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ON THE DEFINITION OF THE SUPG PARAMETER*

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Abstract. We consider the SUPG finite element method for two–dimensional steady scalar convection–diffusion equations and discuss a recently introduced definition of the SUPG stabilization parameter along outflow Dirichlet boundaries for problems containing interior layers.

Key words. convection–diffusion equations, singularly perturbed problems, boundary layers, spurious oscillations, streamline upwind/Petrov–Galerkin (SUPG) method, SOLD methods

AMS subject classifications. 65N30

1. Introduction. In many applications, transport processes are the main mechanism determining distributions of the observed physical quantities. Often, the distributions of some of the quantities are not smooth and contain narrow regions where the quantities change abruptly. Depending on the application, one speaks about layers, shocks or discontinuities. When approximating such quantities numerically, the width of the regions where shocks or layers occur is often much smaller than the resolution of the used mesh. Consequently, the shocks or layers cannot be resolved properly, which usually leads to unwanted spurious (non-physical) oscillations in the numerical solution. The attenuation of these oscillations has been the subject of extensive research for several decades during which a huge number of so–called stabilized methods have been developed. The stabilizing effect can be often interpreted as the addition of some artificial diffusion to a standard (unstable) numerical scheme. On the one hand, this artificial diffusion should damp the oscillations but, on the other hand, it should not smear the numerical solution. Therefore, the design of a proper stabilization is a very difficult task.

In the context of finite element methods, a very popular stabilization technique is the streamline upwind/Petrov–Galerkin (SUPG) method. This method was introduced by Brooks and Hughes [1] for advection–diffusion equations and incompressible Navier–Stokes equations. Later this technique has been applied to various other problems, e.g., coupled multidimensional advective–diffusive systems [8], first–order linear hyperbolic systems [12] or first–order hyperbolic systems of conservation laws [9]. Because of its structural simplicity, generality and the quality of numerical solutions, the SUPG method has attracted considerable attention over the last two decades and many theoretical and computational results have been published. It is not the aim of this paper to provide a review of these results and we only refer to the monograph [16].

For simplicity, we shall confine ourselves to a steady scalar convection–diffusion equation

$$- \varepsilon \, \Delta u + \boldsymbol{b} \cdot \nabla u = f \quad \text{in } \Omega \,, \qquad u = u_b \quad \text{on } \partial\Omega \,. \tag{1.1}$$

We assume that Ω is a bounded domain in $\mathbb{R}^2$ with a polygonal boundary $\partial\Omega$, $\varepsilon > 0$ is the constant diffusivity, $\boldsymbol{b}$ is a given convective field, f is an outer source of u, and u_b represents the Dirichlet boundary condition. In the convection–dominated case $\varepsilon \ll |\boldsymbol{b}|$, the solution u

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typically contains interior and boundary layers. These layers can be divided into characteristic (interior and boundary) layers and outflow boundary layers; see [16].

The SUPG method produces accurate and oscillation–free solutions in regions where no abrupt changes in the solution of (1.1) occur but it does not preclude spurious oscillations (overshooting and undershooting) localized in narrow regions along sharp layers. The magnitude of these oscillations strongly depends on the SUPG stabilization parameter. Unfortunately, a general 'optimal' choice of this parameter is not known. Theoretical investigations of model problems only provide asymptotic behaviour of this parameter (with respect to the mesh width) and certain bounds for which the SUPG method is stable and leads to (quasi–) optimal convergence of the discrete solution. However, it has been reported many times that the choice of the stabilization parameter inside these bounds may dramatically influence the accuracy of the discrete solution.

Recently, a new definition of the SUPG stabilization parameter on elements intersecting an outflow Dirichlet boundary was proposed in [13]. In contrast to other approaches, the parameter on a given element depends on the shape and orientation of neighbouring elements and the convection vector $\boldsymbol{b}$ on these elements. Numerical results in [13] show a significant reduction of spurious oscillations in SUPG solutions in comparison to usual choices of the stabilization parameter while accuracy away from layers is preserved. For simple model problems, even nodally exact solutions are obtained.

The aim of this paper is to discuss the application of the new stabilization parameter to problems involving both boundary and interior layers. Since the choice of the stabilization parameter at interior layers has only a limited influence on the spurious oscillations appearing in these regions (see, e.g., [14]), we shall also apply the discontinuity–capturing crosswind–dissipation method [6] as an additional stabilization. We shall demonstrate that the combination of the new definition of the SUPG stabilization parameter and the discontinuity–capturing crosswind–dissipation method provide fairly satisfactory approximations of solutions to (1.1). Furthermore, we shall show how the quality of a SUPG solution can be improved by small modifications of the mesh.

The plan of the paper is as follows. Section 2 formulates the SUPG method and Section 3 describes the discontinuity–capturing crosswind–dissipation method as an example of spurious oscillations at layers diminishing (SOLD) methods. In Section 4 the SUPG stabilization parameter of [13] is briefly introduced. Section 5 compares this definition of the stabilization parameter with an approach by Madden and Stynes. Finally, various numerical results for problems involving interior layers are presented in Sections 6 and 7. The paper is closed by conclusions in Section 8. Throughout the paper, we use the standard notations $P_1(\Omega)$, $L^2(\Omega)$, $H^1(\Omega) = W^{1,2}(\Omega)$, etc., for the usual function spaces; see, e.g., [5]. For a vector $\boldsymbol{a} \in \mathbb{R}^2$, we denote by $|\boldsymbol{a}|$ its Euclidean norm.

2. The SUPG method. Let $\mathcal{T}_h$ be a triangulation of the domain Ω consisting of a finite number of open triangular elements K. Further, we assume that $\overline{\Omega} = \bigcup_{K\in\mathcal{T}_h} \overline{K}$ and that the closures of any two different elements of $\mathcal{T}_h$ are either disjoint or possess either a common vertex or a common edge.

