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NITSCHE MORTARING FOR PARABOLIC INITIAL-BOUNDARY VALUE PROBLEMS*

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Abstract. This paper is concerned with a method for the numerical solution of parabolic initial-boundary value problems in two-dimensional polygonal domains Ω with or without reentrant corners. The Nitsche finite element method (as a mortar method) is applied for the discretization in space, i.e., non-matching meshes are used. For the discretization in time, the backward Euler method is employed. The rate of convergence in some H^1-like norm and in the L_2-norm is proved for the semidiscrete as well as for the fully discrete problem. In order to improve the accuracy of the method in the presence of singularities arising in case of non-convex domains, meshes with local grading near the reentrant corner are employed for the Nitsche finite element method. Numerical results illustrate the approach and confirm the theoretically expected convergence rates.

Key words. parabolic problem, corner singularity, semidiscrete finite element method, non-matching meshes, Nitsche mortaring, fully discrete method

AMS subject classifications. 65M60, 65N30

1. Introduction. The mathematical modeling of many problems in science and engineering leads to time-dependent differential equations. Therefore, methods for the approximate solution of initial-boundary value problems for parabolic or hyperbolic equations are of special interest. For solving parabolic problems numerically, the finite difference method (see [33] for an overview) as well as combinations of spatial discretization by finite elements with some finite difference time stepping method (see, e.g., [28, 34]) or the discontinuous Galerkin method (see, e.g., [18, 27, 34]) are applied.

In this paper, a combination of the Nitsche finite element method (as a mortar method) with the backward Euler method for solving initial-boundary value problems for the heat equation in 2D-domains is defined and analyzed. The finite element method with Nitsche mortaring has been investigated for several classes of elliptic problems in 2D; see, e.g., [4, 13, 20, 21, 22, 23, 30, 32]. For solving elliptic problems in axisymmetric domains in 3D, a combination of Nitsche mortaring with the approximating Fourier method is presented in [24, 25]. The finite element method with Nitsche mortaring provides several advantages. Since this method is based on a decomposition of the original domain into subdomains with non-matching triangulations, the mesh generation in these subdomains can be carried out independently of each other. Moreover, different discretization techniques in the subdomains are possible. Further, in comparison with the Lagrange multiplier mortar technique (see, e.g., [5, 8, 11, 37]), the saddle point problem, the inf-sup condition and the calculation of additional variables (Lagrange multipliers) are circumvented. Concerning the implementation of the Nitsche finite element method, existing software tools for the standard finite element method can be slightly modified since the bilinear forms in the variational equation differ only by interface terms; see Section 2 for more details.

The convergence analysis of Nitsche mortaring for parabolic problems requires error estimates for the semidiscrete as well as for the fully discrete problem. Further, the convergence estimates obtained for the semidiscrete problem involve norms of the exact solution of the parabolic problem and its derivatives in time.

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The aim of this paper is to derive convergence results for the presented approach which is applied to solve initial-boundary value problems for the heat equation in polygonally bounded domains. Thereby, convex domains as well as domains with reentrant corners are taken into account. In particular, the interface with non-matching meshes may pass the vertex of such a corner. As it is known from [17, Chapter 5], reentrant corners of the domain cause singularities of the solution which can be represented by means of the singularities of the corresponding elliptic problem. The approximation errors of the investigated approach are estimated in the L_2- and $\{1,h\}$-norms. The latter is an H^1-like, mesh-dependent norm which is introduced because of the discontinuity of the approximate solution along the interface of the subdomains provided with non-matching meshes. In order to obtain the error estimates, the Ritz projection (see, e.g., [1, 34, 36]) is now adapted to the bilinear form occurring in the Nitsche finite element discretization. Moreover, the knowledge about singularities of the solutions of elliptic problems in non-convex polygonal domains [16, 17] and their approximation by finite elements is used. Some a priori estimates for the norms of the exact solution of the parabolic problem and its derivatives in time, given in [9, 34], enable us to state the error estimates in such way that only norms of the given data are involved. It can be shown that the presented method yields the same convergence order as the combination of the standard finite element method with the backward Euler method; see [34, Chapter 19]. In case of a solution with singularities, an appropriate grading of the mesh around the reentrant corner leads to the same convergence order of the semidiscretization (in space) and of the fully discrete method as in case of a regular solution. Moreover, the convergence order of discretization in time is not affected by singularities.

In [6], using some results of [1], the Nitsche mortaring for parabolic problems with regular solutions and under more restrictive assumptions than in our paper is considered. It should be mentioned that instead of uniform time meshes, also non-matching meshes in time (see, e.g., [12, 27] and the references therein) could be combined with the Nitsche finite element method. But this would require a completely new analysis and is not considered in this paper. Another method which allows for the use of non-conforming space-time discretizations is the Schwarz Waveform Relaxation; see, e.g., [14, 15, 19].

The paper is organized as follows. In Section 2, the model problem is given and its semidiscretization (in space) by finite elements with Nitsche mortaring is described. The next section considers approximation properties of the Ritz projection and error estimates for the semidiscretization in case of regular solutions (i.e., convex domains). Section 4 contains estimates for solutions with singularities arising in case of non-convex domains, where meshes with local grading are employed. In Section 5, the fully discrete method is defined and its convergence is investigated. Finally, in Section 6 two numerical examples illustrating the approach and the convergence rates are presented.

2. The model problem and its semidiscretization. We consider the following initial-boundary value problem for the heat equation,

$$\begin{aligned} u_t - \Delta u &= f \quad \text{in } \Omega, \quad \text{for } 0 < t \leq T, \\ u &= 0 \quad \text{on } \partial\Omega, \text{ for } 0 < t \leq T, \\ u(\cdot, t=0) &= u_0 \quad \text{in } \Omega, \end{aligned} \tag{2.1}$$

with $u = u(x,t)$, as a model problem, where $\Omega \subset \mathbb{R}^2$ is supposed to be a polygonally bounded domain. In the following we assume that the compatibility condition $u_0 = 0$ on $\partial\Omega$ is satisfied.

Subsequently, for functions defined on X, let $H^s(X)$ ($s \geq 0$, s real, $H^0 = L_2$) denote the usual Sobolev space. The usual scalar product in $L_2(X)$ will be denoted by $(\cdot,\cdot)$. Further, let $v \in H^{-1}(\Omega)$ be the dual space of $H_0^1(\Omega)$, with duality pairing $\langle\cdot\,,\cdot\rangle$ over the space $L_2(\Omega)$. Moreover, we shall need the spaces $\dot{H}^s(\Omega)$; see [9, 34]. For $s \geq -1$, $\dot{H}^s(\Omega)$ denotes the space of functions defined by

$$\|v\|_{\dot{H}^s(\Omega)} = \Big(\sum_{j=1}^{\infty} \lambda_j^s \langle v, \varphi_j\rangle^2\Big)^{1/2} < \infty, \quad v \in H^{-1}(\Omega). \tag{2.2}$$

Here, λ_j $(j = 1, 2, \ldots)$ are the eigenvalues of the operator $-\Delta$ with homogeneous Dirichlet conditions, and φ_j $(j = 1, 2, \ldots)$ are the corresponding eigenfunctions which form an orthonormal basis in $L_2(\Omega)$. The relationship of the spaces $\dot{H}^s(\Omega)$ to the standard Sobolev spaces is as follows. According to [34], for $0 \leq s \leq 1$, the relation $\dot{H}^s(\Omega) = H_0^s(\Omega)$ holds. For $1 \leq s \leq 2$, $\dot{H}^s(\Omega)$ consists of the functions $u \in H_0^1(\Omega)$ such that Δu belongs to the negative order space $H^{s-2}(\Omega)$, with $H^{-\sigma}(\Omega) = [H^{-1}(\Omega), L_2(\Omega)]_{1-\sigma,2}$ for $0 < \sigma < 1$; see [34] for more details.

For an arbitrary Banach space B, let $L_2(0, T; B)$ be the space of functions $u : (0, T) \to B$ satisfying

$$\|u\|_{L_2(0,T;B)} := \Big(\int_0^T \|u(t)\|_B^2\, dt\Big)^{1/2} < \infty. \tag{2.3}$$

For some given $f \in L_2(0, T; L_2(\Omega))$, a function $u = u(x, t)$ is called a weak solution of the problem (2.1) if

$$(u_t, v) + (\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega) \tag{2.4}$$

holds with $u \in L_2(0, T; H_0^1(\Omega))$ and $u_t \in L_2(0, T; H^{-1}(\Omega))$ and if $u(\cdot, t = 0) = u_0 \in L_2(\Omega)$; see, e.g., [18, 28, 34].

In order to define an approximate solution to the problem (2.1) (resp. (2.4)), we first define a semidiscretization in space, i.e., we approximate the solution $u(x, t)$ of (2.1) by means of a function $u_h(x, t)$ which, for each fixed t, belongs to a finite element space. For this semidiscretization, the Nitsche finite element method will be employed. For the characterization of this method we shall need a subdivision of Ω into subdomains. Throughout this paper we restrict ourselves to the case of two subdomains Ω^1, Ω^2, with an interface Γ,

$$\overline{\Omega} = \overline{\Omega}^1 \cup \overline{\Omega}^2, \quad \Omega^1 \cap \Omega^2 = \emptyset, \quad \Gamma = \overline{\Omega}^1 \cap \overline{\Omega}^2.$$

Moreover, we assume that these subdomains are polygonally bounded. There are different cases for the position of the interface Γ: Figure 2.1(a) shows the case $\partial\Omega \cap \Gamma \neq \emptyset$, and in Figure 2.1(b) we have $\partial\Omega \cap \Gamma = \emptyset$.

In view of the subdivision of Ω we introduce the restrictions $v^i := v|_{\Omega^i}$ of some function v on Ω^i as well as the vectorized form $v = (v^1, v^2)$, i.e., $v^i(x) = v(x)$ holds for $x \in \Omega^i$ $(i = 1, 2)$. It should be noted that for simplicity we use here the same symbol v for denoting the function on Ω as well as the vector (v^1, v^2).

