 0,1, \quad K \in \mathcal{T}_h, \\ |\eta|_{m,E} &\le Ch_E^{k+1/2-m}|r|_{k+1,K(E)}, && m = 0,1, \quad E \in \mathcal{E}_h. \end{aligned} \tag{5.14}$$

Now we estimate $a_h(\cdot,\cdot)$ term by term:

$$\begin{aligned} a(\eta, v_h) &\le \varepsilon|\eta|_{1,\Omega}|v_h|_{1,\Omega} + \left\{\|\beta\|_{0,\infty,\Omega}|\eta|_{1,\Omega} + \|\sigma\|_{0,\infty,\Omega}\|\eta\|_{0,\Omega}\right\}\|v_h\|_{0,\Omega} \\ &\le \left\{C\varepsilon h^{k-1} + Ch^k + Ch^{k+1}\right\}|r|_{k+1,\Omega}\|v_h\|_{0,\Omega} \\ &\le C\{\varepsilon h^{k-1} + h^k\}|r|_{k+1,\Omega}\|\!|v_h|\!\|_\Omega, \end{aligned}$$

further using (5.8), (5.9)

$$\begin{aligned} & n_h(\eta, v_h) \\ &= \sum_{E\in\mathcal{E}_h^b} \varepsilon\left\{-\left\langle \frac{\partial \eta}{\partial n_E}, v_h\right\rangle_E - \left\langle \eta, \frac{\partial v_h}{\partial n_E}\right\rangle_E + \frac{\gamma_N}{h_E}\langle \eta, v_h\rangle_E\right\} + \sum_{E\in\mathcal{E}_h^-}\langle|\beta\cdot n_E|\eta, v_h\rangle_E \\ &\le C\left\{\varepsilon\left(h^{k-\frac12}h^{-\frac12} + h^{k+\frac12}h^{-\frac32} + \gamma_N h^{-1}h^{k+\frac12}h^{-\frac12}\right) + h^{k+\frac12}h^{-\frac12}\right\}|r|_{k+1,\Omega}\|v_h\|_{0,\Omega} \\ &\le C\{\varepsilon h^{k-1} + h^k\}|r|_{k+1,\Omega}\|\!|v_h|\!\|_\Omega, \end{aligned}$$

and finally

$$\begin{aligned} s_h(\eta, v_h) &= \sum_{E\in\mathcal{E}_h^0} \gamma h_E^2 \langle [\nabla\eta]_E, [\nabla v_h]_E\rangle_E \\ &\le \sum_{E\in\mathcal{E}_h^0} Ch_E^2 h_E^{k-\frac12}|r|_{k+1,K(E)\cup K'(E)} h_E^{-\frac32}\|v_h\|_{0,K(E)\cup K'(E)} \le Ch^k|r|_{k+1,\Omega}\|\!|v_h|\!\|_\Omega. \end{aligned}$$

Summarizing these estimates, we can conclude (5.13). Note that (5.13) holds also for a locally refined grid, where $\min_{K\in\mathcal{T}_h} h_K \ll h$, since the negative powers of the local mesh size h_K will be cancelled by positive powers.

Step 4. We prove the Galerkin orthogonality

$$a_h(u - u_h, v_h) = 0 \qquad \forall v_h \in V_h^N. \tag{5.15}$$

Let $v_h \in V_h^N$ be a given arbitrary function. Multiplying the first equation of (2.1) by v_h and applying partial integration yields

$$\varepsilon(\nabla u, \nabla v_h)_\Omega + (\beta\cdot\nabla u + \sigma u, v_h)_\Omega - \varepsilon\left\langle \frac{\partial u}{\partial n}, v_h\right\rangle_{\partial\Omega} = (f, v_h)_\Omega.$$

If we use the fact that $u = 0$ in $L^2(\partial\Omega)$, we get

$$\varepsilon\sum_{E\in\mathcal{E}_h^b}\left\{-\left\langle u, \frac{\partial v_h}{\partial n_E}\right\rangle_E + \frac{\gamma_N}{h_E}\langle u, v_h\rangle_E\right\} + \langle|\beta\cdot n|u, v_h\rangle_{\Gamma_-} = 0.$$

