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A COMPARISON OF FOUR- AND FIVE-POINT DIFFERENCE APPROXIMATIONS FOR STABILIZING THE ONE-DIMENSIONAL STATIONARY CONVECTION-DIFFUSION EQUATION*

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Abstract. Some recently developed finite element stabilizations of convection-diffusion problems generate in 1D five-point difference schemes. Because there are only few results on four- and five-point schemes in the literature (in contrast to three-point schemes), we discuss some properties of such schemes with special emphasis on the choice of free parameters for a singularly perturbed problem to avoid oscillations.

Key words. convection-diffusion, difference scheme, stabilized finite element method

AMS subject classifications. 65L10, 65L12, 65L60

1. Introduction.

Let us consider the singularly perturbed boundary value problem

$$-\varepsilon u'' - bu' = f \quad \text{on } (0,1), \qquad u(0) = u(1) = 0, \tag{1.1}$$

under the assumptions $0 < \varepsilon \ll 1$, $b(x) \geq \beta > 0$ and its discretization with finite differences or low order, in general, linear finite elements. It is well known that the use of the central difference scheme or the Galerkin method with linear elements leads to wild oscillations on standard meshes due to the existence of an exponential boundary layer at $x = 0$ unless the mesh width is as small as ε, which makes no sense from the practical point of view.

Therefore, it is quite standard to apply some kind of upwinding in the finite difference framework or to stabilize the Galerkin finite element method (here we do not discuss other approaches, e.g., upwinding in the finite volume method).

For simplicity, we restrict ourselves to uniform meshes with the mesh width h and the mesh points $x_i := ih, i = 0, 1, \ldots, N$, with $x_N = 1$, assuming always $\varepsilon \leq C\,h$ and moderate C. We use the difference operators

$$D^+u_i := \frac{u_{i+1} - u_i}{h}, \quad D^-u_i := \frac{u_i - u_{i-1}}{h}, \quad D_0 u_i := \frac{u_{i+1} - u_{i-1}}{2h}.$$

Then, simple upwinding reads

$$\begin{gathered} -\varepsilon D^+D^-u_i - b_i D^+u_i = f_i, \quad i = 1, \ldots, N-1, \\ u_0 = u_N = 0, \end{gathered} \tag{1.2}$$

and the corresponding simplest stabilization of the Galerkin method based on linear elements is

$$\varepsilon(u_h', v_h') - (bu_h', v_h) + \frac{h}{2}(bu_h', v_h') = (f, v_h). \tag{1.3}$$

Remark that the stabilization term $\frac{h}{2}(bu_h', v_h')$ in the finite element framework has its analogue in the finite difference language; one can rewrite (1.2) in the form

$$-\varepsilon \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - b_i \frac{u_{i+1} - u_{i-1}}{2h} - b_i \frac{h}{2} \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = f_i. \tag{1.4}$$

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Because (1.2) and (1.3) are first-order methods, one is interested in constructing higher-order methods. Here we do not discuss the midpoint upwind finite difference scheme [10] or streamline diffusion based on linear elements [11], because we want to compare approaches leading to four- or five-point schemes. For a survey concerning such schemes, see [8, Chapter I.2]. But this survey shows that theoretical results for such schemes are rare. For simplicity, from now on, we assume b to be constant in the given problem (1.1).

Our renewed interest in such schemes comes from the fact that several recently developed finite element stabilizations of convection-diffusion problems [3, 4, 7] generate in 1D such five-point schemes. Moreover, in most cases, the optimal tuning of parameters involved in the stabilization term is an open problem.

The main purpose of our paper is to explain the close relation of the recently proposed stabilization methods as edge stabilization or local projection to Frjazinov-type difference schemes, which are only weakly monotone. Moreover, it turns out that the "optimal" choice of parameters in these symmetric stabilization methods is more complicated than for four-point upwind schemes.

2. Finite difference schemes. A well-known four-point scheme for solving (1.1) is given by

$$\begin{aligned} &-\varepsilon D^+D^-u_i - bD_0u_i \\ &\qquad + \frac{b\lambda}{h}(-u_{i-1} + 3u_i - 3u_{i+1} + u_{i+2}) = f_i, \quad i = 1, \dots, N-2, \\ &-\varepsilon D^+D^-u_{N-1} - bD^+u_{N-1} = f_{N-1}, \\ &u_0 = u_N = 0, \end{aligned} \tag{2.1}$$

where $\lambda \geq 0$ is a parameter. Remark that the stabilization term is a consistent approximation of the third-order derivative multiplied by h^2.

It is quite interesting that the scheme (2.1) is, for certain values of the parameter λ, inverse-monotone.