Using this notation we obtain that the solution of the BVP (2.1) is equivalent to the solution of the following problem: Find (u^1, u^2), such that

$$\begin{aligned}
u_t^i - \Delta u^i &= f \quad \text{in } \Omega^i && \text{for } i = 1, 2, \text{ and } 0 < t \leq T,\\
u^i &= 0 \quad \text{on } \partial\Omega^i \cap \partial\Omega && \text{for } i = 1, 2, \text{ and } 0 < t \leq T,\\
u^i(\cdot, t = 0) &= u_0 \quad \text{in } \Omega^i && \text{for } i = 1, 2,\\
u^1 &= u^2 \quad \text{on } \Gamma, && \text{for } 0 < t \leq T,\\
\frac{\partial u^1}{\partial n_1} + \frac{\partial u^2}{\partial n_2} &= 0 \quad \text{on } \Gamma, && \text{for } 0 < t \leq T,
\end{aligned} \tag{2.5}$$

are satisfied, where n_i $(i = 1, 2)$ denotes the outward normal to $\partial\Omega^i \cap \Gamma$.

Further we introduce the “broken” space V_0 by

$$V_0 = V_0^1 \times V_0^2, \text{ with } V_0^i := \{v \in H^1(\Omega^i) : v|_{\partial\Omega^i \cap \partial\Omega} = 0\} \text{ for } i = 1, 2$$

(note that $V_0^i = H^1(\Omega^i)$ if $\partial\Omega^i \cap \partial\Omega = \emptyset$).

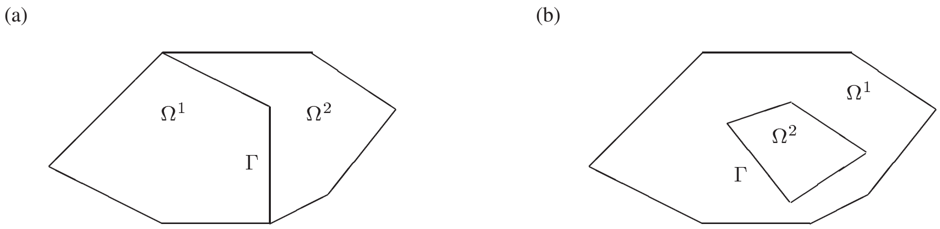

FIG. 2.1. *Domain Ω with subdomains Ω^1, Ω^2.*

Now we describe the finite element discretization of (2.5) with non-matching meshes. We cover Ω^i $(i = 1, 2)$ by a triangulation $\mathcal{T}_h^i$ $(i = 1, 2)$ consisting of triangles K $(K = \overline{K})$. Compatibility of the nodes of $\mathcal{T}_h^1$ and $\mathcal{T}_h^2$ along the “mortar interface” $\Gamma = \partial\Omega^1 \cap \partial\Omega^2$ is not required, i.e., non-matching meshes on Γ are admitted. Let h denote the mesh parameter of the triangulation $\mathcal{T}_h := \mathcal{T}_h^1 \cup \mathcal{T}_h^2$, with $0 < h \le h_0$ and sufficiently small h_0. Take, e.g., $h = \max\{h_K : K \in \mathcal{T}_h\}$, where h_K denotes the diameter of the triangle K. In the sequel, positive constants C occurring in the inequalities are generic constants.

Since in the next section, the case of a regular solution of (2.5) will be considered, we start with quasi-uniform meshes, i.e., we suppose that the following assumption on the triangulations $\mathcal{T}_h^i$ $(i = 1, 2)$ is fulfilled.

ASSUMPTION 2.1.

(i) *For $i = 1, 2$, it holds $\overline{\Omega}^i = \cup_{K \in \mathcal{T}_h^i} K$, and two arbitrary triangles $K, K' \in \mathcal{T}_h^i$ $(K \neq K')$ are either disjoint or have a common vertex, or a common edge.*

(ii) *The mesh in $\overline{\Omega}^i$ $(i = 1, 2)$ is quasi-uniform, i.e.,*

$$\max_{K \in \mathcal{T}_h^i} h_K \big(\min_{K \in \mathcal{T}_h^i} \rho_K\big)^{-1} \le C, \quad i = 1, 2, \tag{2.6}$$

holds for $h \in (0, h_0]$, where ρ_K denotes the diameter of the largest inscribed circle of K.

For $i = 1, 2$ and according to V_0^i we introduce finite element spaces V_{0h}^i of functions v_h^i on $\overline{\Omega}^i$ by

$$V_{0h}^i := \{\, v_h^i \in C(\overline{\Omega}^i) : \ v_h^i \in \mathbf{P}_1(K)\ \forall\, K \in \mathcal{T}_h^i,\ v_h^i|_{\partial\Omega^i \cap \partial\Omega} = 0 \,\},$$

i.e., we employ linear finite elements. The finite element space V_{0h} of vectorized functions v_h with components v_h^i on Ω^i is given by

$$V_{0h} := V_{0h}^1 \times V_{0h}^2 = \{v_h = (v_h^1, v_h^2) : \ v_h^1 \in V_{0h}^1, v_h^2 \in V_{0h}^2\}.$$

It should be pointed out that the functions v_h in V_{0h} are in general not continuous across Γ.

Further we introduce a triangulation $\mathcal{E}_h$ of the mortar interface Γ by intervals E $(E = \overline{E})$, i.e., $\Gamma = \cup_{E \in \mathcal{E}_h} E$, where h_E denotes the diameter of E. We suppose that two segments E, E' are either disjoint or have a common endpoint. A natural choice for the triangulation $\mathcal{E}_h$ is

$\mathcal{E}_h := \mathcal{E}_h^1$ or $\mathcal{E}_h := \mathcal{E}_h^2$, where $\mathcal{E}_h^i$ denotes the triangulation of Γ obtained by restricting $\mathcal{T}_h^i$ to Γ, i.e.,

$$\mathcal{E}_h^i := \{E : E = \partial K \cap \Gamma, \text{ if } E \text{ is a segment, } K \in \mathcal{T}_h^i\} \quad \text{for } i = 1, 2. \tag{2.7}$$

Subsequently we use real parameters α_1 and α_2 with

$$0 \le \alpha_i \le 1, \;\; i = 1, 2, \;\; \text{and} \;\; \alpha_1 + \alpha_2 = 1. \tag{2.8}$$

The asymptotic behaviour of the triangulations $\mathcal{T}_h^1$, $\mathcal{T}_h^2$ and of $\mathcal{E}_h$ should be consistent on Γ in the following sense:

ASSUMPTION 2.2.

(i) *For $E \in \mathcal{E}_h$ and $K \in \mathcal{T}_h^i$ with $\partial K \cap E \neq \emptyset$, $i = 1$ and $i = 2$, there are positive constants C_1 and C_2 independent of h_K, h_E and h $(0 < h \le h_0)$ such that the following condition is satisfied*

$$C_1 h_K \le h_E \le C_2 h_K. \tag{2.9}$$

(ii) *In the special case $\mathcal{E}_h := \mathcal{E}_h^i$ and $\alpha_i := 1$ (see (2.7) and (2.8)), where $i = 1$ or $i = 2$, for $E \in \mathcal{E}_h$ and $K \in \mathcal{T}_h^{3-i}$ with $\partial K \cap E \neq \emptyset$, instead of (2.9) the following condition is required*

$$C_1 h_K \le h_E. \tag{2.10}$$

Relation (2.9) means that the diameter h_K of the triangle K touching the interface Γ at E is asymptotically equivalent to the diameter of the segment E, i.e., the equivalence of h_K, h_E is required only locally. In contrast, condition (2.10) is weaker and admits even locally at Γ different asymptotics of the triangles $K_1 \in \mathcal{T}_h^1$, $K_2 \in \mathcal{T}_h^2$: $K_1 \cap K_2 \neq \emptyset$.

For the Nitsche finite element approximation of the function $u(t) = u(\cdot, t)$ we shall need bilinear forms $\mathcal{B}_h(\cdot,\cdot)$ and functionals $\mathcal{F}(t)$. The definitions of $\mathcal{B}_h(\cdot,\cdot)$ and $\mathcal{F}(t)$ are motivated by the related definitions in case of elliptic problems; cf., [4, 20, 21, 22, 32]. Thus, we introduce

$$\begin{aligned} \mathcal{B}_h(u(t), v) :=& \sum_{i=1}^{2} \big(\nabla u^i(t), \nabla v^i\big) - \Big\langle \alpha_1 \frac{\partial u^1(t)}{\partial n_1} - \alpha_2 \frac{\partial u^2(t)}{\partial n_2}, v^1 - v^2 \Big\rangle_\Gamma \\ & - \Big\langle \alpha_1 \frac{\partial v^1}{\partial n_1} - \alpha_2 \frac{\partial v^2}{\partial n_2}, u^1(t) - u^2(t) \Big\rangle_\Gamma + \gamma \sum_{E \in \mathcal{E}_h} h_E^{-1} \big(u^1(t) - u^2(t), v^1 - v^2\big)_{L_2(E)}, \end{aligned} \tag{2.11}$$

$$\langle \mathcal{F}(t), v \rangle := (f(t), v), \;\; \text{with} \;\; u(t), v \in V_0, \; t \in (0, T]. \tag{2.12}$$