Due to (3.4) we have $s_h(u, v_h) = 0$ since $u \in H^2(\Omega)$. Summation of all three equations implies

$$a_h(u, v_h) = (f, v_h)_\Omega.$$

Together with (3.5) for $u_h = u_h^N$ we obtain the Galerkin orthogonality.
Step 5. Let $v_h := i_h r - u_h$. Then, (5.10), (5.13), (5.6) and the Galerkin orthogonality (5.15) yield

$$\begin{aligned} C_2 |||v_h|||_\Omega^2 &\leq a_h(v_h, v_h) = a_h(i_h r - r, v_h) + a_h(r - u, v_h) \\ &\leq C\{\varepsilon h^{k-1} + h^k + \frac{\varepsilon}{\underline{h}^m}\} ||r||_{k+1,\Omega} |||v_h|||_\Omega, \end{aligned}$$

which gives a bound for $|||v_h|||_\Omega$. Hence, from the triangle inequality we obtain

$$|||r - u_h|||_\Omega \leq |||r - i_h r|||_\Omega + |||i_h r - u_h|||_\Omega \leq C\{\frac{\varepsilon}{\underline{h}^m} + \varepsilon h^{k-1} + h^k\} ||r||_{k+1,\Omega}, \tag{5.16}$$

where the estimate for the interpolation error $\eta = i_h r - r$ directly follows from (5.1), (5.14), $\varepsilon \leq C$, $\eta = 0$ on Γ_-, $h \leq C$ and the estimate

$$\begin{aligned} s_h(\eta, \eta) &= \sum_{E \in \mathcal{E}_h^0} \gamma h_E^2 || [\nabla \eta]_E ||_{0,E}^2 \leq \sum_{E \in \mathcal{E}_h^0} C h_E^2 h_E^{2(k-\frac{1}{2})} |r|_{k+1,K(E)\cup K'(E)}^2 \\ &\leq C h^{2k+1} |r|_{k+1,\Omega}^2. \end{aligned}$$

The estimate (5.16) implies

$$|||r - u_h|||_{\Omega_0} \leq C\{\frac{\varepsilon}{\underline{h}^m} + \varepsilon h^{k-1} + h^k\} ||r||_{k+1,\Omega}$$

for each subdomain $\Omega_0 \subset \Omega$. Using the assumption (5.2), the fact that, for $v := u - r$, we have $v = 0$ on Γ_- and $s_h(v, v) = 0$ due to $v \in H^2(\Omega)$, we get

$$|||u - r|||_{\Omega_0} = \varepsilon^{1/2} |u - r|_{1,\Omega_0} + ||u - r||_{0,\Omega_0} \leq C\varepsilon.$$

Now, the estimate (5.3) follows by the triangle inequality, $\varepsilon \leq h$ and $1 \leq C\underline{h}^{-m}$.
Step 6. We will prove now the estimate (5.5). From (5.16) we obtain with $\varepsilon \leq h$ the estimate

$$||r - u_h||_{0,\Omega_{0h}} \leq C\big(\frac{\varepsilon}{\underline{h}^m} + h^k\big) ||r||_{k+1,\Omega}.$$

This yields

$$||i_h r - u_h||_{0,\Omega_{0h}} \leq ||i_h r - r||_{0,\Omega_{0h}} + ||r - u_h||_{0,\Omega_{0h}} \leq C\big(\frac{\varepsilon}{\underline{h}^m} + h^k\big) ||r||_{k+1,\Omega}.$$

Using an inverse inequality we get

$$\begin{aligned} |r - u_h|_{1,\Omega_{0h}} &\leq |r - i_h r|_{1,\Omega_{0h}} + |i_h r - u_h|_{1,\Omega_{0h}} \\ &\leq C h^k |r|_{k+1,\Omega} + C \underline{h}_0^{-1} ||i_h r - u_h||_{0,\Omega_{0h}} \\ &\leq C \underline{h}_0^{-1} \big(\frac{\varepsilon}{\underline{h}^m} + h^k\big) ||r||_{k+1,\Omega}. \end{aligned}$$