To see that, let us introduce matrices M_1, M_2 of the format $(N+1) \times (N+1)$ by

$$M_1(\lambda) = M_1 = \begin{bmatrix} 1 & 0 & 0 & & & 0 \\ \alpha & \alpha+\beta & -\delta & & & \\ & \ddots & \ddots & \ddots & & \\ & & \alpha & \alpha+\beta & -\delta & 0 \\ & & 0 & r & s & 0 \\ 0 & & 0 & 0 & 0 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & -1 & & 0 \\ & \ddots & \ddots & \\ & & 1 & -1 \\ 0 & & & 1 \end{bmatrix}.$$

Then, with $\gamma = -(\alpha + \beta + \delta)$, the product $\hat{M} = M_1 M_2$ reads

$$\hat{M}(\lambda) = \hat{M} = \begin{bmatrix} 1 & -1 & 0 & 0 & & 0 \\ \alpha & \beta & \gamma & \delta & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & \alpha & \beta & \gamma & \delta \\ & & 0 & r & s-r & -s \\ 0 & & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Except for the first row, this matrix realizes the coefficient matrix of (2.1) (multiplied by h), which is

$$M(\lambda) = M = \begin{bmatrix} 1 & 0 & 0 & 0 & & 0 \\ \alpha & \beta & \gamma & \delta & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & \alpha & \beta & \gamma & \delta \\ & & 0 & r & s-r & -s \\ 0 & & 0 & 0 & 0 & 1 \end{bmatrix},$$

with

$$r = -\frac{\varepsilon}{h}, \quad s = \frac{\varepsilon}{h} + b,$$

and

$$\alpha = -\frac{\varepsilon}{h} + b(\frac{1}{2} - \lambda), \quad \beta = 2\frac{\varepsilon}{h} + 3b\lambda, \quad \gamma = -\frac{\varepsilon}{h} - b(\frac{1}{2} + 3\lambda), \quad \delta = b\lambda.$$

Because $\lambda \geq 0$ and $b > 0$, it follows that $\delta \geq 0$ and $\alpha + \beta = \varepsilon/h + b(2\lambda + 1/2) > 0$. Hence, $M_1(\lambda)$ is an M-matrix if and only if $\alpha \leq 0$. This is equivalent to the condition

$$\lambda \geq \frac{1}{2} - \frac{\varepsilon}{bh}. \tag{2.2}$$

Because M_2 is an M-matrix too, under the condition (2.2) (and $\lambda \geq 0$), the matrices $M_1(\lambda)$ and M_2 and hence the matrix $\hat{M}(\lambda)$ are inverse-monotone.

Let us remark that $\hat{M}(\lambda)$ realizes the coefficient matrix of (2.1) (multiplied by h), if we replace the homogeneous Dirichlet boundary condition $u(x_0) = 0$ by the Neumann boundary condition $u'(x_0) = 0$ and use a common discretization.

Now, because $\hat{M}(\lambda^*)$ is inverse-monotone for $\lambda \geq \lambda^* = \max\{0, \frac{1}{2} - \frac{\varepsilon}{bh}\}$, we can prove for the case $\lambda = \lambda^*$ that the matrix $M(\lambda)$ is inverse-monotone too.

Namely, for the case $\lambda^* > 0$ in which $\alpha = 0$, the relation $M(\lambda^*)v \geq 0$ implies $\hat{M}(\lambda^*)[v_2, v_2, v_3, \ldots, v_{N+1}]^T \geq 0$. Because $\hat{M}(\lambda^*)$ is inverse-monotone, we conclude $v_i \geq 0$ for $i = 2, 3, \ldots, N+1$. Additionally, $v_1 \geq 0$ due to the first inequality of $M(\lambda^*)v \geq 0$. Thus, $M(\lambda^*)$ is inverse-monotone.

REMARK 2.1. Numerical experiments lead to the conjecture that $M(\lambda)$ is inverse-monotone for $\lambda > \lambda^*$ too, but in the moment we are not able to prove it.

Before we discuss other methods with stabilizations, let us prove the inverse-monotonicity by another approach, which is similar as above. We study a modified scheme in which the discretization in x_{N-1} of (2.1) is replaced by

$$-\varepsilon D^+ D^- u_{N-1} - bD_0 u_{N-1} + \frac{b\lambda}{h}(-u_{N-2} + 2u_{N-1} - u_N) = f_{N-1}.$$

If the coefficient matrix of the modified scheme is denoted by $M^{\text{mod}}(\lambda)$, we now have

$$r^{\text{mod}} = \alpha = -\frac{\varepsilon}{h} + b\left(\frac{1}{2} - \lambda\right), \qquad s^{\text{mod}} = \frac{\varepsilon}{h} + b\left(\frac{1}{2} + \lambda\right).$$

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Consequently, we obtain by multiplication of $M^{\text{mod}}(\lambda)$ from the left with the matrix

$$M_3 = \begin{bmatrix} 1 & 0 & 0 & & & & 0 \\ 0 & 1 & 0 & & & & \\ 0 & 1 & 1 & & & & \vdots \\ & \vdots & \ddots & \ddots & & & \\ & 1 & \cdots & 1 & 1 & 0 & 0 \\ 0 & 1 & \cdots & 1 & 1 & 1 & 0 \\ 0 & 0 & & \cdots & & 0 & 1 \end{bmatrix},$$

and we have $M_4(\lambda) = M_4 = M_3 M^{\text{mod}}(\lambda)$, which is an M-matrix for $\lambda \geq \lambda^* = \max\{0, \frac{1}{2} - \frac{\varepsilon}{bh}\}$ and $h \geq \frac{2\varepsilon}{b}$ (which specifies our general assumption $\varepsilon \leq C\,h$). Because $\left(M^{\text{mod}}\right)^{-1} = (M_4)^{-1} M_3$, the matrix M^{mod} is inverse-monotone as well.