Here, $\langle \cdot, \cdot \rangle_\Gamma$ denotes the $H_*^{-\frac{1}{2}}(\Gamma) \times H_*^{\frac{1}{2}}(\Gamma)$-duality pairing, where $H_*^{\frac{1}{2}}(\Gamma)$ is defined as the trace space of $H_0^1(\Omega)$ on Γ provided with the quotient norm (see, e.g., [16]), and $H_*^{-\frac{1}{2}}(\Gamma)$ is the dual space of $H_*^{\frac{1}{2}}(\Gamma)$. If the mortar interface Γ does not intersect the boundary $\partial\Omega$, we have $H_*^{\frac{1}{2}}(\Gamma) = H^{\frac{1}{2}}(\Gamma)$. Otherwise, $H_*^{\frac{1}{2}}(\Gamma)$ is the space $H_{00}^{\frac{1}{2}}(\Gamma)$; see [8]. Moreover, γ is a sufficiently large positive constant (the restriction of γ will be given subsequently) and α_1 as well as α_2 are taken from (2.8). The Nitsche finite element approximation $u_h : [0, T] \to V_{0h}$ of $u(t)$ is defined to be the solution of the equation

$$(u_{h,t}(t), v_h) + \mathcal{B}_h(u_h(t), v_h) = \langle \mathcal{F}(t), v_h \rangle \quad \forall v_h \in V_{0h}, \;\; \forall t \in (0, T], \tag{2.13}$$

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satisfying the initial condition

$$u_h(0) = u_{0h}, \tag{2.14}$$

where u_{0h} is some approximation of u_0 in V_{0h}. In the following we shall need the h-dependent norm $\|\cdot\|_{1,h}$ defined by

$$\|v_h\|^2_{1,h} := \sum_{i=1}^{2} \|\nabla v_h^i\|^2_{L_2(\Omega^i)} + \sum_{E\in\mathcal{E}_h} h_E^{-1} \|v_h^1 - v_h^2\|^2_{L_2(E)}. \tag{2.15}$$

The V_{0h}-boundedness and V_{0h}-ellipticity of the forms $\mathcal{B}_h(\cdot,\cdot)$ is guaranteed if the parameter γ from (2.11) is chosen independently of h and if we impose the restriction $\gamma > C_I$, where the constant C_I is taken from the estimate (see [21, (17) and Theorem 1])

$$\sum_{E\in\mathcal{E}_h} h_E \Big\| \alpha_1 \frac{\partial v_h^1}{\partial n_1} - \alpha_2 \frac{\partial v_h^2}{\partial n_2} \Big\|^2_{L_2(E)} \le C_I \sum_{i=1}^{2} \alpha_i^2 \|\nabla v_h^i\|^2_{L_2(\Omega^i)} \quad \text{for } v_h \in V_{0h}, \tag{2.16}$$

with α_1, α_2 from (2.8). Furthermore, for functions $w \in V_0$ satisfying $\frac{\partial w^i}{\partial n_i} \in L_2(\Gamma)$, $i = 1, 2$, the estimate

$$|\mathcal{B}_h(w, v_h)| \le \mu_3 \|w\|_{h,\Omega} \|v_h\|_{1,h} \tag{2.17}$$

(see also [21]) can be stated for all functions $v_h \in V_{0h}$. Here, the norm $\|\cdot\|_{h,\Omega}$ is defined by

$$\|w\|^2_{h,\Omega} := \sum_{i=1}^{2} \Big\{ \|\nabla w^i\|^2_{L_2(\Omega^i)} + \sum_{E\in\mathcal{E}_h} h_E \Big\| \alpha_i \frac{\partial w^i}{\partial n_i} \Big\|^2_{L_2(E)} \Big\} + \sum_{E\in\mathcal{E}_h} h_E^{-1} \|w^1 - w^2\|^2_{L_2(E)}.$$

3. Convergence of the semidiscretization: case of a regular solution. In the following we also consider the elliptic problem

$$-\Delta\tilde{u} = f \quad \text{in } \Omega, \quad \tilde{u} = 0 \quad \text{on } \partial\Omega, \tag{3.1}$$

which is the corresponding stationary problem associated with (2.1). The variational formulation of (3.1) reads:

$$\text{Find } \tilde{u} \in H_0^1(\Omega) \text{ such that } \quad a(\tilde{u}, v) := (\nabla\tilde{u}, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega). \tag{3.2}$$

Throughout this section we assume that the domain Ω is convex. Then, it is well-known that for $f \in L_2(\Omega)$, the solution $\tilde{u}$ of (3.2) belongs to the space $H^2(\Omega)$.

In order to derive convergence estimates for the semidiscretization, we introduce for $v \in V_0$ the Ritz projection $R_h v \in V_{0h}$. Usually the Ritz projection is defined by means of the bilinear form $a(\cdot,\cdot)$ from (3.2); see, e.g., [28, 34]. But we introduce it by means of the h-dependent bilinear form $\mathcal{B}_h(\cdot,\cdot)$ (see (2.11)) of the Nitsche finite element approach,

$$\mathcal{B}_h(R_h v, v_h) = \mathcal{B}_h(v, v_h) \quad \forall v_h \in V_{0h}. \tag{3.3}$$

Moreover, let $I_h v := (I_h v^1, I_h v^2)$, where $I_h v^i$ $(i = 1, 2)$ denotes the usual Lagrange interpolant of v^i in the space V_{0h}^i. In the next lemma, estimates for the error $R_h v - v$ are given. They generalize the well-know results for the bilinear form $a(\cdot,\cdot)$ discussed in [28, 34].

LEMMA 3.1. *Let Assumptions 2.1 and 2.2 be fulfilled for $\mathcal{T}_h^i$ $(i = 1, 2)$ and for $\mathcal{E}_h$. Moreover, assume that $\gamma > C_I$ holds (with C_I from (2.16)). Then, for $v \in H^2(\Omega) \cap H_0^1(\Omega)$, the function $R_h v$ from (3.3) satisfies the estimates*

$$\|R_h v - v\|_{1,h} \le Ch\|v\|_{H^2(\Omega)}\,, \tag{3.4}$$

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$$\|R_h v - v\|_{L_2(\Omega)} \le Ch^2 \|v\|_{H^2(\Omega)}. \tag{3.5}$$

Proof. For the proof of (3.4) we start from the inequality

$$\|R_h v - v\|_{1,h} \le \|R_h v - I_h v\|_{1,h} + \|v - I_h v\|_{1,h} \le \|R_h v - I_h v\|_{1,h} + \|v - I_h v\|_{h,\Omega}.$$

Then, employing the abbreviation $\chi := R_h v - I_h v$, the first term on the right-hand side of this inequality can be estimated by using (3.3) and (2.17) as well as the ellipticity of $\mathcal{B}_h(\cdot,\cdot)$:

$$\begin{aligned} \|\chi\|_{1,h}^2 &\le C\mathcal{B}_h(\chi,\chi) = C(\mathcal{B}_h(R_h v,\chi) - \mathcal{B}_h(I_h v,\chi)) = C(\mathcal{B}_h(v,\chi) - \mathcal{B}_h(I_h v,\chi)) \\ &= C\mathcal{B}_h(v - I_h v,\chi) \le C\|v - I_h v\|_{h,\Omega}\|\chi\|_{1,h}. \end{aligned} \tag{3.6}$$

The interpolation error can be bounded by $\|v - I_h v\|_{h,\Omega} \le Ch\|v\|_{H^2(\Omega)}$, which follows from [21, proof of Theorem 2]. This, together with previous estimates, leads to (3.4).

In order to prove estimate (3.5), we introduce $\hat{v} \in H_0^1(\Omega)$ as the solution of the auxiliary elliptic problem: find $\hat{v} \in H_0^1(\Omega)$ such that

$$a(\hat{v}, w) = (v - R_h v, w) \quad \forall\, w \in H_0^1(\Omega),$$

with $a(\cdot,\cdot)$ from (3.2). Then we obtain by analogy to [21, proof of Lemma 8],

$$\begin{aligned} \|v - R_h v\|_{L_2(\Omega)}^2 &= (v - R_h v, v) - (v - R_h v, R_h v) = \mathcal{B}_h(\hat{v}, v - R_h v) \\ &= \mathcal{B}_h(v - R_h v, \hat{v} - I_h \hat{v}). \end{aligned} \tag{3.7}$$

Employing the Hölder and Cauchy-Schwarz inequalities, the interpolation error estimate $\|I_h\hat{v} - \hat{v}\|_{h,\Omega} \le Ch\|\hat{v}\|_{H^2(\Omega)}$ as well as the a priori estimate $\|\hat{v}\|_{H^2(\Omega)} \le C\|v - R_h v\|_{L_2(\Omega)}$ for the solution of the auxiliary problem, the term on the right-hand side of (3.7) can be bounded as follows,

$$\begin{aligned} \mathcal{B}_h(v - R_h v, \hat{v} - I_h\hat{v}) &\le \|v - R_h v\|_{h,\Omega}\|\hat{v} - I_h\hat{v}\|_{h,\Omega} \\ &\le Ch\|v - R_h v\|_{h,\Omega}\|v - R_h v\|_{L_2(\Omega)}. \end{aligned} \tag{3.8}$$

By analogy to [21, proof of Lemma 8], the estimate $\|v - R_h v\|_{h,\Omega} \le C\|v - I_h v\|_{h,\Omega}$ can be derived. This, together with the interpolation error estimate $\|v - I_h v\|_{h,\Omega} \le Ch\|v\|_{H^2(\Omega)}$, as well as (3.7) and (3.8), leads to (3.5). □

In the following, the error between the solutions of the semidiscrete and continuous problems is estimated in the L_2-norm and the norm $\|\cdot\|_{1,h}$. These error estimates are based on the splitting of the error (see, e.g., [28, 34]):

$$u_h(t) - u(t) = \theta(t) + \rho(t), \ \text{ with } \theta = u_h - R_h u,\ \rho = R_h u - u, \tag{3.9}$$

and R_h defined by (3.3).

We require in the following that the given data of the parabolic problem has such a regularity that all norms arising on the right-hand sides of the estimates are finite. According to [34, Lemma 19.1], the following regularity result holds for the solution of (2.1).

LEMMA 3.2. *Let Ω be convex and let u be the solution of (2.1). Then for any $\varepsilon \in (0, \frac{1}{2})$ the estimate*

$$\int_0^t (\|u_t\|_{H^2(\Omega)} + \|u_{tt}\|_{L_2(\Omega)})\, d\tau \le C\left\{\|g\|_{H^\varepsilon(\Omega)} + \int_0^t \|f_t\|_{H^\varepsilon(\Omega)}\, d\tau\right\}, \quad t \le T, \tag{3.10}$$

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holds with $C = C(\varepsilon, T)$ and

$$g := u_t(0) = f(0) + \Delta u_0. \tag{3.11}$$

The error estimates for the semidiscretization are given in the following two lemmas.

