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SPHERICAL QUADRATURE FORMULAS WITH EQUALLY SPACED NODES ON LATITUDINAL CIRCLES*

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Abstract. In a previous paper, we constructed quadrature formulas based on some fundamental systems of $(n+1)^2$ points on the sphere ($n+1$ equally spaced points taken on $n+1$ latitudinal circles), constructed by Laín-Fernández. These quadrature formulas are of interpolatory type. Therefore the degree of exactness is at least n. In some particular cases the exactness can be $n+1$ and this exactness is the maximal that can be obtained, based on the above mentioned fundamental system of points. In this paper we try to improve the exactness by taking more equally spaced points at each latitude and equal weights for each latitude. We study the maximal degree of exactness which can be attained with $n+1$ latitudes. As a particular case, we study the maximal exactness of the spherical designs with equally spaced points at each latitude. Of course, all of these quadratures are no longer interpolatory.

Key words. quadrature formulas, spherical functions, Legendre polynomials

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1. Introduction. Let $\mathbb{S}^2 = \{\boldsymbol{x} \in \mathbb{R}^3 : \|\boldsymbol{x}\|_2 = 1\}$ denote the unit sphere of the Euclidean space $\mathbb{R}^3$ and let

$$\begin{aligned}\Psi : [0,\pi] \times [0,2\pi) &\to \mathbb{S}^2,\\ (\rho,\theta) &\mapsto (\sin\rho\cos\theta, \sin\rho\sin\theta, \cos\rho)\end{aligned}$$

be its parametrization in spherical coordinates (ρ,θ). The coordinate ρ of a point $\xi(\Psi(\rho,\theta)) \in \mathbb{S}^2$ is usually called the latitude of ξ. Let $P_k,\ k = 0,1,\ldots,$ denote the Legendre polynomials of degree k on $[-1,1]$ normalized by the condition $P_k(1) = 1$, and let V_n be the space of spherical polynomials of degree less than or equal to n. The dimension of V_n is $\dim V_n = (n+1)^2$ and an orthogonal basis of V_n is given by

$$\left\{ Y_m^l(\theta,\rho) = P_m^{|l|}(\cos\rho) e^{il\theta},\ -m \le l \le m,\ 0 \le m \le n \right\}.$$

Here P_m^ν denotes the associated Legendre functions, defined by

$$P_m^\nu(t) = \left(\frac{(k-\nu)!}{(k+\nu)!}\right)^{1/2} (1-t^2)^{\nu/2} \frac{d^\nu}{dt^\nu} P_m(t),\ \nu = 0,\ldots,m,\ t \in [-1,1].$$

For given functions $f, g : \mathbb{S}^2 \to \mathbb{C}$, the inner product is taken as

$$\langle f, g\rangle = \int_{\mathbb{S}^2} f(\xi)\overline{g(\xi)}\, d\omega(\xi),$$

where $d\omega(\xi)$ stands for the surface element of the sphere. We also denote by Π_n the set of univariate polynomials of degree less than or equal to n.

2. Spherical quadrature. Let $n, p \in \mathbb{N}$, $\boldsymbol{\beta}_n = (\beta_1,\ldots,\beta_{n+1}) \in [0,2\pi)^{n+1}$, $\boldsymbol{\rho}_n = (\rho_1,\ldots,\rho_{n+1})$, $0 < \rho_1 < \rho_2 < \ldots < \rho_{n+1} < \pi$, and let

$$S(\boldsymbol{\beta}_n, \boldsymbol{\rho}_n, p) = \{\xi_{j,k}(\rho_j, \theta_k^j),\ \theta_k^j = \frac{\beta_j + 2k\pi}{p+1},\ j = 1,\ldots,n+1,\ k = 1,\ldots,p+1\}$$

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be a system of $(p+1)$ equally spaced nodes at each of the latitudes ρ_j. We consider the quadrature formula,

$$\int_{\mathbb{S}^2} F(\xi)d\omega(\xi) \approx \sum_{j=1}^{n+1} w_j \sum_{k=1}^{p+1} F(\xi_{j,k}), \tag{2.1}$$

with $\xi_{j,k} \in S(\boldsymbol{\beta}_n, \boldsymbol{\rho}_n, p)$.

A particular case, when n is odd, $p = n$, and

$$\beta_j = \begin{cases} \alpha\pi, & \text{for } j \text{ even}, \\ 0, & \text{for } j \text{ odd}, \end{cases}$$

with $\alpha \in [0,2)$,(see [1, 2]) was already considered in [4]. Here the weights w_j are uniquely determined and are calculated by direct manipulation of some Gram matrices of a local basis associated with the fundamental system of points $S(\boldsymbol{\beta}_n, \boldsymbol{\rho}_n, n)$. The quadrature formulas are interpolatory and therefore the degree of exactness is at least n. In [4] we showed that the degree of exactness is $n+1$ if and only if $\alpha = 1$ and $\sum_{j=1}^{n+1} w_j P_{n+1}(\cos\rho_j) = 0$. In [5] we proved that $n+1$ is the maximal degree of exactness attained in this particular case.

In the following, for a fixed n, we wish to study the maximum degree of exactness which can be achieved with such a formula. This means to impose that (2.1) be exact for the spherical polynomials Y_m^l, for $l = -m, \ldots, m$, and to specify the maximum value of m which makes (2.1) exact.

On the one hand, evaluating the integral in (2.1) for these spherical polynomials, we get

$$\int_{\mathbb{S}^2} P_m^{|l|}(\cos\rho)e^{il\theta}d\omega(\xi) = \int_0^\pi P_m^{|l|}(\cos\rho)\sin\rho\, d\rho \int_0^{2\pi} e^{il\theta}d\theta.$$

However,

$$\int_0^{2\pi} e^{il\theta}d\theta = \begin{cases} 2\pi, & \text{for } l = 0, \\ 0, & \text{otherwise}. \end{cases}$$

On the other hand, evaluating the sum in (2.1) for these spherical polynomials, we get

$$\begin{aligned} \sum_{j=1}^{n+1} w_j \sum_{k=1}^{p+1} P_m^{|l|}(\cos\rho_j)e^{il\theta_k^j} &= \sum_{j=1}^{n+1} w_j P_m^{|l|}(\cos\rho_j) \sum_{k=1}^{p+1} e^{il\frac{\beta_j+2k\pi}{p+1}} \\ &= \sum_{j=1}^{n+1} w_j P_m^{|l|}(\cos\rho_j) e^{il\frac{\beta_j}{p+1}} \sum_{k=1}^{p+1} e^{il\frac{2k\pi}{p+1}}. \end{aligned}$$

The last sum is zero if $l \notin (p+1)\mathbb{Z}$ and is $p+1$ if $l \in (p+1)\mathbb{Z}$.

