Electronic Transactions on Numerical Analysis.
Volume 34, pp. 152-162, 2009.

ISSN 1068-9613.

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MIMETIC SCHEMES ON NON-UNIFORM STRUCTURED MESHES*

E. D. BATISTA† AND J. E. CASTILLO†

Dedicated to Víctor Pereyra on the occasion of his 70th birthday

Abstract. Mimetic operators are approximations that satisfy discrete versions of continuum conservation laws. We propose a technique for constructing mimetic divergence and gradient operators over non-uniform structured meshes based on the application of local transformations and the use of a reference set of cells (RSC). The RSC is not a mesh, but a set of two uniform elements that are used while the operators are being built. The method has been applied to construct second and fourth order gradient and divergence operators over non-uniform 1D meshes. Our approach leaves invariant the boundary operator expressions for uniform and non-uniform meshes, which is a new result and an advantage of our formulation. Finally, a numerical convergence analysis is presented by solving a boundary layer like problem with Robin boundary conditions; this shows that we can obtain the highest order of accuracy when implementing adapted meshes.

Key words. mimetic schemes, summation-by-part operators, non-uniform meshes, partial differential equations, high order, divergence operator, gradient operator, boundary operator.

AMS subject classifications. 65D25, 65M06, 65G99.

1. Introduction.

Mimetic operators or summation-by-part operators (as they are sometimes called) are finite-difference-like approximations that replicate symmetry properties of the continuum operators and could be thought as an effort to construct discrete analogs of vector and tensor calculus. While having a formulation and computational implementation with a complexity like that of standard finite difference schemes, mimetic schemes tend to produce more physically reliable results, because they satisfy discrete versions of continuum conservation laws. Another advantage of mimetic operators is that they simplify the procedure of obtaining energy estimates in computational fluid dynamics and computational aero-acoustics problems [13, 16].

A special class of mimetic schemes was recently presented [1] where the authors develop a way of constructing high order gradient and divergence approximations with mimetic properties for one dimensional problems on uniform grids. These operators will be called Castillo-Grone operators to differentiate them from other mimetic schemes, such as the one studied in [9]. The main attributes of Castillo-Grone operators are that they preserve symmetry properties of the continuum, they have an overall high order accuracy, and no numerical artifacts such as ghost points or extended grids are used in their formulation.

This article focuses on generalizing the Castillo-Grone operators to non-uniform structured meshes and it is organized as follows: Section 2 shows some of the Castillo-Grone operator matrices and presents basic concepts related to the construction of the new operators. In Section 3, we explain the method proposed for expanding high order mimetic divergence and gradient operators to non-uniform meshes, the construction of weight matrices for defining generalized inner products, and the construction of the boundary operator matrix. In the next section we show second and fourth order non-uniform mimetic operators obtained from these ideas and prove that the boundary operator for non-uniform meshes is the same one obtained for the uniform case. Also, the implementation of these operators is tested by solving a boundary-layer-like problem where the numerical convergence rate analysis shows that, when using adapted meshes, they have the same order of accuracy as the corresponding

* Received April 2, 2008. Accepted February 9, 2009. Published online on October 22, 2009. Recommended by Godela Scherer.

†Computational Science Research Center, San Diego State University, San Diego, CA 92182-1245, USA (dbatista@sciences.sdsu.edu, castillo@myth.sdsu.edu).

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operators on uniform meshes. Finally, Section 5 presents the conclusions derived from this work and some of the extensions that will be done in the future.

2. Preliminaries.

2.1. Mimetic schemes. Mimetic methods [1, 6, 15] for solving partial differential equations are based on discretizing the continuum theory underlying the problem in such a way that the scheme obtained tends to replicate much of the behavior found in the continuum problem. Also, as shown in [1, 3, 7], we can build mimetic operators with the same order of approximation at the boundary as in the interior of the domain, achieving the same accuracy for the solution in the whole domain, as is the case for Castillo-Grone operators [1].

Discretizing the continuum theory involves the construction of discrete mathematical analogs for gradient and divergence operators in such a way that they satisfy conservation laws like the general Stokes' theorem (also known as Gauss' theorem) or Green's identity. A generalized discrete version of this law is

$$\langle Dv, f\rangle_Q + \langle v, Gf\rangle_P = \langle Bv, f\rangle_I. \tag{2.1}$$

In this expression D, G, and B stand for the discrete divergence operator, the discrete gradient operator, and the boundary operator, respectively. The brackets represent generalized inner products with weights Q, P, and I.