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Finally, the triangle inequality and the assumption (5.4) imply

$$\begin{aligned}|u-u_h|_{1,\Omega_{0h}} &\le |u-r|_{1,\Omega_{0h}} + |r-u_h|_{1,\Omega_{0h}}\\ &\le C\,\varepsilon \,+\, C\underline{h}_0^{-1}\Big(\frac{\varepsilon}{\underline{h}^m}+h^k\Big)\|r\|_{k+1,\Omega},\end{aligned}$$

which proves (5.5) due to $1 \le C\underline{h}_0^{-1}\underline{h}^{-m}$. ∎

REMARK 5.2. Note that the result of Theorem 5.1 includes also the case of the Galerkin discretization, i.e., the discretization (3.5) with $\gamma = 0$. However, in this case, it holds $s_h(\cdot,\cdot) = 0$ such that the norm $|\!|\!|\cdot|\!|\!|_{\Omega_0}$ is weaker than in the case of CIP stabilization with $\gamma > 0$.

REMARK 5.3. Of course the error bounds in (5.3) and (5.5) blow up for $h \to 0$ and we should apply the error bound of Theorem 5.5 in (5.17). However, since the semi-norm $|u|_{k+1,\Omega\setminus\Omega_0}$ of the exact solution in the layer region $\Omega\setminus\Omega_0$ in general behaves like $\mathcal{O}(\varepsilon^{-k-1/2})$, the error bound in (5.17) gets reasonable not before the local mesh size in the layer region is small compared to ε. As long as $\varepsilon \ll \underline{h}$, i.e., as long as the boundary layers have not been resolved by small local mesh sizes $h_K \sim \underline{h}$ in the layers, the error estimate (5.3) yields a reasonable error bound. The estimate (5.5) gives a valuable error bound if additionally the mesh $\mathcal{T}_h(\Omega_0)$ in the subdomain Ω_0 is quasi uniform, i.e., if $\underline{h}_0^{-1} \le Ch^{-1}$. Note that, for $\varepsilon \ll \underline{h}_0$, the H^1-estimate (5.5) is much better than the one which follows if we divide (5.3) by $\varepsilon^{1/2}$.

REMARK 5.4. Estimates of the error $u - r$, which are needed for the assumptions (5.2) and (5.4), can be obtained in the following way. We create an asymptotic expansion $u_{\rm as} = r + s$ of the exact solution u such that

$$\|u-u_{\rm as}\|_{0,\Omega} \le C\varepsilon \quad\text{and}\quad |u-u_{\rm as}|_{1,\Omega} \le C\varepsilon^{\alpha},$$

where $\alpha = 1/2$ for assumption (5.2) and $\alpha = 1$ for assumption (5.4). The part s contains all boundary layer terms. For the technique to derive such an asymptotic expansion, we refer to [15]. Estimates of the error $u - u_{\rm as}$ in the maximum norm can be found in [11]. In order to obtain estimates of $u - u_{\rm as}$ in the L^2- and H^1-norm we can apply the techique for the proof of Lemma 4.6 in [16] using the estimate $\|u-u_{\rm as}\|_{0,\infty,\Omega} \le C\varepsilon$ from [11]. Depending on the layer part s of the asymptotic expansion $u_{\rm as}$, the subdomain Ω_0 is chosen such that $\|s\|_{0,\Omega_0} \le C\varepsilon$ and $|s|_{1,\Omega_0} \le C\varepsilon^{\alpha}$. If we assume that there are no interior layers, the subdomain Ω_0 can be represented as $\Omega_0 = \Omega\setminus\Omega^+\setminus\Omega^c$ where Ω^+ is a small vicinity of the outflow boundary $\Gamma_+ := \{x\in\partial\Omega : \beta(x)\cdot n(x) > 0\}$ with the thickness $\mathcal{O}(\varepsilon\,|\ln(\varepsilon)|)$ and Ω^c is a small vicinity of the characteristic boundary $\Gamma_0 := \{x\in\partial\Omega : \beta(x)\cdot n(x) = 0\}$ with the thickness $\mathcal{O}(\sqrt{\varepsilon}\,|\ln(\varepsilon)|)$.