We define the finite element spaces

$$W_h = \{v \in H^1(\Omega)\,;\; v|_K \in P_1(K) \;\; \forall\, K \in \mathcal{T}_h\}\,, \qquad V_h = W_h \cap H_0^1(\Omega)\,.$$

Denoting by $u_{bh} \in W_h$ a function whose trace approximates the boundary condition u_b, the SUPG method for the convection–diffusion equation (1.1) reads:

Find $u_h \in W_h$, such that $u_h - u_{bh} \in V_h$ and

$$\varepsilon\,(\nabla u_h, \nabla v_h) + (\boldsymbol{b}\cdot\nabla u_h, v_h + \tau\,\boldsymbol{b}\cdot\nabla v_h) = (f, v_h + \tau\,\boldsymbol{b}\cdot\nabla v_h) \qquad \forall\, v_h \in V_h\,, \tag{2.1}$$

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where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$ or $L^2(\Omega)^2$ and τ is a nonnegative stabilization parameter.

The choice of τ significantly influences the quality of the discrete solution and therefore it has been a subject of an extensive research over the last three decades; see, e.g., the review in the recent paper [10]. Nevertheless, the definitions of τ mostly rely on heuristic arguments and a general 'optimal' way of choosing τ is still not known. Often, the parameter τ is defined, on any element $K \in \mathcal{T}_h$, by

$$\tau|_K = \frac{h_K}{2\,|\boldsymbol{b}|}\left(\coth Pe_K - \frac{1}{Pe_K}\right), \qquad \text{with} \qquad Pe_K = \frac{|\boldsymbol{b}|\,h_K}{2\,\varepsilon}, \tag{2.2}$$

where h_K is the element diameter in the direction of the convection vector $\boldsymbol{b}$. Various justifications of this formula can be found in [10]. Note that, generally, the parameters h_K, Pe_K and $\tau|_K$ are functions of the points $\mathbf{x} \in K$.

3. SOLD methods. In the convection–dominated regime, the SUPG solutions typically contain oscillations in layer regions. Therefore, various stabilizing terms have been proposed to be added to the SUPG discretization in order to obtain discrete solutions in which the local oscillations are suppressed. In [10, 11], such techniques are called spurious oscillations at layers diminishing (SOLD) methods. Other names are shock–capturing methods or discontinuity–capturing methods.

A review of most SOLD methods published in the literature can be found in [10]. According to the numerical and analytical studies in [10, 11], one of the best SOLD methods is a modification of the discontinuity–capturing crosswind–dissipation method by Codina [6] proposed in [10]. This method adds the term

$$(\widetilde{\varepsilon}\,\boldsymbol{b}^\perp\cdot\nabla u_h, \boldsymbol{b}^\perp\cdot\nabla v_h), \qquad \text{with} \qquad \boldsymbol{b}^\perp = \frac{(-b_2, b_1)}{|\boldsymbol{b}|} \tag{3.1}$$

to the left–hand side of the SUPG discretization (2.1) and hence introduces an additional artificial diffusion in the crosswind direction (in the three–dimensional case, the operator $\boldsymbol{b}^\perp\cdot\nabla$ is replaced by the projection of ∇ into the plane orthogonal to $\boldsymbol{b}$). The parameter $\widetilde{\varepsilon}$ is defined, for any $K \in \mathcal{T}_h$, by

$$\widetilde{\varepsilon}|_K = \max\left\{0, C\,\frac{\operatorname{diam}(K)\,|R_h(u_h)|}{2\,|\nabla u_h|} - \varepsilon\right\}, \tag{3.2}$$

where $R_h(u) = \boldsymbol{b}\cdot\nabla u - f$ is the residual and C is a suitable constant. Codina [6] recommended to set $C \approx 0.7$ for linear finite elements and this value was also used in the computations presented in Sections 6 and 7. If $\nabla u_h = 0$ in (3.2), we set $\widetilde{\varepsilon} = 0$. For $f = 0$ (which will be the case in the examples presented in this paper), $\widetilde{\varepsilon}$ is equal to the parameter proposed by Codina [6]. Note that $\widetilde{\varepsilon}$ depends on the unknown discrete solution u_h and hence the resulting method is nonlinear.

There are also SOLD terms for which the validity of the discrete maximum principle can be proved; see, e.g., [2, 3]. Unfortunately, such methods do not attain the quality of the above mentioned method by Codina since they usually lead to considerable smearing of layers; cf., e.g., [10]. Moreover, it is often very difficult to compute the solution of the nonlinear discrete problem.

Numerical tests in [10] revealed that the SOLD methods significantly improve the quality of a SUPG solution only if the SUPG method adds enough artificial diffusion in the streamline direction. This showed the necessity to reconsider the definition of the SUPG stabilization parameter.

4. SUPG stabilization parameter defined using patches of elements. It was demonstrated in [13] that the information available on a particular element of the triangulation is not sufficient for defining the stabilization parameter τ in an optimal way and that the orientation of the neighbouring elements has to be taken into account. Therefore, a new definition of τ appropriate for elements lying at an outflow Dirichlet boundary was proposed in [13] employing information on patches of elements.

Let us mention that an appropriate definition of τ at outflow Dirichlet boundaries is important also in real–life applications although sometimes it is claimed that outflow boundary layers are mainly encountered in academic problems. Of course, it is true that, in computational fluid dynamics (CFD), outflow boundaries are often artificial boundaries at which no layers occur. However, also in CFD applications, outflow boundary layers may occur when problems with moving boundaries are considered. Moreover, there are many other applications leading to convection–diffusion equations whose solutions possess outflow boundary layers in the sense considered in this paper although the vector $\boldsymbol{b}$ often cannot be interpreted as convection. For example, magnetohydrodynamical pipe flow may lead to the convection–diffusion equation (1.1) with $\boldsymbol{b} = (1,0)$ and $u_b = 0$; cf., e.g., [7]. In this case, Ω is the cross–section of the pipe and the parameter ε is the reciprocal of the Hartmann number so that it can be very small.