LEMMA 3.3. *Let the assumptions of Lemma 3.1 be satisfied. Then, for the solutions u and u_h from (2.1) and (2.13), with $u_{0h} = R_h u_0$, the following error estimate holds,*

$$\|u_h(t) - u(t)\|_{L_2(\Omega)} \le Ch^2 \Big\{ \|u_0\|_{H^2(\Omega)} + \int_0^t \|u_t\|_{H^2(\Omega)} d\tau \Big\}, \quad \text{for } t \le T. \tag{3.12}$$

Proof. In view of Lemma 3.1 and the fact that $u \in H^2(\Omega)$, the summand $\rho(t)$ occurring in the splitting (3.9) can be bounded by

$$\begin{aligned} \|\rho(t)\|_{L_2(\Omega)} &= \|R_h u(t) - u(t)\|_{L_2(\Omega)} \le Ch^2 \|u(t)\|_{H^2(\Omega)} \\ &\le Ch^2 \Big(\|u(0)\|_{H^2(\Omega)} + \int_0^t \|u_t\|_{H^2(\Omega)} d\tau \Big). \end{aligned} \tag{3.13}$$

In order to find an estimate for the remaining summand $\theta(t)$, we use (2.13) and (3.3) leading to

$$\begin{aligned} (\theta_t, v_h) + \mathcal{B}_h(\theta, v_h) &= (u_{h,t}, v_h) + \mathcal{B}_h(u_h, v_h) - ((R_h u)_t, v_h) - \mathcal{B}_h(R_h u, v_h) \\ &= (f, v_h) - (R_h u_t, v_h) - \mathcal{B}_h(u, v_h) = (u_t, v_h) - (R_h u_t, v_h) = -(\rho_t, v_h) \end{aligned} \tag{3.14}$$

for $v_h \in V_{0h}$. With the special choice $v_h := \theta$, this yields

$$(\theta_t, \theta) + \mathcal{B}_h(\theta, \theta) = -(\rho_t, \theta),$$

and by means of the ellipticity of $\mathcal{B}_h(\cdot,\cdot)$ as well as the Cauchy–Schwarz inequality we get

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{L_2(\Omega)} = \|\theta\|_{L_2(\Omega)} \frac{d}{dt}\|\theta\|_{L_2(\Omega)} \le \|\rho_t\|_{L_2(\Omega)} \|\theta\|_{L_2(\Omega)}.$$

After dividing by $\|\theta\|_{L_2(\Omega)}$ and integrating, this implies, in consideration of the assumption $u_{0h} = R_h u_0$ (i.e., $\theta(0) = 0$),

$$\|\theta(t)\|_{L_2(\Omega)} \le \|\theta(0)\|_{L_2(\Omega)} + \int_0^t \|\rho_t\|_{L_2(\Omega)} d\tau = \int_0^t \|\rho_t\|_{L_2(\Omega)} d\tau, \tag{3.15}$$

and thanks to (3.5), the norm of ρ_t can be bounded as follows,

$$\|\rho_t\|_{L_2(\Omega)} = \|R_h u_t - u_t\|_{L_2(\Omega)} \le Ch^2 \|u_t\|_{H^2(\Omega)}. \tag{3.16}$$

Finally, the assertion is a result of (3.9) and (3.13)–(3.16). ∎

LEMMA 3.4. *Under the assumptions of Lemma 3.1, the solutions u and u_h from (2.1) and (2.13), with $u_{0h} = R_h u_0$, satisfy the following error estimate,*

$$\|u_h(t) - u(t)\|_{1,h} \le Ch \Big\{ \|u_0\|_{H^2(\Omega)} + \int_0^t \|u_t\|_{H^2(\Omega)} d\tau \Big\}, \quad \text{for } t \le T. \tag{3.17}$$

Proof. Taking into account (2.15) we obtain for any $v_h \in V_{0h}$ the inequality

$$\|u_h(t) - u(t)\|_{1,h}^2 \le 2\sum_{i=1}^{2}(S_1^i + S_2^i + S_3^i + S_4^i), \quad \text{with} \tag{3.18}$$

$$\begin{aligned} S_1^i &:= \|\nabla(u_h^i(t) - v_h^i)\|_{L_2(\Omega^i)}^2, \quad S_2^i := \|\nabla(v_h^i - u^i(t))\|_{L_2(\Omega^i)}^2 \\ S_3^i &:= \sum_{E\in\mathcal{E}_h} h_E^{-1}\|u_h^i(t) - v_h^i\|_{L_2(E)}^2, \quad S_4^i := \sum_{E\in\mathcal{E}_h} h_E^{-1}\|v_h^i - u^i(t)\|_{L_2(E)}^2, \ i = 1, 2. \end{aligned} \tag{3.19}$$

In the following, we set $v_h = I_h u(t)$, i.e., $v_h^i = I_h u^i(t)$, $i = 1, 2$. Since the mesh is supposed to be quasi-uniform in $\overline{\Omega}^i$, the term S_1^i from (3.19) may be bounded by means of an inverse inequality; see, e.g., [10]. This, together with Lemma 3.3 and some interpolation error estimate, leads to

$$\begin{aligned} S_1^i &\le Ch^{-2}(\|u_h^i(t) - u^i(t)\|_{L_2(\Omega^i)}^2 + \|u^i(t) - I_h u^i(t)\|_{L_2(\Omega^i)}^2) \\ &\le Ch^2\left\{\|u_0^i\|_{H^2(\Omega^i)}^2 + \Big[\int_0^t \|u_t^i\|_{H^2(\Omega^i)}d\tau\Big]^2 + \|u^i(t)\|_{H^2(\Omega^i)}^2\right\}, \ i = 1, 2. \end{aligned} \tag{3.20}$$

Further, the terms S_2^i and S_4^i from (3.19) can be bounded by using the estimate for $\|u - I_h u\|_{h,\Omega}$ from [21, proof of Theorem 2] so that we arrive at

$$\sum_{i=1}^{2}(S_2^i + S_4^i) \le Ch^2\sum_{i=1}^{2}\|u^i(t)\|_{H^2(\Omega^i)}^2 \le Ch^2\|u(t)\|_{H^2(\Omega)}^2. \tag{3.21}$$

Hence, it remains to find an estimate for S_3^i. The summation over $E \in \mathcal{E}_h$ can be rewritten such that the estimates of $u_h^i(t) - I_h u^i(t)$, $i = 1$ or $i = 2$, involve the side F of the triangle $K \subset \overline{\Omega}^i$ $(K = K_F)$ with $K_F \cap \Gamma = F \in \mathcal{E}_h^i$ ($\mathcal{E}_h^i$ from (2.7)):

$$S_3^i = \sum_{E\in\mathcal{E}_h} h_E^{-1}\|u_h^i(t) - I_h u^i(t)\|_{L_2(E)}^2 \le C\sum_{F\in\mathcal{E}_h^i} h_F^{-1}\|u_h^i(t) - I_h u^i(t)\|_{L_2(F)}^2. \tag{3.22}$$

Then we get by means of [35, Theorem 3], for $i = 1, 2$,

$$\|u_h^i(t) - I_h u^i(t)\|_{L_2(F)}^2 \le C(h_F^{\perp})^{-1}\|u_h^i(t) - I_h u^i(t)\|_{L_2(K_F)}^2, \tag{3.23}$$

where $h_F^{\perp}$ is the height of the triangle K_F over the side F. From $h_F^{-1} \le Ch^{-1}$ and $(h_F^{\perp})^{-1} \le Ch^{-1}$, (3.22) and (3.23) imply

$$\begin{aligned} S_3^i &\le C\sum_{\substack{K\in\mathcal{T}_h^i:\\ K\cap\Gamma\neq\emptyset}} h^{-2}\|u_h^i(t) - I_h u^i(t)\|_{L_2(K_F)}^2 \le Ch^{-2}\|u_h^i(t) - I_h u^i(t)\|_{L_2(\Omega_i)}^2 \\ &\le Ch^{-2}\{\|u_h^i(t) - u^i(t)\|_{L_2(\Omega_i)}^2 + \|u^i(t) - I_h u^i(t)\|_{L_2(\Omega_i)}^2\}, \end{aligned} \tag{3.24}$$

and we deduce that estimate (3.20) also holds with S_3^i instead of S_1^i. This, together with (3.18), (3.20), (3.21), as well as

$$\|u^i(t)\|_{H^2(\Omega^i)} \le \|u_0^i\|_{H^2(\Omega^i)} + \int_0^t \|u_t^i\|_{H^2(\Omega^i)}d\tau,$$

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leads to the desired estimate. □

The terms on the right-hand sides of (3.12) and (3.17) still comprise norms of the derivative of the solution u. In order to establish estimates in terms of data of the problem, we apply Lemma 3.2 leading to the following result.

THEOREM 3.5. *Let Assumptions 2.1 and 2.2 be fulfilled for $\mathcal{T}_h^i$ ($i = 1, 2$) and for $\mathcal{E}_h$. Moreover, assume that $\gamma > C_I$ holds (with C_I from (2.16)), and let the function g be defined by (3.11). Then the solutions u and u_h from (2.1) and (2.13), with $u_{0h} = R_h u_0$, satisfy the following error estimates,*

$$\|u_h(t) - u(t)\|_{L_2(\Omega)} \le Ch^2\Big\{\|u_0\|_{H^2(\Omega)} + \|g\|_{H^\varepsilon(\Omega)} + \int_0^t \|f_t\|_{H^\varepsilon(\Omega)}\, d\tau\Big\}, \tag{3.25}$$

$$\|u_h(t) - u(t)\|_{1,h} \le Ch\Big\{\|u_0\|_{H^2(\Omega)} + \|g\|_{H^\varepsilon(\Omega)} + \int_0^t \|f_t\|_{H^\varepsilon(\Omega)}\, d\tau\Big\}, \tag{3.26}$$

for any $\varepsilon \in (0, \frac{1}{2})$ and with $t \le T$, $C = C(\varepsilon, T)$.