In order to prove the convergence of the method for fixed ε and $h \to 0$ we introduce the following global "energy" norm

$$\|v\|_a := \varepsilon^{1/2}|v|_{1,\Omega} + \|v\|_{0,\Omega} + \|\,|\beta\cdot n|^{1/2}v\,\|_{0,\Gamma_-} + \Big\{\sum_{E\in\mathcal{E}_h^b}\frac{\varepsilon\gamma_N}{2h_E}\|v\|_{0,E}^2\Big\}^{1/2} + s_h(v,v)^{1/2}.$$

THEOREM 5.5 (Global estimate). *Let V_h^N be the space V_h defined in (3.2) with the polynomial degree $k \ge 1$ and $u_h = u_h^N$ the solution of the discrete problem (3.5). Assume (2.2), $\gamma \ge 0$, $\varepsilon \le C$ and $u \in H^{k+1}(\Omega)$ for the solution u of (2.1). Then, for sufficiently large γ_N, it holds*

$$\|u-u_h\|_a \le C\Big(\sum_{K\in\mathcal{T}_h} h_K^{2k}|u|_{k+1,K}^2\Big)^{1/2}. \tag{5.17}$$

Proof. As in the proof of Theorem 5.1 we get the Galerkin orthogonality (see (5.15))

$$a_h(u-u_h, v_h) = 0 \qquad \forall v_h \in V_h^N$$

and the estimate (see (5.12))

$$C_2 \|v_h\|_a^2 \le a_h(v_h, v_h) \qquad \forall v_h \in V_h^N.$$

Let $i_h u \in V_h^N$ be a standard finite element interpolate of the solution u which satisfies the boundary condition $i_h u = 0$ on $\partial\Omega$ and let $\eta := i_h u - u$ be the interpolation error. If we apply the local estimates (5.14), where r is replaced by u, we obtain

$$\begin{aligned} a(\eta, v_h) &\le \varepsilon^{1/2} |\eta|_{1,\Omega}\, \varepsilon^{1/2} |v_h|_{1,\Omega} + \big\{ \|\beta\|_{0,\infty,\Omega} |\eta|_{1,\Omega} + \|\sigma\|_{0,\infty,\Omega} \|\eta\|_{0,\Omega} \big\} \|v_h\|_{0,\Omega} \\ &\le C \Big\{ \sum_{K\in\mathcal{T}_h} (\varepsilon h_K^{2k} + h_K^{2k} + h_K^{2k+2}) |u|_{k+1,K}^2 \Big\}^{1/2} \|v_h\|_a \\ &\le C \Big\{ \sum_{K\in\mathcal{T}_h} h_K^{2k} |u|_{k+1,K}^2 \Big\}^{1/2} \|v_h\|_a , \end{aligned}$$

where the last inequality follows from $\varepsilon \le C$ and $h \le C$. Furthermore, by means of $\eta = 0$ on $\partial\Omega$ and (5.14) it follows

$$\begin{aligned} n_h(\eta, v_h) &= \sum_{E\in\mathcal{E}_h^b} -\varepsilon \left\langle \frac{\partial \eta}{\partial n_E}, v_h \right\rangle_E \le \sum_{E\in\mathcal{E}_h^b} C\varepsilon h_E^{k-1/2} |u|_{k+1,K(E)} \|v_h\|_{0,E} \\ &\le C \Big\{ \sum_{E\in\mathcal{E}_h^b} \varepsilon h_E^{2k} |u|_{k+1,K(E)}^2 \Big\}^{1/2} \|v_h\|_a \le C \Big\{ \sum_{K\in\mathcal{T}_h} h_K^{2k} |u|_{k+1,K}^2 \Big\}^{1/2} \|v_h\|_a . \end{aligned}$$

Finally, we have

$$\begin{aligned} s_h(\eta, v_h) &= \sum_{E\in\mathcal{E}_h^0} \gamma h_E^2 \left\langle [\nabla\eta]_E , [\nabla v_h]_E \right\rangle_E \\ &\le \sum_{E\in\mathcal{E}_h^0} C h_E^2 h_E^{k-\frac{1}{2}} |u|_{k+1,K(E)\cup K'(E)} h_E^{-\frac{3}{2}} \|v_h\|_{0,K(E)\cup K'(E)} \\ &\le C \Big\{ \sum_{K\in\mathcal{T}_h} h_K^{2k} |u|_{k+1,K}^2 \Big\}^{1/2} \|v_h\|_a . \end{aligned}$$