Let us introduce the outflow Dirichlet boundary

$$\Gamma = \overline{\{\mathbf{x} \in \partial\Omega\,;\ (\boldsymbol{b}\cdot\boldsymbol{n})(\mathbf{x}) > 0\}}\,,$$

where $\boldsymbol{n}$ is the unit outward normal vector to $\partial\Omega$. For simplicity, we assume that Γ is connected and consists of whole boundary edges of $\mathcal{T}_h$. We set

$$G_h = \text{interior} \bigcup_{K\in\mathcal{G}_h} \overline{K}, \qquad \text{where} \qquad \mathcal{G}_h = \{K \in \mathcal{T}_h\,;\ \overline{K}\cap\Gamma \neq \emptyset\},$$

and denote by $\varphi_1, \dots, \varphi_{M_h}$ all standard basis functions of V_h satisfying

$$\operatorname{supp}\varphi_i \cap G_h \neq \emptyset, \qquad i = 1, \dots, M_h\,.$$

For $i = 1, \dots, M_h$, let $\mathbf{x}_i$ be the vertex associated with the basis function φ_i, i.e., $\varphi_i(\mathbf{x}_i) = 1$ and $\varphi_i(\mathbf{x}) = 0$ for any vertex $\mathbf{x} \neq \mathbf{x}_i$.

The idea of defining τ is to require that

$$\int_{G_h} v_h + \tau\,\boldsymbol{b}\cdot\nabla v_h\,\mathrm{d}\mathbf{x} = 0 \qquad \forall\ v_h \in V_h\,,$$

which can be equivalently written in the form

$$\int_{G_h} \varphi_i + \tau\,\boldsymbol{b}\cdot\nabla\varphi_i\,\mathrm{d}\mathbf{x} = 0, \qquad i = 1, \dots, M_h\,.$$

This suppresses the influence of the Dirichlet boundary condition onto the values of the SUPG solution at interior vertices near Γ. In other words, it increases the upwind character of the method near Γ. The efficiency of this approach also depends on the used triangulation, in particular, on its alignment with the boundary.

To obtain a method which is applicable also for small values of the Péclet number, we set, on any element $K \in \mathcal{G}_h$,

$$\tau|_K = \tau_0|_K \left(\coth Pe_K - \frac{1}{Pe_K}\right), \qquad \text{with} \qquad Pe_K = \frac{|\boldsymbol{b}_K|\,h_K}{2\,\varepsilon}\,, \tag{4.1}$$

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where

$$\boldsymbol{b}_K = \frac{1}{|K|} \int_K \boldsymbol{b} \, \mathrm{d}\mathbf{x}$$

and τ_0 is a piecewise constant function on G_h satisfying

$$\int_{G_h} \varphi_i + \tau_0 \, \boldsymbol{b} \cdot \nabla \varphi_i \, \mathrm{d}\mathbf{x} = 0, \qquad i = 1, \dots, M_h \,. \tag{4.2}$$

On elements $K \in \mathcal{T}_h \setminus \mathcal{G}_h$, we define τ by (2.2) with $\boldsymbol{b}$ replaced by $\boldsymbol{b}_K$.

The relations (4.2) do not determine τ_0 uniquely. However, it was shown in [13] that there always exists τ_0, such that (4.2) holds at least for vertices $\mathbf{x}_i$ which are not contained in elements sharing with Γ only the end points of Γ. A detailed algorithm for computing τ_0 applicable to general triangulations can be found in [13]. Here we mention only the basic idea.

Since it is generally not possible to fulfil (4.2) elementwise, we first determine τ_0 on elements having only one vertex on Γ. Consider any vertex $\mathbf{z} \in \Gamma$ and let $G_{\mathbf{z}}$ be the union of all elements sharing with Γ only the vertex $\mathbf{z}$. If $\mathbf{z}$ is not an end point of Γ, then it is possible to define τ_0 on $G_{\mathbf{z}}$ in such a way that

$$\int_{G_{\mathbf{z}}} \varphi_i + \tau_0 \, \boldsymbol{b} \cdot \nabla \varphi_i \, \mathrm{d}\mathbf{x} = 0$$

at least for all $\mathbf{x}_i$ which are not contained in elements sharing two vertices with Γ. Now it is easy to define τ_0 on elements sharing two vertices with Γ in such a way that (4.2) holds.

5. Comparison with the approach by Madden and Stynes. In some cases the parameter τ defined in the preceding section coincides with the stabilization parameter introduced by Madden and Stynes [14]. In this section, we compare these two choices for the following very simple model problem.

EXAMPLE 5.1. We consider the problem (1.1) with

$$\Omega = (0,1)^2, \qquad \varepsilon = 10^{-8}, \qquad \boldsymbol{b} = (\cos(\pi/3), -\sin(\pi/3)), \qquad f = 0, \tag{5.1}$$

and

$$u_b(x,y) = \begin{cases} 0 & \text{for } x = 1 \text{ or } y = 0, \\ 1 & \text{else.} \end{cases}$$

If we use a triangulation of the type from Figure 5.1(a) with the same mesh width h in both the horizontal and the vertical directions, the SUPG solution with τ defined by (2.2) contains large spurious oscillations along both outflow boundary layers; see [13]. On the other hand, if we define τ as in the preceding section, we obtain a nodally exact solution. This also can be verified by simple theoretical considerations.

Usually, there are many possibilities how to define a piecewise constant function τ_0 satisfying (4.2). Particularly, in the present example, we can use τ_0 which is constant for $x < 1 - 2\,h$ and for $y > 2\,h$. Then $\tau_0 = \frac{1}{2} h / |b_2| = h/\sqrt{3}$ in the former case and $\tau_0 = \frac{1}{2} h / b_1 = h$ in the latter case. These values also can be obtained by the approach of Madden and Stynes [14] who adjusted the SUPG parameter in boundary layer regions in such a way that the artificial diffusion added by the SUPG method in the normal direction to an outflow boundary equals to the optimal value known from the one–dimensional case. Consequently, the approach of Madden and Stynes leads to a discrete solution which is nodally exact except in a small neighbourhood of the corner $(1,0)$.

