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STRUCTURAL AND RECURRENCE RELATIONS FOR HYPERGEOMETRIC-TYPE FUNCTIONS BY NIKIFOROV-UVAROV METHOD*

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Abstract. The functions of hypergeometric-type are the solutions $y = y_\nu(z)$ of the differential equation $\sigma(z)y'' + \tau(z)y' + \lambda y = 0$, where σ and τ are polynomials of degrees not higher than 2 and 1, respectively, and λ is a constant. Here we consider a class of functions of hypergeometric type: those that satisfy the condition $\lambda + \nu\tau' + \frac{1}{2}\nu(\nu-1)\sigma'' = 0$, where ν is an arbitrary complex (fixed) number. We also assume that the coefficients of the polynomials σ and τ do not depend on ν. To this class of functions belong Gauss, Kummer, and Hermite functions, and also the classical orthogonal polynomials. In this work, using the constructive approach introduced by Nikiforov and Uvarov, several structural properties of the hypergeometric-type functions $y = y_\nu(z)$ are obtained. Applications to hypergeometric functions and classical orthogonal polynomials are also given.

Key words. hypergeometric-type functions, recurrence relations, classical orthogonal polynomials

AMS subject classifications. 33C45, 33C05, 33C15

1. Introduction. When solving numerous theoretical and applied quantum mechanical problems, one is led to potentials that can be solved analytically; see, e.g., [3, 13, 19, 21, 22, 23]. In most cases the Schrödinger equation for such potentials can be transformed into the *generalized hypergeometric-type* differential equation [15] which has the form

$$u''(z) + \frac{\tilde{\tau}(z)}{\sigma(z)}u'(z) + \frac{\tilde{\sigma}(z)}{\sigma^2(z)}u(z) = 0,$$

where σ, $\tilde{\sigma}$ and $\tilde{\tau}$ are polynomials, $\deg[\sigma] \le 2$, $\deg[\tilde{\sigma}] \le 2$ and $\deg[\tilde{\tau}] \le 1$. By a certain change of dependent variable (see [15, Section 1, pp. 1–3]) this equation can be transformed into the hypergeometric-type equation

$$\sigma(z)y''(z) + \tau(z)y'(z) + \lambda y(z) = 0\,, \tag{1.1}$$

where σ and τ are polynomials of degrees not higher than two and one, respectively, and λ is a constant. Their solutions are known as *hypergeometric-type functions* and to this class belong the Bessel, Airy, Weber, Whittaker, Gauss, Kummer, and Hermite functions, the classical orthogonal polynomials, among others.

The class of functions $y = y_\nu(z)$ we are dealing with in this work corresponds to the solutions of the hypergeometric equation (1.1) under the condition

$$\lambda + \nu\tau' + \frac{\nu(\nu-1)}{2}\sigma'' = 0,$$

where ν is a complex number. One basic important property of this class of functions is that *their derivatives are again hypergeometric-type functions*. The converse is also true when $\deg[\sigma(s)] = 2 \;\vee\; \deg[\tau(s)] = 1$: *any hypergeometric-type function is the derivative of a hypergeometric-type function*. More precisely, we have the following results [15, Section 2, p. 6].

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1. if $y = y(z)$ is a solution of (1.1) then, the n-th derivative of $y(z)$, $v_n(z) := y^{(n)}(z)$, is a solution of

$$\sigma(z)v_n''(z) + \tau_n(z)v_n'(z) + \mu_n v_n(z) = 0, \tag{1.2}$$

where

$$\tau_n(z) = \tau(z) + n\sigma'(z) \tag{1.3}$$

and

$$\mu_n = \mu_n(\lambda) = \lambda + n\tau' + \frac{n(n-1)}{2}\sigma'';$$

2. if $v_n(z)$ is a solution of (1.2) and $\mu_k \neq 0$ for $k = 1, \dots, n-1$, then $v_n = y^{(n)}(z)$ where $y = y(z)$ is a solution of (1.1).

Joining these two properties it is possible to derive many other properties [15, pp. 14, 207, and 265]. Numerous structural properties of this class of functions have been studied in the last two decades [5, 6, 7, 8, 9, 25, 26].

