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A NOTE ON SYMPLECTIC BLOCK REFLECTORS*

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Dedicated to Gérard Meurant on the occasion of his 60th birthday

Abstract. A symplectic block reflector is introduced. The parallel with the Euclidean block reflector is studied. Some important features of symplectic block reflectors are given. Algorithms to compute a symplectic block reflector that introduces a desired block of zeros into a matrix are developed.

Key words. Skew-symmetric scalar product, symplectic Householder transformation, block algorithm, symplectic QR-factorization

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1. Introduction.

Let $V \in \mathbb{R}^{n\times r}$ with $n \geq r$ be a nonzero matrix. A block reflector is a matrix of the form

$$P = I - 2V(V^TV)^\dagger V^T, \tag{1.1}$$

where $(V^TV)^\dagger$ is the pseudo-inverse of V^TV. It is easy to see that P is orthogonal and symmetric and satisfies $PV = -V$, and $Px = x$ for all vectors x orthogonal to range(V). The matrix P is a natural block version of the standard Householder reflectors [4, 5], i.e., matrices of the form $I - 2(vv^T)/(v^Tv)$, where v is a nonzero vector.

Block reflectors are used to introduce blocks of zeros in selected parts of a rectangular matrix and provide, in particular, a stable and efficient block QR-factorization of a given matrix. As with many block algorithms in linear algebra, an advantage of our block extension is the possibility of intensive use of matrix-matrix operations, which are efficient with regard to memory management and parallelism. Theoretical and algorithmic aspects of block reflectors are well developed in [11]; see also [8] for an application. In practice, it is more convenient and always possible to find a block reflector of the form $P = I - 2VV^T$ with $V^TV = I$; see [11].

Symplectic block reflectors are intended to play analogous role as P when the Euclidean scalar product is replaced by an indefinite scalar product induced by a skew-symmetric and nonsingular matrix. This entails changes in the notion of Euclidean orthogonality and symmetry on which the construction of P is based. The point of this note is to clarify these changes and to study the extent to which symplectic block reflectors analogous to (1.1) can be constructed.

Throughout this note, the skew-symmetric and nonsingular matrix mentioned above is given by

$$J_{2n} = \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix}, \tag{1.2}$$

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where 0_n and I_n are the zero and identity matrices of order n, respectively. Note that $J^T = J^{-1} = -J$. The induced skew-symmetric scalar product is defined by

$$(x, y)_J = x^J y, \quad \forall x, y \in \mathbb{R}^{2n}, \tag{1.3}$$

where $x^J = x^T J_{2n}$ is the adjoint of x with respect to $(\cdot,\cdot)_J$. The adjoint of a matrix $A \in \mathbb{R}^{2n\times 2m}$ with respect to $(\cdot,\cdot)_J$ is defined by

$$A^J = J_{2m}^T A^T J_{2n} \in \mathbb{R}^{2m\times 2n}.$$

It is the unique matrix that satisfies

$$(Ax, y)_J = \left(x, A^J y\right)_J, \quad \forall x \in \mathbb{R}^{2m}, \forall y \in \mathbb{R}^{2n}.$$

It is easy to see that:

- If $A, B \in \mathbb{R}^{2n\times 2m}$, then $(A+B)^J = A^J + B^J$, $\left(A^J\right)^T = \left(A^T\right)^J$, $\left(A^J\right)^J = A$, and $\left(A^J\right)^\dagger = \left(A^\dagger\right)^J$.
- If $A \in \mathbb{R}^{2n\times 2k}$, $B \in \mathbb{R}^{2k\times 2m}$, $x \in \mathbb{R}^{2k}$, then $(AB)^J = B^J A^J$ and $(Ax)^J = x^J A^J$.

A matrix $A \in \mathbb{R}^{2n\times 2m}$ such that $A^J A = I_{2m}$ is said to be symplectic. A matrix $A \in \mathbb{R}^{2n\times 2n}$ such that $A^J = A$ is said to be skew-Hamiltonian.

2. Symplectic block reflector. A natural generalization of (1.1) to the symplectic case would be

$$H = I - 2V(V^J V)^\dagger V^J, \tag{2.1}$$

where $V \in \mathbb{R}^{2n\times 2r}$, $n \geq r$. It also can be viewed as a block generalization of the symplectic Householder transformations given in [10] and [6]. It is clear that H remains unchanged when V is replaced by VK, where $K \in \mathbb{R}^{2r\times 2r}$ is an arbitrary nonsingular matrix.

