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ETNA
Kent State University
etna@mcs.kent.edu

NUMERICAL ANALYSIS OF STOKES EQUATIONS WITH IMPROVED LBB DEPENDENCY*

M. WOHLMUTH† AND M. DOBROWOLSKI‡

Abstract. We provide a-priori bounds with improved domain dependency for the solution of Stokes equations and the numerical error of an approximation by conforming finite element methods. The domain dependency appears primarily in terms of the LBB-constant L, and several previous works have shown that L degenerates with the aspect ratio of the domain. We explain the LBB dependency of common a-priori bounds on Du and p and improve most of these estimates by avoiding a global inf-sup condition and assuming *locally-balanced flow*, which is in particular satisfied if $g = 0$. In this case, all error bounds on $u - u_h$ and $p - p_h$, except for $\|p - p_h\|_{L^2(\Omega)}$, prove to be completely independent of L.

Key words. LBB-constant, inf-sup condition, Stokes equations, a-priori estimates, finite elements

AMS subject classifications. 65N30, 76D07

1. Introduction.

Let Ω be a bounded domain of Euclidean $\mathbb{R}^n$, $n = 2, 3$. The Stokes equations are given by

$$-\Delta u + Dp = f \text{ in } \Omega, \quad \operatorname{div} u = g \text{ in } \Omega,$$

$$u = 0 \text{ on } \partial\Omega.$$

This system of partial differential equations describes the special case of a stationary, laminar, and viscous flow in Ω with velocity $u(x) \in \mathbb{R}^n$ and inner pressure $p(x) \in \mathbb{R}$. Here, f models boundary effects and external forces like gravity. By neglecting any physical parameters (for instance viscosity) we assume them to be constant in Ω and set them equal to 1. In our case, any in- and outflow is described by sources and drains, which are modeled by the function g.

Throughout this paper, we adopt the standard notation for Lebesgue and Sobolev spaces $L^2(\Omega)$ and $H^m(\Omega)$; see Section 2.1. $H_0^1(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. For the analysis of the Stokes equations we need the spaces

$$X = H_0^1(\Omega)^n, \quad Y = L_0^2(\Omega) = \left\{ p \in L^2(\Omega) \;\middle|\; \int_\Omega p \, dx \;=\; 0 \right\},$$

equipped with the L^2-norm and H^1-seminorm

$$\|p\|_Y^2 = \|p\|^2 = \int_\Omega |p|^2 \, dx, \quad \|v\|_X = |v|_1 = \|Dv\|.$$

The weak formulation of Stokes equations is defined as follows: For $f \in L^2(\Omega)$, $g \in Y$, seek $(u, p) \in X \times Y$, such that

$$(Du, Dv) - (\operatorname{div} v, p) = (f, v) \quad \forall v \in X, \tag{1.1a}$$

$$(\operatorname{div} u, q) = (g, q) \quad \forall q \in Y. \tag{1.1b}$$

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†Department of Computer Science 10 and Erlangen Graduate School in Advanced Optical Technologies (SAOT), Friedrich-Alexander University Erlangen-Nuremberg, Cauerstr. 6, D-91058 Erlangen, Germany (wohlmuth@informatik.uni-erlangen.de).

‡Department of Mathematics, University Wuerzburg, Am Hubland, D-97074 Wuerzburg, Germany (dobro@mathematik.uni-wuerzburg.de).

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Kent State University
etna@mcs.kent.edu

The existence of a weak solution is connected to the so-called LBB-constant $L = L(\Omega)$ defined by

$$L = \inf_{p \in Y} \sup_{v \in X} \frac{(\operatorname{div} v, p)}{|v|_1 \, \|p\|}. \tag{1.2}$$

We remark that $L \leq 1$, which follows from $\|\operatorname{div} v\| \leq |v|_1$ for all $v \in X$.

The following results stem from Galdi's book [7], which uses previous works by Bogovskij [2, 3] and Gagliardo [6]. In [7] it is proved that $L > 0$ for a large class of domains, including Lipschitz domains, which by a simple variational analysis (see [8]) implies that the weak solution of (1.1a) and (1.1b) is uniquely determined. Furthermore, it is proved that for domains Ω, which are star-shaped with respect to a ball of radius R,

$$L \geq c \left(\frac{\delta(\Omega)}{R} \right)^n,$$

where $\delta(\Omega)$ denotes the diameter of Ω. This lower bound is very convenient in sphere-like cases, where $\delta(\Omega)$ and R are of similar size. Unfortunately, many common domains in fluid mechanics, such as large and flat water reservoirs, the atmosphere or tubes and pipelines, have a huge aspect ratio. The results in [4, 5] show that the LBB-constant is not only allowed to, but in fact does degenerate with increasing aspect ratio.

THEOREM 1.1 (See [4]). *Let Ω be a fixed domain with LBB-constant L and $\alpha \in \mathbb{R}^n$ with*

$$1 = \alpha_1 \leq \alpha_i \leq \alpha_n, \quad 1 \leq i \leq n.$$

Let further Ω_α be the associated stretched domain, defined by

$$\Omega_\alpha = \left\{ y \in \mathbb{R}^n : \left(\frac{y_1}{\alpha_1}, \ldots, \frac{y_n}{\alpha_n} \right) \in \Omega \right\},$$

with aspect ratio proportional to $a = \alpha_n$. Then the LBB-constant L_α of Ω_α satisfies

$$\frac{L}{a} \leq L_\alpha \leq \frac{c(\Omega)}{a}.$$

Special cases of this theorem are channel domains $\Omega = \omega \times (0, a)$, $\omega \subset \mathbb{R}^{n-1}$, and plates $\Omega = \omega \times (0, 1/a) \subset \mathbb{R}^3$.

Let $X_h \subset X$ and $Y_h \subset Y$ be finite-dimensional spaces depending on a discretization parameter $h > 0$. For the analysis of the standard finite element approximation of the Stokes equations in the spaces X_h, Y_h, a discrete LBB-constant is defined by

$$L_h = \inf_{p_h \in Y_h} \sup_{v_h \in X_h} \frac{(\operatorname{div} v_h, p_h)}{|v_h|_1 \, \|p\|}. \tag{1.3}$$

The uniform stability of the method is equivalent to the existence of a constant $c_\Pi > 0$, such that $L_h \geq L/c_\Pi$; see Lemma 4.1. On the other hand, if the finite element method is convergent, we have $\liminf_h L_h \leq L$. Hence, for a uniformly stable and convergent method

$$\frac{L}{c_\Pi} \leq \liminf_h L_h \leq L.$$

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etna@mcs.kent.edu

This inclusion implies that for stable methods small discrete LBB-constants are only caused by small L.