Consequently, if for the semidiscretization of the initial-boundary value problem (2.1) the Nitsche finite element method is applied, then the same convergence rate as in case of a semidiscretization with the standard finite element method is achieved; see [34, Chapter 19].

4. Convergence of the semidiscretization: solution with singularities. Throughout this section we consider non-convex domains Ω. For simplicity we assume that there is only one reentrant corner P, with angle β, $\pi < \beta < 2\pi$. Then, according to [16, 17], the solution $\tilde{u}$ of the elliptic problem (3.2) in general does not belong to $H^2(\Omega)$, but admits a splitting into a regular and a singular part:

$$\tilde{u} = w + s, \quad \text{with } w \in H^2(\Omega), \quad s = \eta(r) r^\lambda \sin(\lambda\theta), \quad \lambda := \frac{\pi}{\beta}, \quad \frac{1}{2} < \lambda < 1. \tag{4.1}$$

In (4.1), (r, θ) denotes polar coordinates with respect to the reentrant corner, and $\eta(r)$ is a smooth cut-off function with $0 \le \eta \le 1$, $\eta = 1$ for $0 \le r \le \frac{r_0}{3}$, $\eta = 0$ for $r \ge \frac{2r_0}{3}$. For the singular part s of the solution holds $s \in H^{1+\lambda-\varepsilon}(\Omega)$ with any $\varepsilon > 0$. Furthermore, according to [9, 34], the singular part s belongs to the space $\dot{H}^2(\Omega)$ defined by (2.2). This regularity statement will be essentially needed for subsequent estimates.

As it is shown, e.g., in [2, 3, 7, 29, 31], the convergence rate of the standard finite element method on quasi-uniform meshes is reduced when this method is applied for the solution of boundary value problems with singularities of the type (4.1). This is a reason to modify the assumptions on the meshes given in Section 2 so that meshes with some local grading are admitted, where the mortar interface may pass the vertex of the reentrant corner. Instead of Assumption 2.1 we suppose from now on that the following assumption is fulfilled.

ASSUMPTION 4.1.

(i) *For $i = 1, 2$, it holds $\overline{\Omega}^i = \cup_{K \in \mathcal{T}_h^i} K$, and two arbitrary triangles $K, K' \in \mathcal{T}_h^i$ ($K \ne K'$) are either disjoint or have a common vertex, or a common edge.*

(ii) *The mesh in $\overline{\Omega}^i$ ($i = 1, 2$) is shape regular, i.e., the following relation holds (ρ_K : radius of the largest inscribed circle of K),*

$$h_K {\rho_K}^{-1} \le C \text{ for } K \in \mathcal{T}_h^i, \; h \in (0, h_0]. \tag{4.2}$$

Relation (4.2) means that the triangulations $\mathcal{T}_h^i$, $i = 1, 2$, do not have to be quasi-uniform in general. Moreover, for providing a framework for graded meshes, we introduce the real grading parameter μ, $0 < \mu \leq 1$, the grading function R_i, $i = 0, 1, \ldots, n$, with some real constant $d > 0$, and the step size h_i for the mesh associated with layers $[R_{i-1}, R_i] \times [0, \theta_0]$ around the reentrant corner P,

$$R_i := d(ih)^{\frac{1}{\mu}}, \text{ for } i = 0, 1, \ldots, n; \quad h_i := R_i - R_{i-1}, \text{ for } i = 1, 2, \ldots, n. \tag{4.3}$$

Here $n := n(h)$ denotes an integer of order h^{-1}, $n := [\delta h^{-1}]$ for some real $\delta > 0$ ($[\cdot]$ means the integer part). We shall choose d and δ such that $\frac{2}{3}r_0 < R_n < r_0$ holds.

Using the step size h_i, $i = 1, 2, \ldots, n$, from (4.3) we define a mesh which is graded locally in the neighbourhood of the vertex P of the reentrant corner and quasi-uniform in the remaining part of the domain Ω. The diameter h_K of a triangle $K \in \mathcal{T}_h$ is now characterized by the mesh size h ($0 < h \leq h_0$), by the distance R_K of K from P, and by the grading parameter μ, with fixed μ, $0 < \mu \leq 1$. The properties of $\mathcal{T}_h$ are summarized in the following assumption.

ASSUMPTION 4.2. *Let the triangulation $\mathcal{T}_h$ satisfy Assumption 4.1. Furthermore, let $\mathcal{T}_h$ be provided with a local grading around the vertex P of the reentrant corner, such that $h_K :=$ diam K depends on the distance R_K of K from P, $R_K :=$ dist (K, P) in the following way:*

$$\begin{aligned} c_1 h^{\frac{1}{\mu}} \leq h_K \leq c_1^{-1} h^{\frac{1}{\mu}} \qquad & \text{for } K \in \mathcal{T}_h : R_K = 0, \\ c_2 h R_K^{1-\mu} \leq h_K \leq c_2^{-1} h R_K^{1-\mu} \quad & \text{for } K \in \mathcal{T}_h : 0 < R_K < R_g, \\ c_3 h \leq h_K \leq c_3^{-1} h \qquad & \text{for } K \in \mathcal{T}_h : R_g \leq R_K, \end{aligned} \tag{4.4}$$

with some constants c_i, $0 < c_i \leq 1$, $i = 1, 2, 3$, and some real constant R_g, $0 < \underline{R}_g < R_g < \overline{R}_g$, where $\underline{R}_g, \overline{R}_g$ are fixed and independent of h.

In (4.4), R_g is the radius of the sector with mesh grading, and without loss of generality, we may assume $R_g = R_n$. Examples of meshes with local grading as described in Assumption 4.2 will be given in §6. The value $\mu = 1$ yields a quasi-uniform mesh in the whole domain Ω. Further, it should be noted that the total number of nodes of $\mathcal{T}_h$ satisfying Assumption 4.2 is always of the order $\mathcal{O}(h^{-2})$. In [7, 21, 29, 31], related methods for mesh grading are given. In [9], a mesh grading is described which guarantees an optimal convergence rate even in the $C(\overline{\Omega})$-norm.

Under Assumptions 4.1, 2.2, and 4.2, the definitions of the spaces V_{0h}^i and V_{0h} remain the same as in Section 2. Moreover, the Nitsche finite element approximation $u_h : [0, T] \to V_{0h}$ of the solution u is defined by (2.13), (2.14) as before. The statement concerning boundedness and ellipticity of $\mathcal{B}_h(\cdot, \cdot)$ (see Section 2) is also valid in case of graded meshes; cf., [21].

Now we turn to error estimates of the semidiscretization. In view of the splitting (3.9), we need error estimates for the Ritz projection in the case when the solution has singularities.

LEMMA 4.3. *Let $\tilde{u}$ be the solution of (3.2), where the representation (4.1) holds. Further, let Assumptions 4.1, 2.2, and 4.2 be satisfied for $\mathcal{T}_h^i$ ($i = 1, 2$) and for $\mathcal{E}_h$. If $\gamma > C_I$ (with C_I from (2.16)), then the inequalities*

$$\|R_h \tilde{u} - \tilde{u}\|_{1,h} \leq C \kappa(h, \mu) \|\Delta \tilde{u}\|_{L_2(\Omega)}, \tag{4.5}$$

$$\|R_h \tilde{u} - \tilde{u}\|_{L_2(\Omega)} \leq C \kappa^2(h, \mu) \|\Delta \tilde{u}\|_{L_2(\Omega)}, \tag{4.6}$$

hold, with

$$\kappa(h, \mu) = \begin{cases} h^{\frac{\lambda}{\mu}} & \text{for } \lambda < \mu \leq 1, \\ h |\ln h|^{1/2} & \text{for } \mu = \lambda, \\ h & \text{for } 0 < \mu < \lambda, \end{cases} \tag{4.7}$$

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and $\lambda := \frac{\pi}{\beta}$, where β is the angle of the reentrant corner.

Proof. By analogy to the proof of Lemma 3.1, we obtain the inequalities

$$\|R_h\tilde{u} - \tilde{u}\|_{1,h} \le C\|\tilde{u} - I_h\tilde{u}\|_{h,\Omega},$$
$$\|R_h\tilde{u} - \tilde{u}\|^2_{L_2(\Omega)} = \mathcal{B}_h(\tilde{u} - R_h\tilde{u}, \hat{u} - I_h\hat{u}) \le C\|\tilde{u} - I_h\tilde{u}\|_{h,\Omega}\|\hat{u} - I_h\hat{u}\|_{h,\Omega},$$

now with $\hat{u}$ as the solution of the auxiliary elliptic problem with the right-hand side $\tilde{u} - R_h\tilde{u}$. Then, using [21, Lemma 7] and $\|f\|_{L_2(\Omega)} = \|\Delta\tilde{u}\|_{L_2(\Omega)}$, we are led to (4.5) and (4.6). □

According to [34, Lemma 19.3], the estimate

$$\|\Delta\tilde{u}\|_{L_2(\Omega)} \le C\|\tilde{u}\|_{\dot{H}^2(\Omega)} \tag{4.8}$$

holds with $\|\cdot\|_{\dot{H}^2(\Omega)}$ defined by (2.2). This, together with Lemma 4.3 yields

$$\|R_h\tilde{u} - \tilde{u}\|_{1,h} \le C\kappa(h,\mu)\|\tilde{u}\|_{\dot{H}^2(\Omega)}, \quad \|R_h\tilde{u} - \tilde{u}\|_{L_2(\Omega)} \le C\kappa^2(h,\mu)\|\tilde{u}\|_{\dot{H}^2(\Omega)}. \tag{4.9}$$

For subsequent estimates we need the following regularity result; see [34, Lemma 19.5] or [9, Lemma 3.1].