Let us finally not that the technique just used corresponds to Kopteva's approach [6] to estimate the discrete Green's function for a modified discretization with $\lambda = 1/2$.

Instead of using a stabilization term of the form $h^2 u'''$, one can also use $h^3 u^{(4)}$. A well-known stencil to approximate the fourth-order derivative is given by

$$\frac{u(x_{i-2}) - 4u(x_{i-1}) + 6u(x_i) - 4u(x_{i+1}) + u(x_{i+2})}{h^4} = u^{(4)}(x_i) + O(h^2).$$

Therefore, it is natural to stabilize the central scheme in the interior mesh points by

$$\begin{aligned} &-\varepsilon D^+ D^- u_i - b D_0 u_i \\ &\qquad + \frac{b\gamma}{h}(u_{i-2} - 4u_{i-1} + 6u_i - 4u_{i+1} + u_{i+2}) = f_i, \quad i = 2, \ldots, N-2. \end{aligned} \tag{2.3}$$

Here $\gamma \geq 0$ is a parameter.

It is clear that the same discretization and hence also stabilization cannot be used for all interior mesh points. For the five-point scheme (2.3) in the mesh points x_1 and x_{N-1}, some modification is necessary. In the points x_1, x_{N-1}, we follow the idea of Frjazinov [10] and choose the stabilization and, consequently, the discretization in such a way that the matrix M corresponding to the stabilization term is symmetric and positive semi definite.

Because the stabilization (2.3) consists of a difference approximation of the fourth-order derivative, the following question arises: which results are known for symmetric and positive semi definite difference schemes for fourth-order differential operators?

It turns out that the authors of [1], in contrast to many other books, discussed this question. Let us consider the boundary value problem

$$(p(x)y'')'' + r(x)y = f(x), \quad \text{on } (0,1),$$

subject to one of the three types of boundary conditions

$$\begin{array}{lrr} \text{(i)} & y(x_b) = \alpha, & y'(x_b) = \beta, \\ \text{(ii)} & y(x_b) = \alpha, & y''(x_b) = \beta, \\ \text{(iii)} & y''(x_b) = \alpha, & (py'')'(x_b) = \beta, \end{array}$$

with $x_b = 0$ and $x_b = 1$.

For instance, a discretization of the boundary conditions (i) leads to the matrix M of format $(N-1)\times(N-1)$ (in the case $p\equiv 1$, $r\equiv 0$),

$$M=\begin{bmatrix} 7 & -4 & 1 & & & & 0\\ -4 & 6 & -4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & -4 & 6 & -4 & 1\\ & & & 1 & -4 & 6 & -4\\ 0 & & & & 1 & -4 & 7 \end{bmatrix}.$$

With $v=(v_1,\dots,v_{N-1})^T$ and $z_0=-2v_1$, $z_N=-2v_{N-1}$, $z_i=-v_{i-1}+2v_i-v_{i+1}$, $i=1,\dots,N-1$, $v_0=v_N=0$, the matrix M satisfies (see [1])

$$(M\,v,v)=\frac{1}{2}z_0^2+\sum_{i=1}^{N-1}z_i^2+\frac{1}{2}z_N^2, \tag{2.4}$$

where $(\cdot,\cdot)$ denotes the Euclidean scalar product.

Alternatively, based on the boundary conditions (ii), the generated matrix M^* is almost identical with M (with the exception that the number 7 is to replace by 5) and satisfies

$$(M^*\,v,v)=\sum_{i=1}^{N-1}z_i^2, \tag{2.5}$$

instead of (2.4).

Thus, we can, e.g., complete the discretization of (1.1) based on (2.3) by

$$\begin{aligned} &-\varepsilon D^+D^-u_1-bD_0u_1+\frac{b\gamma}{h}(\tau u_1-4u_2+u_3)=f_1,\\ &-\varepsilon D^+D^-u_i-bD_0u_i\\ &\qquad+\frac{b\gamma}{h}(u_{i-2}-4u_{i-1}+6u_i-4u_{i+1}+u_{i+2})=f_i,\quad i=2,\dots,N-2,\\ &-\varepsilon D^+D^-u_{N-1}-bD_0u_{N-1}\\ &\qquad\qquad+\frac{b\gamma}{h}(u_{N-3}-4u_{N-2}+\tau u_{N-1})=f_{N-1},\\ &\qquad\qquad\qquad u_0=u_N=0, \end{aligned} \tag{2.6}$$

and we call it Frjasinov-type difference scheme. So far, the parameter τ admits the value 7 or 5; later we will still generate a scheme with $\tau=6$.