The recurrence relations for the special functions (and thus for the associated wave functions) are interesting not only from the theoretical point of view (they are useful for computing the values of matrix elements of certain physical quantities; see, e.g., [5] and references therein), but also they can be used to numerically compute the values of the functions as well as their derivatives, as is shown in our previous paper [2] for the case of Laguerre polynomials and the associated wave functions of the harmonic oscillator and of the hydrogen atom. Nevertheless, we need to point out that, although the recurrence relations seem to be more useful for the evaluation of the corresponding functions than other direct methods, one should be very careful when using them; see, e.g., the nice surveys [11, 24] on numerical evaluation and the convergence problem that appears when dealing with recurrence relations, or the most recent results [17, 18] for the case of the hypergeometric function ${}_2F_1$.

In the present paper we obtain, in a unified way, several algebraic characteristics (recurrence and structural relations) for the solutions of the hypergeometric equation (1.1), i.e., for the hypergeometric-type functions. In particular, we find several new recurrence and structural relations for the solutions of the hypergeometric equation (1.1) in terms of the polynomial coefficients σ and τ, namely, Theorems 3.3 and 3.5; Corollaries 3.8, 3.9, 3.10, 3.12, and 3.14. For the particular cases of *hypergeometric*, *confluent hypergeometric*, and *Hermite* equations, some of these results can be obtained also by using the properties of the hypergeometric functions; see [16, Section 33]. However, here we will use an alternative and more general *direct* approach based on the Nikiforov and Uvarov method [15, §3–§4], which allows us to obtain recurrences relations *à la carte*. In this case we concentrate our effort on the not so well known fourth-term recurrence relations for the hypergeometric-type functions. Another advantage of this method is that it enables derivation of the recurrences in terms of the coefficients of the differential equation (1.1), and therefore it can be easily implemented in any computer algebra system; see, e.g., [12].

Finally, let us mention that many of the recurrence relations involving derivatives can be obtained by appropriate combinations of three recurrence relations of the hypergeometric-type functions; namely, the three-term recurrence relation and the two differentiation formulas of these functions given in [7]; see identities (5), (27), and (30), pp. 713, 716, and 717, respectively. This technique in not recommended, since the corresponding computations are cumbersome and the resulting coefficients are not, in general, as simple as the ones we present here. Let us also mention that for the case when the coefficients σ and τ of the equation (1.1)

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change with the spectral parameter λ, a general theory that extents the Nikiforov-Uvarov one is presented in [27].

The structure of the paper is as follows: In Section 2 the preliminary results are presented. The main results of the paper are in Section 3, where three four-term recurrence relations are obtained and from them, several three-term recurrence relations are explicitly written down. Finally, in Section 4, applications to the hypergeometric, confluent hypergeometric, and Hermite functions, as well as to the classical orthogonal polynomials are given.

2. Preliminaries. Here we will follow the notation and results of [15]. The above properties 1 and 2 allow us to construct a family of particular solutions of (1.1) for a given λ. In fact, when $\mu_n = 0$, (1.2) has the particular solution $v_n(z) = C$, (constant). By property 2, $v_n(z) = y^{(n)}$ where $y = y(z)$ is a solution of (1.1). This means that when

$$\lambda = \lambda_n := -n\tau' - \frac{n(n-1)}{2}\sigma'', \tag{2.1}$$

the equation (1.1) has a (particular) polynomial solution $y(z) = y_n(z)$, with $\deg[y_n(z)] = n$. Such polynomials are known as *polynomials of hypergeometric type* and correspond to the case when $\lambda = \lambda_n$ is given by (2.1). In particular, for them we have the *Rodrigues formula*

$$y_n(z) = \frac{B_n}{\rho(z)}\left[\sigma^n(z)\rho(z)\right]^{(n)}, \tag{2.2}$$

where the B_n, $n = 0, 1, 2, \ldots$, are normalizing constants and $\rho(z)$ is a solution of the Pearson equation

$$\left[\sigma(z)\rho(z)\right]' = \tau(z)\rho(z)\,. \tag{2.3}$$

Assuming that ρ is an analytic function on and inside a closed contour $\mathcal{C}$ surrounding the point $s = z$ and making use of the Cauchy's integral theorem (see, e.g., [10]), we may write

$$y_n(z) = \frac{C_n}{\rho(z)}\int_{\mathcal{C}} \frac{\sigma^n(s)\rho(s)}{(s-z)^{(n+1)}}ds\,, \tag{2.4}$$

where the $C_n = n!B_n/(2\pi i)$ is a normalizing constant and $\rho(z)$ satisfies (2.3). This suggests to look for a particular solution of (1.1) of the form

$$y_\nu(z) = \frac{C_\nu}{\rho(z)}\int_{\mathcal{C}} \frac{\sigma^\nu(s)\rho(s)}{(s-z)^{(\nu+1)}}ds\,, \tag{2.5}$$

where C_ν is a normalizing constant and ν is an arbitrary complex parameter connected with λ by

$$\lambda = \lambda_\nu = -\nu\tau' - \frac{\nu(\nu-1)}{2}\sigma''\,. \tag{2.6}$$