The matrix H is skew-Hamiltonian and symplectic, $H^J = H = H^{-1}$. It satisfies $Hx = x$ if x is J-orthogonal to range(V), i.e., $(x, v)_J = 0\ \forall v \in$ range(V). However, it may be that $HV \neq -V$, since in general $V(V^J V)^\dagger V^J V \neq V$, in contrast with $V(V^T V)^\dagger V^T V = V$. In the case $HV \neq -V$, H is no longer a reflector. When the matrix $V^J V$ is nonsingular, then $H = I - 2V(V^J V)^{-1} V^J$ and H satisfies $HV = -V$. In this case, it will be shown that H can be represented in the following interesting form: $H = I - 2WW^J$, where W is some matrix satisfying $W^J W = I$. Furthermore, necessary and sufficient conditions will be given. This form is then used to derive a block version of the SR algorithm [3]. Note that the matrix $V^J V$ may be singular even if V has full rank. These constraints will necessarily result in some difficulties in the construction of H.

Let $E, F \in \mathbb{R}^{2n\times 2r}$. Assume that a symplectic block reflector (SBR) H of the form (2.1) exists and satisfies $HE = \pm F$. Then $(HE)^J HE = F^J F$ and $E^J HE = \pm E^J F$. Hence,

$$E^J E = F^J F \ \text{ and } \ E^J F \ \text{ is skew-Hamiltonian.} \tag{2.2}$$

Unfortunately, the conditions (2.2) alone do not guarantee the existence of an SBR of the form (2.1) such that $HE = \pm F$.

EXAMPLE 2.1. Consider the symplectic linear space $\mathbb{R}^8$ and let $E = [e_1, e_2, e_3, e_7]$ and $F = [\frac{1}{2}e_1 + \frac{1}{4}e_2, \frac{1}{3}e_1 + \frac{2}{5}e_2, -e_3, -e_7]$, where $\{e_1, \ldots, e_8\}$ denotes the canonical basis of $\mathbb{R}^8$. We obtain

$$E^J E = F^J F = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \ \text{and} \ E^J F = F^J E = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, the conditions (2.2) are satisfied.

Suppose now that there exists $V \in \mathbb{R}^{8\times 4}$ of full rank, such that $HE = F$, where H is the SBR given by $H = I - 2V(V^JV)^\dagger V^J$. Since rank$(E-F) = 4$, it follows that $V = (E-F)K$, where K is any 4×4 nonsingular real matrix. There is no loss of generality to take $K = I$ and $H = I - 2(E-F)((E-F)^J(E-F))^\dagger(E-F)^J$. A simple calculation gives

$$HE = [e_1, e_2, -e_3, -e_4] \neq F.$$

Note that the matrix $V = E - F$ satisfies

$$\begin{aligned} V(V^JV)^\dagger V^JV &= [0, 0, 2e_3, 2e_7] \\ &\neq V = \left[\tfrac{1}{2}e_1 - \tfrac{1}{4}e_2, -\tfrac{1}{3}e_1 + \tfrac{3}{5}e_2, 2e_3, 2e_7\right]. \end{aligned}$$

Some necessary and/or sufficient conditions that ensure the existence of H are given in the following propositions.

PROPOSITION 2.2. *Let $Z \in \mathbb{R}^{2n\times 2r}$ and $\mathcal{Z} = range(Z)$ be such that*

$$\mathcal{Z} \cap \mathcal{Z}^{\perp_J} = \{0\}, \tag{2.3}$$

where $\perp_J$ denotes orthogonality with respect to $(.,.)_J$. Then

$$Z(Z^JZ)^\dagger Z^JZ = Z.$$

Moreover, if Z is full rank, then Z^JZ is nonsingular.