The algebraic aspect of a small LBB-constant is visible. Most of the methods for solving the discrete linear system (Uzawa, CG for the Schur complement, etc.) deteriorate for domains with high aspect ratios.

But the analytical aspect of a small LBB-constant is not visible in practice. For instance, if we solve the Stokes equations on a channel $\omega \times (0, a)$, we usually obtain similar errors for the velocity for all a in spite of the fact that standard finite element error estimates depend on L_h or L; see [8].

The object of the paper is to refine the standard error estimates by avoiding the global LBB-condition (1.3). In Section 5, error estimates independent of L in the H^1- and L^2-norm are given for the velocity u. From these estimates it follows that the error estimate for the pressure gradient is also independent of L. Only the estimate for the pressure in L^2 depends on L which is also confirmed by numerical experience.

For error estimates independent of L using only the data f and g, refined regularity results are required. This is carried out in Section 3 for locally balanced flows (see Definition 3.4) which include the common case $g = 0$. For locally balanced flows it can be proved that most quantities in the standard regularity results are independent of L, the only exception is the L^2-norm of the pressure; see Corollary 3.10. In an example in Section 3.3 it is shown that, if the condition of a locally balanced flow is violated, then the estimates for the velocity depend on L.

2. Technical preliminaries.

2.1. Notation. We use the following standard notation for Lebesgue and Sobolev spaces $L^2(\Omega)$ and $H^m(\Omega)$, together with their related (semi-)norms.

$$\|v\|_\Omega^2 = \int_\Omega |v|^2 \ dx,$$
$$\|v\|_{m;\Omega}^2 = \sum_{\alpha \le m} \|D^\alpha v\|_\Omega^2 \,,$$
$$|v|_{m;\Omega}^2 = \sum_{\alpha = m} \|D^\alpha v\|_\Omega^2 \,.$$

Here $H_0^1(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. Furthermore, we use

$$X_g = \{v \in X \ | \ (\operatorname{div} v, q) = g(q) \quad \forall q \in Y\}, \quad g \in Y'.$$

We assume $\Omega \subset \mathbb{R}^n$ to be a bounded domain with diameter $\delta(\Omega)$ and Lebesgue measure $\mu(\Omega)$. If we omit the domain index by writing $\|\cdot\|, \|\cdot\|_m, |\cdot|_m$, we always assume Ω to be the considered domain.

We denote the dual space of $H_0^1(\Omega)$ by $H^{-1}(\Omega)$ equipped with the negative norm

$$\|f\|_{-1} = \sup_{v \in H_0^1(\Omega)} \frac{f(v)}{|v|_1}.$$

2.2. Poincaré's inequality. Note that $|\cdot|_1$ is equivalent to the standard norm $\|\cdot\|_1$ in X by Poincaré's inequality.

LEMMA 2.1 (Poincaré's inequality). *For all $v \in X$ we have*

$$\|v\| \le c_P \, |v|_1 \,,$$

where c_P only depends on the smallest dimension of Ω, i.e., c_P is independent of the aspect ratio a of Ω and is finite for $\Omega = \omega \times \mathbb{R}$, $\omega \subset \mathbb{R}^{n-1}$.

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2.3. The negative norm of Dp. The LBB-condition (1.2) also reads

$$L\,\|p\| \le \|-Dp\|_{-1} = \|Dp\|_{-1} \quad \forall p \in Y,$$

where we understand $-Dp \in X'$ as a functional in terms of

$$v \in X \longmapsto (\operatorname{div} v, p)\,.$$

According to the Riesz representation theorem, the functional $-Dp$ can be represented by a function $w \in X$ satisfying

$$(Dw, Dv) = -Dp(v) = (\operatorname{div} v, p)\,, \quad |w|_1 = \|Dp\|_{-1}\,.$$

Hence, the supremum in (1.2) is attained at w. This yields

$$L\,\|p\| \le \frac{(\operatorname{div} w, p)}{|w|_1} = \|Dp\|_{-1} = |w|_1\,. \tag{2.1}$$

LEMMA 2.2. *If Ω_i, $i = 1, \dots I$, are disjoint open subsets of Ω then*

$$\sum_{i=1}^{I} \|Dq\|^2_{-1;\Omega_i} \le \|Dq\|^2_{-1;\Omega} \quad \forall q \in Y.$$

Proof. Let $w_i \in X(\Omega_i) := H^1_0(\Omega_i)^n$ and $w \in X$ be the solutions of the problems

$$(Dw_i, D\phi)_{\Omega_i} = (\operatorname{div} \phi, q)_{\Omega_i} \quad \forall \phi \in X(\Omega_i), \quad (Dw, D\phi) = (\operatorname{div} \phi, q) \quad \forall \phi \in X.$$

Using (2.1) and extending the functions w_i by 0, we obtain

$$\sum_{i=1}^{I} \|Dq\|^2_{-1;\Omega_i} = \sum_{i=1}^{I} |w_i|^2_{1,\Omega_i} = \sum_{i=1}^{I} \int_{\Omega_i} q \operatorname{div} w_i\,dx = \sum_{i=1}^{I} \int_{\Omega_i} DwDw_i\,dx$$
$$\le \frac{1}{2}|w|_1^2 + \frac{1}{2}\sum_{i=1}^{I} |w_i|^2_{1;\Omega_i} = \frac{1}{2}\|Dq\|^2_{-1} + \frac{1}{2}\sum_{i=1}^{I} \|Dq\|^2_{-1;\Omega_i}. \qquad \square$$

2.4. The solvability of the equation $\operatorname{div} v = f$**.**

THEOREM 2.3 (Bogovskij's Theorem, [3, 7]). *Let*

$$\Omega = \cup_{i=1}^{N} \Omega_i, \quad N \ge 1,$$

where each Ω_i is star-shaped with respect to some open ball B_i with $\overline{B}_i \subset \Omega_i$, where R is the smallest radius of the balls B_i. Then, for any $f \in Y$ there exists $v \in X$ satisfying

$$\operatorname{div} v = f,$$
$$|v|_1 \le c_B\,\|f\|\,,$$

where the constant c_B obeys the upper bound condition

$$c_B \le c_0 C \left(\frac{\delta(\Omega)}{R}\right)^n \cdot \left(1 + \frac{\delta(\Omega)}{R}\right),$$

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Kent State University
etna@mcs.kent.edu

with constants $c_0 = c_0(n)$ and $C = C(\mu(\Omega_i))$. An explicit expression for C is given in [7].