With the above remarks, the quadrature formula (2.1) is exact for Y_m^l with $l \neq 0$, in the case when $m < p+1$. In order to be exact for $l = 0$ we should have

$$\int_{\mathbb{S}^2} P_m(\cos\rho)d\omega(\xi) = \sum_{j=1}^{n+1} w_j \sum_{k=1}^{p+1} P_m(\cos\rho_j),$$

which yields

$$\int_{-1}^{1} P_m(x)dx = \frac{p+1}{2\pi} \sum_{j=1}^{n+1} w_j P_m(\cos\rho_j).$$

With the notation $\cos\rho_j = r_j$, $a_j = \frac{p+1}{2\pi} w_j$, we arrive at

$$\int_{-1}^{1} P_m(x)dx = \sum_{j=1}^{n+1} a_j P_m(r_j). \tag{2.2}$$

In conclusion, we proved the following result.

PROPOSITION 2.1. *Let $n, p, s \in \mathbb{N}$ such that $s < p+1$, and consider the spherical quadrature formula (2.1) with $\xi_{j,k} \in S(\boldsymbol{\beta}_n, \boldsymbol{\rho}_n, p)$. This formula is exact for the spherical polynomials in V_s if and only if the quadrature formula*

$$\int_{-1}^{1} f(x)dx \approx \sum_{j=1}^{n+1} a_j f(r_j) \tag{2.3}$$

is exact for all polynomials in Π_s.

Let us remark that, taking $m = 0, 1, \ldots, p$ in (2.2) (or, equivalently, taking $f = 1, x, \ldots, x^p$ in (2.3)), we obtain the system

$$\sum_{j=1}^{n+1} a_j r_j^{\lambda} = \left((-1)^{\lambda} + 1\right) \frac{1}{\lambda + 1}, \tag{2.4}$$

for $\lambda = 0, \ldots, p$. This system has $p+1$ equations and $2n+2$ unknowns, a_j, r_j, $j = 1, \ldots, n+1$.

Next it is natural to ask when formula (2.1) is exact for spherical polynomials in V_s with $s \geq p+1$. If we further impose that formula (2.1) is exact for the spherical polynomials Y_{p+1}^{l}, $l = -p-1, \ldots, p+1$, then we have

$$\sum_{j=1}^{n+1} a_j r_j^{p+1} = \left((-1)^{p+1} + 1\right) \frac{1}{p+2}, \tag{2.5}$$

$$\sum_{j=1}^{n+1} a_j (\sin\rho_j)^{p+1} e^{i\beta_j} = 0. \tag{2.6}$$

Equation (2.5) follows from the fact that (2.1) is exact for Y_{p+1}^{0}, while equation (2.6) results from the fact that formula (2.1) is exact for the spherical polynomials Y_{p+1}^{p+1} and Y_{p+1}^{-p-1}. For $l = -p, \ldots, -1, 1, \ldots, p$, both sides of quadrature (2.1) are zero, therefore it is exact.

In conclusion the following proposition holds.

PROPOSITION 2.2. *Let $n, p \in \mathbb{N}$. Then formula (2.1) is exact for all spherical polynomials in V_p if and only if conditions (2.4) are satisfied for $\lambda = 0, \ldots, p$. Moreover, formula (2.1) is exact for all spherical polynomials in V_{p+1} if and only if supplementary conditions (2.5) and (2.6) are fulfilled.*

3. Maximal degree of exactness which can be attained with equally spaced nodes at $n+1$ latitudes. In this section we establish which is the maximum degree of exactness that can be obtained by taking the same number of equally spaced nodes on each of the $n+1$ latitudinal circles and then we construct quadrature formulas with maximal degree of exactness.

What is well known is that the system (2.4) is solvable for a maximal number of conditions $2n+2$ (for $\lambda = 0, 1, \ldots, 2n+1$), when it solves uniquely. This is the case of the univariate Gauss quadrature formula. In this case, the maximal value for p which can be taken

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in (2.4) is $p = 2n + 1$, implying that (2.1) is exact for all spherical polynomials in V_{2n+1}. In conclusion, the following result holds.

PROPOSITION 3.1. *Let $n \in \mathbb{N}$ and consider the quadrature formula (2.1). Its maximal degree of exactness is $2n + 1$ and if we want it to be attained, then we must take the cosines of the latitudes, $\cos \rho_j = r_j$, as the roots of the Legendre polynomial P_{n+1} and the weights as [3]*

$$w_j = \frac{2\pi}{p+1} a_j, \text{ with } a_j = \frac{2(1 - r_j^2)}{(n+2)^2 (P_{n+2}(r_j))^2} > 0. \tag{3.1}$$

One possible case when it can be attained is by taking $2n + 2$ equally spaced nodes at each latitude and arbitrary deviations $\beta_j \in [0, 2\pi)$.

The question which naturally arises is whether we can obtain degree of exactness $2n+1$ with fewer than $2n + 2$ points at each latitude.