Proof. The proof follows from the fact that $I - (Z^JZ)^\dagger(Z^JZ)$ is the orthogonal projection onto Null(Z^JZ), see, e.g., [12, Chap. III], and that, due to (2.3), Null$(Z^JZ) \subset$ Null(Z). Note that the inclusion Null$(Z) \subset$ Null(Z^JZ) is always true. Therefore Null$(Z^JZ) =$ Null(Z), and hence $Z(I - (Z^JZ)^\dagger(Z^JZ)) = 0$. Let $x \in \mathbb{R}^{2r}$ be such that $Z^JZx = 0$. From (2.3), it follows that $Zx = 0$; and since Z is full rank, one gets $x = 0$. □

REMARK 2.3. It is interesting to note that (2.3) implies that

$$\mathbb{R}^{2n} = \mathcal{Z} \oplus \mathcal{Z}^{\perp_J}. \tag{2.4}$$

In fact, any $x \in \mathbb{R}^{2n}$ can be expressed as

$$x = Z(Z^JZ)^\dagger Z^Jx + (I - Z(Z^JZ)^\dagger Z^J)x.$$

The first term of the sum belongs to $\mathcal{Z}$. From (2.3), we have $Z(Z^JZ)^\dagger Z^JZ = Z$, and hence $Z^J(I - Z(Z^JZ)^\dagger Z^J) = 0$. Thus, the second term of the sum is in $\mathcal{Z}^{\perp_J}$.

COROLLARY 2.4. *Let $H = I - 2Z(Z^JZ)^\dagger Z^J$, with $Z \in \mathbb{R}^{2n\times 2r}$, $\mathcal{Z} = range(Z)$, $2r' = dim(\mathcal{Z})$ and $\mathcal{Z} \cap \mathcal{Z}^{\perp_J} = \{0\}$. Then there exists $W \in \mathbb{R}^{2n\times 2r'}$, such that $W^JW = I$ and $H = I - 2WW^J$.*

Proof. Using Proposition 2.2 and Remark 2.3, H is completely determined by the following properties: $Hx = x$ for $x \in \mathcal{Z}^{\perp_J}$ and $Hx = -x$ for $x \in \mathcal{Z}$. Due to (2.3), the induced inner product on the subspace $\mathcal{Z}$ is not degenerate, and hence there exists a symplectic matrix, see e.g., [1, 10], $W \in \mathbb{R}^{2n\times 2r'}$ (i.e., $W^JW = I$), such that $\mathcal{Z} = range(W)$. It follows that the matrix $H' = I - 2WW^J$ satisfies $H'x = x$ for $x \in \mathcal{Z}^{\perp_J}$ and $H'x = -x$ for $x \in \mathcal{Z}$. Thus $H = H'$. □

PROPOSITION 2.5. *Let $E, F \in \mathbb{R}^{2n\times 2r}$ satisfy the conditions (2.2).*

- *If $\mathcal{Z} = range(D)$ satisfies (2.3) with $D = E - F$, then the SBR $H = I - 2D(D^JD)^\dagger D^J$ is such that $HE = F$.*

- *If $\mathcal{Z} = range(S)$ satisfies (2.3) with $S = E+F$, then the SBR $H = I-2S(S^J S)^\dagger S^J$ is such that $HE = -F$.*

Proof. Since E and F satisfy (2.2) and $\mathcal{Z}$ satisfies (2.3), it follows that $D^J S = 0$, $D(D^J D)^\dagger D^J D = D$ and $S(S^J S)^\dagger S^J S = S$. □

PROPOSITION 2.6. *Let $E, F \in \mathbb{R}^{2n\times 2r}$ and let H be an SBR of the form (2.1). Let $D = E - F$ and $S = E + F$. Then the conditions*

$$V(V^J V)^\dagger V^J D = D \;\text{ and }\; (V^J V)^\dagger V^J S = 0 \tag{2.5}$$

are necessary and sufficient for HE to equal F.

Proof. Under the conditions (2.5), we have

$$\begin{aligned}2HE &= (I - 2V(V^J V)^\dagger V^J)(S + D)\\ &= (S + D) - 2D = S - D = 2F.\end{aligned}$$

Conversely, if $HE = F$, then $E = HF$ and therefore $HS = S$ and $HD = -D$, which gives the conditions (2.5). □

PROPOSITION 2.7. *Let $E, F \in \mathbb{R}^{2n\times 2r}$ satisfy the conditions (2.2). If $(E-F)^J(E-F)$ is nonsingular, then*

$$H = I - 2(E - F)\left((E - F)^J(E - F)\right)^{-1}(E - F)^J \tag{2.6}$$

is the unique SBR of the form (2.1) such that $HE = F$.