In [1], the authors describe a method for constructing high-order mimetic operators on uniform 1D grids that satisfy (2.1) and have the same order of accuracy everywhere. Among other properties these operators are centro-skew-symmetric; they satisfy the row sum condition, meaning that the sum of every row is equal to zero; they also satisfy the column sum condition, given by: $\langle e, PGf\rangle = f_n - f_1$, where $e, f \in \mathbb{R}^n$, and $e = \begin{bmatrix}1 & 1 & \dots & 1\end{bmatrix}^T \in \mathbb{R}^{n+1}$. As an example, expressions (2.2) and (2.3) below show Castillo-Grone second order operators, and (2.4) and (2.5) below show fourth order operators.

$$D = \frac{1}{h}\begin{bmatrix} -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \end{bmatrix} \tag{2.2}$$

$$G = \frac{1}{h}\begin{bmatrix} \frac{-8}{3} & 3 & \frac{-1}{3} & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \end{bmatrix} \tag{2.3}$$

$$D = \frac{1}{h}\begin{bmatrix} \frac{-4751}{5192} & \frac{909}{1298} & \frac{6091}{15576} & \frac{-1165}{5192} & \frac{129}{2596} & \frac{-25}{15576} & 0 & \cdots & 0 \\ \frac{1}{24} & \frac{-9}{8} & \frac{9}{8} & \frac{-1}{24} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{24} & \frac{-9}{8} & \frac{9}{8} & \frac{-1}{24} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{24} & \frac{-9}{8} & \frac{9}{8} & \frac{-1}{24} & 0 & \cdots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & & \vdots \end{bmatrix} \tag{2.4}$$

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$$G = \frac{1}{h}\begin{bmatrix} \frac{-1152}{407} & \frac{10063}{3256} & \frac{2483}{9768} & \frac{-3309}{3256} & \frac{2099}{3256} & \frac{-697}{4884} & 0 & \cdots & 0 \\ 0 & \frac{-11}{12} & \frac{17}{14} & \frac{3}{8} & \frac{-5}{24} & \frac{1}{24} & 0 & \cdots & 0 \\ 0 & \frac{1}{24} & \frac{-9}{8} & \frac{9}{8} & \frac{-1}{24} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{24} & \frac{-9}{8} & \frac{9}{8} & \frac{-1}{24} & 0 & \cdots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & & \vdots \end{bmatrix}. \tag{2.5}$$

These operators have a factor $1/h$ that depends on the spacing of the uniform mesh and a factor that depends on the order of accuracy, the matrix at the right. This matrix will be called the fixed part of the operator, since its basic structure does not depend on the discretization.

In addition, there is an important fact associated with these operators: the treatment of boundary conditions does not involve ghost points or any other numerical artifacts, as sometimes happens with standard finite difference methods. Instead, a boundary operator B is used.

Here we present a technique that uses these operators to build mimetic schemes on non-uniform structured meshes while preserving their valuable attributes. Other attempts have been made seeking this goal, as in [12], where the authors construct mimetic operators for non-uniform staggered meshes from a matrix-analysis-based method. They explain the method in a general way and present the explicit construction of second order operators. However, the gradient and boundary operators obtained in [12] are more complex than the operators obtained with our technique.

We propose a finite-element-like technique for constructing the operators, based on the implementation of local transformations of the mesh cells. We also present numerical examples of the implementation of second and fourth order mimetic operators.

2.2. Staggered mesh. Consider a discretization of the domain by a geometric mesh U. In Figure 4.1, the geometric mesh is given by the points x_i with $i = 0, 1, \ldots, n$, and the cells of this mesh are the intervals $[x_{i-1}, x_i]$ with $i = 1, 2, \ldots, n$. Define the divergence operator at the centers of these cells, and call these points D-points. Define the gradient operator at the edges of the cells in U, x_i with $i = 0, 1, \ldots, n$, and call these points G-points. The set of all G-points and D-points will be called a *staggered mesh*. Staggered meshes are widely used, particularly when solving problems related to fluid dynamics.