The following theorem asserts that the above suggestion is true.

Theorem A [15, p. 10]. *Let $\rho(z)$ satisfy the Pearson equation (2.3), where ν is a solution of (2.6), and let D be a region of the complex plane which contains the piecewise smooth curve $\mathcal{C}$ of finite length. Then, equation (1.1) has a particular solution of the form (2.5) provided that the functions $\frac{\sigma^\nu(s)\rho(s)}{(s-z)^{(\nu+k)}}$, for $k = 1, 2$,*

- *are continuous as functions of the variables $s \in \mathcal{C}$, $z \in D$;*

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- *for each fixed $s \in \mathcal{C}$, they are analytic as functions of $z \in D$;*

and $\mathcal{C}$ is such that $\left.\frac{\sigma^{\nu+1}(s)\rho(s)}{(s-z)^{\nu+2}}\right|_{s_1}^{s_2} = 0$, where s_1 and s_2 are the endpoints of $\mathcal{C}$.

If the integral in (2.5) is an improper one, then the result remains valid if the convergence of the integral is uniform [10, p. 188].

In the next sections, generalizing (1.3), we will use the notation

$$\tau_\nu(z) = \tau(z) + \nu\sigma'(z) = \tau'_\nu\, z + \tau_\nu(0), \quad \nu \in \mathbb{C}, \tag{2.7}$$

and, in order to keep valid property 2 of Section 1, we will restrict ourselves to the condition $\deg[\sigma(s)] = 2 \ \vee\ \deg[\tau(s)] = 1$.

3. Recurrence relations for the hypergeometric-type functions. Now we are ready to establish the main results of this paper.

3.1. Four-term recurrence relations. First, we prove the following theorem.

THEOREM 3.1. *Consider the hypergeometric-type functions $y^{(k)}_{\nu-1}(z)$, $y^{(k)}_\nu(z)$, $y^{(k+1)}_\nu(z)$, and $y^{(k+1)}_{\nu+1}(z)$ defined by (2.5). Suppose that $\rho(z)$ is a solution of (2.3) and*

$$\left.\frac{\sigma^\nu(s)\rho(s)}{(s-z)^{\nu-k-1}}s^m\right|_{s_1}^{s_2} = 0, \quad m = 0, 1, 2, ..., \tag{3.1}$$

where s_1 and s_2 are the end points of $\mathcal{C}$. Then, there exist polynomial coefficients $A_{ik}(z)$, $i = 1, 2, 3, 4$, not all identically zero, such that

$$A_{1k}(z)y^{(k)}_{\nu-1}(z) + A_{2k}(z)y^{(k)}_\nu(z) + A_{3k}(z)y^{(k+1)}_\nu(z) + A_{4k}(z)y^{(k+1)}_{\nu+1}(z) = 0. \tag{3.2}$$