It is obvious that with the property (2.4) of the matrix M we obtain better stability properties of the scheme. While we only have

$$(Lu_h,u_h)_{0,h}=\varepsilon|u_h|_1^2,$$

for central difference, where L denotes the difference operator generating the corresponding scheme and $(\cdot,\cdot)_{0,h}$ denotes the discrete L_2 scalar product, we have instead for the scheme (2.6) and $\gamma>0$ an improved stability, because

$$(Lu_h,u_h)_{0,h}=\varepsilon|u_h|_1^2+b\gamma(M\,u_h,u_h).$$

Also with the improved stability of the scheme, oscillations of the discrete solution are possible as our numerical experiments show.

In [9], a scheme of the type (2.6) for $\gamma = 1/4$ is called weakly monotone, because the difference equation

$$a_4 y_{i-2} - a_3 y_{i-1} + a_2 y_i - a_1 y_{i+1} + a_0 y_{i+2} = 0, \tag{2.7}$$

with

$$a_0 = a_4 = \frac{b}{4h}, \quad a_{3,1} = \pm\frac{b}{2h} + \frac{\varepsilon}{h^2} + \frac{b}{h}, \quad a_2 = \frac{2\varepsilon}{h^2} + \frac{3b}{2h},$$

admits the following property: all roots of the characteristic equation of (2.7) are real and positive or have a positive real part [9]. It seems that this property excludes wild oscillations, but the influence of the discretization in the grid points near to the boundary is so far not absolutely clear.

Further, we do not know error estimates for Frjasinov-type schemes with respect to the maximum norm in the singularly perturbed case.

3. Related schemes generated by stabilizing linear finite elements. As mentioned above, the difference scheme (2.1) is based on a stabilization term, which is a consistent approximation of the third-order derivative multiplied by h^2. In a finite element context, one could realize that perturbation by the discretization:

Find $u_h \in V_h$, such that

$$\varepsilon(u_h', v_h') - (bu_h', v_h) + \frac{b}{2}h \sum_{i=1}^{N-1} [u_h']_i (v_h(x_{i-1}) - v_h(x_i)) = (f, v_h) \quad \forall v_h \in V_h. \tag{3.1}$$

Here $V_h \subset H_0^1(0,1)$ denotes the space of linear finite elements and $[\cdot]_i$ the jump of a discontinuous function in the point x_i. The scheme generated by (3.1) coincides with (2.1) and $\lambda = 1/2$. So far, to the best of our knowledge, nobody observed the possibility to generate that scheme based on (3.1) and there does not exist an error analysis for the finite element approach.

It is much more popular to stabilize based on approximations of the fourth-order derivative. Burman and Hansbo introduced in [3] the edge stabilization of the Galerkin method. For problem (1.1) with constant coefficients, the method has the form

$$\varepsilon(u_h', v_h') - (bu_h', v_h) + b\,\gamma\, h^2 \sum_{i=1}^{N-1} [u_h']_i [v_h']_i = (f, v_h) \quad \forall v_h \in V_h. \tag{3.2}$$

It is not difficult to see that (3.2) is equivalent to the difference scheme

$$\begin{aligned}
&-\varepsilon D^+D^- u_1 - bD_0 u_1 + \frac{b\gamma}{h}(5u_1 - 4u_2 + u_3) = f_1,\\
&-\varepsilon D^+D^- u_i - bD_0 u_i\\
&\quad + \frac{b\gamma}{h}(u_{i-2} - 4u_{i-1} + 6u_i - 4u_{i+1} + u_{i+2}) = f_i, \quad i = 2,\dots,N-2,\\
&\qquad\qquad u_0 = u_N = 0
\end{aligned}$$

(we omit the corresponding equation in x_{N-1}), which is scheme (2.6) with $\tau = 5$.

The method (3.2) belongs to the class of symmetric stabilization FEMs of the general form

$$a_G(u_h, v_h) + S(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h. \tag{3.3}$$

Here $a_G(\cdot,\cdot)$ denotes the bilinear form of the pure Galerkin approach and $S(\cdot,\cdot)$ the symmetric stabilization. Remember that in the finite difference framework Frjasinov-type schemes are based on a similar idea.

Introducing

$$\|w\|_E^2 = \varepsilon|w|_1^2 + |w|_0^2 + S(w,w),$$

typical error estimates for symmetric stabilization FEMs do have the form (for linear finite elements)

$$\|u - u_h\|_E \le c\left(\varepsilon^{1/2}h + h^{3/2}\right)|u|_2. \tag{3.4}$$

Because different methods are analyzed in different norms, a fair comparison of different methods is not easy. We simply use the maximum norm in our numerical experiments presented later.

Next we present two variants of projection methods. In the first class of projection methods, one uses a projection π into V_h. The discretization is given by

$$a_G(u_h, v_h) + b\,\tilde\gamma h\,\langle u_h{}' - \pi(u_h{}'), v_h{}' - \pi(v_h{}')\rangle = (f, v_h) \qquad \forall v_h \in V_h. \tag{3.5}$$

Here $\langle\cdot,\cdot\rangle$ denotes some arbitrary scalar product.