Proof. The conditions on E and F ensure that

$$H(E + F) = E + F \text{ and } H(E - F) = F - E.$$

Hence, $HE = F$. Now let $V \in \mathbb{R}^{2n\times r}$ and let $\widetilde{H} = I - 2V\left(V^J V\right)^\dagger V^J$ be an SBR, such that $HE = F$. Then

$$E - F = 2V\left(V^J V\right)^\dagger V^J E$$

and

$$(E - F)^J(E - F) = 2E^J(E - F) = 4(V^J E)^J\left(V^J V\right)^\dagger V^J E.$$

We conclude that $V^J E$ and $V^J V$ are nonsingular. Let $K = 2\left(V^J V\right)^\dagger V^J E$. Then K is nonsingular and $E - F = VK$. Therefore, $H = \widetilde{H}$. □

2.1. The standard task. Given $E = \left[\begin{smallmatrix} E_1^T & E_2^T & E_3^T & E_4^T\end{smallmatrix}\right]^T \in \mathbb{R}^{2n\times 2r}$ with $E_1, E_3 \in \mathbb{R}^{k\times 2r}$ and $E_2, E_4 \in \mathbb{R}^{(n-k)\times 2r}$, the standard task consists of finding an SBR H, such that HE is of the form $F = \left[\begin{smallmatrix} F_1^T & 0 & F_3^T & 0\end{smallmatrix}\right]^T$ with $F_1, F_3 \in \mathbb{R}^{k\times 2r}$.

We present two approaches that accomplish this task. The first one requires an SBR of the form (2.1) and only can be applied under certain conditions. The second approach uses the product of two SBRs of the form (2.1) and always can be applied.

2.1.1. A limited approach. Using the symplectic Gram-Schmidt algorithm [9], we may decompose E as

$$E = SR, \tag{2.7}$$

where $S \in \mathbb{R}^{2n\times 2q}$, $q \le r$, is symplectic and $R \in \mathbb{R}^{2q\times 2r}$ is block upper triangular.

Note that it is suffices to determine H, such that $HS = Q$, where Q has the same structure as F, since we will have $HE = QR$, and QR has the same structure as F.

The properties of H yield that $S^J S = Q^J Q = I$ and that $S^J Q$ is skew-Hamiltonian. If we partition S and Q conformally with E and F, i.e., $S = \left[\, S_1^T \; S_2^T \; S_3^T \; S_4^T \,\right]^T \in \mathbb{R}^{2n\times 2q}$ with $S_1, S_3 \in \mathbb{R}^{r\times 2q}$, $S_2, S_4 \in \mathbb{R}^{(n-r)\times 2q}$ and $Q = \left[\, Q_1^T \; 0 \; Q_3^T \; 0 \,\right]^T$ with $Q_1, Q_3 \in \mathbb{R}^{r\times 2r}$, then the conditions on S and Q become:

$$\begin{bmatrix} S_1 \\ S_3 \end{bmatrix}^J \begin{bmatrix} S_1 \\ S_3 \end{bmatrix} + \begin{bmatrix} S_2 \\ S_4 \end{bmatrix}^J \begin{bmatrix} S_2 \\ S_4 \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_3 \end{bmatrix}^J \begin{bmatrix} Q_1 \\ Q_3 \end{bmatrix} = I \tag{2.8}$$

and

$$\begin{bmatrix} S_1 \\ S_3 \end{bmatrix}^J \begin{bmatrix} Q_1 \\ Q_3 \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_3 \end{bmatrix}^J \begin{bmatrix} S_1 \\ S_3 \end{bmatrix}. \tag{2.9}$$

Assume that $\left[\begin{smallmatrix} S_1 \\ S_3 \end{smallmatrix}\right]^J \left[\begin{smallmatrix} S_1 \\ S_3 \end{smallmatrix}\right]$ has no eigenvalues on $\mathbb{R}^- = \{x \in \mathbb{R} : \;\; x \leq 0\}$. Then $\left[\begin{smallmatrix} S_1 \\ S_3 \end{smallmatrix}\right]$ has a generalized polar decomposition:

$$\begin{bmatrix} S_1 \\ S_3 \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_3 \end{bmatrix} T, \tag{2.10}$$

where $\left[\begin{smallmatrix} Q_1 \\ Q_3 \end{smallmatrix}\right]$ is symplectic and T is skew-Hamiltonian with eigenvalues that have positive real part [7]. Since the matrix $I+T$ is skew-Hamiltonian, it admits a Cholesky-like decomposition of the form:

$$I + T = L^J L; \tag{2.11}$$

see [2]. Moreover, since $I + T$ is nonsingular, so is the matrix L.