Moreover, the functions A_{ik}, $i = 1, 2, 3, 4$ are given by

$$\begin{cases} A_{1k}(z) = -\tau'_\nu\tau'_{\frac{\nu+k-1}{2}}\tau'_{\frac{\nu+k-2}{2}}\left[\tau^2_{\nu-1}(0)\dfrac{\sigma''}{2} - \tau'_{\nu-1}\Big(\tau_{\nu-1}(0)\sigma'(0) - \tau'_{\nu-1}\sigma(0)\Big)\right] \\ \qquad\qquad \times\dfrac{C_\nu}{C_{\nu-1}}\Big(R(z) - 2\sigma'(z)\Big), \\ A_{2k}(z) = (\nu-k)\tau'_{\frac{\nu}{2}-1}\tau'_\nu\tau'_{\frac{\nu+k-1}{2}}\left[\Big(R(z)-2\sigma'(z)\Big)\Big(\tau_{\nu-1}(z)\dfrac{\sigma''}{2}-\sigma'(z)\tau'_{\nu-1}\Big)-\tau'_{\nu-1}\sigma''\sigma(z)\right], \\ A_{3k}(z) = \left[\tau'_\nu\tau'_{\frac{\nu+k-1}{2}}R(z) + (\nu-k)\dfrac{(\sigma'')^2}{2}\tau_\nu(z) - 2\tau'_{\nu-\frac{1}{2}}\tau'_\nu\sigma'(z)\right]\tau'_{\frac{\nu}{2}-1}\tau'_{\nu-1}\sigma(z), \\ A_{4k}(z) = (\nu-k)\dfrac{C_\nu}{C_{\nu+1}}\tau'_{\frac{\nu}{2}-1}\tau'_{\frac{\nu-1}{2}}\tau'_{\nu-1}\sigma''\sigma(z), \end{cases} \tag{3.3}$$

where $R(z)$ is an arbitrary function of z.

Proof. From [15, Eq. (9), p. 17], we have

$$y^{(k)}_\nu(z) = \frac{C^{(k)}_\nu}{\sigma^k(z)\rho(z)}\int_{\mathcal{C}}\frac{\sigma^\nu(s)\rho(s)}{(s-z)^{\nu-k+1}}ds, \qquad C^{(n)}_\nu = \left(\prod_{j=0}^{n-1}\tau'_{\frac{\nu+j-1}{2}}\right)C_\nu. \tag{3.4}$$

Now, using [15, Eqs. (4), p. 16 and (9), p. 17],

$$y^{(k)}_\nu(z) = \frac{C^{(k)}_\nu}{\sigma^k(z)\rho(z)}\frac{1}{\nu-k}\int_{\mathcal{C}}\frac{\tau_{\nu-1}(s)\sigma^{\nu-1}(s)\rho(s)}{(s-z)^{\nu-k}}ds. \tag{3.5}$$

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Substituting the above expressions (3.4) and (3.5) in

$$S(z) = A_{1k}(z)y_{\nu-1}^{(k)}(z) + A_{2k}(z)y_{\nu}^{(k)}(z) + A_{3k}(z)y_{\nu}^{(k+1)}(z) + A_{4k}(z)y_{\nu+1}^{(k+1)}(z),$$

we obtain

$$S(z) = \frac{1}{\sigma^{k+1}(z)\rho(z)} \int_{\mathcal{C}} \frac{\sigma^{\nu-1}(s)\rho(s)}{(s-z)^{\nu-k}} P(s)ds\,, \tag{3.6}$$

where

$$P(s) = A_{1k}(z)C_{\nu-1}^{(k)}\sigma(z) + A_{2k}(z)\frac{C_{\nu}^{(k)}}{\nu-k}\sigma(z)\tau_{\nu-1}(s) + A_{3k}(z)C_{\nu}^{(k+1)}\sigma(s) +$$
$$A_{4k}(z)\frac{C_{\nu+1}^{(k+1)}}{\nu-k}\tau_{\nu}(s)\sigma(s)\,. \tag{3.7}$$

Let us define a function $Q(z, s)$ which is, for every fixed z, a polynomial in s such that

$$\frac{\sigma^{\nu-1}(s)\rho(s)}{(s-z)^{\nu-k}} P(s) = \frac{\partial}{\partial s}\left[\frac{\sigma^{\nu}(s)\rho(s)}{(s-z)^{\nu-k-1}}Q(z,s)\right].$$

If such a function Q exists, then the integral (3.6) vanishes by the boundary conditions (3.1), and therefore (3.2) holds. Let us show that the aforementioned function Q always exists.

Taking the derivative of the right hand side of the last equality, one gets

$$P(s) = [\tau_{\nu-1}(s)(s-z) - (\nu-k-1)\sigma(s)]\,Q(z,s) + \sigma(s)(s-z)\frac{\partial Q}{\partial s}(z,s)\,. \tag{3.8}$$