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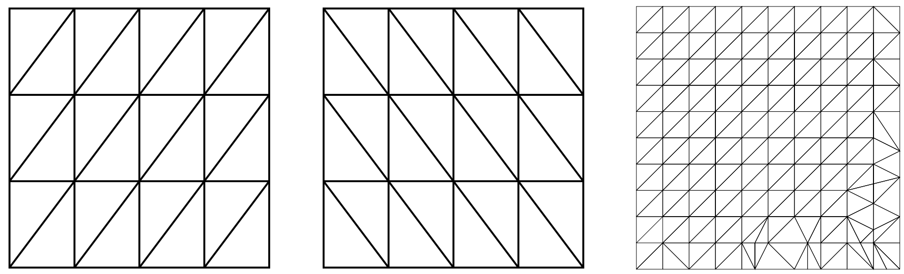
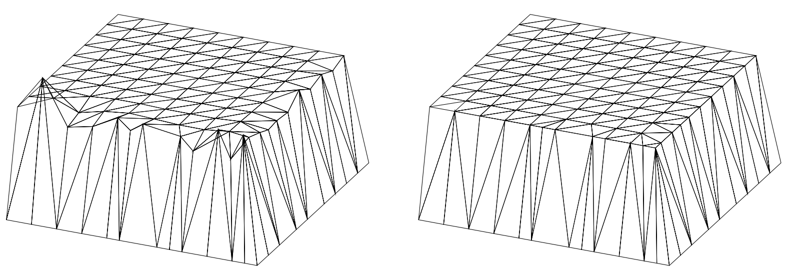


(a) (b) (c)

FIG. 5.1. *Triangulations of the unit square.*

If we use a triangulation which is irregular along the outflow boundary, simple approaches like the one of Madden and Stynes typically do not work properly. As an example, let us consider the triangulation of Figure 5.1(c). Figure 5.2(a) shows that the approach of Madden and Stynes does not give a satisfactory solution, which is due to the fact that the irregular triangulation does not allow to locally reduce the problem to the one–dimensional case. Nevertheless, the solution in Figure 5.2(a) is much better than for τ defined by (2.2). The discrete solution corresponding to τ defined in Section 4 is still nodally exact; see Figure 5.2(b).

A tuning of the SUPG parameter on elements intersecting an outflow boundary was also proposed by do Carmo and Alvarez [4]. However, on uniform triangulations like in Figure 5.1(a), the parameter τ would have the same value on all elements intersecting the outflow boundary, which does not enable the computation of both boundary layers of Example 5.1 sharply.



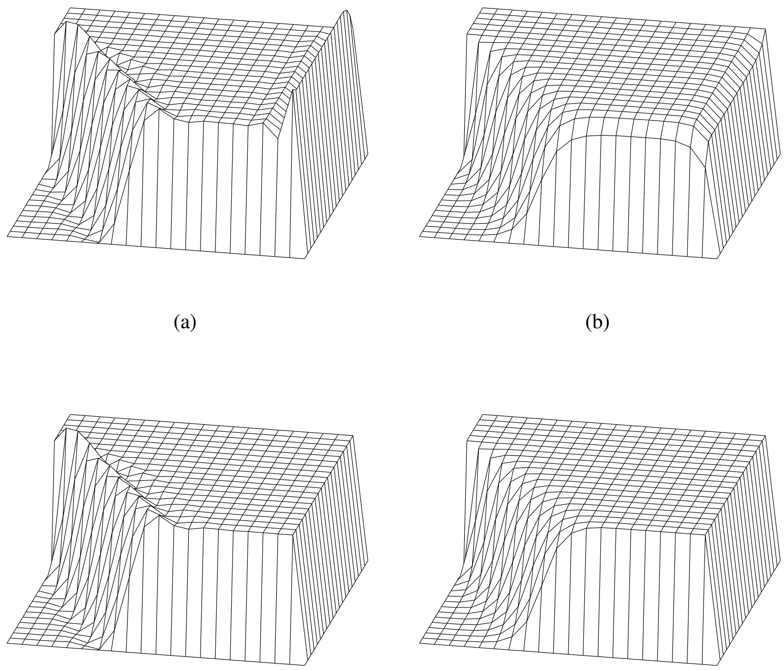
(a) (b)

FIG. 5.2. *Example 5.1, SUPG solutions computed on the triangulation from Figure 5.1(c): (a) τ defined according to Madden and Stynes [14], (b) τ defined by (4.1).*

6. Example with an interior layer originating from a discontinuous boundary condition. In this section, we investigate the problem of Example 5.1 with another discontinuous boundary condition:

EXAMPLE 6.1. We consider the problem (1.1) with (5.1) and

$$u_b(x,y) = \begin{cases} 0 & \text{for } x = 1 \text{ or } y \leq 0.7, \\ 1 & \text{else.} \end{cases}$$



(a) (b)

(c) (d)

FIG. 6.1. *Example 6.1, 21×21 triangulation of the type from Figure 5.1(b): (a) SUPG method with τ from (2.2), (b) SOLD method with τ from (2.2) and SOLD term (3.1), (3.2), (c) SUPG method with τ defined by (4.1), (d) SOLD method with τ defined by (4.1) and SOLD term (3.1), (3.2) applied away from the boundary layers.*

The solution u possesses an interior (characteristic) layer in the direction of the convection starting at $(0, 0.7)$. On the boundary $x = 1$ and on the right-hand part of the boundary $y = 0$, exponential layers are developed.