2.3. The Reference Set of Cells (RSC). The RSC, as its name suggests, is a set of uniform reference elements where the actual approximations are carried out. This set has two uniform objects: one called CD, used for the estimation of the divergence operator and another one, called CG, used for the estimation of the gradient operator.

The RSC is independent of the number of points of the mesh, though its elements could change depending on the order of the approximation desired, on the type of elements employed for discretizing the physical domain, and on the dimension of the physical problem. However, once these aspects are established, the RSC can be constructed and it will remain unchanged during the process of constructing the mimetic operators.

3. Mimetic operators on non-uniform structured meshes.

3.1. Idea proposed. *Develop local transformations of the cells by using a reference set of cells in order to obtain local, mimetic operators G and D.*

Notice that this is not the same as using traditional curvilinear coordinates for the following reasons.

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(i) We *locally* transform the cells instead of transforming the entire mesh at once.
(ii) We use a reference set of cells instead of a reference mesh.
(iii) The idea can be implemented either for structured or unstructured meshes, at least in the case of second order operators. This would not be possible using a reference mesh.

It is worth mentioning that in [16], the authors proved that it is not possible to construct a coordinate transformation operator X_ξ such that PX_ξ can be used to define a norm and keep the order of accuracy, p, for $p \geq 3$. In this case, P defines a norm for the mimetic operators or summation-by-part operators discussed in [10, 11]. P is a diagonal matrix except at the upper left and lower right corners. However, as we shall show, the use of the operators in [1] along with local transformations does solve this problem, allowing us to construct high-order mimetic operators on non-uniform meshes. These new operators maintain the same accuracy as their uniform counterpart when adapted meshes are used.

3.2. Operators D and G. To calculate the divergence, we transform each cell, one at a time, by using the element CD of the RSC. In, CD we can implement the approximations presented in [1] for uniform grids. Then, we go back with the transformation to obtain the divergence at the D-points of the non-uniform staggered mesh.

To calculate the gradient, we take groups of juxtaposed cells and transform them by using the element CG of the RSC. As we did for the divergence, in CG we can calculate the gradient by using the approximations presented in [1] and then reverse the transformation. Note the difference between calculating the gradient and the divergence operator: for the former, we have to take into account groups of juxtaposed cells because we consider the gradient defined at the edges of the geometric mesh.

3.3. Weights P and Q. As shown in the previous section, we would like our operators to satisfy a discrete conservation law of the form

$$\langle \widehat{D}v, f\rangle_Q + \langle v, Gf\rangle_P = \langle Bv, f\rangle_I, \tag{3.1}$$

which ensures that our approximation is fully conservative and mimics the physical properties of the problem. Here, the matrix $\widehat{D}$ is the extended divergence operator – the matrix D with rows of zeros added, one row per boundary node. This is because we seek to approximate the solution of a given PDE at the D-points as well as at the boundary points. Therefore, the number of null rows added and the places where they are incorporated into the matrix D will depend on the numbering of boundary nodes that we have and the numbering of the nodes, respectively.

For the construction of the weight matrix P, we use the conservation law (3.1), setting $v = e_v = \begin{bmatrix}1 & 1 & \dots & 1\end{bmatrix}^T \in \mathbb{R}^{n+1}$; we get

$$\begin{aligned} \langle \widehat{D}e_v, f\rangle_Q + \langle e_v, Gf\rangle_P &= \langle Be_v, f\rangle_I, \\ \langle e_v, Gf\rangle_P &= \langle Be_v, f\rangle_I. \end{aligned} \tag{3.2}$$

This is true because the divergence of a constant function ($\widehat{D}e_v$) is equal to zero.

At this point, we do not know what the boundary operator for non-uniform meshes is, but we note that the left hand side of (3.2) is a weighted discrete version of the integral

$$\int_\Omega \operatorname{grad} f \, d\Omega.$$

Then, by using the fundamental theorem of calculus we get that

$$\langle e_v, Gf\rangle_P = f_n - f_1, \tag{3.3}$$

and solve it for P.