Comparing the expressions (3.7) and (3.8), we may conclude that, with respect to the variable s, $\deg\,[Q(z,s)] = \deg_s\,[Q(z,s)] \leq 1$. Thus, using the expansions

$$Q(z,s) = Q(z,z) + \frac{\partial Q}{\partial s}(z,z)(s-z),$$

$$\tau_{\nu}(s) = \tau_{\nu}(z) + \tau_{\nu}'(s-z), \quad \sigma(s) = \sigma(z) + \sigma'(z)(s-z) + \frac{\sigma''}{2}(s-z)^2\,, \tag{3.9}$$

as well as (2.7), we have

$$\left\{\begin{aligned}
&A_{1k}(z)C_{\nu-1}^{(k)}\sigma(z) + A_{2k}(z)\frac{C_{\nu}^{(k)}}{\nu-k}\tau_{\nu-1}(z)\sigma(z) + A_{3k}(z)C_{\nu}^{(k+1)}\sigma(z) + \\
&\qquad + A_{4k}(z)\frac{C_{\nu+1}^{(k+1)}}{\nu-k}\tau_{\nu}(z)\sigma(z) = -(\nu-k-1)\sigma(z)Q(z,z), \\
&A_{2k}(z)\frac{C_{\nu}^{(k)}}{\nu-k}\tau_{\nu-1}'\sigma(z) + A_{3k}(z)C_{\nu}^{(k+1)}\sigma'(z) + A_{4k}(z)\frac{C_{\nu+1}^{(k+1)}}{\nu-k}\left[\tau_{\nu}(z)\sigma'(z) + \tau_{\nu}'\sigma(z)\right] = \\
&\qquad \tau_k(z)Q(z,z) - (\nu-k-2)\sigma(z)\frac{\partial Q}{\partial s}(z,z), \\
&A_{3k}(z)C_{\nu}^{(k+1)}\frac{\sigma''}{2} + A_{4k}(z)\frac{C_{\nu+1}^{(k+1)}}{\nu-k}\left[\tau_{\nu}(z)\frac{\sigma''}{2} + \tau_{\nu}'\sigma'(z)\right] = \tau_{\frac{\nu+k-1}{2}}'Q(z,z) + \\
&\qquad \tau_{k+1}(z)\frac{\partial Q}{\partial s}(z,z), \\
&A_{4k}(z)\frac{C_{\nu+1}^{(k+1)}}{\nu-k}\tau_{\nu}'\sigma'' = 2\tau_{\frac{\nu+k}{2}}'\frac{\partial Q}{\partial s}(z,z)\,.
\end{aligned}\right. \tag{3.10}$$

Therefore, we have an indeterminate linear system of four equations with six unknowns: the functions $A_{ik}(z)$, $i = 1, 2, 3, 4$, and the coefficients in the variable z of the polynomial (on s) $Q(z, s)$. This guarantees not only the existence of the functions $A_{ik}(z)$, $i = 1, 2, 3, 4$, but also of the polynomial $Q(z, s)$ introduced above. Assuming that $\sigma'' \neq 0$, the above system can be written as

$$\begin{cases} A_{1k}(z)C^{(k)}_{\nu-1}\sigma(z) = -\dfrac{\tau_{\nu-1}(z)}{\tau'_{\nu-1}}\left(Q(z,z) - \dfrac{2}{\sigma''}\sigma'(z)\dfrac{\partial Q}{\partial s}(z,z)\right)\times \\ \qquad\qquad\qquad \left[\tau_{\nu-1}(z) + \dfrac{2}{\sigma''}\tau'_{\nu-1}\times\left(\dfrac{\tau'_{\nu-1}}{\tau_{\nu-1}(z)}\sigma(z) - \sigma'(z)\right)\right], \\ A_{2k}(z)\dfrac{C^{(k)}_{\nu}}{\nu-k} = \dfrac{1}{\sigma(z)\tau'_{\nu-1}}\left(\tau_{\nu-1}(z) - \dfrac{2}{\sigma''}\sigma'(z)\tau'_{\nu-1}\right)\left(Q(z,z) - \dfrac{2}{\sigma''}\sigma'(z)\dfrac{\partial Q}{\partial s}(z,z)\right) \\ \qquad\qquad\qquad -\dfrac{2}{\sigma''}\dfrac{\partial Q}{\partial s}(z,z), \\ A_{3k}(z)C^{(k+1)}_{\nu} = \dfrac{2}{\sigma''}\left[\tau'_{\frac{\nu+k-1}{2}}Q(z,z) + \left(\dfrac{\tau_\nu(z)}{\tau'_\nu}(\nu-k)\dfrac{\sigma''}{2} - \dfrac{2}{\sigma''}\sigma'(z)\tau'_{\nu-\frac{1}{2}}\right)\dfrac{\partial Q}{\partial s}(z,z)\right], \\ A_{4k}(z)\dfrac{C^{(k+1)}_{\nu+1}}{\nu-k} = \dfrac{2}{\sigma''}\dfrac{\tau'_{\frac{\nu+k}{2}}}{\tau'_\nu}\dfrac{\partial Q}{\partial s}(z,z)\,. \end{cases}$$