3.1. Maximal exactness $2n + 1$ with only $2n + 1$ nodes at each latitude. Consider $2n+1$ equally spaced nodes at each latitude. If we suppose that conditions (2.4) are satisfied for $\lambda = 0, 1, \dots, 2n$, then formula (2.1) will be exact for all spherical polynomial in V_{2n}. From Proposition 2.2 we deduce that, if we want it to be exact for all polynomials in V_{2n+1}, then we should add the conditions

$$\sum_{j=1}^{n+1} a_j r_j^{2n+1} = 0, \tag{3.2}$$

$$\sum_{j=1}^{n+1} a_j (\sin \rho_j)^{2n+1} e^{i\beta_j} = 0. \tag{3.3}$$

In this case the quadrature formula (2.2) becomes the Gauss quadrature formula. Thus, r_j will be the roots of the Legendre polynomial P_{n+1} and a_j are given in (3.1). Since $a_{n+2-j} = a_j$ and $\rho_j = \pi - \rho_{n+2-j}$ for $j = 1, \dots, n+1$ and $r_{\frac{n}{2}+1} = 0$ for even n, condition (3.3) can be written as

$$\sum_{j=1}^{(n+1)/2} a_j (\sin \rho_j)^{2n+1} (e^{i\beta_j} + e^{i\beta_{n+2-j}}) = 0, \text{ for } n \text{ odd}, \tag{3.4}$$

$$a_{\frac{n}{2}+1} e^{i\beta_{\frac{n}{2}+1}} + \sum_{j=1}^{n/2} a_j (\sin \rho_j)^{2n+1} (e^{i\beta_j} + e^{i\beta_{n+2-j}}) = 0, \text{ for } n \text{ even}. \tag{3.5}$$

For n odd, equation (3.4) is always solvable and possible solutions are discussed in Appendix A. For n even the solvability of equation (3.5) is discussed in Appendix B. Numerical tests performed for $n \le 100$ show that inequality (B.3) in Appendix B holds only for $n \ge 12$. Therefore, the equation (3.5) is not solvable for $n \in \{2, 4, \dots, 10\}$ and solvable for $12 \le n \le 100$. In conclusion, the following result holds.

PROPOSITION 3.2. *Let $n \in \mathbb{N}$ and consider the quadrature formula (2.1) with $2n + 1$ equally spaced nodes at each latitude. For $n \in \{2, 4, 6, 8, 10\}$ one cannot attain exactness $2n + 1$. For n odd and for $n \in \{12, 14, \dots, 100\}$, if $\cos \rho_j$ are the roots of the Legendre polynomial P_{n+1}, the weights are as in (3.1), the numbers β_j are solutions of equation (3.3) (given in Appendices 1 and 2), then the quadrature formula (2.1) has the degree of exactness $2n + 1$.*

We further want to know if it is possible to obtain the maximal degree of exactness $2n+1$ with fewer points at each latitude.

3.2. Maximal exactness $2n+1$ with $2n$ points at each latitude. Let us consider $2n$ points ($p = 2n-1$) at each latitude. If we suppose that conditions (2.4) are satisfied for $\lambda = 0, 1, \dots, 2n-1$, then formula (2.1) will be exact for all polynomials in V_{2n-1}. If we want it to be exact for Y_{2n}^l, for $l = -2n, \dots, 2n$, then we should add the conditions

$$\sum_{j=1}^{n+1} a_j r_j^{2n} = \frac{2}{2n+1}, \tag{3.6}$$

$$\sum_{j=1}^{n+1} a_j (\sin\rho_j)^{2n} e^{i\beta_j} = 0. \tag{3.7}$$

Further, if we want the formula (2.1) to be exact for all Y_{2n+1}^l, for $l = -2n-1, \dots, 2n+1$, then we should impose the conditions

$$\sum_{j=1}^{n+1} a_j r_j^{2n+1} = 0, \tag{3.8}$$

$$\sum_{j=1}^{n+1} a_j (\sin\rho_j)^{2n} \cos\rho_j e^{i\beta_j} = 0. \tag{3.9}$$

From conditions (3.6) and (3.8) we get again that $\cos\rho_j = r_j$ are the roots of the Legendre polynomial P_{n+1} and a_j are as in (3.1). Therefore, formula (2.1) has the degree of exactness $2n+1$ if and only if equations (3.7) and (3.9) are simultaneously satisfied. Due to the symmetry, they reduce to the system

$$\sum_{j=1}^{(n+1)/2} a_j (\sin\rho_j)^{2n} (e^{i\beta_j} + e^{i\beta_{n+2-j}}) = 0, \tag{3.10}$$

$$\sum_{j=1}^{(n+1)/2} a_j (\sin\rho_j)^{2n} \cos\rho_j (e^{i\beta_j} - e^{i\beta_{n+2-j}}) = 0, \tag{3.11}$$

for n odd, and to the system

$$a_{\frac{n}{2}+1} e^{i\beta_{\frac{n}{2}+1}} + \sum_{j=1}^{n/2} a_j (\sin\rho_j)^{2n} (e^{i\beta_j} + e^{i\beta_{n+2-j}}) = 0,$$
$$\sum_{j=1}^{n/2} a_j (\sin\rho_j)^{2n} \cos\rho_j (e^{i\beta_j} - e^{i\beta_{n+2-j}}) = 0,$$

for n even.

For n odd, we give some conditions on the solvability or non-solvability of this system in Appendix C (Proposition C.1). Numerical tests performed for $n \in \{1, 3, 5, \dots, 99\}$ show that the hypotheses (C.4) in Appendix C are fulfilled only for $n \in \{1, 3, \dots, 13\}$, in each of these cases the index k being $k = (n+1)/2$. In conclusion, for these values of n, the above system has no solution and therefore the quadrature formula cannot have maximal exactness $2n+1$.

For $n \in \{15, 17, \dots, 41\}$ the system is solvable since hypotheses (C.7)-(C.8) in Appendix C are fulfilled, each time for $v = (n+1)/2$. In the proof of Proposition C.1, 3

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in Appendix C, we give a possible solution of the system. For $n \in \{43, 45, \ldots, 99\}$, the solvability is not clear yet. In this case, both sequences $\{\alpha_j, j = 1, \ldots, (n+1)/2\}$ and $\{\mu_j, j = 1, \ldots, (n+1)/2\}$ satisfy the triangle inequality.

In Table 3.1 we summarize all the cases discussed above.

TABLE 3.1
Some choices for which the maximal degree of exactness $2n+1$ is attained, for $P_{n+1}(\cos\rho_j) = 0$, $j \in \{1, \ldots, n+1\}, n \leq 100$.

number of nodes at each latitude	n	β_j
$2n+2$	$\mathbb{N}$	$[0, 2\pi)$
$2n+1$	odd	Appendix A
	$\{2, 4, 6, 8, 10\}$	$\emptyset$ (cf. Appendix B)
	$\{12, 14, \ldots, 100\}$	Appendix B
$2n$	$\{1, 3, \ldots, 13\}$	$\emptyset$ (cf. Appendix C, Prop. C.1, 1)
	$\{15, 17, \ldots, 41\}$	Appendix C, Prop. C.1, 3
	$\{43, 45, \ldots, 99\}$	no answer
	even	no answer

As a final remark, we mention that the improvement brought to the interpolatory quadrature formulas in [4], which were established only for n odd, is the following: In [4], for attaining the degree of exactness $2n+1$ one needs $(2n+2)^2$ nodes. The quadrature formulas presented here can attain this degree of exactness with only $(2n+2)(n+1)$ nodes (for arbitrary choices of the deviations β_j) and with only $(2n+1)(n+1)$ nodes or only $2n(n+1)$ nodes (for some special cases summarized in Table 3.1).