For the construction of the weight matrix Q, we again use (3.1) setting $f = e_f = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}^T \in \mathbb{R}^{n+2}$; we get

$$\begin{aligned} \langle \widehat{D}v, e_f \rangle_Q + \langle v, Ge_f \rangle_P &= \langle Bv, e_f \rangle_I, \\ \langle \widehat{D}v, e_f \rangle_Q &= \langle Bv, e_f \rangle_I, \end{aligned} \tag{3.4}$$

which holds because $Ge_f = 0$. The right hand side of (3.4) is a weighted discrete version of the flux across the boundary

$$\int_{\partial\Omega = S} \vec{v} \cdot \hat{n} \, dS.$$

Assuming we have a stationary problem, we obtain the conservation law

$$\langle \widehat{D}v, e_f \rangle_Q = 0, \tag{3.5}$$

for the calculation of Q.

3.4. Boundary operator B. From Green's identity (3.1) we obtain a relationship between the boundary operator and the matrices G, $\widehat{D}$, P, and Q that allows us to construct the operator B, i.e.,

$$B = (\widehat{D}^T Q + PG)^T. \tag{3.6}$$

4. Numerical implementation and results.

4.1. Construction of the operators. With the following 1D example we show how to construct mimetic operators over non-uniform meshes using the idea proposed in Section 3.

We use the following notation to indicate what is going to be calculated at each location of the staggered mesh: $\times$ for the calculation of the gradient G, $\bigtriangledown$ for the calculion of the divergence D, and $\bullet$ for the calculation of the scalar function f (see Figure 4.1). Here, f represents the scalar function in the given 1D, non-uniform staggered mesh, $\tilde{f}$ represents the scalar function in the RSC, and the cell $[x_{i-1}, x_i]$ is referred to as having cell number $i - \frac{1}{2}$.

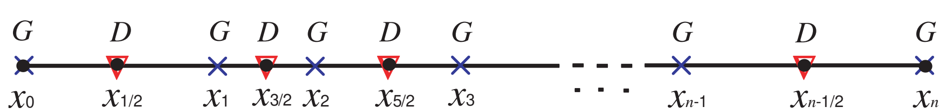

FIGURE 4.1. *1D, non-uniform staggered mesh.*

Since we are going to calculate second order, one dimensional operators, the element CD will be a single segment and the element CG of the RSC will be two, equally large, juxtaposed segments. We have taken CD$= [0, 1]$ and CG$= [-1, 0] \cup [0, 1]$.

To calculate the divergence at $x_{i+1/2}$, $i = 0, 1, \dots, n-1$, we first transform the cell $i + \frac{1}{2}$ to CD according to the following,

$$x = \begin{cases} 2(x_{i+1/2} - x_i)\xi + x_i, & \text{for} \quad 0 \le \xi < \frac{1}{2}, \\ (x_{i+1} - x_i)\xi + x_i, & \text{for} \quad \frac{1}{2} \le \xi \le 1. \end{cases}$$

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Now let us consider $\tilde{v}(\xi) = v(x(\xi))$, so that $\tilde{v}_\xi = v_x x_\xi$ and $v_x = \tilde{v}_\xi / x_\xi$. Then the divergence at $x_{i+1/2}$ is given by

$$\begin{aligned}(Dv)(x_{1/2}) = v_x(x_{i+1/2}) = \left.\frac{\tilde{v}_\xi}{x_\xi}\right|_{\xi=1/2} &= \frac{\tilde{v}(1) - \tilde{v}(0)}{x(1) - x(0)} \\ &= \frac{v(x(1)) - v(x(0))}{x(1) - x(0)} \\ &= (v(x_{i+1}) - v(x_i))\,\frac{1}{J_{x_{i+1/2}}},\end{aligned}$$

where $J_{x_{i+1/2}} = x_{i+1} - x_i$.

Our divergence operator for non-uniform, 1D meshes will look like

$$Dv = \begin{bmatrix} \frac{-1}{J_{x_{1/2}}} & \frac{1}{J_{x_{1/2}}} & 0 & 0 & \cdots & 0 \\ 0 & \frac{-1}{J_{x_{3/2}}} & \frac{1}{J_{x_{3/2}}} & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & \cdots & 0 & \frac{-1}{J_{x_{n-3/2}}} & \frac{1}{J_{x_{n-3/2}}} & 0 \\ 0 & \cdots & 0 & 0 & \frac{-1}{J_{x_{n-1/2}}} & \frac{1}{J_{x_{n-1/2}}} \end{bmatrix} \begin{bmatrix} v_{1/2} \\ v_{3/2} \\ \vdots \\ v_{n-3/2} \\ v_{n-1/2} \end{bmatrix}.$$