Summarizing these estimates, we can conclude

$$a_h(i_h u - u, v_h) \le C \Big\{ \sum_{K\in\mathcal{T}_h} h_K^{2k} |u|_{k+1,K}^2 \Big\}^{1/2} \|v_h\|_a \qquad \forall v_h \in V_h^N,$$

which yields for $w_h := i_h u - u_h$ by means of the Galerkin orthogonality

$$C_2 \|w_h\|_a^2 \le a_h(w_h, w_h) = a_h(i_h u - u, w_h) \le C \Big\{ \sum_{K\in\mathcal{T}_h} h_K^{2k} |u|_{k+1,K}^2 \Big\}^{1/2} \|w_h\|_a .$$

For the interpolation error $\eta = i_h u - u$ we have

$$\begin{aligned} s_h(\eta, \eta) &= \sum_{E\in\mathcal{E}_h^0} \gamma h_E^2 \| [\nabla\eta]_E \|_{0,E}^2 \le \sum_{E\in\mathcal{E}_h^0} C h_E^2 h_E^{2(k-\frac{1}{2})} |u|_{k+1,K(E)\cup K'(E)}^2 \\ &\le C \sum_{K\in\mathcal{T}_h} h_K^{2k+1} |u|_{k+1,K}^2 , \end{aligned}$$

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and

$$\begin{aligned}\|\eta\|_a &= \varepsilon^{1/2}|\eta|_{1,\Omega} + \|\eta\|_{0,\Omega} + s_h(\eta,\eta)^{1/2}\\ &\le C\Big\{\sum_{K\in\mathcal{T}_h}\big(\varepsilon h_K^{2k} + h_K^{2k+2} + h_K^{2k+1}\big)|u|^2_{k+1,K}\Big\}^{1/2} \le C\Big\{\sum_{K\in\mathcal{T}_h} h_K^{2k}|u|^2_{k+1,K}\Big\}^{1/2}.\end{aligned}$$

This implies

$$\|u-u_h\|_a \le \|\eta\|_a + \|w_h\|_a \le C\Big\{\sum_{K\in\mathcal{T}_h} h_K^{2k}|u|^2_{k+1,K}\Big\}^{1/2}. \qquad \square$$

REMARK 5.6. The estimate (5.17) gives a reasonable error bound if the local mesh size h_K is very small for all elements K located in the boundary layer where the local norm $|u|_{k+1,K}$ is large. On the other hand, the global error estimate (5.17) makes no sense in the case $\varepsilon \ll h$ for a quasi uniform mesh, where $h_K \ge Ch$ for all $K \in \mathcal{T}_h$, since the exact solution u in genral (if boundary layers occur) has the property that $|u|_{k+1,\Omega}$ behaves like $C\varepsilon^{-(k+1/2)}$ when ε tends to zero. However, in this situation, the estimate (5.3) in Theorem 5.1 guarantees that the error is small in the subdomain Ω_0 outside the boundary layer.

REMARK 5.7. In the literature on CIP methods [9, 4, 7, 6], it has been proven that, for small ε, the optimal order of the discretization error in the L^2-norm is $\mathcal{O}(h^{k+1/2})$ where k is the polynomial degree of the finite element space. However, in all the mentioned papers [9, 4, 7, 6], the final estimate is not in the locally adaptive fashion like in (5.17) where a power of the local mesh size h_K of an element K is multiplied with a Sobolev norm of the exact solution on K. Even in [7], which seems to be in a locally adaptive fashion, a look into the estimate of Theorem 4.9 shows that the error bound contains a term where a power of the maximum mesh size h is multiplied with a Sobolev norm of the solution on the global domain. Thus, the application of locally adapted meshes is not justified by this estimate. However, in the case of a near constant flow field β, the analysis in [7] is completely local and therefore sharper than the analysis of Theorem 5.5 in this paper since it proves the optimal order. In order to overcome the problem to justify the use of locally adapted meshes in the case of a genral flow field β, we have proved as a first step in this paper the "nearly" optimal order $\mathcal{O}(h^k)$ but in a complete locally adaptive fashion. A method to prove the optimal order $\mathcal{O}(h^{k+1/2})$ in a completely local form can be found in [5] although the estimates are not written in this way. Here the use of a global L^2-projection is avoided in the analysis which has been the main problem so far.