Proof. See [7, Chapter III.3]. □

Bogovskij's Theorem 2.3 bears directly on the divergence operator and the solvability of the constraint $\operatorname{div} u = g$, whereas the LBB-condition is related to its adjoint operator, namely the gradient $-D$, and is essential for the existence of a pressure function p. As we show in the following, these two points of view are mutually equivalent.

Considering the divergence and gradient operators on the quotient space $\tilde{X} = X/\ker(\operatorname{div})$, we obtain

$$T = \operatorname{div}: \ \tilde{X} \to Y,$$
$$T' = -D: \ Y' \to \tilde{X}',$$

where both T and T' are bijective operators with norms

$$\|T\| = c_B, \qquad \|T'\| = \frac{1}{L}.$$

Since T and T' are adjoint operators, these norms have to be equal,

$$\frac{1}{L} = c_B. \tag{2.2}$$

Equation (2.2) is noteworthy since the domain dependency of c_B (expressed by its upper bound in Theorem 2.3) transfers to L.

By definition, the LBB-condition assures that the gradient $-D$ has a closed range $\mathcal{R}(D)$. According to the closed range theorem, $\mathcal{R}(\operatorname{div})$ is also closed and given by

$$\mathcal{R}(\operatorname{div}) = \ker(D)^0 \cong Y,$$

where $\ker(D)^0$ denotes the annihilator of $\ker(D)$. With these results in mind, one can easily prove the unique solvability of the Stokes problem, as for example done in [7]. We omit these details and continue with deriving some a-priori bounds.

3. Regularity results.

3.1. Standard regularity results. For $g \in Y$, according to Theorem 2.3 there exists $u_g \in X$ with

$$\operatorname{div} u_g = g, \quad |u_g|_1 \le c_B \|g\|.$$

Then, the solution u of the Stokes equations (1.1) for given $g \in Y$ can be split into $u = u_0 + u_g$ with $u_0 \in X_0$. By the Riesz representation theorem such a u_0 exists and is uniquely defined by the solution of

$$(Du_0, Dv) = (f, v) - (Du_g, Dv), \quad \forall v \in X_0.$$

Taken together, we obtain the following estimate for $|u|_1$,

$$\begin{aligned} |u|_1 &\le |u_0|_1 + |u_g|_1 \\ &\le c_P \|f\| + 2c_B \|g\|. \end{aligned}$$

Now (2.2) provides the same bound in terms of the LBB-constant,

$$|u|_1 \le c_P \|f\| + \frac{2}{L} \|g\|. \tag{3.1}$$

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Kent State University
etna@mcs.kent.edu

The next inequality follows immediately from (1.2), (1.1a), and (3.1). It shows the LBB dependency of the pressure bound.

$$\begin{aligned}
\|p\| &\le \frac{1}{L} \sup_{v\in X} \frac{(\operatorname{div} v, p)}{|v|_1} \\
&\le \frac{1}{L} \sup_{v\in X} \frac{(Du, Dv) - (f, v)}{|v|_1} \\
&\le \frac{1}{L} \left(|u|_1 + c_P \|f\|\right) \\
&\le \frac{c}{L} \left(c_P \|f\| + \frac{1}{L} \|g\|\right).
\end{aligned}$$

We cite, without proof, important results on regularity of higher order, $m \ge 2$, which can be found in [7].

THEOREM 3.1 (m-Regularity). *If $f \in H^{m-2}(\Omega)$, $g \in H^{m-1}(\Omega)$ and*

1. *Ω is a convex polygon ($n = 2$ and $m = 2$) or*
2. *$\partial\Omega \in C^m$,*

then the Stokes problem is m-regular, i.e., it obeys

$$\|u\|_m + \|p\|_{m-1} \le \frac{c}{L} \left(\|f\|_{m-2} + \|g\|_{m-1}\right).$$

In the following, especially the case $m = 2$ is important to us since it is used in the duality approach on $\|u - u_h\|$. All common a-priori estimates of the solution (u, p) depend on L; therefore, we now want to elucidate the domain dependency of L.

3.2. Refined regularity results. We first define a modified 2-regularity that allows us to give a 2-regularity estimate which solely depends on local LBB-constants L_i, i.e., we may think of Ω as an open covering of its parts Ω_i, chosen such that all constants L_i are independent of the aspect ratio of Ω.

DEFINITION 3.2 (Local 2-Regularity). *Let $\{\tilde\Omega_i\}$ be an open covering of $\overline{\Omega}$, such that Ω_i, which is given by $\Omega_i = \tilde\Omega_i \cap \Omega$, has diameter $\delta(\Omega_i)$ and is star-shaped with respect to a ball of radius R_i. Furthermore, let $\{\phi_i\}$ be a partition of unity with respect to $\{\tilde\Omega_i\}$. The open covering $\{\tilde\Omega_i\}$ of Ω is called **locally 2-regular** if and only if the Stokes problems*

$$\begin{aligned}
-\Delta u_i + Dp_i &= f_i \quad \text{in } \Omega_i, \\
\operatorname{div} u_i &= g_i \quad \text{in } \Omega_i, \\
u_i &= 0 \quad \text{on } \partial\Omega_i,
\end{aligned}$$

are regular in the following sense: There exists a constant c_i, such that for all $f_i \in L^2(\Omega_i)^n$ and $g_i \in H^1(\Omega_i) \cap Y(\Omega_i)$ the weak solution $(u_i, p_i) \in H_0^1(\Omega_i)^n \times L_0^2(\Omega_i)$ is in $H^2(\Omega_i)^n \times H^1(\Omega_i)$ and satisfies

$$\|u_i\|_{2;\Omega_i} + \|p_i\|_{1;\Omega_i} \le c_i(\|f_i\|_{\Omega_i} + \|g_i\|_{1;\Omega_i}).$$

REMARK 3.3. For instance, for a channel domain $\omega \times (0, k)$, $k \in \mathbb{N}$, we can choose $\Omega_i = \tilde\omega \times (i - 1, i + 1)$, $0 \le i \le k$, where $\omega \subset\subset \tilde\omega$. Then $\max_i c_i$ is independent of k.