Let $V = \frac{1}{\sqrt{2}}(S + Q)L^{-1}$. Then it is easy to check that $V^J V = I$, $V^J S = \frac{1}{\sqrt{2}} L$. Therefore, the SBR $H = I - 2VV^J$ is of the form (2.1) and satisfies

$$HE = F \;\text{ with }\; F = \begin{array}{c} 2r \\ \begin{bmatrix} -Q_1 R \\ 0 \\ -Q_3 R \\ 0 \end{bmatrix} \end{array} \begin{array}{c} \\ r \\ n-r \\ r \\ n-r. \end{array}$$

2.1.2. A more general approach. First, using for example Algorithm 2 in [11], we find $G \in \mathbb{R}^{n\times p}$ with orthonormal columns and a block reflector $P = I - 2GG^T$, such that

$$P \begin{bmatrix} E_3 \\ E_4 \end{bmatrix} = \begin{bmatrix} E_3' \\ 0 \end{bmatrix}, \text{ with } E_3' \in \mathbb{R}^{2r\times 2r}. \tag{2.12}$$

Let

$$H_1 = \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix}. \tag{2.13}$$

The matrix H_1 is an SBR of the form (2.1). In fact, we have

$$H_1 = I - 2V_1 V_1^J \;\text{ with }\; V_1 = \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix} \in \mathbb{R}^{2n\times 2p} \text{ and } V_1^J V_1 = I. \tag{2.14}$$

Then, using the symplectic Gram-Schmidt algorithm, we may decompose H_1E as

$$H_1E = SR, \tag{2.15}$$

where $S \in \mathbb{R}^{2n\times 2q}$, $q \le r$ is symplectic and $R \in \mathbb{R}^{2q\times 2r}$ is block upper triangular. Due to (2.12), the matrix S is of the form $S = \begin{bmatrix} S_1^T & S_2^T & S_3^T & 0 \end{bmatrix}^T$ with $S_1, S_3 \in \mathbb{R}^{2r\times 2q}$ and $S_2 \in \mathbb{R}^{(n-2r)\times 2q}$. Furthermore, the symplecticity of S reduces to that of $\begin{bmatrix} S_1^T & S_3^T \end{bmatrix}^T$.

Let

$$V_2 = \begin{bmatrix} S_1^T & \frac{1}{2}S_2^T & S_3^T & 0 \end{bmatrix}^T \in \mathbb{R}^{2n\times 2q}.$$

Then it is easy to verify that

$$V_2^J V_2 = V_2^J S = \begin{bmatrix} S_1 \\ S_3 \end{bmatrix}^J \begin{bmatrix} S_1 \\ S_3 \end{bmatrix} = I.$$

Therefore the SBR $H_2 = I - 2V_2V_2^J$ is of the form (2.1) and satisfies $H_2S = \begin{bmatrix} -S_1^T & 0 & -S_3^T & 0 \end{bmatrix}^T$. Finally, the product $H = H_2H_1$ is an SBR, such that

$$HE = F \;\; \text{with} \;\; F = \begin{array}{c} 2r \\ \begin{bmatrix} -S_1R \\ 0 \\ -S_3R \\ 0 \end{bmatrix} \end{array} \begin{array}{c} \\ 2r \\ n-2r \\ 2r \\ n-2r. \end{array}$$

Note that H is not necessarily of the form (2.1). However, it can be written as

$$H = I - 2V\Sigma V^J, \tag{2.16}$$

where

$$V = [V_1, V_2] \in \mathbb{R}^{2n\times 2(p+q)} \;\; \text{and} \;\; \Sigma = \begin{bmatrix} 0 & -J_{2p} \\ J_{2q} & 2V_2^J V_1 J_{2p} \end{bmatrix} \in \mathbb{R}^{2(p+q)\times 2(p+q)}.$$

3. Application: block symplectic QR-factorization. The technique presented in Section 2.1.2 can be used to partition a matrix into block upper triangular form. It can be considered a block variant of the SR decomposition [3]. We illustrate the reduction where, for clarity, the matrix A is partitioned as follows