If we rewrite the matrix D as a product of two matrices,

$$D = \begin{bmatrix} \frac{1}{J_{x_{1/2}}} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{J_{x_{3/2}}} & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & \frac{1}{J_{x_{n-3/2}}} & 0 \\ 0 & \cdots & 0 & 0 & \frac{1}{J_{x_{n-1/2}}} \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & \cdots & 0 & -1 & 1 & 0 \\ 0 & \cdots & 0 & 0 & -1 & 1 \end{bmatrix}, \tag{4.1}$$

we can see that the resulting divergence operator for non-uniform meshes is a diagonal matrix with the inverse of the Jacobians of the local transformations (which we call J_D) times the fixed part of the divergence operator for uniform meshes presented in [1].

To calculate the gradient at the interior point x_i, $i = 1, 2, \ldots, n-1$, we first transform cells $i - 1/2$ and $i + 1/2$ to CG according to the following,

$$x = \begin{cases} (x_i - x_{i-1})\xi + x_i, & \text{for} \quad -1 \le \xi < 0, \\ (x_{i+1} - x_i)\xi + x_i, & \text{for} \quad 0 \le \xi \le 1. \end{cases}$$

Now let us consider $\tilde{f}(\xi) = f(x(\xi))$, so that $\tilde{f}_\xi = f_x x_\xi$ and $f_x = \frac{\tilde{f}_\xi}{x_\xi}$. Then the gradient at x_i is given by

$$\begin{aligned}(Gf)(x_i) = f_x(x_i) = \left.\frac{\tilde{f}_\xi}{x_\xi}\right|_{\xi=0} &= \frac{\tilde{f}(.5) - \tilde{f}(-.5)}{x(.5) - x(-.5)} \\ &= \frac{f(x(.5)) - f(x(-.5))}{x(.5) - x(-.5)} \\ &= \left(f(x_{i+\frac{1}{2}}) - f(x_{i-\frac{1}{2}})\right)\frac{1}{J_{x_i}},\end{aligned}$$

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where $J_{x_i} = x_{i+1/2} - x_{i-1/2}$.

Both $\tilde{f}_\xi$ and x_ξ have been calculated using the approach presented in [1].

To calculate the gradient at the boundary point x_0, we transform cells $\frac{1}{2}$ and $\frac{3}{2}$ to CG according to the following,

$$x = \begin{cases} (x_1 - x_0)\xi + x_1, & \text{for} \quad -1 \le \xi < 0, \\ (x_2 - x_1)\xi + x_1, & \text{for} \quad 0 \le \xi \le 1. \end{cases}$$

Then, using the approach presented in [1], we obtain

$$\begin{aligned} (Gf)(x_0) = f_x(x_0) = \left.\frac{\tilde{f}_\xi}{x_\xi}\right|_{\xi=-1} &= \frac{-\frac{8}{3}\tilde{f}(-1) + 3\tilde{f}(-.5) - \frac{1}{3}\tilde{f}(.5)}{-\frac{8}{3}x(-1) + 3x(-.5) - \frac{1}{3}x(.5)} \\ &= \frac{-\frac{8}{3}f(x(-1)) + 3f(x(-.5)) - \frac{1}{3}f(x(.5))}{-\frac{8}{3}x(-1) + 3x(-.5) - \frac{1}{3}x(.5)} \\ &= \left(-\frac{8}{3}f(x_0) + 3f(x_{1/2}) - \frac{1}{3}f(x_{3/2})\right)\frac{1}{J_{x_0}}, \end{aligned}$$

where $J_{x_0} = -\frac{8}{3}x_0 + 3x_{\frac{1}{2}} - \frac{1}{3}x_{\frac{3}{2}}$.

Analogously, the gradient at the boundary point x_n is obtained by transforming the cells $n - \frac{3}{2}$ and $n - \frac{1}{2}$ to CG.

We note that the element CG of the RSC does not change during the calculation of the gradient. Rather the derivatives $\tilde{f}_\xi$ and x_ξ are evaluated at different points, depending on whether we are calculating the gradient at interior points or at boundary points.