To discretize this example, we use a triangulation of Ω of the type shown in Figure 5.1(b) containing 21×21 vertices. If we solve Example 6.1 using the SUPG method with the stabilization parameter τ defined by (2.2), we obtain a solution with spurious oscillations along the interior layer and along the boundary layer at $x = 1$; see Figure 6.1(a). One possibility to suppress these oscillations is to apply a SOLD method. If we add the SOLD term (3.1) with $\widetilde{\varepsilon}$ defined by (3.2) to the SUPG discretization just applied, we obtain the solution depicted in Figure 6.1(b). This solution is oscillation–free, however, the boundary layers are smeared. It is possible to adjust the constant C in (3.2) in such a way that this smearing is avoided; cf. [11]. But, in general, the appropriate value of C is not known. On the other hand, if we apply the SUPG method with τ defined in Section 4, the discrete solution possesses sharp oscillation–free boundary layers; see Figure 6.1(c). Of course, along the interior layer, the solution is the same as in Figure 6.1(a) since we use the same values of τ in this region. The oscillations along the interior layer can be suppressed by using the additional SOLD term (3.1), (3.2). However, since we now know that the parameter τ from Section 4 suppresses oscillations along boundary layers, it suffices to add the SOLD term only on elements which do not intersect the outflow boundary. Then we obtain the oscillation–free solution depicted in Figure 6.1(d) with sharp boundary layers and an acceptable smearing of

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the interior layer.

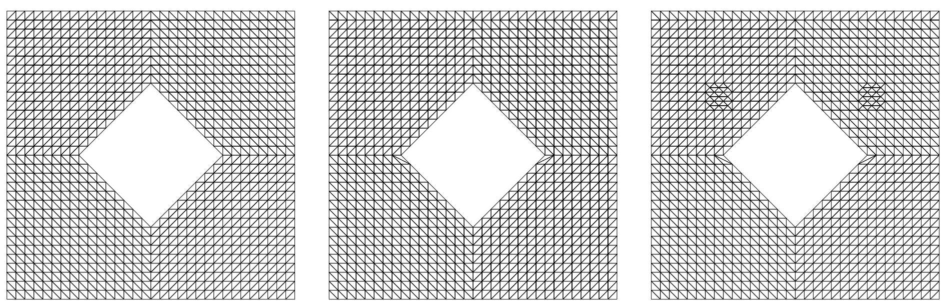
For more general problems and triangulations, we can not guarantee that the SUPG method with τ from Section 4 completely removes oscillations at boundary layers. Nevertheless, numerical results in [13] show that the oscillations are significantly suppressed. Therefore, the SOLD term would be applied also in the boundary layer region but with a much smaller parameter C than in the interior of the computational domain.

Let us mention that it cannot be generally expected that the oscillations along interior (or more generally characteristic) layers will be significantly suppressed by an appropriate choice of the stabilization parameter τ. Indeed, characteristic layers follow the streamlines and the SUPG method contains no mechanism for stabilization in the direction perpendicular to streamlines where spurious oscillations occur. Therefore, an oscillation–free SUPG approximation of a characteristic layer can be obtained only by introducing an additional crosswind diffusion like above or by using a layer–adapted mesh; see, e.g., [15].

7. Examples with interior layers behind an obstacle. In this section, we shall consider the computational domain

$$\Omega = \{(x,y) \in (-1,1)^2\,;\ |x| + |y| > \tfrac{1}{2}\}\,.$$

Three structured triangulations of Ω which will be discussed in this section are depicted in Figure 7.1. The square hole in Ω can be viewed as an obstacle inside the computational domain. We shall start with the following setting.



(a) (b) (c)

FIG. 7.1. *Structured triangulations of the domain Ω from Section 7.*

EXAMPLE 7.1. We consider the problem (1.1) with the above domain Ω and $\varepsilon = 10^{-8}$, $\boldsymbol{b} = (1,2)$, $f = 0$, and

$$u_b(x,y) = \begin{cases} 1 & \text{for } |x| + |y| = \frac{1}{2}, \\ 0 & \text{else.} \end{cases}$$

In view of the boundary conditions, the obstacle inside the flow field gives rise to two interior layers. Moreover, there is a boundary layer at the front part of the obstacle (with respect to the flow) and a boundary layer at a part of the boundary of $(-1,1)^2$ behind the obstacle.

We shall first consider the triangulation depicted in Figure 7.1(a). If we compute an approximation of the solution to Example 7.1 using the SUPG method with τ defined by (2.2), we obtain a solution with spurious oscillations at all four layers; see Figure 7.2(a). An application of the SOLD method (3.1), (3.2) is now not able to suppress the oscillations at $y = 1$

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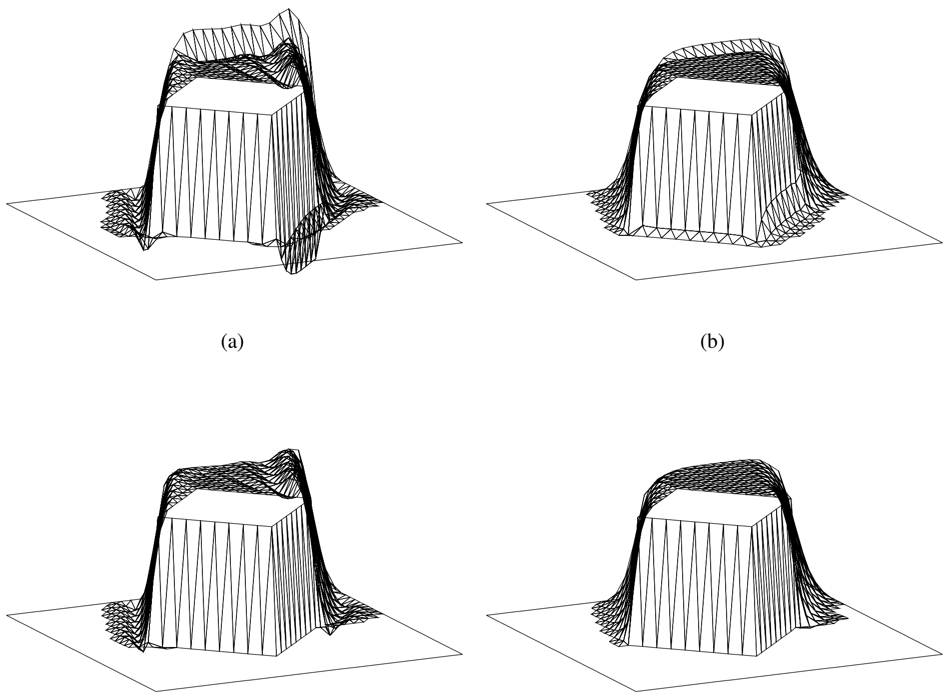