At the end of this section, we will draw from Theorem 5.5 some conclusions about the error $u-u_h$ at the boundary. For a sufficiently small constant $C_3 > 0$, we define the following subset of the inflow boundary Γ_-

$$\Gamma_-(C_3) := \{x \in \Gamma_- : |\beta(x)\cdot n(x)| \ge C_3\}.$$

Then, we get from (5.17) the estimate

$$\|u-u_h\|_{0,\Gamma_-(C_3)} \le C_3^{-1/2}\, C\Big(\sum_{K\in\mathcal{T}_h} h_K^{2k}|u|^2_{k+1,K}\Big)^{1/2},$$

which says that the discretization error at the inflow boundary is of the order $\mathcal{O}(h^k)$ uniformly in ε if the mesh $\mathcal{T}_h$ is locally adapted in such a way that

$$\sum_{K\in\mathcal{T}_h}\left(\frac{h_K}{h}\right)^{2k}|u|^2_{k+1,K} \le C_4, \tag{5.18}$$

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with an (ε, h)-uniformly bounded constant C_4. Let us denote by $\Gamma_{0+} := \partial\Omega \setminus \Gamma_-$ the non-inflow boundary. Under the assumption that the inflow boundary Γ_- is completely resolved by the mesh, i.e., $\overline{\Gamma_-} = \bigcup_{E \in \mathcal{E}_h^-} \overline{E}$, we get the estimate

$$\|u - u_h\|_{0,\Gamma_{0+}} \leq C \left(\frac{2\bar{h}_{0+}}{\varepsilon \gamma_N} \right)^{1/2} \Big(\sum_{K \in \mathcal{T}_h} h_K^{2k} |u|_{k+1,K}^2 \Big)^{1/2},$$

where $\bar{h}_{0+} := \max_{E \in \mathcal{E}_h^b \setminus \mathcal{E}_h^-} h_E$ is the maximum mesh size at the non-inflow boundary. This says that the discretization error at the non-inflow boundary Γ_{0+} is of the order $\mathcal{O}(h^k)$ if the local mesh size of the elements at Γ_{0+} is in the range of ε and the mesh $\mathcal{T}_h$ is locally adapted such that (5.18) is satisfied with an (ε, h)-uniformly bounded constant C_4.

6. Numerical study of the order of convergence. In this section, we consider again the test problem (4.1) and investigate the numerical order of convergence for the discretization (3.5) with the weak imposition of the Dirichlet boundary conditions. We consider both, the case of a very small ε and the case of the "moderate" value $\varepsilon = 10^{-3}$. For all computations, we use an equidistant mesh with the mesh size h, the polynomial degree $k = 3$, the parameter $\gamma_N = 10$ and the subdomain $\Omega_0 = (0, x_t)$ with $x_t := 1 - 4\varepsilon|\ln(\varepsilon)|$. The error $u - u_h$ is measured on the discrete subdomain $\Omega_{0h} = (0, x_{th}) \subset \Omega_0$ with $x_{th} := 1 - k_0 h$, where, for a given h, k_0 denotes the smallest integer $k \geq 1$ such that $kh \geq 1 - x_t = 4\varepsilon|\ln(\varepsilon)|$. The parameter γ for the CIP stabilization has been chosen as $\gamma = 0.01$ which was the best one in some numerical experiments with different values.