Let us introduce the special scalar product (with $g(x^\pm) = \lim_{t\to x\pm} g(t)$)

$$\langle w, v\rangle := \sum_{i=1}^{N} h\frac{(wv)(x_{i-1}^+) + (wv)(x_i^-)}{2}$$

and denote by $\pi w \in V_h$ the orthogonal projection with respect to that discrete scalar product, i.e., $\pi w \in V_h$ is defined by

$$\langle \pi w, v_h\rangle = \langle w, v_h\rangle \quad \forall v_h \in V_h.$$

Then, the method (3.5) generates the scheme (again $u_0 = u_N = 0$ and we omit the equation for $i = N-1$)

$$\begin{aligned}
&-\varepsilon D^+D^-u_1 - bD_0u_1 + \frac{b\,\tilde\gamma}{4h}(7u_1 - 4u_2 + u_3) = f_1,\\
&-\varepsilon D^+D^-u_i - bD_0u_i\\
&\quad + \frac{b\,\tilde\gamma}{4h}(u_{i-2} - 4u_{i-1} + 6u_i - 4u_{i+1} + u_{i+2}) = f_i, \quad i = 2,\dots,N-2,
\end{aligned}$$

which is nothing but scheme (2.6) with $\tau = 7$ and $\gamma = \tilde\gamma/4$.

REMARK 3.1. Codina [4] proposed a nonsymmetric variant of (3.5) based on the L_2 scalar product $(\cdot,\cdot)$,

$$a_G(u_h, v_h) + b\gamma h(u_h' - \pi(u_h'), v_h') = (f, v_h) \quad \forall v_h \in V_h. \tag{3.6}$$

In his original version, the orthogonal L_2-projection is used (which is practically bad). Then, the symmetric and nonsymmetric versions coincide, because

$$\left(u_h' - \pi(u_h'), \pi(v_h')\right) = 0.$$

But if we replace π in (3.6) by some other local projection and, e.g., use the Oswald projector (or the Clemént projector), the symmetric and nonsymmetric versions become different. It turns out that we generate a seven-point difference scheme with the symmetrized version, but we generate the difference scheme (2.6) when using the nonsymmetric version (3.6).

The second class of symmetric projection methods uses a macro mesh $\mathcal{T}_{\mathcal{M}}$ and a second finite element space M_h of possibly discontinuous finite elements. Now the projector π_h projects into M_h and the method reads

$$a_G(u_h, v_h) + b\gamma h \sum_M \left(u_h' - \pi_h(u_h'), v_h' - \pi_h(v_h')\right)_M = (f, v_h) \quad \forall v_h \in V_h. \tag{3.7}$$

We denote by $(\cdot,\cdot)_M$ the L_2 scalar product restricted to some $M \in \mathcal{T}_{\mathcal{M}}$. For linear elements it is standard to choose M_h as the space of piecewise constants on the macro mesh and to define the projection as the piecewise orthogonal L_2-projection (we do not discuss schemes based on enrichment of approximation spaces; see [7]).

Often a $2h$-mesh is proposed to be the macro mesh. Then the following stencils are generated by the stabilization term:

$$\begin{aligned}
&-\varepsilon D^+D^-u_i - bD_0u_i + \frac{b\gamma}{h}(-u_{i-1} + 2u_i - u_{i+1}) = f_i, \qquad \text{if } i \text{ is odd},\\
&-\varepsilon D^+D^-u_i - bD_0u_i\\
&\qquad + \frac{b\gamma}{2h}(u_{i-2} - 2u_{i-1} + 2u_i - 2u_{i+1} + u_{i+2}) = f_i, \qquad \text{if } i \text{ is even},\\
&\qquad\qquad u_0 = u_N = 0.
\end{aligned} \tag{3.8}$$

That means the stencils generated are comparable with (1.4), i.e., the stabilization term is a consistent approximation of the second-order derivative, but only multiplied by h. Therefore, this variant of the local projection method is not recommended in comparison to the methods discussed so far.

It seems better to use the piecewise constant projection onto the Voronoi boxes related to x_i. This method generates, for interior mesh points not close to the boundary, again the stencil $\{1, -4, 6, -4, 1\}$, but for the points near the boundary, e.g., x_1,

$$-\varepsilon D^+D^-u_1 - bD_0u_1 + \frac{b\gamma}{4h}(6u_1 - 4u_2 + u_3) = f_1. \tag{3.9}$$

4. How to choose the parameters?. If N is even, for central differencing, it is known that $\lim_{\varepsilon\to 0} u_1 = \infty$, thus we have unrealistic, even unbounded oscillations. Any stabilization method should avoid oscillations or at least guarantee that occurring oscillations are small.

For the scheme (2.1), we have inverse-monotonicity if $\lambda = \max\{0, 1/2 - \varepsilon/(bh)\}$, which is optimal for avoiding oscillations. Further, the scheme (2.6) is weakly monotone in the sense of Stoyan [9] for certain parameters γ. In the following, we propose a new strategy for the choice of the stabilization parameter to reduce possible oscillations in the case of layers.