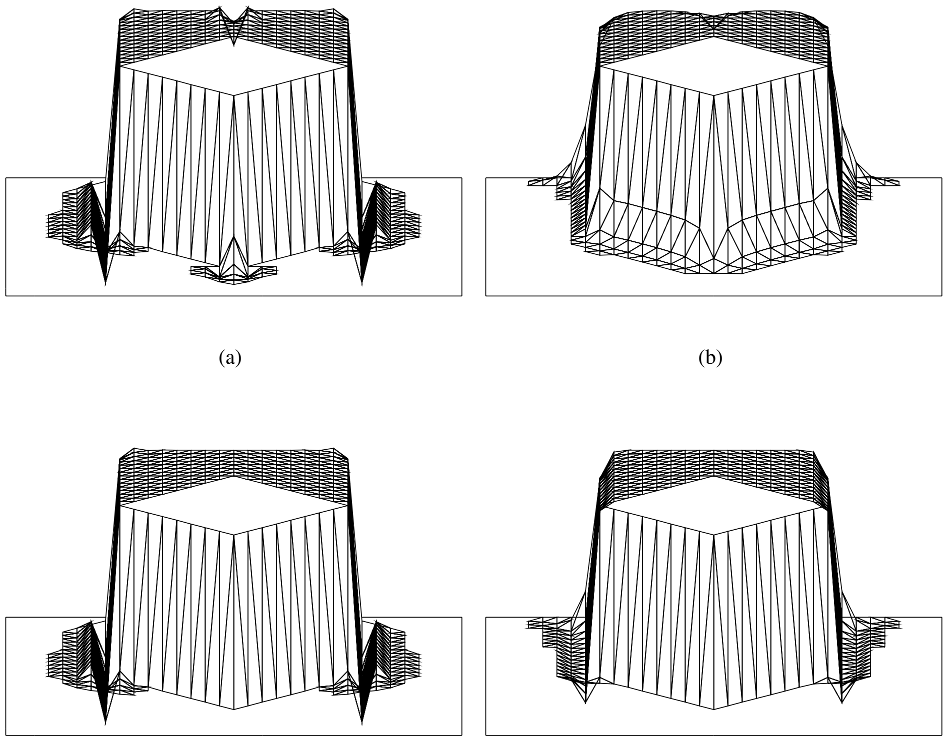
(a) (b)

(c) (d)

FIG. 7.2. *Example 7.1, triangulation from Figure 7.1(a): (a) SUPG method with τ from (2.2), (b) SOLD method with τ from (2.2) and SOLD term (3.1), (3.2), (c) SUPG method with τ defined by (4.1), (d) SOLD method with τ defined by (4.1) and SOLD term (3.1), (3.2).*

sufficiently; see Figure 7.2(b). At the remaining three layers the oscillations are removed. On the other hand, if we apply the SUPG method with τ defined in Section 4, we obtain sharp approximations of both boundary layers without any oscillations; see Figure 7.2(c). An addition of the SOLD term (3.1), (3.2) removes to a large extent also the oscillations at the interior layers; see Figure 7.2(d). We observe that both boundary layers are approximated significantly better than in case of the solutions from Figures 7.2(a) and 7.2(b).

The results in Figure 7.2 demonstrate that it is essential to define the parameter τ (and also the mesh as we shall see in the following) in such a way that the spurious oscillations in the SUPG solution are as small as possible. Otherwise the addition of a SOLD term cannot be expected to lead to an oscillation–free solution (unless we use a very diffusive method, which typically leads to an excessive smearing of the layers). This is true for all the SOLD methods reviewed and investigated in [10, 11]. Note also that it is generally not possible to remove spurious oscillations at outflow Dirichlet boundaries by simply increasing the parameter τ since the oscillations are influenced not only by the magnitude of τ but also by the relation between values of τ on neighbouring elements. Moreover, such simple approaches are usually not able to suppress spurious oscillations without smearing the layers. Therefore, more complicated definitions of τ, such as the one described in Section 4, seem to be unavoidable.



(a) (b)

(c) (d)

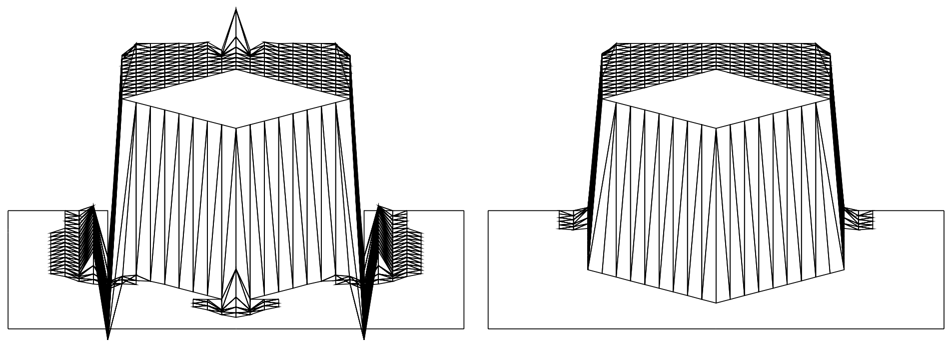
FIG. 7.3. *Example 7.2, triangulation from Figure 7.1(a): (a) SUPG method with τ from (2.2), (b) SOLD method with τ from (2.2) and SOLD term (3.1), (3.2), (c) SUPG method with τ defined by (4.1), (d) SOLD method with τ defined by (4.1) and SOLD term (3.1), (3.2).*

Now we investigate the following simpler situation.

EXAMPLE 7.2. We consider the problem (1.1) with the same data as in Example 7.1 except for $\boldsymbol{b}$ which is defined by $\boldsymbol{b} = (0, 1)$.