ETNA
Kent State University
etna@mcs.kent.edu

Later on, the next definition proves to be a good criterion for an improved LBB dependency.

DEFINITION 3.4 (Locally Balanced Flow). *The Stokes problem has* ***locally balanced flow*** *if and only if there exists a partition* $\{\bar{\Omega}_i\}$ *of* Ω *with LBB-constants* L_i *independent of the aspect ratio* a *of* Ω *and* g *satisfies*

$$\int_{\Omega_i} g\,dx = 0 \quad \forall \Omega_i.$$

We now discuss a lemma about the class of Stokes problems with locally balanced flow to emphasize its advantage.

LEMMA 3.5. *If the Stokes problem has locally balanced flow, then* $|u|_1$ *can be estimated as follows:*

$$|u|_1 \le c_{LB}(\|f\| + \|g\|),$$

where c_{LB} *depends solely on* Ω_i *and* L_i*, used in Definition 3.2, and Poincaré's constant* c_P.

Proof. Setting $v = u$, $q = p$ in (1.1) results in

$$|u|_1^2 = (f, u) + (g, p)\,.$$

The conditions on g allow us to insert $p_i := \frac{1}{\mu(\Omega_i)} \int_{\Omega_i} p\,dx$ in the following derivation.

$$\begin{aligned}(g, p) &= \int_\Omega pg\,dx = \sum_i \int_{\Omega_i} pg\,dx = \sum_i \int_{\Omega_i} (p - p_i) g\,dx \\ &\le \sum_i \|p - p_i\|_{\Omega_i} \|g\|_{\Omega_i}\,.\end{aligned}$$

Since $(p - p_i) \in L_0^2(\Omega_i)$, we apply the LBB-condition (1.2) for $p - p_i$ on Ω_i and then use Lemma 2.2 and (1.1a) to obtain

$$\begin{aligned}(g, p) &\le \sum_i \frac{1}{L_i} \|g\|_{\Omega_i} \|Dp\|_{-1;\Omega_i} \\ &\le \max_i \left(\frac{1}{L_i}\right) \|g\| \left(\sum_i \|Dp\|_{-1;\Omega_i}^2\right)^{1/2} \\ &\le c\,\|g\| \sup_{v \in X} \frac{(\operatorname{div} v, p)}{|v|_1} \\ &\le c\,\|g\| \sup_{v \in X} \frac{(Du, Dv) - (f, v)}{|v|_1} \\ &\le c\,\|g\| \left(|u|_1 + c_P \|f\|\right).\end{aligned}$$

Using the above inequality together with Young's and Poincaré's inequality we finally get the following desired result

$$|u|_1^2 = (f, u) + (g, p) \le c_{\rm LB}\left(\|f\|^2 + \|g\|^2\right). \qquad \square$$

Later in this paper, we will need estimates for dual problems according to Aubin and Nitsche. These problems, in general, do not have locally balanced flow. For these cases, we provide the following definitions.

DEFINITION 3.6. *Let $c_1 = c_1(f, g)$ be defined by*

$$c_1 = \frac{|u|_1}{(\|f\| + \|g\|)},$$

for given Stokes problems with data f, g and solution u, p.

DEFINITION 3.7 (Worst Case Regularity Constant $c_R(\Omega)$). *Let $c_R(\Omega)$ be the supremum of the constants c_1 with respect to all possible Stokes problems in Ω.*

REMARK 3.8. In general, $c_R(\Omega)$ is bounded by $\frac{c}{L}$, yet in the case of a simple channel, the example of Section 3.3 suggests the dependency

$$c_R(\text{channel}) \sim \frac{1}{\sqrt{L}}.$$

In case of locally 2-regular domains, the following theorem shows that the LBB dependency appears only in terms of $|u|_1$.

THEOREM 3.9. *Assuming that Ω is locally 2-regular according to Definition 3.2, the weak solution $(u, p) \in X \times Y_0$ of the Stokes problem*

$$-\Delta u + Dp = f, \quad \operatorname{div} u = g,$$

with boundary condition $u = 0$ on $\partial\Omega$ is in $H^2(\Omega)^n \times H^1(\Omega)$. Furthermore, we obtain

$$\|u\|_2 + |p|_1 \le c\,(\|f\| + \|g\|_1 + |u|_1)\,,$$

with c independent of the LBB-constant L.

Proof. Let $\{\phi_i\}$ be the partition of unity from Definition 3.2. Then, we define f_i, g_i as the data of the Stokes problem with solution $(u\phi_i, (p - p_i)\phi_i)$ on Ω_i and $p_i := \frac{1}{\mu(\Omega_i)} \int_{\Omega_i} p\,dx$:

$$\begin{aligned} -\Delta(u\phi_i) + D((p - p_i)\phi_i) &= f_i, \\ \operatorname{div}(u\phi_i) &= g_i. \end{aligned}$$

Then f_i, g_i satisfy the bounds

$$\begin{aligned} \|f_i\| &\le \|f\|_{\Omega_i} + c\,\|u\|_{1;\Omega_i} + c\,\|p - p_i\|_{\Omega_i}\,, \\ \|g_i\|_1 &\le c\,\|u\|_{1;\Omega_i} + c\,\|g\|_{1;\Omega_i}\,. \end{aligned}$$

Since $(p - p_i) \in L_0^2(\Omega_i)$, $\|p - p_i\|_{\Omega_i}$ can be estimated using a local inf-sup condition on Ω_i as follows:

$$\begin{aligned} \|p - p_i\|_{\Omega_i} &\le \frac{1}{L_i} \sup_{v \in H_0^{1,2}(\Omega_i)} \frac{(v, D(p - p_i))}{|v|_1} \\ &= \frac{1}{L_i} \sup_{v \in H_0^{1,2}(\Omega_i)} \frac{(v, f + \Delta u)}{|v|_1} \;\le\; \frac{1}{L_i}\left(|u|_{1;\Omega_i} + c_{P_i}\,\|f\|_{\Omega_i}\right). \end{aligned}$$

Now, using Definition 3.2, we have

$$\|u\phi_i\|_{2;\Omega_i}^2 + \|(p - p_i)\phi_i\|_1^2 \;\le\; c\left(\|f\|_{\Omega_i}^2 + \|u\|_{1;\Omega_i}^2 + \|g\|_{1;\Omega_i}^2\right).$$

Summation over i results in

$$\begin{aligned} \|u\|_2^2 + |p|_1^2 &\leq \sum_i \left(\|u\phi\|_{2;\Omega_i}^2 + \|(p-p_i)\phi_i\|_1^2 \right) \\ &\leq c' \left(\|f\|^2 + \|g\|_1^2 + \|u\|_1^2 \right) \\ &\leq c \left(\|f\|^2 + \|g\|_1^2 + |u|_1^2 \right), \end{aligned}$$

with $c = c' \cdot (1 + c_p)^2$ independent of the LBB-constant L and hence, also independent of the aspect ratio of the domain Ω. ◻

Clarified by Theorem 3.9 and purged by virtue of Lemma 3.5 we finally obtain the primary result of this section as follows.