Because the behavior of the discrete solution near the layer is very important, let us look on the discrete equation in the first mesh point. In all cases (except (3.8)), it has the form

$$\frac{2\varepsilon}{h}u_1 - \frac{\varepsilon}{h}u_2 - b\frac{u_2}{2} + b\,\mu(Au_1 - Bu_2 + u_3) = f_1 h,$$

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with constants A, B characterizing the scheme and some parameter μ. Because for $\varepsilon \ll h$ the first mesh point has some distance to the layer, $u_2 = u_3 = U$ imply $u_1 = U$. This leads to (assuming h to be small)

$$\mu = \frac{\frac{1}{2} - \frac{\varepsilon}{bh}}{A - B + 1}. \tag{4.1}$$

This gives $\lambda = 1/2 - \varepsilon/(bh)$ for scheme (2.1) and $\gamma = 1/4 - \varepsilon/(2bh)$ for scheme (2.6) with $\tau = 5$, e.g. If we neglect $\varepsilon/(bh)$, this gives $\lambda = 1/2$ and $\gamma = 1/4$, respectively. But the numerical experiments of the next section will show that the optimal choice of the parameters is very important.

Equivalently to the above derivation of (4.1), we can write the scheme in the first mesh point in the form

$$-\frac{\varepsilon}{h}(u_0 - 2u_1 + u_2) - b\frac{u_2 - u_0}{2} + b\,\mu\big[(B - A - 1)u_0 + Au_1 - Bu_2 + u_3\big] = f_1 h$$

and require that u_0 (if different from zero) has no influence on the other u_i, $i = 1, 2, \ldots$. That gives

$$-\frac{\varepsilon}{h} + b/2 + b\mu(B - A - 1) = 0.$$

and hence (4.1) too.

Analogously, considering five-point schemes like (2.3) in the second mesh point, it follows that it is necessary to switch off the stabilization. That is verified in our experiments too.

REMARK 4.1. Our approach corresponds to a necessary convergence condition for $h \to 0$ and $\varepsilon \ll h$ for a problem without a layer component in the solution decomposition. It is also possible in the usual way [8] to derive a necessary condition for uniform convergence of u_1 towards $u(x_1)$, but this results in a complicated formula for $\mu = \mu(q)$ with $q = (bh)/(2\varepsilon)$. Because exponential fitting in 2D, especially for problems with characteristic layers, is useless, we do not follow this approach.

5. Numerical experiments. In our numerical experiments, we set $b = 1$ and study

$$-\,\varepsilon\, u'' - u' = f(x), \quad x \in (0,1), \qquad u(0) = u(1) = 0$$

for equidistant meshes characterized by $N = N_i = 10 \cdot 2^{i-1}$, $h_i = \frac{1}{N_i}$, $i = 1, 2, \ldots, 20$, in general, for $\varepsilon = 10^{-5}$ (except in Figure 5.4), and present all results in the max-norm and in dependence on N.

First we verified for a problem without layers and a smooth solution the convergence rate of the four-point scheme (2.1) and the five-point scheme (2.6). We fixed f in such a way that $u(x) = \sin(\pi x)$ solves the problem.

Figure 5.1 shows convergence of order 2 for the four-point scheme (2.1) for different values of λ, coinciding with the theory.

For the five-point schemes (2.6), we consider $\gamma = 1/4$ and the cases $\tau = 5, 6, 7$. If $\tau = 5$, we have consistency of order 1 in the mesh points close to the boundary; in the other cases this order is 0. We hope for convergence of order 2 (but do not have a proof in the maximum norm); but only the scheme with $\tau = 5$ clearly shows this behavior; see Figure 5.2.

This little surprising fact can be explained with the consistency order near the boundary: in the nonsingularly perturbed case for kth order equations, consistency of order m in the

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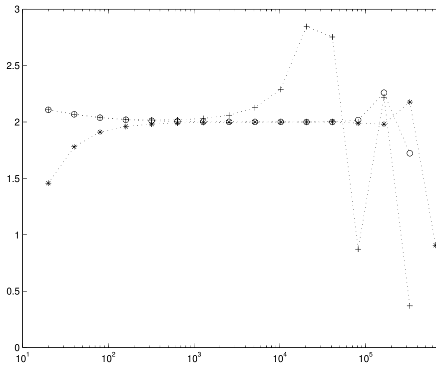

FIG. 5.1. *Order of convergence for the scheme (2.1) with* $\circ \ldots\ \lambda = 1/2$, $* \ldots\ \lambda = 2$, $+ \ldots \lambda = \max\{0, 1/2 - \varepsilon/(bh)\}$.