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix},$$

where $A_{1i}, A_{3i} \in \mathbb{R}^{2r\times r}$, $A_{2i}, A_{4i} \in \mathbb{R}^{(n-2r)\times r}$, and $1 \le i \le 4$. In the first step of the reduction, we let

$$A_1 = \begin{bmatrix} A_{11} & A_{13} \\ A_{21} & A_{23} \\ A_{31} & A_{33} \\ A_{41} & A_{43} \end{bmatrix}$$

and generate two SBRs, $H_1^{(1)} = I_{2n} - 2V_1^{(1)}(V_1^{(1)})^J$ and $H_2^{(1)} = I_{2n} - 2V_2^{(1)}(V_2^{(1)})^J$, such that

$$H_2^{(1)} H_1^{(1)} A_1 = \begin{bmatrix} F_{11} & F_{13} \\ 0 & 0 \\ F_{31} & F_{33} \\ 0 & 0 \end{bmatrix} \quad \text{with} \;\; F_{ij} \in \mathbb{R}^{2r\times r} \; i,j = 1,4.$$

The matrix $H_2^{(1)} H_1^{(1)} A$ is partitioned as

$$H_2^{(1)} H_1^{(1)} A = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ 0 & F_{22} & 0 & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ 0 & F_{42} & 0 & F_{44} \end{bmatrix} \quad \text{with} \;\; F_{2i}, F_{4i} \in \mathbb{R}^{(n-2r)\times r}, \; i = 2,4.$$

In the second step, the same process is applied to the submatrix

$$\tilde{A}_2 = \begin{bmatrix} F_{22} & F_{24} \\ F_{42} & F_{44} \end{bmatrix}$$

to generate $\tilde{H}_1^{(2)} = I_{2(n-2r)} - 2\tilde{V}_1^{(2)}(\tilde{V}_1^{(2)})^J$ and $\tilde{H}_2^{(2)} = I_{2(n-2r)} - 2\tilde{V}_2^{(2)}(\tilde{V}_2^{(2)})^J$, such that

$$H_2^{(2)} H_1^{(2)} \tilde{A}_2 = \begin{bmatrix} \tilde{F}_{22} & \tilde{F}_{24} \\ 0 & 0 \\ \tilde{F}_{42} & \tilde{F}_{44} \\ 0 & 0 \end{bmatrix}, \;\; \text{where} \;\; \tilde{F}_{ij} \in \mathbb{R}^{2r\times r}, \; i,j = 2,4.$$

Let

$$V_1^{(2)} = \begin{bmatrix} 0_{2r\times 2p} \\ \tilde{V}_1^{(2)}(1{:}n-2r, 1{:}2p) \\ 0_{2r\times 2p} \\ \tilde{V}_1^{(2)}(n-2r+1{:}2(n-2r), 1{:}2p) \end{bmatrix}, \;\; V_2^{(2)} = \begin{bmatrix} 0_{2r\times 2q} \\ \tilde{V}_2^{(2)}(1{:}n-2r, 1{:}2q) \\ 0_{2r\times 2q} \\ \tilde{V}_2^{(2)}(n-2r+1{:}2(n-2r), 1{:}2q) \end{bmatrix},$$

and

$$H_1^{(2)} = I_{2n} - V_1^{(2)}(V_1^{(2)})^J, \;\; H_2^{(2)} = I_{2n} - V_2^{(2)}(V_2^{(2)})^J.$$

Then the matrix $H = H_2^{(2)} H_1^{(2)} H_2^{(1)} H_1^{(1)}$ is symplectic and satisfies

$$HA = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ 0 & \begin{bmatrix} \tilde{F}_{22} \\ 0 \end{bmatrix} & 0 & \begin{bmatrix} \tilde{F}_{24} \\ 0 \end{bmatrix} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ 0 & \begin{bmatrix} \tilde{F}_{42} \\ 0 \end{bmatrix} & 0 & \begin{bmatrix} \tilde{F}_{44} \\ 0 \end{bmatrix} \end{bmatrix}.$$

4. Conclusion. The purpose of this short note is to discuss the construction of block symplectic reflectors analogous to those developed in [11], where the Euclidean scalar product is replaced by a skew-symmetric scalar product. This change introduces difficulties in the construction. We investigated necessary and/or sufficient conditions for the existence and uniqueness of such reflectors. We discussed algorithms for computing a symplectic block reflector that introduces a block of zeros into a matrix and showed how to obtain a symplectic block QR factorization.

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