COROLLARY 3.10. *If Ω is locally 2-regular and the Stokes problem has locally balanced flow, then the estimate*

$$\|u\|_2 + |p|_1 \leq c \left(\|f\| + \|g\|_1 \right)$$

holds independent of the LBB-constant L and the aspect ratio a of Ω.

Proof. This corollary follows immediately from Lemma 3.5 and Theorem 3.9. ◻

REMARK 3.11. Besides its obvious advantage of estimating u and p, we obtain an important application for the case $g = 0$. This allows us to use the dual problem according to Aubin and Nitsche with data $f = u - u_h$, $g = 0$, and hence to refine the L^2-approximation of u. Unfortunately, a similar approach to the L^2-approximation of p would require a dual problem with $f = 0$ and $g = p - p_h$. Since we cannot assume this dual problem to have locally balanced flow, the improvement fails at this point.

The following example shows a Stokes problem which does not have locally balanced flow. Yet even in this case an explicitly calculated bound for $|u|_1$ in combination with Theorem 3.9 delivers an improved LBB dependency.

3.3. Example: Free channel flow. We now take a look at an example without locally balanced flow, i.e., with c_{LB} depending on L. For the sake of simplicity, we choose the stretched domain $\Omega = (0, a) \times (0, 1) \subset \mathbb{R}^2$ and the subdomain $\Omega_0 = \left(\frac{a - a_0}{2}, a - \frac{a - a_0}{2} \right) \times (0, 1) \subset \mathbb{R}^2$ according to Figure 3.1. In particular, we are interested in very long domains with $a \approx a_0$. Since fluids are almost incompressible, a realistic possibility of a non-vanishing constraint term g is in the form of sources g_+ and drains g_- to model inflow and outflow. Obviously, these sources and drains mainly appear at the ends of the channel and thus, are separated by a distance of about the aspect ratio $a \approx a_0$. Here, source g_+ and drain g_- can be of arbitrary shape, but we assume g_+ to vanish outside the subdomain Ω_+ and g_- to vanish outside the subdomain Ω_-. Thus we get

$$g = \begin{cases} g_+, & \text{in } \Omega_+, \\ g_-, & \text{in } \Omega_-, \\ g_0, & \text{in } \bar{\Omega}_0, \end{cases}$$

and g_- and g_+ satisfy the relation

$$\int_{\Omega_-} g_- \, d(x,y) = \int_{\Omega} g_- \, d(x,y) = -\int_{\Omega} g_+ \, d(x,y) = -\int_{\Omega_+} g_+ \, d(x,y),$$

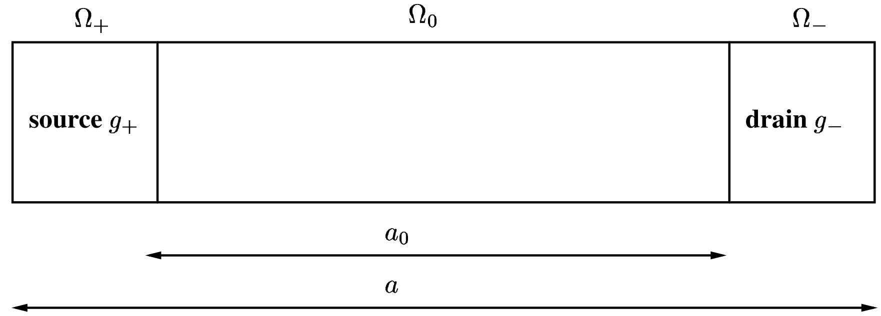

FIG. 3.1. *Stokes problem with maximally separated in- and outflow on a channel domain with aspect ratio a and $a \approx a_0$ for long channels.*

where the chosen nomenclature suggests predominantly positive values for the source g_+ and negative values for the drain g_-.

Note that the chosen case promises a very large constant c_{LB} , because it has a constraint term g that vanishes on most parts of the channel domain Ω. Consequently, the support of g does not increase together with the integration area and the aspect ratio a of Ω, when the length of the channel is increased. In fact, we will see that the enlargement of the integration area lets $|u|_1$ increase with the aspect ratio, which has to be compensated by an LBB dependency of the constant c_{LB} , since g is chosen independent of a_0.

Within this setup, we want to model a force-free flow $u_x(x,y) = u_x(y)$, $u_y(x,y) \equiv 0$ in Ω_0 with zero boundary conditions on both vertical ends ($y = 0 \vee y = 1$) of the channel. At both horizontal ends of Ω_0 we only require

$$\int_{\partial\Omega_0} u_x d\sigma = 0.$$

The latter is satisfied since the flow profile does not change along the channel. In order to obtain zero boundary conditions all over $\partial\Omega$ we add domains Ω_+ and Ω_- with appropriate source g_+ and drain g_- to model the connection between zero boundary conditions on $\partial\Omega$ and the flow profile at both ends of Ω_0. By virtue of $u(x,y) = (u_x(y), 0)^T$ we are able to explicitly calculate the solution of the present Stokes equations in Ω_0. The first Stokes equation

$$-\Delta u + Dp = 0$$

simplifies to

$$\begin{aligned} -D_{yy}u_x(y) + D_x p &= 0, \\ D_y p &= 0, \\ \Longrightarrow \; -D_{yyy}u_x(y) + D_{xy} p &= 0, \end{aligned}$$

where we assumed $u_x \in C^3(\Omega), p \in C^2(\Omega)$. Hence, by virtue of $u(0) = u(1) = 0$, we have

$$\begin{aligned} u = u_x(y) &= Ay(y-1), \\ p = p(x) &= -2Ax + c, \end{aligned}$$