LEMMA 4.4. *Let u be the solution of (2.1), and let g be defined by (3.11). Then we have, for $\varepsilon \in (0, \frac{1}{2})$ and $C = C(\varepsilon, T)$,*

$$\int_0^t (\|u_t\|_{\dot{H}^2(\Omega)} + \|u_{tt}\|_{L_2(\Omega)})\, d\tau \le C\left\{\|g\|_{H^\varepsilon(\Omega)} + \int_0^t \|f_t\|_{H^\varepsilon(\Omega)}\, d\tau\right\}, \quad t \le T. \tag{4.10}$$

Now we are ready to give estimates of the error between the solutions of the semidiscrete and continuous problems.

THEOREM 4.5. *Let Assumptions 4.1, 2.2, and 4.2 be fulfilled for $\mathcal{T}_h^i$ $(i = 1, 2)$ and for $\mathcal{E}_h$, and let $\gamma > C_I$ (with C_I from (2.16)) as well as $u_{0h} = R_hu_0$. Then, for the solutions u and u_h from (2.1) and (2.13), the error estimate*

$$\|u_h(t) - u(t)\|_{L_2(\Omega)} \le C\kappa^2(h,\mu)\Big\{\|\Delta u_0\|_{L_2(\Omega)} + \|g\|_{H^\varepsilon(\Omega)} + \int_0^t \|f_t\|_{H^\varepsilon(\Omega)}\, d\tau\Big\} \tag{4.11}$$

holds for $t \le T$, with $\kappa(h,\mu)$ from (4.7), g from (3.11), and any $\varepsilon \in (0, \frac{1}{2})$. The constant C in (4.11) depends on ε and T.

Proof. Taking into account (3.9) and (3.15), the error can be bounded as follows,

$$\begin{aligned} &\|u_h(t) - u(t)\|_{L_2(\Omega)} \\ &\quad\le \|\rho(t)\|_{L_2(\Omega)} + \|\theta(t)\|_{L_2(\Omega)} \le \|\rho(0)\|_{L_2(\Omega)} + 2\int_0^t \|\rho_t\|_{L_2(\Omega)} d\tau. \end{aligned} \tag{4.12}$$

In view of Lemma 4.3 and the assumption $u_{0h} = R_hu_0$, we obtain

$$\|\rho(0)\|_{L_2(\Omega)} = \|u(0) - R_hu(0)\|_{L_2(\Omega)} \le C\kappa^2(h,\mu)\|\Delta u_0\|_{L_2(\Omega)}, \tag{4.13}$$

such that it remains to estimate the integral on the right-hand side of (4.12). Using the second equation of (4.9) we get

$$\int_0^t \|\rho_t\|_{L_2(\Omega)} d\tau = \int_0^t \|u_t - R_hu_t\|_{L_2(\Omega)} d\tau \le C\kappa^2(h,\mu)\int_0^t \|u_t\|_{\dot{H}^2(\Omega)} d\tau. \tag{4.14}$$

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Applying Lemma 4.4, the right-hand side of this inequality can be bounded in terms of data. Then, the assertion of the theorem can be deduced from (4.12)–(4.14). □

THEOREM 4.6. *Under the assumptions of Theorem 4.5 we have for $t \leq T$ the error estimate*

$$\|u_h(t) - u(t)\|_{1,h} \leq C\kappa(h,\mu)\Big\{\|\Delta u_0\|_{L_2(\Omega)} + \|g\|_{H_0^1(\Omega)} + \Big(\int_0^t \|f_t\|^2_{L_2(\Omega)}\,d\tau\Big)^{1/2} + \int_0^t \|f_t\|_{H^\varepsilon(\Omega)}\,d\tau\Big\},$$

with $\varepsilon \in (0, \frac{1}{2})$ *and* $C = C(\varepsilon, T)$.

Proof. We use the splitting (3.9) of the error and derive estimates for the norms of $\rho(t)$ and $\theta(t)$. First we obtain, by Lemma 4.3 and (4.5),

$$\|\rho(t)\|_{1,h} \leq C\kappa(h,\mu)\|\Delta u(t)\|_{L_2(\Omega)} \leq C\kappa(h,\mu)\Big\{\|\Delta u_0\|_{L_2(\Omega)} + \int_0^t \|\Delta u_t\|_{L_2(\Omega)}\,d\tau\Big\}.$$

Using inequalities (4.10) and (4.8), we are led to

$$\|\rho(t)\|_{1,h} \leq C\kappa(h,\mu)\Big\{\|\Delta u_0\|_{L_2(\Omega)} + \|g\|_{H^\varepsilon(\Omega)} + \int_0^t \|f_t\|_{H^\varepsilon(\Omega)}\,d\tau\Big\}. \tag{4.15}$$

Further, equation (3.14) with $v_h := \theta_t$ yields $(\theta_t, \theta_t) + \mathcal{B}_h(\theta, \theta_t) = -(\rho_t, \theta_t)$, and by means of the Cauchy–Schwarz inequality we obtain

$$\|\theta_t\|^2_{L_2(\Omega)} + \frac{1}{2}\frac{d}{dt}\mathcal{B}_h(\theta,\theta) \leq \frac{1}{2}\|\rho_t\|^2_{L_2(\Omega)} + \frac{1}{2}\|\theta_t\|^2_{L_2(\Omega)},$$

which implies

$$\frac{1}{2}\|\theta_t\|^2_{L_2(\Omega)} + \frac{1}{2}\frac{d}{dt}\mathcal{B}_h(\theta,\theta) \leq \frac{1}{2}\|\rho_t\|^2_{L_2(\Omega)}.$$

After omitting the first term on the left-hand side, integrating and using the ellipticity of $\mathcal{B}_h(\cdot,\cdot)$ as well as the assumption $u_{0h} = R_h u_0$ (i.e., $\theta(0) = 0$), we arrive at

$$\|\theta(t)\|^2_{1,h} \leq C\mathcal{B}_h(\theta(t),\theta(t)) \leq C\int_0^t \|\rho_t\|^2_{L_2(\Omega)}d\tau. \tag{4.16}$$

In order to bound the integral on the right-hand side of (4.16), we use Lemma 4.3, (4.6), and [34, Lemma 19.6]. This gives

$$\|\theta(t)\|^2_{1,h} \leq C\kappa^4(h,\mu)\Big\{\|g\|^2_{H_0^1(\Omega)} + \int_0^t \|f_t\|^2_{L_2(\Omega)}d\tau\Big\}. \tag{4.17}$$

Finally, we deduce from (3.9), (4.15), and (4.17) that the assertion of Theorem 4.6 holds. □

It should be mentioned that the error estimate in Theorem 4.6 contains the additional term $\Big(\int_0^t \|f_t\|^2_{L_2(\Omega)}\,d\tau\Big)^{1/2}$, when compared with the estimate (3.26) in Theorem 3.5. This term stems from the application of [34, Lemma 19.6]. In case of a regular solution, we apply [34, Lemma 19.1] (see also Lemma 3.2) instead.

5. Estimates for the fully discrete method. For the discretization in time of the spatially semidiscrete problem (2.13), the backward Euler method is applied. The constant time step is denoted by k. Further we use the notation $U^n = U_h^n$, where U_h^n means the approximation in V_{0h} of the exact solution $u(t) = u(\cdot, t)$ from (2.1) at time $t = t_n = nk$, $n = 0, 1, \dots, N_T$. The fully discrete problem then reads,

$$(5.1)\qquad \begin{aligned} (\overline{\partial} U^n, v_h) + \mathcal{B}_h(U^n, v_h) &= \langle \mathcal{F}(t_n), v_h\rangle \quad \forall\, v_h \in V_{0h},\ 1 \le n \le N_T,\\ U^0 &= u_{0h} = R_h u_0, \end{aligned}$$

with $\overline{\partial} U^n = (U^n - U^{n-1})/k$.

First we give error estimates in case of a convex domain, i.e., the results from Section 3 for the semidiscretization will be used. The L_2- and $\{1, h\}$-norms of the error for the fully discrete method can be bounded as follows.

THEOREM 5.1. *Let Ω be a convex domain and let the assumptions of Lemma 3.1 be satisfied. Then, for the solution U^n of (5.1) and the solution $u(t_n)$ of (2.1), with $0 \le n \le N_T$, we obtain the error estimates*

$$(5.2)\qquad \|U^n - u(t_n)\|_{L_2(\Omega)} \le C(h^2 + k)\Big\{\|u_0\|_{H^2(\Omega)} + \|g\|_{H^\varepsilon(\Omega)} + \int_0^{t_n} \|f_t\|_{H^\varepsilon(\Omega)} d\tau\Big\},$$

$$(5.3)\qquad \begin{aligned} &\|U^n - u(t_n)\|_{1,h}\\ &\le C(h + k)\Big\{\|u_0\|_{H^2(\Omega)} + \|g\|_{H_0^1(\Omega)} + \Big(\int_0^{t_n} \|f_t\|^2_{L_2(\Omega)} d\tau\Big)^{1/2} + \int_0^{t_n} \|f_t\|_{H^\varepsilon(\Omega)} d\tau\Big\}, \end{aligned}$$

with g from (3.11), $\varepsilon \in (0, \frac{1}{2})$, and $C = C(\varepsilon, T)$.