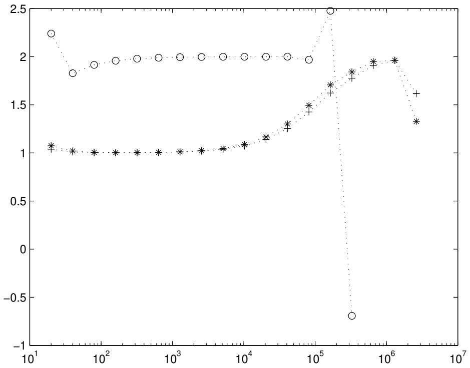

FIG. 5.2. *Order of convergence for the scheme (2.6) with* $\circ \ldots \tau = 5$, $* \ldots \tau = 6$, $+ \ldots \tau = 7$
.

interior and $m - k$ near to the boundary gives convergence of order m; see [2]. But for singularly perturbed problems this property does not hold uniformly with respect to ε.

Remark that in Figures 5.1 and 5.2 we stop the output for some value of N, because for larger N the influence of roundoff error is dominant.

To study the numerical behavior of our schemes for a problem with a layer, we choose $f(x) = \mathrm{e}^{x-1}$ and obtain an exact solution of the structure

$$u(x) = C_1 - \frac{1}{1+\varepsilon}\,\mathrm{e}^{x-1} + C_2\,\mathrm{e}^{-x/\varepsilon}.$$

Because we want to study equidistant meshes, it makes no sense to measure convergence rates. Instead, we observe the error behavior for fixed small ε and decreasing h.

It is well known that for upwind schemes the error at the layer can increase for decreasing h in a certain region depending on ε; see [8, Chapter I, Figure 2.1]. We expect the same principal behavior for our schemes but want to study this effect and its dependence on the parameters λ and γ in the schemes.

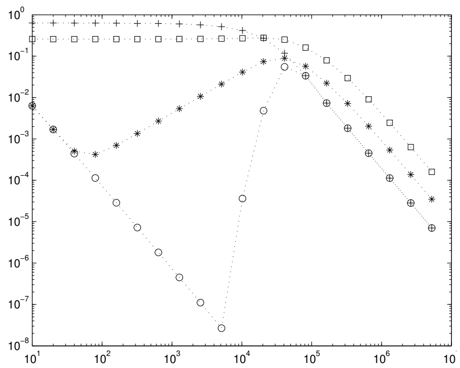

FIG. 5.3. *Error behavior for the four-point scheme (2.1) with* $+ \ldots \lambda = 0$, $* \ldots \lambda = 1/2$, $\square \ldots \lambda = 2$, $\circ \ldots \lambda = \max\{0, 1/2 - \varepsilon/(bh)\}$.

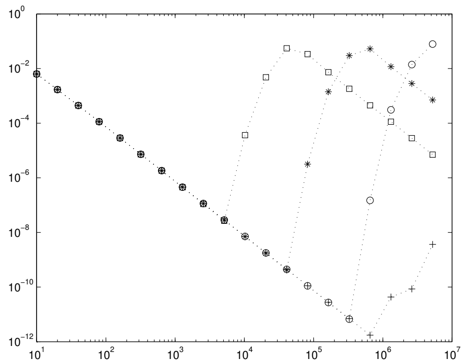

FIG. 5.4. *Error behavior for the four-point scheme (2.1) in dependence on* $\square \ldots \varepsilon = 10^{-5}$, $* \ldots \varepsilon = 10^{-6}$, $\circ \ldots \varepsilon = 10^{-7}$, $+ \ldots \varepsilon = 10^{-8}$.

Figure 5.3 shows the error behavior for the four-point scheme (2.1) for different values of λ.

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It turns out that the choice $\lambda = \max\{0, 1/2 - \varepsilon/(bh)\}$ is the best, theoretically, to explain by the fact that for this choice the scheme does not employ the "outflow" boundary value u_0.

Figure 5.4 shows the results for $\lambda = \max\{0, 1/2 - \varepsilon/(bh)\}$ and different ε.

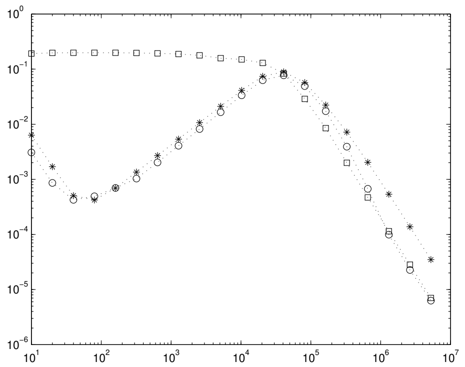

FIG. 5.5. *Error behavior for* $\circ$ *... one-point upwind scheme (1.2),* $*$ *... four-point scheme (2.1) with* $\lambda = 1/2$, $\square$ *... five-point scheme (2.6) with* $\gamma = 1/4$.