ETNA
Kent State University
etna@mcs.kent.edu

for any desired flow rate A and constant pressure c. That is, we obtain a quadratic flow profile u independent of x and a pressure p depending linearly on x. Therefore $|u|_{1;\Omega_0}$ is proportional to $\sqrt{a_0}$:

$$|u|^2_{1;\Omega_0} \leq \int_{\Omega_0} c\, dx \sim a_0 \sim \frac{1}{L}.$$

By this result, we conclude

$$|u|^2_1 \sim a \sim \frac{1}{L}.$$

Since $\|f\|, \|g\|_1$ are independent of L, we obtain

$$c_1 \sim \frac{1}{\sqrt{L}}.$$

Hence, even in this disadvantageous case an improved 2-regularity result holds by virtue of Theorem 3.9:

$$\|u\|_2 + |p|_1 \leq \frac{c}{\sqrt{L}} \left(\|f\| + \|g\|_1\right),$$

where the LBB dependency reduces from $\frac{1}{L}$ to $\sqrt{\frac{1}{L}}$.

Having established all these results, we now concentrate on a discrete approximation of Stokes equations by a finite element method and its associated a-priori error bounds.

4. FEM setup. In this section, we are concerned with the approximation of weak Stokes equations (1.1) by a conforming finite element method (FEM), i.e., we would like to determine the solution $(u_h, p_h) \in X_h \times Y_h$ of

$$(Du_h, Dv_h) - (\operatorname{div} v_h, p_h) = (f, v_h) \quad \forall v_h \in X_h, \tag{4.1a}$$

$$(\operatorname{div} u_h, q_h) = (g, q_h) \quad \forall q_h \in Y_h, \tag{4.1b}$$

with finite dimensional spaces $X_h \subset X, Y_h \subset Y$. In the following, all constants including the generic constant c are assumed to be independent of h.

Similar to the continuous problem, a discrete LBB-condition ensures the solvability of the constraint (4.1b) and the existence of a discrete solution of the FEM. Therefore (X_h, Y_h) is said to be *uniformly stable*, if there exists a constant m, such that (see [8])

$$0 < m \leq L_h = \inf_{q_h \in Y_h} \sup_{v_h \in X_h} \frac{(\operatorname{div} v_h, q_h)}{|v_h|_1 \|q_h\|}. \tag{4.2}$$

The following lemma (see [8]) provides a criterion for the existence of such a constant m, although we only need its statement in one direction, namely to ensure the discrete LBB-condition. For this purpose, we consider a linear approximation operator $\Pi_h \in \mathcal{L}(X, X_h)$, where $\mathcal{L}(X, X_h)$ denotes the space of linear operators from X to X_h. This operator $\Pi_h \in \mathcal{L}(X, X_h)$ is used in the following lemma and will be an essential part in the derivation of the a-priori estimates in Section 5. Its properties are specified in (5.2a) and (5.2b) in detail.

LEMMA 4.1 (Discrete LBB Condition). *Assume the inf-sup condition holds for the continuous Stokes problem* (1.1). *Then the following two statements are equivalent:*

i) The condition (4.2) holds.

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Kent State University
etna@mcs.kent.edu

ii) There exists an operator $\Pi_h \in \mathcal{L}(X, X_h)$ *and a constant* c_Π *satisfying*

$$(\operatorname{div}(v - \Pi_h v), q_h) = 0,$$
$$|\Pi_h v|_1 \le c_\Pi \, |v|_1 \,,$$

for all $v \in X$, $q_h \in Y_h$.

Proof. To prove $i) \Leftarrow ii)$, we have, for any $q_h \in Y_h$,

$$\sup_{v_h \in X_h} \frac{(\operatorname{div} v_h, q_h)}{|v_h|_1} \ge \sup_{v \in X} \frac{(\operatorname{div} \Pi_h v, q_h)}{|\Pi_h v|_1} = \sup_{v \in X} \frac{(\operatorname{div} v, q_h)}{|\Pi_h v|_1}$$
$$\ge \sup_{v \in X} \frac{1}{c_\Pi} \frac{(\operatorname{div} v, q_h)}{|v|_1} \ge \frac{L}{c_\Pi} \|q_h\| \,.$$

The proof of $i) \Rightarrow ii)$ can be found in [7]. □

With these preparations we can now approach the a-priori estimates of the approximation errors of the FEM.

5. A-priori error bounds. In this section, we investigate the convergence of the discrete FEM solution. Hereby, the error relations

(5.1a) $$(D(u - u_h), Dv_h) = (\operatorname{div} v_h, p - p_h) \quad \forall v_h \in X_h,$$

(5.1b) $$(\operatorname{div}(u - u_h), q_h) = 0 \quad \forall q_h \in Y_h,$$

which we obtain by subtracting the FEM formulation (4.1) from the Stokes problem (1.1), are an important tool to estimate error bounds for both $u - u_h$, $p - p_h$ and their derivatives. The derived results are independent of the chosen conforming FEM as long as it is based on a uniformly regular discretization. For example, the latter is required to have approximation operators with the desired properties. The operator $\Pi_h \in \mathcal{L}(X, X_h)$ was already used in Lemma 4.1, and a similar one is needed for the pressure spaces Y and Y_h. We therefore assume that there exist approximation operators $\Pi_h \in \mathcal{L}(X, X_h)$, $S_h \in \mathcal{L}(Y, Y_h)$, and $k_{\max} \ge 1$, which comply with the requirements

(5.2a) $$(\operatorname{div}(v - \Pi_h v), q_h) = 0 \quad \forall q_h \in Y_h,$$

(5.2b) $$|v - \Pi_h v|_l \le c h^k \, |v|_{k+l} \quad \forall k = 0, \ldots, k_{\max}, \; l = 0, 1,$$

(5.2c) $$|q - S_h q|_l \le c h^k \, |q|_{k+l} \quad \forall k = 0, \ldots, k_{\max}, \; l = 0, 1,$$

for all $v \in X$, $q \in Y$.

REMARK 5.1. Note that Condition (5.2a), together with error relation (5.1b), involves

(5.3) $$(\operatorname{div}(\Pi_h u - u_h), q_h) = 0 \quad \forall q_h \in Y_h.$$

The existence of such operators is discussed in [9], including an explicit expression for these operators in case of low-degree finite elements. For example, in case of the Mini-Element [1] one can use the approximation operator by Scott-Zhang [10] on each macro element to obtain the properties (5.2b)-(5.2c) and locally add appropriate constants to fulfill condition (5.2a).