TABLE 6.1
$\varepsilon = 10^{-10}$, $\Omega_0 = (0, 1 - 0.92 \cdot 10^{-8})$, $\Omega_{0h} = (0, x_{th})$, top table: CIP ($\gamma = 0.01$), bottom table: Galerkin ($\gamma = 0$).

h	x_{th}	$\|u - u_h\|_{0,\Omega_{0h}}$	order	$\|u - u_h\|_{1,\Omega_{0h}}$	order
1/10	0.90000	1.54e-07		1.20e-05	
1/20	0.95000	1.03e-08	3.90	1.58e-06	2.93
1/40	0.97500	6.98e-10	3.88	2.06e-07	2.94
1/80	0.98750	2.50e-10	1.48	1.21e-07	0.76
1/160	0.99375	3.46e-10	-0.47	3.36e-07	-1.47
1/10	0.90000	5.45e-08		9.26e-06	
1/20	0.95000	4.85e-09	3.49	1.49e-06	2.64
1/40	0.97500	3.01e-09	0.69	1.57e-06	-0.08
1/80	0.98750	5.63e-09	-0.90	5.84e-06	-1.90
1/160	0.99375	1.13e-08	-1.00	2.33e-05	-2.00

In Table 6.1 we show, for the case of the very small parameter $\varepsilon = 10^{-10}$, the results for the CIP stabilization and the standard Galerkin discretization ($\gamma = 0$). The CIP stabilization shows a nearly optimal order of convergence in the L^2- and the H^1-norm in the range $h \geq 1/40$ whereas the error stagnates for $h \leq 1/80$. This is in agreement with estimate (5.3) in Theorem 5.1 which says that for small h the error term $\varepsilon / \underline{h}^m$ becomes dominant. The standard Galerkin discretization, in principle shows the same qualitative behavior. However, it is less accurate for smaller mesh sizes.

In Table 6.2, we present the results for the case $\varepsilon = 10^{-3}$. Here, already for the first mesh size $h = 1/10$, the L^2-error is in the range of ε such that the error estimate (5.3) cannot give a realistic error bound anymore. However, when h gets smaller the L^2-error decreases,

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initially with a small order but then very rapidly. We conjecture that for very small mesh sizes $h < 1/640$, when the boundary layer will be resolved, the optimal order of 4 for the L^2-norm and 3 for the H^1-norm will show up. Again the standard Galerkin discretization, in principle shows the same qualitative behavior. However, we observe that the CIP stabilization is more accurate for larger mesh sizes $h \geq 1/40$ but less accurate for small mesh sizes $h \leq 1/80$.

TABLE 6.2
$\varepsilon = 10^{-3}$, $\Omega_0 = (0, 0.9724)$, $\Omega_{0h} = (0, x_{th})$, *top table: CIP* ($\gamma = 0.01$), *bottom table: Galerkin* ($\gamma = 0$).

h	x_{th}	$\|\|u - u_h\|\|_{0,\Omega_{0h}}$	order	$\|u - u_h\|_{1,\Omega_{0h}}$	order
1/10	0.90000	7.99e-04		4.43e-02	
1/20	0.95000	1.03e-03	-0.36	1.07e-01	-1.28
1/40	0.95000	2.30e-04	2.16	4.42e-02	1.28
1/80	0.96250	5.55e-05	2.05	2.31e-02	0.94
1/160	0.96875	4.31e-07	7.01	3.84e-04	5.91
1/320	0.97187	3.60e-13	20.19	5.07e-10	19.53
1/640	0.97187	3.92e-15	6.52	2.16e-11	4.55
1/10	0.90000	2.86e-03		3.69e-01	
1/20	0.95000	2.51e-03	0.19	6.37e-01	-0.79
1/40	0.95000	7.29e-04	1.78	3.50e-01	0.87
1/80	0.96250	2.62e-05	4.80	2.05e-02	4.09
1/160	0.96875	1.95e-10	17.04	1.88e-07	16.74
1/320	0.97187	4.32e-14	12.14	1.72e-10	10.09
1/640	0.97187	3.37e-15	3.68	2.11e-11	3.03

As a summary we can say that these numerical experiments confirm the theoretical results of Theorem 5.1 and Theorem 5.5. Moreover, they indicate that the estimates are one order suboptimal with respect to the power of h. Although the results for the standard Galerkin discretization in case of weakly imposed Dirichlet boundary conditions look quite nice let us mention that we do not recommend to use this method since it is not robust in cases where the solution r of the reduced problem is not smooth. What we recommend is the CIP method in combination with adaptive grid refinement. Numerical experiments with an error indicator based on gradient recovery have shown that the CIP method works very well in practice for locally refined grids.

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