4. A particular case: spherical designs. A spherical design is a set of points of $\mathbb{S}^2$ which generates a quadrature formula with equal weights which is exact for spherical polynomials up to a certain degree. For a fixed $n \in \mathbb{N}$, we intend to specify the maximal degree of exactness that can be attained with the points in $S(\boldsymbol{\beta}_n, \boldsymbol{\rho}_n, p)$ and show for which choices of the parameters $\boldsymbol{\beta}_n, \boldsymbol{\rho}_n, p$ this maximal degree can be attained. Therefore, let us consider the quadrature formula

$$\int_{\mathbb{S}^2} F(\xi) d\omega(\xi) \approx w_{n,p} \sum_{j=1}^{n+1} \sum_{k=1}^{p+1} F(\xi_{j,k}), \text{ with } \xi_{j,k} \in S(\boldsymbol{\beta}_n, \boldsymbol{\rho}_n, p). \tag{4.1}$$

If we require that this formula is exact for constant functions, we obtain

$$w_{n,p} = \frac{4\pi}{(n+1)(p+1)}.$$

As in the general case, we obtain that formula (4.1) is exact for the spherical polynomials Y_m^l for $m < p+1$ and $-m \leq l \leq m,\ l \neq 0$. In order to be exact for Y_m^0 for $m < p+1$, we should have

$$\int_{-1}^{1} P_m(x) dx = \frac{2}{n+1} \sum_{j=1}^{n+1} P_m(r_j),$$

where $r_j = \cos\rho_j$, for $j = 1, \ldots, n+1$. In conclusion, if the quadrature formula

$$\int_{-1}^{1} f(x) dx \approx \frac{2}{n+1} \sum_{j=1}^{n+1} f(r_j) \tag{4.2}$$

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is exact for all univariate polynomials in $\Pi_s,\ s < p+1$, then the quadrature formula (4.1) will be exact for all spherical polynomials in V_s. If in (4.2) we take $f(x) = x^m$ for $m = 1, \ldots, p$, we obtain the system

$$\sum_{j=1}^{n+1} r_j^{\lambda} = \frac{(-1)^{\lambda}+1}{\lambda+1} \cdot \frac{n+1}{2}, \tag{4.3}$$

with $\lambda = 1, \ldots, p$. This system has $n+1$ unknowns. The maximal degree of exactness of the quadrature formula (4.2) (respectively, the maximal value of p) is obtained in the classical case of Chebyshev one-dimensional quadrature formula, when the system (4.3) has a unique solution. In this case $p = n+1$, since the number of conditions needed to solve the quadrature formula uniquely is $n+1$. More precisely, in the one-dimensional case of Chebyshev quadrature, it is known that $r_j = r_{n+2-j}$ for $j = 1, \ldots, [n/2]$ and that system (4.3) has no solution for $n = 7$ and $n > 8$. For $n \in \{2, 4, 6, 8\}$, the quadrature formula (4.2) has the degree of exactness $n+1$ if the conditions in (4.3) are fulfilled for $\lambda = 1, \ldots, n+1$. For $n \in \{1, 3, 5\}$, if the same conditions are fulfilled, the degree of exactness is $n+2$ since one additional condition in (4.3) for $\lambda = n+2$ is satisfied.

In conclusion, the following result holds.

PROPOSITION 4.1. *Let $n \in \{1, 2, 3, 4, 5, 6, 8\}$ and consider the quadrature formula (4.1) with $p+1$ equally spaced nodes at each latitude. Its maximal degree of exactness is*

$$\mu_{max} = \begin{cases} n+1, & \text{for } n \in \{2,4,6,8\}, \\ n+2, & \text{for } n \in \{1,3,5\}. \end{cases} \tag{4.4}$$

It can be attained, for example, by taking $n+2$ equally spaced nodes at each latitude ($p = n+1$), for all choices of the deviations β_j in $[0, 2\pi)$ and for $\cos\rho_j$ the nodes of the classical one-dimensional Chebyshev quadrature formula.

We wish to investigate if the maximal degree of exactness μ_{max} can be obtained with fewer than $n+2$ points at each latitude.

4.1. Maximal degree of exactness attained with only $n+1$ points at each latitude. Suppose $p = n$ and suppose (4.3) is fulfilled for $\lambda = 1, \ldots, n$. This implies that (4.1) is exact for the spherical polynomials Y_λ^0, for $\lambda = 1, \ldots, n$. We want again to investigate if the maximal degree of exactness μ_{max} can be attained with only $n+1$ points at each latitude.

Case 1: n even. If we want formula (4.1) to be exact for all spherical polynomials in $V_{n+1} = V_{\mu_{max}}$, it remains to impose the condition that (4.1) is exact for Y_{n+1}^0 and $Y_{n+1}^{\pm(n+1)}$. Exactness for Y_{n+1}^0 means $\sum_{j=1}^{n+1} r_j^{n+1} = 0$, which, together with (4.3) fulfilled for $\lambda = 1, \ldots, n$, leads finally to the system in the classical one-dimensional Chebyshev case. Thus $r_j = r_{n+2-j}$, for $j = 1, \ldots, n/2, r_{\frac{n}{2}+1} = 0$ and a solution exists only for $n \in \{2, 4, 6, 8\}$. Further, exactness for $Y_{n+1}^{\pm(n+1)}$ reduces to

$$e^{i\beta_{\frac{n}{2}+1}} + \sum_{j=1}^{n/2} (\sin\rho_j)^{n+1} (e^{i\beta_j} + e^{i\beta_{n+2-j}}) = 0. \tag{4.5}$$

Numerical tests show that condition (B.3) in Appendix B is fulfilled for $n \in \{2, 4, 6, 8\}$. Therefore, equation (4.5) is solvable.