In the following, we implicitly assume that Ω, and consequently u, p, provide sufficient regularity. Especially, Ω is assumed to be a Lipschitz domain. This guarantees the existence of a locally 2-regular open covering of Ω.

ETNA
Kent State University
etna@mcs.kent.edu

5.1. Approximation of Du and u. The above setup, consisting of the refined regularity estimates and appropriate approximation operators, allows us to state the following error bounds for our discrete solutions.

THEOREM 5.2. *Let the conditions of* (5.2) *be fulfilled, then the discrete solution satisfies*

$$|u-u_h|_1 \le ch^k(|u|_{k+1}+|p|_k),$$
$$\|u-u_h\| \le ch\,|u-u_h|_1 + ch^{k+1}\,|p|_k \le ch^{k+1}\left(|u|_{k+1}+|p|_k\right).$$

In both cases the constants c are independent of L.

Proof. We start by employing error relations (5.1) with $v_h = \Pi_h u - u_h$, and we have

$$\begin{aligned}|u-u_h|_1^2 &= (D(u-u_h), D(u-\Pi_h u)) + (D(u-u_h), D(\Pi_h u - u_h))\\ &= (D(u-u_h), D(u-\Pi_h u)) + (\operatorname{div}(\Pi_h u - u_h), p).\end{aligned}$$

Using (5.1), approximation properties (5.2b) and (5.2c), and finally Young's inequality, we obtain

$$\begin{aligned}|u-u_h|_1^2 &= (D(u-u_h), D(u-\Pi_h u)) + (\operatorname{div}(\Pi_h u - u_h), p - S_h p)\\ &\le |u-\Pi_h u|_1\,|u-u_h|_1 + (|u-\Pi_h u|_1 + |u-u_h|_1)\,\|p-S_h p\|\\ &\le \frac{1}{2}\,|u-u_h|_1^2 + ch^{2k}\left(|u|_{k+1}^2 + |p|_k^2\right),\end{aligned}$$

which proves the first part of the theorem. To approach the L^2-estimate and hence to prove the second part, we define the dual problem with associated solution $(\omega,\phi)\in X\times Y$ according to Aubin and Nitsche:

(5.4a) $$(Dv, D\omega) - (\operatorname{div} v, \phi) = (u-u_h, v) \quad \forall v\in X,$$

(5.4b) $$(\operatorname{div}\omega, q) = 0 \quad \forall q\in Y.$$

Here, we start with (5.4a) and $v = u-u_h$,

$$\|u-u_h\|^2 = (D(u-u_h), D\omega) - (\operatorname{div}(u-u_h), \phi),$$

and then employ error relation (5.1a) for the dual problem (5.4) and error relation (5.1b) for the original Stokes problem (4.1) to yield

$$\begin{aligned}\|u-u_h\|^2 &= (D(u-u_h), D(\omega-\Pi_h\omega))\\ &\qquad + (\operatorname{div}\Pi_h\omega, p-p_h) - (\operatorname{div}(u-u_h), \phi - S_h\phi).\end{aligned}$$

Using (5.2a) and (5.4b) results in

$$\begin{aligned}(\operatorname{div}\Pi_h\omega, p-p_h) &= (\operatorname{div}(\Pi_h\omega-\omega), p-p_h)\\ &= (\operatorname{div}(\Pi_h\omega-\omega), p-S_h p).\end{aligned}$$

Taken together, we obtain

$$\begin{aligned}\|u-u_h\|^2 &\le |u-u_h|_1\,(|\omega-\Pi_h\omega|_1 + \|\phi-S_h\phi\|) + |\omega-\Pi_h\omega|_1\,\|p-S_h p\|\\ &\le |u-u_h|_1\,ch\,(|\omega|_2 + |\phi|_1) + ch^{k+1}\,|\omega|_2\,|p|_k,\end{aligned}$$

ETNA
Kent State University
etna@mcs.kent.edu

which finally reveals

$$\|u-u_h\|^2 \le c\,\|u-u_h\|\left(h\,|u-u_h|_1 + h^{k+1}\,|p|_k\right),$$

due to (5.2) and Corollary 3.10. ☐

As we have seen, the result for $|u-u_h|_1$ was obtained solely by using the properties of the previously defined approximation operators. However, the estimate of $\|u-u_h\|$ requires an auxiliary problem with homogeneous constraint $g=0$. Therefore we can eliminate the dependency on L for arbitrary g as soon as Ω is 2-regular. As it will become clear below, a similar result for $\|p-p_h\|$ would require Corollary 3.10 without any assumptions on g.

5.2. Approximation of p and Dp. Firstly, we look for more promising norms to measure the pressure error with improved LBB dependency. Secondly, we use these results to estimate the L^2-norm of the pressure error.

Since the LBB-condition suggests

$$\|p-p_h\| \le \frac{1}{L}\,\|D(p-p_h)\|_{-1},$$

the negative norm seems to be a reasonable candidate for an LBB-free error bound. Indeed, we find a result comparable to the one for $|u-u_h|_1$ and obtain the following lemma for arbitrary $g\in L_0^2(\Omega)$.

LEMMA 5.3. *The approximation error of the pressure gradient Dp in its negative norm can be estimated by*

$$\|D(p-p_h)\|_{-1} \le ch^k\left(|p|_k + |u|_{k+1}\right).$$

Proof. Just by the definition of the negative norm, we obtain

$$\begin{aligned}\|D(p-p_h)\|_{-1} &= \sup_{v\in X}\frac{(\operatorname{div} v, p-p_h)}{|v|_1}\\ &\le \sup_{v\in X}\left(\frac{(\operatorname{div}(v-\Pi_h v), p-p_h)}{|v|_1} + \frac{(\operatorname{div}\Pi_h v, p-p_h)}{|v|_1}\right)\\ &\le \sup_{v\in X}\left(\frac{\|v-\Pi_h v\|\,|p-S_h p|_1}{|v|_1} + c_\Pi\frac{(\Pi_h v, D(p-p_h))}{|\Pi_h v|_1}\right),\end{aligned}$$

where the last line follows from (5.2b) and was already used for Lemma 4.1. The last steps follow by error relation (5.1a), approximation properties (5.2b) and (5.2c), and Theorem 5.2:

$$\begin{aligned}\|D(p-p_h)\|_{-1} &\le ch^k\,|p|_k + \sup_{v_h\in X_h}\frac{(Dv_h, D(u-u_h))}{|v_h|_1}\\ &\le ch^k(|p|_k + |u|_{k+1}). \qquad ☐\end{aligned}$$