In this case the convection and hence also the interior layers are aligned with the mesh and we may expect better properties of discrete solutions. Indeed, the SUPG solution for τ defined by (2.2) (see Figure 7.3(a)) approximates the boundary layers much better and all oscillations can be removed by introducing the SOLD term considered above (see Figure 7.3(b)). The SUPG solution with τ defined by (4.1) now differs from the SUPG solution with τ defined by (2.2) only by better approximations in the middle of boundary layers; see Figure 7.3(c). However, after adding the SOLD term, the difference between the two choices of τ is much larger; cf. Figures 7.3(b) and 7.3(d). For τ defined by (4.1), the approximation of boundary layers is much better but, at the same time, the suppression of oscillations at the beginning of the interior layers is worse. This is probably connected with the fact that the definition of τ from Section 4 tries to assure that the piecewise linear interpolate of u is (at least locally) the solution of the SUPG discretization, which is not allowed by the used triangulation.

Let us now look closer at the oscillations in the SUPG solutions along the interior layers.



(a) (b)

FIG. 7.4. *Example 7.2, triangulation from Figure 7.1(b), SUPG method: (a) τ defined by (2.2), (b) τ defined by (4.1).*

We denote by $\mathbf{x}_\pm = (\pm\frac{1}{2}, 0)$ the two vertices of the obstacle where the interior layers begin and by Γ^o the part of the boundary of the obstacle consisting of points with a nonpositive vertical coordinate. Then Γ^o represents an outflow Dirichlet boundary and $\mathbf{x}_\pm$ are the end points of Γ^o. Furthermore, we denote by G_h^o the set G_h corresponding to Γ^o; see Section 4. A careful inspection of elements near the vertices $\mathbf{x}_\pm$ shows that the interpolant of u cannot be the SUPG solution for any choice of τ. Moreover, the conditions (4.2) cannot be satisfied for some vertices $\mathbf{x}_i$ connected by an edge with $\mathbf{x}_+$ or $\mathbf{x}_-$. To improve the quality of the SUPG solution in a neighbourhood of $\mathbf{x}_+$ (say), we proceed in the following way. First, we denote by $\mathbf{x}_+^i$ the neighbouring vertex of $\mathbf{x}_+$ lying on Γ^o and by $\mathbf{x}_+^e$ the remaining vertex of the element possessing the vertices $\mathbf{x}_+$ and $\mathbf{x}_+^i$. Then we go through the vertices on $\partial G_h^o \setminus \Gamma^o$ in the order in which they are connected by edges, starting with the vertex connected with $\mathbf{x}_+$ by an edge lying on ∂G_h^o, and we find the first vertex $\overline{\mathbf{x}}$ for which the open triangle with the vertices $\mathbf{x}_+$, $\mathbf{x}_+^i$, $\overline{\mathbf{x}}$ lies in Ω and satisfies the required minimal angle condition. If $\overline{\mathbf{x}} \neq \mathbf{x}_+^e$, we change the triangulation in such a way that we delete the edges connecting the vertex $\mathbf{x}_+$ with the vertex $\mathbf{x}_+^e$ and the vertices on ∂G_h^o between $\mathbf{x}_+^e$ and $\overline{\mathbf{x}}$ and we introduce new edges which connect the vertex $\mathbf{x}_+^i$ with the vertex $\overline{\mathbf{x}}$ and the vertices on ∂G_h^o between $\mathbf{x}_+^e$ and $\overline{\mathbf{x}}$. The elements containing any of the vertices lying on ∂G_h^o between $\mathbf{x}_+$ and $\overline{\mathbf{x}}$ are removed from the definition of the set G_h^o. Analogously, we proceed for the vertex $\mathbf{x}_-$. Then, using the algorithm from [13], we can compute a piecewise constant function τ_0 on G_h^o, such that the requirement (4.2) is satisfied. In case of the triangulation from Figure 7.1(a), the described changes of the triangulation concern two elements at each of the vertices $\mathbf{x}_+$ and $\mathbf{x}_-$; see the modified triangulation in Figure 7.1(b). Note that such modifications of the triangulation can be performed a priori in the framework of a computer code.

Further improvements of the SUPG solution can be achieved a posteriori at places where the computer code detects that an interior layer meets a boundary layer. In the present case this happens at the boundary $y = 1$. Here it is desirable to change the direction of the 'diagonal' edges. For simplicity, we made this change along the whole boundary $y = 1$ although it would be sufficient only in a neighbourhood of the interior layer; see again Figure 7.1(b).

On the modified triangulation shown in Figure 7.1(b), the solution of the SUPG method with τ defined by (4.1) is almost nodally exact; see Figure 7.4(b). The only discrepancies appear in the neighbourhood of points where the interior layers meet the boundary $y = 1$. In

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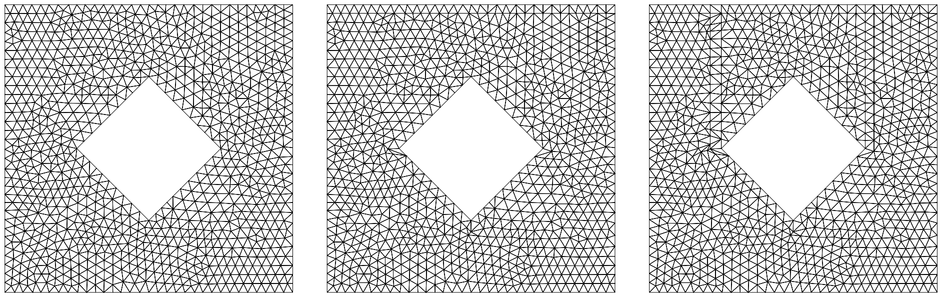

(a) (b) (c)

FIG. 7.5. *Unstructured triangulations of the domain Ω from Section 7.*

fact, the definition of τ could be modified in such a way that the discrete solution is nodally exact also in these regions, however, such modifications cannot be easily performed in an automatic way in the framework of a computer code. Let us also mention that the SUPG solution for τ defined by (2.2) is worse on the triangulation from Figure 7.1(b) than on the triangulation from Figure 7.1(a); see Figure 7.4(a). Moreover, the SOLD method (3.1), (3.2) is not able to remove the overshoot in the neighbourhood of the point $(0, 1)$.