As a result, our gradient operator for non-uniform, 1D meshes will look like

$$Gf = \begin{bmatrix} \frac{-8/3}{J_{x_0}} & \frac{3}{J_{x_0}} & \frac{-1/3}{J_{x_0}} & 0 & \cdots & 0 \\ 0 & \frac{-1}{J_{x_1}} & \frac{1}{J_{x_1}} & \cdots & 0 & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & \cdots & 0 & \frac{-1}{J_{x_{n-1}}} & \frac{1}{J_{x_{n-1}}} & 0 \\ 0 & \cdots & 0 & \frac{1/3}{J_{x_n}} & \frac{-3}{J_{x_n}} & \frac{8/3}{J_{x_n}} \end{bmatrix} \begin{bmatrix} f_0 \\ f_{\frac{1}{2}} \\ f_{\frac{3}{2}} \\ \vdots \\ f_{n-\frac{1}{2}} \\ f_n \end{bmatrix},$$

so that after rewriting G as a product of two matrices we get

$$G = \begin{bmatrix} \frac{1}{J_{x_0}} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{J_{x_1}} & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & \frac{1}{J_{x_{n-1}}} & 0 \\ 0 & \cdots & 0 & 0 & \frac{1}{J_{x_n}} \end{bmatrix} \begin{bmatrix} \frac{-8}{3} & 3 & \frac{-1}{3} & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & \cdots & 0 & -1 & 1 & 0 \\ 0 & \cdots & 0 & \frac{1}{3} & -3 & \frac{8}{3} \end{bmatrix}. \tag{4.2}$$

We can clearly appreciate the effect of the local transformations. The resulting gradient operator for non-uniform meshes is a diagonal matrix with the inverse of the Jacobians of the local transformations (that will be called J_G) times the fixed part of the gradient operator for uniform meshes presented in [1].

If we implement (4.1) and (4.2) over an uniform mesh, then D and G reduce to the discrete operators presented in [3, 4, 6, 7]. For non-uniform meshes, the operator G is simpler than that obtained in [12].

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Let us now intoduce the following notation: A_u indicates a matrix A as it appears for the uniform mesh case, whereas A_{nu} indicates a matrix A as it appears for the non-uniform mesh case. For instance, G_u denotes the gradient operator for uniform meshes and G_{nu} denotes the gradient operator for non-uniform meshes, like (4.2). We will use this notation when necessary; otherwise, G and D will denote the gradient and divergence operator (respectively) for the non-uniform case. Moreover, as we saw in the previous construction of the operators, these can be written as a product of two matrices: one fixed matrix, which depends on the order of the approximation and comes from the schemes presented in [1], and another matrix with the inverse of the Jacobians of the local transformations, which depends on the staggered mesh. G' will denote the fixed part of the gradient operator matrix whereas G will be the whole operator. For example, G'_{nu} will be the fixed part of the gradient operator for non-uniform meshes.

The weight matrices P_{nu} and Q_{nu} obtained by solving (3.3) and (3.5) are $P_{nu} = P'_u J_G^{-1}$ and $Q_{nu} = \hat{J}_D^{-1} Q'_u$. The hat over the matrix J_D^{-1} means that we are considering the augmented matrix which has some extra rows, as many as the augmented divergence has and in the same position. These extras rows are zero everywhere except at the main diagonal position, where they are equal to one.

Finally, the boundary operator can be built by using expression (3.6), as follows,

$$\begin{aligned} B_{nu} &= (\hat{D}_{nu}^T Q_{nu} + P_{nu} G_{nu})^T = ((\hat{J}_D \hat{D}'_u)^T \hat{J}_D^{-1} Q'_u + P'_u J_G^{-1} J_G G'_u)^T \\ &= ((\hat{D}'_u)^T \hat{J}_D^T \hat{J}_D^{-1} Q'_u + P'_u G'_u)^T = ((\hat{D}'_u)^T Q'_u + P'_u G'_u)^T \\ &= (B_u^T)^T = B_u. \end{aligned}$$

That is, the second order boundary operator for non-uniform meshes is the same operator obtained for the uniform case. The boundary operator for non-uniform meshes, B_{nu}, obtained with our method is simpler than that obtained in [12]. A similar situation was found with the gradient operator. Thus, our approach provides an alternative to generalize Castillo-Grone operators to non-uniform meshes that is simpler to implement than that proposed in [12].