Furthermore we must control the approximation $|p-p_h|_1$ of the pressure gradient. We therefore have to assume an inverse inequality for Y_h:

$$\|Dq_h\| \le ch^{-1}\,\|Dq_h\|_{-1}. \tag{5.5}$$

REMARK 5.4. Equation (5.5) holds for all finite elements, i.e piecewise polynomial functions on elements Λ of diameter $O(h)$. Denoting the mean value of q_h over Λ by q_Λ, we

ETNA
Kent State University
etna@mcs.kent.edu

obtain from the inverse relation, the LBB-condition on each Λ and Lemma 2.2

$$\|Dq_h\|^2 = \sum_{\Lambda} \|D(q_h - q_\Lambda)\|_\Lambda^2 \le ch^{-2} \sum_{\Lambda} \|q_h - q_\Lambda\|_\Lambda^2$$
$$\le ch^{-2} \sum_{\Lambda} \|Dq_h\|_{-1;\Lambda}^2 \le ch^{-2} \|Dq_h\|_{-1}^2.$$

Clearly, the constant c in this estimate does not depend on the LBB-constant of Ω.

The above preparations now lead to the desired estimate of the pressure gradient as follows.

THEOREM 5.5. *The error of the pressure gradient can be estimated by*

$$|p - p_h|_1 \le ch^{k-1} \left(|u|_{k+1} + |p|_k\right), \qquad k \ge 1.$$

Proof. We insert the approximation operator S_h and use its property (5.2c) together with the above inverse inequality (5.5) to obtain

$$\begin{aligned}
|p - p_h|_1 &\le |p - S_h p|_1 + \|D(S_h p - p_h)\| \\
&\le ch^{k-1} |p|_k + ch^{-1} \|D(S_h p - p_h)\|_{-1} \\
&= ch^{k-1} |p|_k + ch^{-1} \sup_{v \in X} \frac{(\operatorname{div} v, S_h p - p_h)}{|v|_1} \\
&= ch^{k-1} |p|_k + ch^{-1} \sup_{v \in X} \frac{(\operatorname{div} \Pi_h v, S_h p - p_h)}{|v|_1} \\
&\le ch^{k-1} |p|_k + ch^{-1} \left(\|p - S_h p\| + \sup_{v \in X} \frac{(\operatorname{div} \Pi_h v, p - p_h)}{|v|_1} \right) \\
&\le ch^{k-1} |p|_k + ch^{-1} \sup_{v \in X} \frac{(\operatorname{div} \Pi_h v, p - p_h)}{|v|_1}.
\end{aligned}$$

Again, employing error relations (5.1), Inequality (5.2c), and Theorem 5.2 yields

$$\begin{aligned}
|p - p_h|_1 &\le ch^{k-1} |p|_k + ch^{-1} \sup_{v \in X} \frac{(D\Pi_h v, D(u - u_h))}{|v|_1} \\
&\le ch^{k-1} (|p|_k + |u|_{k+1}). \qquad \square
\end{aligned}$$

Finally, we consider the L^2-error $\|p - p_h\|$. For this purpose, we define the dual problem

(5.6a) $$(D\omega, D\phi) + (\operatorname{div} \phi, q) = 0 \qquad \forall \phi \in X,$$

(5.6b) $$(\operatorname{div} \omega, \psi) = (p - p_h, \psi) \qquad \forall \psi \in Y,$$

with solution $(\omega, q) \in X \times Y$. By testing the second equation with $p - p_h$ and by virtue of error relations (5.1), we obtain for its L^2-bound

$$\begin{aligned}
\|p - p_h\|^2 &\le (\operatorname{div}(\omega - \Pi_h \omega), p - p_h) + (\operatorname{div} \Pi_h \omega, p - p_h) \\
&\le |\omega - \Pi_h \omega|_1 \|D(p - p_h)\|_{-1} + (D(u - u_h), D\Pi_h \omega) \\
&\quad - (\operatorname{div}(u - u_h), S_h q) + (D(u - u_h), -D\omega + D\omega),
\end{aligned}$$

where the last line is equal to 0. By reordering these terms and using (5.4a), this results in

$$\begin{aligned}\|p-p_h\|^2 &\le |\omega-\Pi_h\omega|_1 \cdot \|D(p-p_h)\|_{-1}\\ &\quad + (D(u-u_h), D(\Pi_h\omega-\omega)) + (\mathrm{div}(u-u_h), q-S_hq),\end{aligned}$$

and finally yields by Lemma 5.3, approximation properties (5.2b), (5.2c), and by Young's inequality

$$\begin{aligned}\|p-p_h\|^2 &\le ch^2 |\omega|_2^2 + ch^2 |q|_1^2 + c\,|u-u_h|_1^2\\ &\le cc_1^2h^2 |p-p_h|_1^2 + ch^2 |u|_2^2\\ &\le cc_1^2h^2 \left(|p|_1^2 + |u|_2^2\right),\end{aligned}$$

where the last inequalities follow from Theorems 5.2 and 5.5.

The problem of the L^2-error bound is, that the dual problem requires $p-p_h$ as a source term which does not have locally balanced flow. Thus, we only obtain

$$c_1 \le c_R \le \frac{1}{L}.$$

But for instance, if we consider the example of Section 3.3, then p is linear and is exactly approximated in the free flow part Ω_0 of the domain Ω. Hence, $p-p_h$ corresponds to the source terms g_+ and g_- which we assumed in the example. In this case we obtain the partial improvement

$$c_1 \sim \frac{1}{\sqrt{L}}.$$

Since in general, we can not improve the first order L^2-error of the pressure, we do not want to consider any higher order estimates of $\|p-p_h\|$. To obtain higher order estimates comparable to those we provided for the other errors, an immediate generalization of the above proof would require estimates for $|w|_{k+1}, |q|_k$, and $|p-p_h|_k$. The required estimates for $|w|_{k+1}, |q|_k$ can be obtained by a generalization of Corollary 3.10.

Finally, if the Stokes problem has locally-balanced flow or even satisfies $g=0$, the right-hand side $c\left(|u|_2+|p|_1\right)$ of all our first order error bounds can be estimated by $\|f\|, \|g\|_1$ without any further LBB dependency.

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