Proof. For the proof of (5.2), we follow the techniques of [34, proof of Theorem 1.5], with the necessary modifications due to $\mathcal{B}_h(\cdot, \cdot)$. By analogy to (3.9), we employ the following splitting of the error

$$(5.4)\qquad U^n - u(t_n) = (U^n - R_h u(t_n)) + (R_h u(t_n) - u(t_n)) =: \theta^n + \rho^n.$$

The estimate of the L_2-norm of $\rho^n = \rho(t_n)$ can be deduced from (3.13) and (3.10). Further, in order to bound the norm of the term θ^n, we introduce ω^n as follows,

$$(5.5)\qquad \omega^n = \omega_1^n + \omega_2^n = (R_h - I)\overline{\partial} u(t_n) + (\overline{\partial} u(t_n) - u_t(t_n)).$$

Calculations analogously to (3.14) show that ω^n satisfies

$$(5.6)\qquad (\overline{\partial}\theta^n, v_h) + \mathcal{B}_h(\theta^n, v_h) = -(\omega^n, v_h) \quad \forall\, v_h \in V_{0h},\ 1 \le n \le N_T,$$

where $\overline{\partial}\theta^n$ is defined by $\overline{\partial}\theta^n = (\theta^n - \theta^{n-1})/k$. Taking θ^n for v_h in (5.6) and using the Cauchy-Schwarz inequality we are led to $(\overline{\partial}\theta^n, \theta^n) \le \|\omega^n\|_{L_2(\Omega)}\|\theta^n\|_{L_2(\Omega)}$. Then, by means of the definition of $\overline{\partial}\theta^n$ we get the inequalities

$$(5.7)\qquad \begin{aligned} \|\theta^n\|^2_{L_2(\Omega)} - (\theta^{n-1}, \theta^n) &\le k\|\omega^n\|_{L_2(\Omega)}\|\theta^n\|_{L_2(\Omega)},\\ \|\theta^n\|_{L_2(\Omega)} &\le \|\theta^{n-1}\|_{L_2(\Omega)} + k\|\omega^n\|_{L_2(\Omega)}. \end{aligned}$$

Applying the second equation in (5.7) repeatedly and using $\theta(0) = 0$ (since $u_{0h} = R_h u_0$) as well as (5.5), we arrive at

$$\|\theta^n\|_{L_2(\Omega)} \le \|\theta^0\|_{L_2(\Omega)} + k\sum_{j=1}^{n}\|\omega^j\|_{L_2(\Omega)} \le k\sum_{j=1}^{n}\|\omega_1^j\|_{L_2(\Omega)} + k\sum_{j=1}^{n}\|\omega_2^j\|_{L_2(\Omega)} =: S_1 + S_2. \tag{5.8}$$

Then, by analogy to [34, proof of Theorem 1.5], we obtain

$$S_1 \le Ch^2 \int_0^{t_n} \|u_t\|_{H^2(\Omega)}\, d\tau, \quad S_2 \le k \int_0^{t_n} \|u_{tt}\|_{L_2(\Omega)} d\tau. \tag{5.9}$$

Relations (5.8) and (5.9), together with (3.10), yield the estimate for $\|\theta^n\|_{L_2(\Omega)}$ which completes the proof of (5.2).

Further, the norm $\|\rho^n\|_{1,h}$ can be estimated by means of (3.4) and (3.10). In order to find an estimate for $\|\theta^n\|_{1,h}$, we use (5.6), now with the special choice $v_h := \overline{\partial}\theta^n$. This, together with the linearity of $\mathcal{B}_h(\cdot,\cdot)$ and the Cauchy-Schwarz inequality, leads to $\overline{\partial}\mathcal{B}_h(\theta^n,\theta^n) \le \|\omega^n\|^2_{L_2(\Omega)}$. A repeated application of this estimate, taking into account $\theta(0) = 0$ and the ellipticity of $\mathcal{B}_h(\cdot,\cdot)$, yields

$$\|\theta^n\|^2_{1,h} \le C\Big\{k\sum_{j=1}^{n}\|\omega_1^j\|^2_{L_2(\Omega)} + k\sum_{j=1}^{n}\|\omega_2^j\|^2_{L_2(\Omega)}\Big\}. \tag{5.10}$$

Then, using (5.9), we are led to

$$k\sum_{j=1}^{n}\|\omega_1^j\|^2_{L_2(\Omega)} + k\sum_{j=1}^{n}\|\omega_2^j\|^2_{L_2(\Omega)} \le Ch^4 \int_0^{t_n} \|u_t\|^2_{H^2(\Omega)}\, d\tau + Ck^2 \int_0^{t_n} \|u_{tt}\|^2_{L_2(\Omega)} d\tau. \tag{5.11}$$

For the integrand in the first term on the right-hand side of (5.11) holds (see [34, proof of Lemma 19.1]): $\|u_t\|^2_{H^2(\Omega)} \le C(\|u_{tt}\|^2_{L_2(\Omega)} + \|f_t\|^2_{L_2(\Omega)})$. Therefore, it remains to find a bound for the second integral on the right-hand side of (5.11). Here we make use of [34, Lemma 19.6] which yields, together with inequality (5.10),

$$\|\theta^n\|_{1,h} \le C(h^2 + k)\Big\{\|g\|_{H_0^1(\Omega)} + \Big(\int_0^{t_n} \|f_t\|^2_{L_2(\Omega)} d\tau\Big)^{1/2}\Big\}. \tag{5.12}$$

This completes the proof of (5.3). $\square$

Now we turn to the case of a non-convex domain, i.e., consider solutions with singularities. For this case, the convergence of the semidiscretization has been analyzed in Section 4.

THEOREM 5.2. *Let Ω be a non-convex domain, with the interior angle β at the reentrant corner, and let the assumptions of Theorem 4.5 be satisfied. Then, for the solutions U^n and $u(t_n)$ from (5.1) and (2.1), the error estimates*

$$\|U^n - u(t_n)\|_{L_2(\Omega)} \le C(\kappa^2(h,\mu) + k)\Big\{\|\Delta u_0\|_{L_2(\Omega)} + \|g\|_{H^\varepsilon(\Omega)} + \int_0^{t_n} \|f_t\|_{H^\varepsilon(\Omega)} d\tau\Big\},$$

$$\|U^n - u(t_n)\|_{1,h}$$
$$\leq C(\kappa(h,\mu)+k)\Big\{\|\Delta u_0\|_{L_2(\Omega)} + \|g\|_{H_0^1(\Omega)} + \Big(\int_0^{t_n} \|f_t\|_{L_2(\Omega)}^2 d\tau\Big)^{1/2} + \int_0^{t_n} \|f_t\|_{H^\varepsilon(\Omega)} d\tau\Big\}$$

hold, with $\varepsilon \in (0, \frac{1}{2})$*,* $C = C(\varepsilon, T)$*, and* $\kappa(h,\mu)$ *from (4.7) (with* $\lambda := \frac{\pi}{\beta}$*).*

Proof. We will only give a sketch of the proof since it is again based on the splitting (5.4). In order to bound the L_2-norm of ρ^n, we now use (4.12)–(4.14) and Lemma 4.4. Further, by analogy with the proof of Theorem 5.1, the L_2-norm of θ^n can be estimated by means of (5.8), with ω^j given in (5.5). For the term S_1 on the right-hand side of (5.8), we obtain

$$S_1 \leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} C\kappa^2(h,\mu)\|u_t\|_{\dot{H}^2(\Omega)} d\tau = C\kappa^2(h,\mu) \int_0^{t_n} \|u_t\|_{\dot{H}^2(\Omega)} d\tau, \tag{5.13}$$

whereas the second inequality in (5.9) remains valid for the term S_2. Summing up these estimates for S_1 and S_2 and applying Lemma 4.4 yields the desired bound for θ^n. This leads to the first assertion of the theorem.

The $\{1, h\}$-norm of ρ^n can be bounded by means of (4.9) and (4.10). In order to estimate $\|\theta^n\|_{1,h}$, we start from inequality (5.10). Then, by means of [34, (1.51)], (4.5), (5.9), and [34, Lemma 19.6], we obtain

$$k\sum_{j=1}^{n} \|\omega_1^j\|_{L_2(\Omega)}^2 + k\sum_{j=1}^{n} \|\omega_2^j\|_{L_2(\Omega)}^2 \leq C\kappa^4(h,\mu)\int_0^{t_n} \|\Delta u_t\|_{L_2(\Omega)}^2\, d\tau + Ck^2 \int_0^{t_n} \|u_{tt}\|_{L_2(\Omega)}^2 d\tau$$
$$\leq C(\kappa^4(h,\mu)+k^2)\Big\{\|g\|_{H_0^1(\Omega)}^2 + \int_0^{t_n} \|f_t\|_{L_2(\Omega)}^2 d\tau\Big\}.$$

leading to the second assertion of the theorem. ◻

Theorem 5.2 implies that for solutions with singularities, the presented method has the same convergence order as in case of regular solutions if an appropriate mesh grading parameter is chosen.

6. Numerical results. In order to illustrate the convergence rate of the investigated approach, some numerical experiments were carried out. Some details concerning the implementation can be found in [6].

First we observe the convergence in case of a convex domain, i.e., for regular solutions. We consider an initial-boundary value problem of type (2.1) in the domain $\Omega = (-1,1) \times (0,1)$. Let the two subdomains Ω_i, $i = 1, 2$, be given by $\Omega_1 = (-1,0) \times (0,1)$ and $\Omega_2 = (0,1) \times (0,1)$; see Figure 6.1. We take $t \in [0,1]$ for the time interval. The right-hand side f and the initial function u_0 are chosen such that the solution of (2.1) is

$$u(x,t) = (1-x_1^2)(1-x_2)x_2(1+t)^{-0.8}.$$

The initial mesh, shown in Figure 6.1, is used for the semidiscretization in space. This mesh is refined globally by dividing each triangle into four equal triangles, so that the mesh parameters form a sequence $\{h_1, h_2, h_3, h_4\}$ given by $h_{i+1} = 0.5\, h_i$. The mortar parameters (see

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Section 2) are chosen as follows: $\alpha_1 = \alpha_2 = 0.5$ and $\gamma = 6$. The triangulation $\mathcal{E}_h$ of the mortar interface Γ is defined as

$$\mathcal{E}_h := \{E : E = \partial K_1 \cap \partial K_2, \text{if } E \text{ is a segment; } K_i \in \mathcal{T}_h^i \text{ with } K_i \cap \Gamma \neq \emptyset \text{ for } i = 1, 2\},$$

i.e., the nodes of both triangulations $\mathcal{T}_h^1, \mathcal{T}_h^2$ lying on Γ establish the endpoints of the intervals $E \in \mathcal{E}_h$. For the discretization in time we employ three levels k_i, $i = 1, 2, 3$, where $k_1 = \frac{1}{10}$, and $k_{i+1} = 0.5k_i$.