Some other properties of the mimetic operators are:

(i) D'_{nu} and G'_{nu} are centro-skew-symmetric,
(ii) D_{nu} and G_{nu} are banded matrices,
(iii) the global conservation law (3.1) on non-uniform meshes is satisfied.

4.2. Implementation. We aim to solve the Boundary Value Problem (BVP)

$$-\nabla^2 f(x) = F(x), \qquad x \in [0,1], \tag{4.3}$$
$$\alpha f(0) + \beta f'(0) = b_1, \tag{4.4}$$
$$\alpha f(1) + \beta f'(1) = b_2, \tag{4.5}$$

where

$$F(x) = \frac{2 \times 10^6 x}{\arctan(100)(1 + 1 \times 10^4 x^2)^2}, \quad \alpha = 1, \quad \beta = 1, \tag{4.6}$$

$$b_1 = \frac{100}{\arctan(100)}, \quad b_2 = 1 + \frac{100}{\arctan(100)\,(1 + 1 \times 10^4)}. \tag{4.7}$$

To solve this problem, we use second and fourth order mimetic operators on non-uniform meshes obtained by the technique described earlier.

First, we compute the solution of (4.3)–(4.5) by using the second order operators previously constructed. The discrete Laplacian is

$$L = \hat{D} G \in \mathbb{R}^{(n+2)\times(n+2)},$$

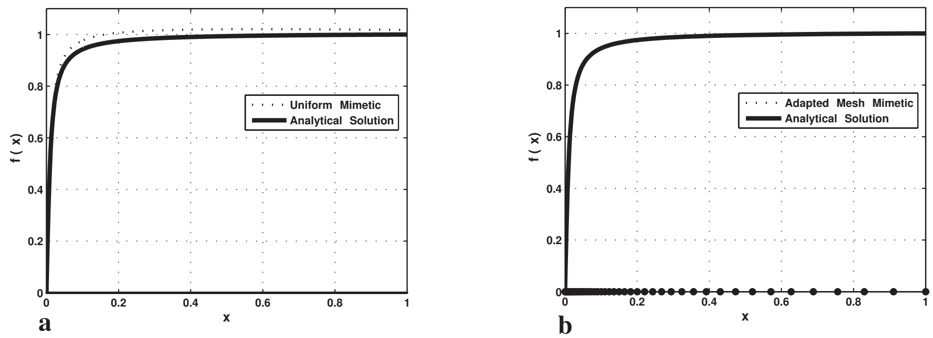

FIGURE 4.2. *(a) Numerical solution obtained with a uniform mesh, (b) Numerical solution obtained with an adapted mesh.*

where $\hat{D}$ is the augmented divergence operator, which has two rows of zeros added (one at the top and one at the bottom). For the boundary conditions (4.4) and (4.5), we have that

$$\alpha A\hat{f} + \beta BG\hat{f} = \begin{bmatrix} b_1 & 0 & \cdots & 0 & b_2 \end{bmatrix}^T. \tag{4.8}$$

The matrix A is such that $A_{1,1} = A_{n+2,n+2} = 1$ and all other entries are zero. A discrete form of the boundary value problem (4.3)–(4.5) can be written as

$$(\alpha A + \beta BG - L)\hat{f} = F. \tag{4.9}$$

In (4.8) and (4.9), $\hat{f}$ represents the numerical solution of the problem.

Figure 4.2 shows the solution of (4.9) (dotted line) along with the analytical solution $f(x) = \frac{\arctan(100x)}{\arctan(100)}$ (solid line). For this boundary-layer-like problem we have used a uniform mesh with 1000 points, shown in Figure 4.2(a), and an adapted mesh with just 50 points, shown in Figure 4.2(b). For the latter, the points have been clustered to the left using the mesh-size function $\mathrm{fms}(x) = \arctan\left(\frac{x}{10}\right) + 1 \times 10^{-3}$.

The BVP (4.3)–(4.5) is notoriously difficult to solve. It is well known that traditional finite difference methods tend to produce approximations with non-physical oscillations close to the boundary $x = 0$. Some other options exist that avoid the appearance of oscillations, such as the upwind difference scheme and the donor cell scheme; but both these schemes suffer from a drop in accuracy [5, 8]. With mimetic schemes, on the other hand, we obtain solutions that agree with the physics of the problem in the sense that no oscillations occur. Moreover, by using non-uniform mimetic operators and an adapted mesh, we obtained very good results with just a few points (Figure 4.2(b)).