Case 2: n odd. In this case, if we want formula (4.1) to be exact for all spherical polynomials in $V_{n+2} = V_{\mu_{max}}$, it remains to require that it is exact for $Y_{n+1}^0, Y_{n+2}^0, Y_{n+1}^{\pm(n+1)}$ and $Y_{n+2}^{\pm(n+1)}$.

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Exactness for the spherical polynomial Y_{n+1}^{0} reduces to the condition

$$\sum_{j=1}^{n+1} r_j^{n+1} = \frac{n+1}{n+2},$$

which, added to conditions (4.3) for $\lambda = 1, \dots, n$, leads again to the system in the classical one-dimensional Chebyshev case (which is uniquely solvable).

Exactness for Y_{n+2}^{0} reduces to condition

$$\sum_{j=1}^{n+1} r_j^{n+2} = 0,$$

which is automatically satisfied.

Further, exactness for $Y_{n+1}^{\pm(n+1)}$ and $Y_{n+2}^{\pm(n+1)}$ means, respectively,

$$\sum_{j=1}^{(n+1)/2} (\sin\rho_j)^{n+1}(e^{i\beta_j} + e^{i\beta_{n+2-j}}) = 0. \tag{4.6}$$

$$\sum_{j=1}^{(n+1)/2} (\sin\rho_j)^{n+1}\cos\rho_j(e^{i\beta_j} - e^{i\beta_{n+2-j}}) = 0. \tag{4.7}$$

In conclusion, the maximal degree of exactness $n+2$ is attained if and only if r_j are the nodes in univariate Chebyshev quadrature and the system (4.6)-(4.7) is solvable. The solvability of this system is discussed in Appendix C in the general case. For $n = 1$, the non-solvability is clear. For $n = 3$, the system is again not solvable (cf. Proposition C.1, Appendix C), since $\mu_1 < \mu_2$. For $n = 5$, it is solvable since the hypotheses (C.5)-(C.6) in Proposition C.1 are satisfied, with $v = 2$.

To summarize the above considerations, we state the following result.

PROPOSITION 4.2. *Let $n \in \{1, 2, 3, 4, 5, 6, 8\}$ and consider the quadrature formula (4.1) with $n+1$ equally spaced nodes at each latitude. Then the maximal degree of exactness μ_{max} given in Proposition 4.1 can be attained for $n = 2, 4, 6, 8$, if $\cos\rho_j$ are chosen as nodes of the classical one-dimensional Chebyshev quadrature formula and the numbers β_j are chosen as described in Appendix B. For $n = 1, 3$, the maximal degree of exactness cannot be attained, while for $n = 5$ it can be attained if the deviations β_j, $j = 1, \dots, 6$, are taken as described in Appendix C, Proposition C.1, 2.*

The natural question which arises now is: Is it possible to have maximal degree of exactness $n + 1$ with only n points at each latitude? The answer is given in the following section.

4.2. Maximal degree of exactness with only n points at each latitude. Let us consider n points at each latitude ($p = n - 1$) and suppose (4.3) holds for $\lambda = 1, \dots, n - 1$. We want to see if the maximal degree of exactness μ_{max} can be attained with only n points at each latitude.

Case 1: n odd. In this case, if we want formula (4.1) to be exact for all spherical polynomials in $V_{n+2} = V_{\mu_{max}}$, it remains to impose that it is exact for Y_{n+1}^{0}, Y_{n+2}^{0}, $Y_{n}^{\pm n}$, $Y_{n+1}^{\pm n}$ and $Y_{n+2}^{\pm n}$. Altogether, they imply that $r_j = \cos\rho_j$ are the abscissa in the classical univariate Chebyshev case, and the deviations β_j should satisfy the system

$$\sum_{j=1}^{(n+1)/2} (\sin\rho_j)^n (e^{i\beta_j} + e^{i\beta_{n+2-j}}) = 0, \tag{4.8}$$

$$\sum_{j=1}^{(n+1)/2} (\sin\rho_j)^n \cos\rho_j (e^{i\beta_j} - e^{i\beta_{n+2-j}}) = 0, \tag{4.9}$$

$$\sum_{j=1}^{(n+1)/2} (\sin\rho_j)^n P_{n+2}^{(n)}(\cos\rho_j)(e^{i\beta_j} + e^{i\beta_{n+2-j}}) = 0.$$

Since $P_{n+2}^{(n)}(\cos\rho)$ is an even polynomial of degree two in $\cos\rho$, using equation (4.8), we can replace the last equation by

$$\sum_{j=1}^{(n+1)/2} (\sin\rho_j)^n (\cos\rho_j)^2 (e^{i\beta_j} + e^{i\beta_{n+2-j}}) = 0. \tag{4.10}$$

For $n = 1$, the system is clearly not solvable.
For $n = 3$, the system is solvable since $\sin^3\rho_1 \cos\rho_1 = \sin^3\rho_2\cos\rho_2$. A solution can be written as

$$\beta_1 \in [0, 2\pi),\ \beta_3 = \beta_1,\ \beta_2 = \beta_4 = \beta_1 + \pi \ (\text{mod } 2\pi).$$

For $n = 5$, up to now we do not have a result regarding the solvability of the system.

TABLE 4.1
Some choices for which the maximal degree of exactness $\mu_{\max}$ is attained, for $\cos\rho_j,\ j \in \{1,\ldots,n+1\}$, the nodes in the case of classical Chebyshev quadrature.

number of nodes at each latitude	n	β_j
$n+2$	$\{1,2,3,4,5,6,8\}$	$[0,2\pi)$
$n+1$	$\{2,4,6,8\}$	$[0,2\pi)$
	$\{1,3\}$	$\emptyset$ (cf. Appendix C, Prop. C.1, 2)
	5	no answer
n	1	$\emptyset$
	3	$\beta_1 \in [0,2\pi), \beta_3 = \beta_1,\ \beta_2 = \beta_4 = \beta_1 + \pi$
	{2,4,6,8}	no answer