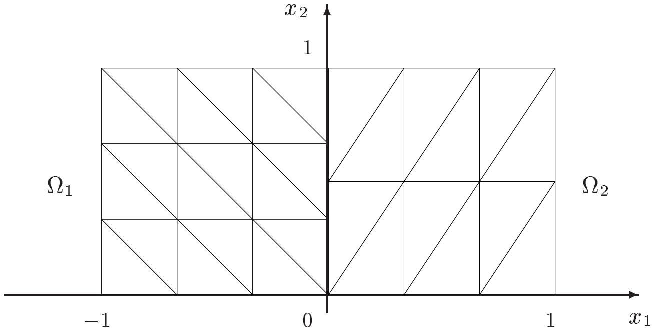

FIG. 6.1. *Initial triangulation (first example).*

For the approximate measurement of the convergence rates stated in Theorem 5.1, the hypothesis for the tests is that

$$\|U^n - u(t_n)\|_{L_2(\Omega)} \approx C_1^{(0)} h^{\sigma_0} + C_2^{(0)} k^{\tau_0}, \quad \|U^n - u(t_n)\|_{1,h} \approx C_1^{(1)} h^{\sigma_1} + C_2^{(1)} k^{\tau_1}, \tag{6.1}$$

with U^n defined in Section 5 ($n = 0, 1, \ldots, N_T$). The parameters $C_1^{(i)}$ and $C_2^{(i)}$, $i = 0, 1$, are assumed to be approximately constant for three consecutive levels of h and k. Then, the exponents σ_0 and τ_0 in (6.1) can be approximately computed by

$$\sigma_0 = \log_2 \frac{e_0(h_i, k_j) - e_0(h_{i+1}, k_j)}{e_0(h_{i+1}, k_j) - e_0(h_{i+2}, k_j)}, \quad \tau_0 = \log_2 \frac{e_0(h_i, k_j) - e_0(h_i, k_{j+1})}{e_0(h_i, k_{j+1}) - e_0(h_i, k_{j+2})}, \tag{6.2}$$

where $e_0(h_i, k_j)$ denotes the L_2-norm of the approximation error for discretization parameters h_i for the Nitsche finite element method and k_j for the backward Euler method. Analogous relations hold for the convergence rates σ_1 and τ_1 if in (6.2) the errors $e_0(\cdot, \cdot)$ are replaced by $e_{1,h}(\cdot, \cdot)$, the approximation errors in the $\{1, h\}$-norm.

Table 6.1 shows the approximation errors $e_0(h_i, k_j)$ and $e_{1,h}(h_i, k_j)$ for $i = 1, \ldots, 4$, and $j = 3$ at the level $t_n = T$ as well as the observed convergence rates σ_0^{obs} and σ_1^{obs} which are obtained by using σ_0 in (6.2) and its analogue for σ_1, with $i = 2$ and $j = 3$. According to Theorem 5.1, the theoretically expected convergence rates are $\sigma_0 = 2$ and $\sigma_1 = 1$, and the observed rates σ_i^{obs} from Table 6.1 are approximately equal to σ_i, $i = 1, 2$. In Table 6.2 we represent the approximation errors $e_0(h_i, k_j)$ and $e_{1,h}(h_i, k_j)$ for $i = 4$ and $j = 1, 2, 3$ at the level $t_n = T$ as well as the observed convergence rates τ_0^{obs} and τ_1^{obs} obtained by using τ_0 in (6.2) and its analogue for τ_1, with $i = 4$ and $j = 1$. The observed values confirm approximately the expected convergence rates $\tau_0 = \tau_1 = 1$; cf., Theorem 5.1.

In order to investigate the convergence rate in the presence of singularities caused by non-convex domains, we consider the initial-boundary value problem (2.1) in the L-shaped domain $\Omega = (-1, 1) \times (-1, 1) \setminus [0, 1) \times (-1, 0]$. We choose $\Omega_1 = (-1, 0) \times (-1, 1)$ and

TABLE 6.1
Error norms and convergence rates for $h = h_1, \dots, h_4$, and $k = k_3$.

level	$e_0(h_i, k_3)$	$e_{1,h}(h_i, k_3)$
h_1	3.13072e-2	1.88140e-1
h_2	1.01964e-2	9.59374e-2
h_3	5.07436e-3	4.92381e-2
h_4	3.88784e-3	2.68348e-2
	$\sigma_0^{\text{obs}} = 2.11$	$\sigma_1^{\text{obs}} = 1.06$

TABLE 6.2
Error norms and convergence rates for $h = h_4$, and $k = k_1, k_2, k_3$.

level	$e_0(h_4, k_j)$	$e_{1,h}(h_4, k_j)$
k_1	3.92313e-3	2.68924e-2
k_2	3.89884e-3	2.68527e-2
k_3	3.88783e-3	2.68348e-2
	$\tau_0^{\text{obs}} = 1.14$	$\tau_1^{\text{obs}} = 1.15$

$\Omega_2 = (0,1) \times (0,1)$ as subdomains. The time interval is again $[0,1]$. Let the right-hand side f and the initial function u_0 be given such that the solution of problem (2.1) is

$$u(x,t) = (1 - x_1^2)(1 - x_2^2)\, r^\lambda \sin(\lambda\theta)(1 + \frac{1}{2}\,\mathrm{e}^{-t}), \quad \lambda = \frac{2}{3},$$

where (r, θ) are polar coordinates centered at $(0,0)$; cf., Section 4.

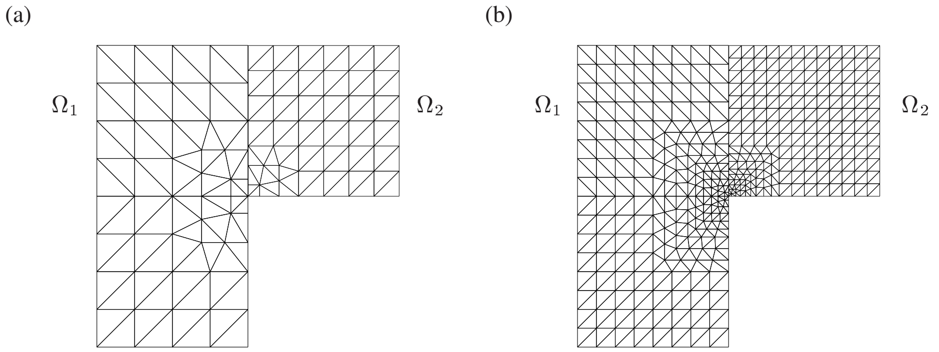

FIG. 6.2. *Triangulations on the levels $h = h_1$ and $h = h_2$ (second example).*

For the Nitsche finite element discretization, the initial mesh (with mesh parameter h_1) depicted in Figure 6.2(a) is employed. Near the reentrant corner, this mesh is provided with local grading as defined in Section 4, the grading parameter is $\mu = 0.7\lambda \approx 0.47$. The subsequent meshes (with mesh parameters h_i, $i = 2, 3, 4$, $h_{i+1} = 0.5h_i$) arise by dividing each triangle into four equal triangles in the quasi-uniform part of the mesh and by local grading with $\mu = 0.7\lambda \approx 0.47$ near the reentrant corner; see Figure 6.2(b) for the mesh with $h = h_2$. The mortar parameters and the triangulation $\mathcal{E}_h$ are defined as in the first example. For the discretization in time we take the three levels k_1, k_2, and k_3, with $k_1 = \frac{1}{20}$, $k_{i+1} = 0.5k_i$. For the computation of approximate convergence rates we use again (6.2) and

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analogous relations for σ_1, τ_1.

Table 6.3 contains the approximation errors $e_0(h_i, k_j)$ and $e_{1,h}(h_i, k_j)$ for $i = 1, \ldots, 4$, and $j = 3$, at the level $t_n = T$ as well as the observed convergence rates σ_0^{obs} and σ_1^{obs}. The observed values σ_0^{obs}, σ_1^{obs} are approximately equal to the expected convergence rates $\sigma_0 = 2$ and $\sigma_1 = 1$, as stated in Theorem 5.2.

TABLE 6.3
Error norms and convergence rates for $h = h_1, \ldots, h_4$, and $k = k_3$.

level	$e_0(h_i, k_3)$	$e_{1,h}(h_i, k_3)$
h_1	3.83180e-2	3.82840e-1
h_2	1.46901e-2	1.90226e-1
h_3	9.48033e-3	9.84876e-2
h_4	8.23570e-3	5.32338e-2
	$\sigma_0^{\text{obs}} = 2.07$	$\sigma_1^{\text{obs}} = 1.02$

The errors $e_0(h_i, k_j)$ and $e_{1,h}(h_i, k_j)$ for $i = 4$ and $j = 1, 2, 3$ at the level $t_n = T$ are reported in Table 6.4. The convergence rates resulting from these errors are very close to the expected values $\tau_0 = \tau_1 = 1$.

TABLE 6.4
Error norms and convergence rates for $h = h_4$, and $k = k_1, k_2, k_3$.

level	$e_0(h_4, k_j)$	$e_{1,h}(h_4, k_j)$
k_1	8.26364e-3	5.32740e-2
k_2	8.24468e-3	5.32467e-2
k_3	8.23570e-3	5.32338e-2
	$\tau_0^{\text{obs}} = 1.08$	$\tau_1^{\text{obs}} = 1.08$

The numerical examples presented in this paper illustrate that Nitsche mortaring combined with the backward Euler method is a suitable approach for the numerical treatment of initial-boundary value problems for the heat equation in polygonally bounded domains. In particular, for solutions with singularities, the use of meshes with a grading parameter $\mu < \lambda$ leads to the same convergence rates as for regular solutions.

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