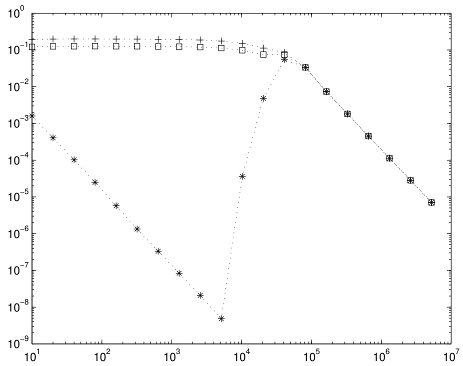

FIG. 5.6. *Error behavior for the five-point scheme (2.6) with* $\gamma_1 = \max\{0, 1/4 - \varepsilon/(2bh)\}$ *and* $+ \ldots \gamma_2 = \gamma_1$, $* \ldots \gamma_2 = 0$, $\square \ldots \gamma_2 = \max\{0, (1/2 - \varepsilon/(bh))/3\}$.

For the five-point scheme (2.6) (now we always use the variant with $\tau = 5$), the choice of the parameter γ is extremely important. If we simply choose some positive value, say $\gamma = 1/4$, the result for $\varepsilon \ll h$ is bad. Figure 5.5 shows a comparison with the four-point scheme (2.1) for $\lambda = 1/2$ and with one-point upwinding (central differencing with upwinding only in the nearest mesh point to the layer).

To eliminate the influence of u_0, it is necessary to choose in the first mesh point $\gamma_1 = \max\{0, 1/4 - \varepsilon/(2bh)\}$, but additionally $\gamma_2 = 0$ in the second mesh point. Figure 5.6 clearly demonstrates that the choice $\gamma_1 = \max\{0, 1/4 - \varepsilon/(2bh)\}$ alone is not sufficient.

If $\gamma_1 = \max\{0, 1/4 - \varepsilon/(2bh)\}$ and $\gamma_2 = 0$, the choice of the stabilization parameter in the remaining mesh points is not important.

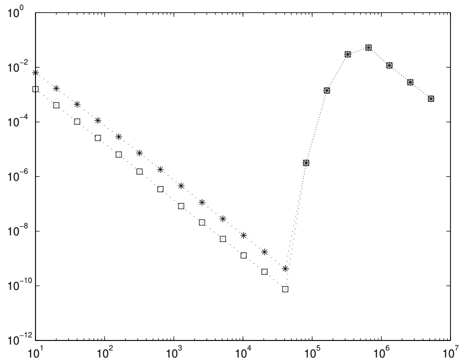

FIG. 5.7. *Error behavior for optimal parameters and* $* \ldots$ *scheme (2.1),* $\square \ldots$ *scheme (2.6).*

Figure 5.7 shows that the four-point scheme and the five-point scheme yield similar results if the parameters are chosen in an optimal way, i.e., for the scheme (2.1) $\lambda = \max\{0, 1/2 - \varepsilon/(bh)\}$ and for the scheme (2.6) $\gamma_1 = \max\{0, 1/4 - \varepsilon/(2bh)\}$ and $\gamma_2 = 0$.

Finally, we still study scheme (3.8). If we do not choose $\gamma_1 = \max\{0, 1/4 - \varepsilon/(2bh)\}$ and $\gamma_2 = 0$, the scheme is similarly bad as other five-point schemes. But for $\gamma = \max\{0,$

$1/4 - \varepsilon/(2bh)\}$ for all odd i, $\gamma = 0$ for $i = 2$, and $\gamma = 1/4$ for all other even i, the scheme is not so good as the schemes (2.1) and (2.6) for optimal parameters, compare Figures 5.7 and 5.8.

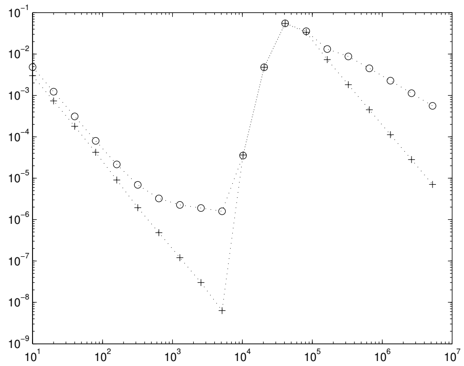

FIG. 5.8. *Error behavior for the scheme (3.8) and* $+ \ldots \gamma = \max\{0, 1/4 - \varepsilon/(2bh)\}$ *for* $i = 1$, $\gamma = 0$ *for all other* i, $\circ \ldots \gamma = \max\{0, 1/4 - \varepsilon/(2bh)\}$ *for all odd* i, $\gamma = 0$ *for* $i = 2$, *and* $\gamma = 1/4$ *for all other even* i.

To summarize, we observed that the schemes (2.1) and (2.6) only beat the one-point upwind scheme if the stabilization parameters are extremely carefully chosen, especially for the five-point scheme.

Of course, in 2D it is much more complicated to tune the parameters in such a way that the outflow boundary values do not influence the numerical solution. Recently, Knobloch [5] studied this question for the SUPG stabilization (which generates simpler stencils than edge stabilization or local projection based on macro elements). Knobloch proved that it is in 2D generally not possible to define the SUPG parameter in such a way that the difference scheme does not employ the outflow boundary values. Additionally, Knobloch proposed a new strategy for defining the SUPG parameter, which approximately reflects the wish to minimize the influence of the outflow boundary values.

For symmetric stabilization methods, however, so far in 2D this is an open question for further research.

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