Case 2: n even. If we want formula (4.1) to be exact for all spherical polynomials in $V_{n+1} = V_{\mu_{max}}$, it remains to impose that (4.1) is exact for $Y_n^0, Y_{n+1}^0, Y_n^{\pm n}$ and $Y_{n+1}^{\pm n}$. Exactness for Y_n^0 and Y_{n+1}^0 means $\sum_{j=1}^{n+1} r_j^n = 1$ and $\sum_{j=1}^{n+1} r_j^{n+1} = 0$, respectively. Together with (4.3) fulfilled for $\lambda = 1,\ldots,n-1$, they lead to the system in the classical one-dimensional Chebyshev case. Thus $r_j = r_{n+2-j}$, for $j = 1,\ldots,n/2$, $r_{\frac{n}{2}+1} = 0$ and a solution exists only for $n \in \{2,4,6,8\}$. Further, using again the symmetry of the latitudes, exactness for

$Y_n^{\pm n}$ and $Y_{n+1}^{\pm n}$ reduces to

$$e^{i\beta_{\frac{n}{2}+1}} + \sum_{j=1}^{n/2} (\sin\rho_j)^n (e^{i\beta_j} + e^{i\beta_{n+2-j}}) = 0, \tag{4.11}$$

$$\sum_{j=1}^{n/2} (\sin\rho_j)^n \cos\rho_j (e^{i\beta_j} - e^{i\beta_{n+2-j}}) = 0. \tag{4.12}$$

In conclusion, the maximal degree of exactness $\mu_{max} = n+1$ can be attained if and only if the system (4.11)-(4.12) is solvable. Unfortunately we could not give a result regarding the solvability of this system.

All these cases are summarized in Table 4.1.

5. Numerical examples. In order to demonstrate the efficiency of our formulas, we consider the quadrature formula

$$\int_{\mathbb{S}^2} F(\xi) d\omega(\xi) \approx \sum_{j=1}^{m+1} w_j \sum_{k=1}^{p+1} F(\xi_{j,k}),$$

with $\xi_{j,k}(\rho_j, \theta_k^j) \in \mathbb{S}^2$, in the following cases:

1. The classical Gauss-Legendre quadrature formula, with $m = n$, $p = 2n+1$, $\cos\rho_j = r_j$, the roots of Legendre polynomial P_{n+1},

$$\theta_k^j = \frac{k\pi}{n+1},$$
$$w_j = \frac{2\pi}{2n+2} a_j, \text{ with } a_j = \frac{2(1-r_j^2)}{(n+2)^2 (P_{n+2}(r_j))^2},$$

   $j = 1, \ldots, n+1$, $k = 1, \ldots, 2n+2$. This formula has $2n^2 + 4n + 2$ nodes and is exact for polynomials in V_{2n+1}. It is in fact a particular case of the quadratures given in Proposition 3.1, when all deviations β_j are zero.

2. The Clenshaw-Curtis formula[1], with $m = 2n$, $p = 2n+1$,

$$\theta_k^j = \frac{k\pi}{n+1}, \quad \rho_j = \frac{(j-1)\pi}{2n} \text{ for } j = 1, \ldots, 2n+1,\ k = 1, \ldots, 2n+2,$$
$$w_j = w_{2n+1-j} = \frac{4\pi \varepsilon_j^{2n+1}}{n(n+1)} \sum_{l=0}^{n} \varepsilon_{l+1}^{n+1} \frac{1}{1-4l^2} \cos\frac{(j-1)l\pi}{n}, \text{ for } j = 1, \ldots, n,$$

   where

$$\varepsilon_j^J = \begin{cases} \frac{1}{2} & \text{if } j = 1 \text{ or } j = J, \\ 1 & \text{if } 0 < j < J. \end{cases}$$

   This formula has $4n^2 + 6n + 2$ nodes and is exact for polynomials in V_{2n+1}.

[1]This formula is sometimes called Chebyshev formula, since in the one-dimensional case it is based on the expansion of a function in terms of Chebyshev polynomials T_i of the first kind. The nodes $\cos j\pi/2n$ are the extrema of the Chebyshev polynomial T_{2n} of degree $2n$.

In our numerical experiments we have considered the following test functions:

$$f_1(\mathbf{x}) = -5\sin(1+10x_3),$$
$$f_2(\mathbf{x}) = \|\mathbf{x}\|_1/10,$$
$$f_3(\mathbf{x}) = 1/\|\mathbf{x}\|_1,$$
$$f_4(\mathbf{x}) = \exp(x_1^2),$$

where $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{S}^2$.

From the quadrature formulas constructed in this paper, we consider those from Section 3.1 and we compare them with the Gauss-Legendre and Clenshaw-Curtis quadratures mentioned above. We do not present here quadratures from Proposition 3.1 for deviations β_j different from zero, since in this case, for the above test functions, the errors are comparable with the ones obtained for Gauss-Legendre (when all β_j are equal to zero).

Figure 5.1 shows the interpolation errors (logarithmic scale) for each of the functions f_1, f_2, f_3, and f_4, respectively.

Appendix A. For n odd, we provide solutions of the equation

$$\sum_{j=1}^{q} \alpha_j (e^{i\beta_j} + e^{i\beta_{n+2-j}}) = 0, \tag{A.1}$$

with $q = (n+1)/2$, $\alpha_j > 0$ given and the unknowns β_j, $j = 1, \ldots, n+1$. For this we need the following result.

LEMMA A.1. *Let $A > 0$ be given. Then, for every $z = \tau e^{i\theta} \in \mathbb{C}$ with $0 \le \tau \le 2A$, $\theta \in [0, 2\pi)$, there exist $\omega_j = \omega_j(\tau, \theta) \in [0, 2\pi)$, $j = 1, 2$, such that*

$$A(e^{i\omega_1} + e^{i\omega_2}) = z. \tag{A.2}$$

Proof. Indeed, denoting

$$\gamma = \arccos \frac{\tau}{2A} \in \left[0, \frac{\pi}{2}\right],$$

a possible choice of the ω_1, ω_2 which satisfy relation (A.2) is the following:

1. If $\theta - \gamma \ge 0$ and $\theta + \gamma < 2\pi$, then $(\omega_1, \omega_2) \in \{(\theta+\gamma, \theta-\gamma), (\theta-\gamma, \theta+\gamma)\}$;
2. If $\theta - \gamma < 0$, then $(\omega_1, \omega_2) \in \{(\theta+\gamma, \theta-\gamma+2\pi), (\theta-\gamma+2\pi, \theta+\gamma)\}$;
3. If $\theta + \gamma \ge 2\pi$, then $(\omega_1, \omega_2) \in \{(\theta+\gamma-2\pi, \theta-\gamma), (\theta-\gamma, \theta+\gamma-2\pi)\}$,

or, shorter,

$$\begin{cases} \omega_1 = \theta + \varepsilon\gamma \,(\mathrm{mod} 2\pi), \\ \omega_2 = \theta - \varepsilon\gamma \,(\mathrm{mod} 2\pi), \end{cases} \quad \text{with } \varepsilon \in \{-1, 1\}.$$

Equality (A.2) can be verified by direct calculations. ∎

Let us come back to equation (A.1). For $j = 1, \ldots, q$, we consider $z_j = \tau_j e^{i\theta_j} \in \mathbb{C}$ with $0 \le \tau_j \le 2\alpha_j$, such that

$$z_1 + \ldots + z_q = 0.$$

In fact, we take $q-1$ arbitrary complex numbers $z_j^* = \tau_j^* e^{i\theta_j}$, $\tau_j^* \ge 0$, $j = 1, \ldots, q-1$, and then consider $z_q^* = -z_1^* - \ldots - z_{q-1}^*$. The numbers $z_j = \tau_j e^{i\theta_j}$, $j = 1, \ldots, q$, satisfying

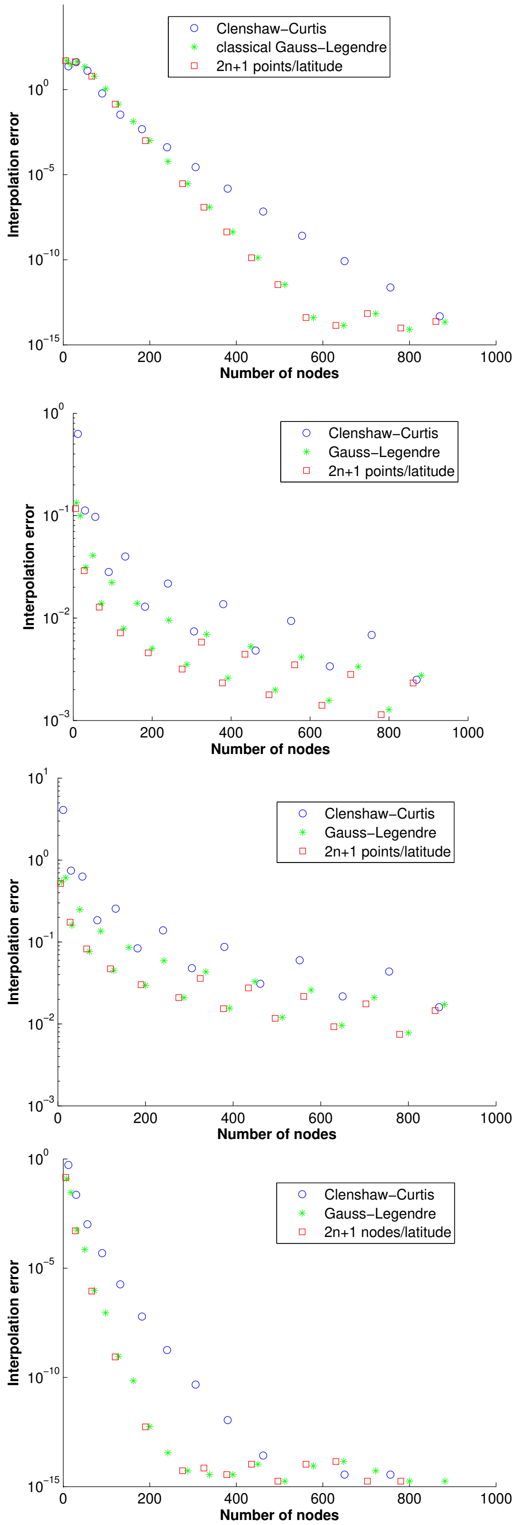

FIG. 5.1. *Interpolation errors (logarithmic scales) for the test functions* f_1, f_2, f_3, f_4.

the inequalities $\tau_j \leq 2\alpha_j$ are taken such that

$$\tau_j = \tau_j^* B, \text{ with } B = \min_{\substack{k=1,\dots,q, \\ \tau_k^* > 0}} \frac{2\alpha_k}{\tau_k^*}.$$

Denoting

$$\gamma_j = \arccos \frac{\tau_j}{2\alpha_j}, \ j = 1, \dots, q,$$

and applying Lemma A.1, we can write a solution of equation (A.1) as

$$\begin{cases} \beta_j = \theta_j + \varepsilon_j \gamma_j \pmod{2\pi}, \\ \beta_{n+2-j} = \theta_j - \varepsilon_j \gamma_j \pmod{2\pi}, \end{cases} \quad \text{with } \varepsilon_j \in \{-1, 1\}.$$

Appendix B. For n even, we discuss the equation

$$\alpha_{q+1} e^{i\beta_{q+1}} + \sum_{j=1}^{q} \alpha_j (e^{i\beta_j} + e^{i\beta_{n+2-j}}) = 0, \tag{B.1}$$

with $q = n/2$, $\alpha_j > 0$ given and the unknowns β_j, $j = 1, \dots, q+1$. For determining a non-trivial solution we need the following result.

LEMMA B.1. *Let $a, b_1, \dots, b_q > 0$ such that $a \leq b_1 + \ldots + b_q$. Then there exist numbers $t_j \in [0, 1]$ (not all of them equal) for $j \in \{1, \dots, q\}$, such that*

$$a = \sum_{j=1}^{q} t_j b_j. \tag{B.2}$$

Proof. Of course, a trivial solution, when all t_j are equal, is

$$t_j = t^* = \frac{a}{b_1 + \ldots + b_q} \in (0, 1], \text{ for } j = 1, 2, \dots, q+1,$$

and it leads to a trivial solution of (3.5).