Table 4.1 shows that the mimetic approximations implemented are second order with asymptotic truncation errors E_h estimated by

$$E_h = ch^p + O(h^{p+1}),$$

where p is the order of the approximation, c is the convergence-rate constant, n is the number of points in the mesh, and $h = 1/n$. The order of convergence was estimated using the maximum norm

$$\|\hat{f} - f\|_\infty = \max\{|\hat{f}_{i+\frac{1}{2}} - f_{i+\frac{1}{2}}|, \quad i = 0, \ldots, n-1\},$$

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and the mean-square norm

$$\|\hat{f} - f\|_{L_2} = \sqrt{\sum_{i=0}^{n-1} (\hat{f}_{i+\frac{1}{2}} - f_{i+\frac{1}{2}})^2 (J_D^{-1})_{i,i}},$$

where $(J_D^{-1})_{i,i}$ is the Jacobian of the local transformation or the volume associated with the cell $i + \frac{1}{2}$.

TABLE 4.1
Convergence analysis for second order mimetic schemes.

Scheme	E_h ($\|\cdot\|_\infty$)	E_h ($\|\cdot\|_{L_2}$)
Uniform	$24480.7 \times h^{1.94}$	$18637.6 \times h^{1.94}$
Adapted	$3.44 \times h^{2.08}$	$.45 \times h^{1.93}$
Random	$410.57 \times h^{.88}$	$335.97 \times h^{.89}$

Numerical results show that mimetic operators over adapted meshes are not only second order, but also have convergence-rate constants smaller than the corresponding constants for the uniform scheme. Similar results are reported in [14], where scheme [12] was implemented.

Next, we consider the solution of the BVP (4.3)–(4.5) obtained with fourth order uniform and non-uniform (adapted) mimetic operators. These operators are equal to the product of a diagonal matrix with the inverse of the Jacobians of the local transformations times the fixed part of the Castillo-Grone fourth order operators. We note that operators G and D are fourth order accurate at the images of ξ_i and $\xi_{i+\frac{1}{2}}$, respectively [2], and that these points have been approximated by Lagrange interpolations of fourth order. For this case we have clustered the points by using the same mesh-size function as before. The convergence results are presented in Table 4.2.

As with the second order case, we note an improvement in the numerical solution when using adapted meshes, suggesting a much smaller convergence-rate constant. Moreover, this example illustrates that we can construct, from the operators in [1], fourth order mimetic approximations on adapted meshes and preserve their accuracy.

Thus, the use of adapted meshes with the mimetic schemes constructed in this paper presents a very good option for solving boundary-layer problems.

TABLE 4.2
Convergence analysis for fourth order mimetic schemes.

Scheme	E_h ($\|\cdot\|_\infty$)	E_h ($\|\cdot\|_{L_2}$)
Uniform	$1.18 \times 10^9 \times h^{4.17}$	$9.10 \times 10^8 \times h^{4.17}$
Adapted	$11029.6 \times h^{3.98}$	$8105.84 \times h^{3.98}$
Random	$1699.21 \times h^{1.05}$	$1342.42 \times h^{1.06}$

5. Conclusions and future works. We have established a technique that allows us to construct, from the schemes presented in [1], mimetic approximations over non-uniform meshes. In this process, we introduced new elements to the theory of mimetic methods: the RSC and the use of local transformations as the basic tool for constructing mimetic operators.

The operators obtained are local and satisfy discrete conservation laws. We showed that they can be expressed as the product of a fixed part, dependent on the order of accuracy, and a factor that depends on the discretization.

Also, the technique reproduces the operators used in earlier publications [4, 6] when applied to uniform meshes; it extends the Castillo-Grone operators to non-uniform meshes; and it keeps invariant the boundary operator for the uniform and non-uniform case, which makes its computation and implementation simpler than the method proposed in [12]. It is worth mentioning that the last two provide completely new results.

Future work includes the implementation of this method to construct mimetic operators of higher order, for higher dimension problems, and for general polygonal discretizations of the physical domain.

Acknowledgments. Thanks to Dany De Cecchis for his advice with the formatting of this document.

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