It should be emphasized that a SUPG solution like in Figure 7.4(b) can be obtained only for special triangulations. As soon as the interior layers will cross elements of the triangulation, like in case of the triangulation in Figure 7.1(c), spurious oscillations will appear and the application of a SOLD method will be necessary.

Finally, let us discuss the application of the techniques treated in this paper on unstructured meshes. We shall consider the triangulation depicted in Figure 7.5(a) which contains approximately the same number of elements as the structured triangulation in Figure 7.1(a). Figures 7.6(a) and 7.6(b) show the SUPG solutions of Example 7.2 for τ defined by (2.2) and (4.1), respectively, computed on this unstructured triangulation and we observe that both solutions contain unacceptable spurious oscillations. In case of τ defined by (4.1), the oscillations are partially caused by the fact that the unstructured triangulation does not satisfy the assumptions used in [13] for deriving the conditions (4.2). More precisely, the triangulation should be constructed in such a way that the part of the boundary of the set G_h lying in Ω copies the outflow boundary Γ. This requirement can be easily satisfied by shifting some of the vertices of the triangulation shown in Figure 7.5(a). In addition, we modify the triangulation in the neighbourhoods of the vertices $\mathbf{x}_\pm$ as described above, which leads to the triangulation depicted in Figure 7.5(b). The corresponding SUPG solution with τ defined by (4.1) approximates very well the boundary layers but possesses still spurious oscillations along the interior layers as we can observe in Figure 7.6(c). These oscillations can be removed by aligning edges of the triangulation with the interior layers; see Figures 7.5(c) and 7.6(d). The quality of the triangulation in Figure 7.5(c) could be improved but our aim was only to show that simple shifting of vertices of the triangulation leads to an almost perfect SUPG solution. Let us mention that, for τ defined by (2.2), the magnitude of spurious oscillations in the SUPG solution even increases if the triangulation from Figure 7.5(a) is replaced by the triangulations from Figures 7.5(b) or 7.5(c).

The above results show that the construction or adaptation of the triangulation is very important for the quality of the discrete solution. Although small deviations from an optimal mesh alignment do not lead to a dramatic deterioration of the discrete solution, it is difficult

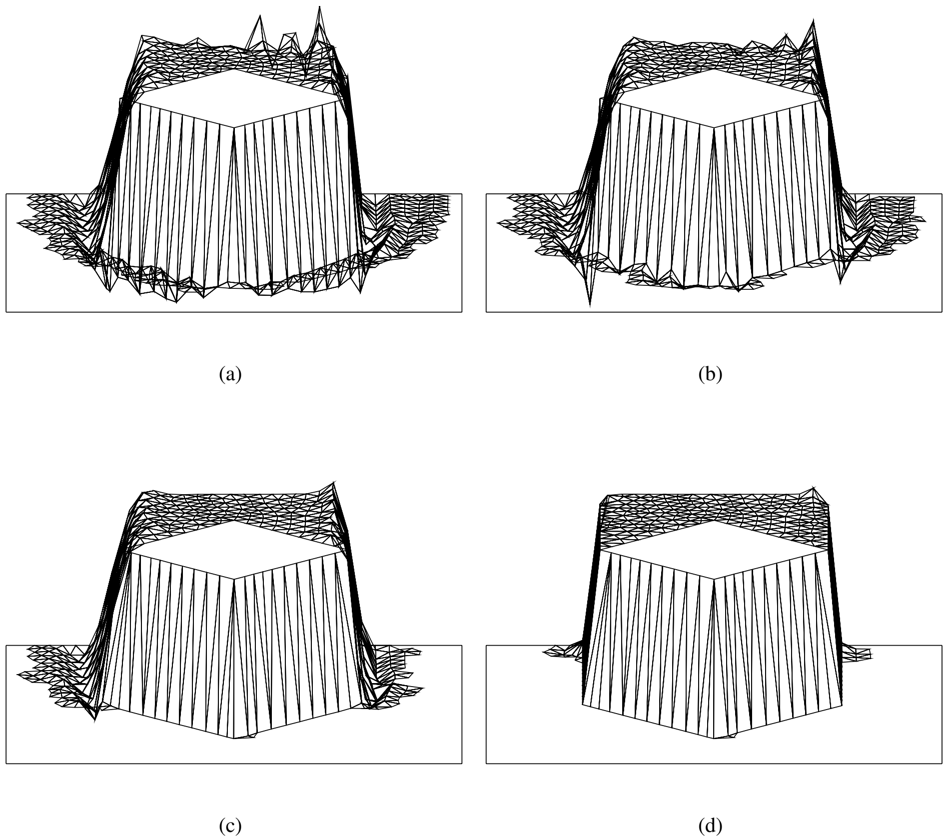

FIG. 7.6. *Example 7.2, SUPG method: (a) τ defined by (2.2), triangulation from Figure 7.5(a), (b) τ defined by (4.1), triangulation from Figure 7.5(a), (c) τ defined by (4.1), triangulation from Figure 7.5(b), (d) τ defined by (4.1), triangulation from Figure 7.5(c).*

to quantify the sensitivity of the discrete solution to the mesh since spurious oscillations are significantly influenced by mutual orientation of neighbouring elements of the triangulation.

Finally, let us mention that it is completely open to what extent the presented techniques can be extended to the three–dimensional case.

8. Conclusions. In this paper, we discussed properties of the SUPG finite element method applied to the numerical solution of two–dimensional steady scalar convection–diffusion equations. We demonstrated that the choice of the SUPG stabilization parameter proposed in [13] together with an application of the discontinuity–capturing crosswind–dissipation method [6] leads to satisfactory discrete solutions in the convection–dominated case. Further numerical results show that the quality of the SUPG solution can be significantly improved if an appropriate mesh is used.

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