For non-trivial solutions, let $t = a(b_1 + \ldots + b_q)^{-1} \in (0, 1]$. There exist $\varepsilon_j \in [0, t]$, $j = 1, \dots, q-1$ such that

$$c := \frac{\sum_{j=1}^{q-1} \varepsilon_j b_j}{b_q} \leq 1 - t.$$

The numbers t_j, defined as

$$t_\nu = \begin{cases} t - \varepsilon_\nu, & \text{for } \nu \neq q, \\ t + c, & \text{for } \nu = q, \end{cases}$$

satisfy the equality (B.2). ☐

We will prove that equation (B.1) is solvable if and only if

$$\alpha_{q+1} \leq 2 \sum_{j=1}^{q} \alpha_j. \tag{B.3}$$

Indeed, if the equation is solvable, (B.3) follows immediately by applying the triangle inequality. Conversely, suppose that (B.3) holds. From the previous lemma, there exist numbers $t_j \in [0,1]$ such that $\alpha_{q+1} = 2\sum_{j=1}^{q} \alpha_j t_j$. Then a solution of equation (B.1) is

$$\beta_j = \arccos t_j,\ \beta_{n+2-j} = 2\pi - \beta_j \,(\mathrm{mod}\, 2\pi),\ \text{for } j = 1,\ldots,q,$$
$$\beta_{q+1} = \pi.$$

Appendix C. For n odd, we discuss the solutions of the system

$$\sum_{j=1}^{q} \alpha_j (e^{ix_j} + e^{iy_j}) = 0, \tag{C.1}$$

$$\sum_{j=1}^{q} \mu_j (e^{ix_j} - e^{iy_j}) = 0, \tag{C.2}$$

with $q = \frac{n+1}{2}$, $\alpha_j, \mu_j > 0$ given and $x_j, y_j \in [0, 2\pi)$ unknowns. Due to our particular problems (systems (3.10)-(3.11) and (4.6)-(4.7)), we will also suppose that

$$\frac{\alpha_{j+1}}{\mu_{j+1}} \geq \frac{\alpha_j}{\mu_j} \quad \text{for all } j = 1,\ldots,q-1. \tag{C.3}$$

For $n = 1$ the incompatibility is immediate, so let us suppose in the sequel that $n \geq 3$.

PROPOSITION C.1. *Under the above assumptions, the following statements are true:*

1. *If there exists $k \in \{1,\ldots,q\}$ such that*

$$\alpha_k \mu_k > \alpha_k \sum_{j=1}^{k-1} \mu_j + \mu_k \sum_{j=k+1}^{q} \alpha_j, \tag{C.4}$$

then the system (C.1)-(C.2) is not solvable.

2. *If there exists $v \in \{1,\ldots,q\}$ such that*

$$\mu_v \geq \sum_{j=1,\, j\neq v}^{q} \mu_j, \tag{C.5}$$

$$\alpha_v \leq \sum_{j=1,\, j\neq v}^{q} \alpha_j, \tag{C.6}$$

then the system is solvable.

3. *If there exists $v \in \{1,\ldots,q\}$ such that*

$$\alpha_v \geq \sum_{j=1,\, j\neq v}^{q} \alpha_j, \tag{C.7}$$

$$\mu_v \leq \sum_{j=1,\, j\neq v}^{q} \mu_j, \tag{C.8}$$

then the system is solvable.

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Proof.

1. We suppose that the system is solvable and let $x_j, y_j, j = 1, \ldots, q$, be a solution. If we multiply the equations (C.1)-(C.2) by μ_k and α_k, respectively, and then we add them, we get, for all $k = 1, \ldots, q$,

$$2\alpha_k\mu_k e^{ix_k} = \sum_{j=1,\, j\neq k}^{q} -(\alpha_k\mu_j + \alpha_j\mu_k)e^{ix_j} + (\alpha_k\mu_j - \alpha_j\mu_k)e^{iy_j}.$$

Using the triangle inequality and the identity $a + b + |a - b| = 2\max\{a, b\}$, we obtain

$$\alpha_k\mu_k \leq \sum_{j=1,\, j\neq k}^{q} \max\{\alpha_k\mu_j, \alpha_j\mu_k\}.$$

Using now the hypothesis (C.3), this inequality can be written as

$$\alpha_k\mu_k \leq \alpha_k \sum_{j=1}^{k-1} \mu_j + \mu_k \sum_{j=k+1}^{q} \alpha_j,$$

which contradicts (C.4). In conclusion, the system is incompatible.

2. Applying Lemma B.1, there are numbers $t_j \in [0, 1],\ j = 1, \ldots, q,\ j \neq v$, such that

$$\alpha_v = \sum_{j=1,\, j\neq v} \alpha_j t_j.$$

We define the function $\varphi : [0, 2] \to \mathbb{R}$,

$$\varphi(t) = \sum_{j=1,\, j\neq v}^{q} \mu_j \sqrt{4 - t_j^2 t^2} - \mu_v \sqrt{4 - t^2}.$$

Since $\varphi(0) \cdot \varphi(2) \leq 0$, there exists $t_0 \in [0, 2]$ such that $\varphi(t_0) = 0$. A simple calculation shows that a solution of the system can be written as

$$x_j = \arccos \frac{t_0 t_j}{2},\ y_j = 2\pi - x_j \pmod{2\pi},\ \text{for } j \neq v,$$
$$x_v = \pi + \arccos \frac{t_0}{2},\ y_v = \pi - \arccos \frac{t_0}{2}.$$

3. Let $t_1 = \alpha_v^{-1} \sum_{j=1,\, j\neq v}^{q} \alpha_j \leq 1$ and define the function $\varphi : [0, 1] \to \mathbb{R}$,

$$\varphi(t) = \sqrt{1 - t^2} \sum_{j=1,\, j\neq v}^{q} \mu_j - \mu_v \sqrt{1 - t_1^2 t^2}.$$

Since $\varphi(0) \cdot \varphi(1) \leq 0$, there exists $t_0 \in [0, 1]$ such that $\varphi(t_0) = 0$. Then we define

$$\delta_\nu = \begin{cases} 2\alpha_\nu t_0, & \text{for } \nu \neq v, \\ 2t_0 \sum_{j=1,\, j\neq v}^{q} \alpha_j, & \text{for } \nu = v. \end{cases}$$

A simple calculation shows that a solution of the system can be written as

$$x_j = \arccos \frac{\delta_j}{2\alpha_j},\ y_j = 2\pi - x_j \pmod{2\pi},\ \text{for } j \neq v,$$
$$x_v = \pi + \arccos \frac{\delta_v}{2\alpha_v},\ y_v = \pi - \arccos \frac{\delta_v}{2\alpha_v}. \qquad \square$$

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