Substituting the values A_{ik}, $i = 1, 2, 3, 4$, from above in the equation (3.2), which is a homogeneous linear equation, choosing $Q(z, z) = \frac{\partial Q}{\partial s}(z, z)R(z)/\sigma''$, where $R(z)$ is an arbitrary function of z, using relations (3.4) for the constants $C^{(n)}_\nu$, and simplifying the common factors, we get the non-trivial solution (3.3). □

Notice that formulae (3.3) are still valid for $\sigma'' = 0$. This is a consequence of (3.2) and the principle of analytic continuation. Moreover, if one chooses $R(z)$ to be a polynomial in z then the corresponding expressions for the coefficients $A_{ik}(z)$, $i = 1, 2, 3, 4$, in (3.3) are polynomials in z. Finally, let us mention that this method enables one to construct other types of solutions, not necessarily polynomials, since $R(z)$ is an arbitrary function of z.

REMARK 3.2. Let us briefly analyze the cases when $\sigma'' = 0$. In this case, from (3.3), we have two possibilities.

1. If $\deg[\sigma(s)] = 1 \wedge \deg[\tau(s)] = 1$, then $A_{4k} = 0$.
2. If $\deg[\sigma(s)] = 0 \wedge \deg[\tau(s)] = 1$, then $A_{2k} = 0 = A_{4k}$.

Since we are looking for solutions with coefficients $A_{ik}(z)$, $i = 1, 2, 3, 4$, which do not vanish at the same time, we need to compare the expressions (3.7) and (3.8). From this analysis follows that $deg_s[Q(z, s)] = 0$. In the first case $Q(z, s)$ is a constant $\neq 0$, while in the second one $Q(z, s)$ is identically zero. Notice that in both cases the resulting systems are not equivalent to (3.10). The coefficients in these two cases are given as follows,

- when $\deg\sigma = 1$, $\sigma(s) = \sigma(z) + \sigma'(s - z)$ and $\tau(s) = \tau(z) + \tau'(s - z)$, by

$$\begin{cases} A_{1k}(z) = \dfrac{2C_\nu}{C_{\nu-1}}\tau'\Big(\tau'\sigma(z) - \sigma'\tau_{\nu-1}(z)\Big), \\ A_{2k}(z) = (\nu-k)\tau'\sigma', \\ A_{3k}(z) = -(\nu-k)\big(\sigma'\big)^2 - 2\tau'\sigma(z), \\ A_{4k}(z) = \dfrac{C_\nu}{C_{\nu+1}}(\nu-k)\sigma'\,; \end{cases}$$

- when $\deg\sigma = 0$, $\sigma(s) = \sigma(z)$ and $\tau(s) = \tau(z) + \tau'(s - z)$, by

$$\begin{cases} A_{1k}(z) = -\dfrac{C_\nu}{C_{\nu-1}}\tau' A_{3k}(z), \\ A_{2k}(z) = -\dfrac{C_{\nu+1}}{C_\nu}\tau' A_{4k}(z), \end{cases}$$

  where $A_{3k}(z)$ and $A_{4k}(z)$ are arbitrary polynomials in z.

In a similar fashion we can prove the following theorem.

THEOREM 3.3. *Consider the functions of hypergeometric type $y_{\nu-1}^{(k+1)}(z)$, $y_\nu^{(k)}(z)$, $y_\nu^{(k+1)}(z)$, and $y_{\nu+1}^{(k+1)}(z)$. Suppose that $\rho(z)$ is a solution of (2.3), satisfying condition (3.1). Then, there exist polynomial coefficients $B_{ik}(z)$, $i = 1,2,3,4$, not all identically zero, such that*

$$B_{1k}(z)y_{\nu-1}^{(k+1)}(z) + B_{2k}(z)y_\nu^{(k)}(z) + B_{3k}(z)y_\nu^{(k+1)}(z) + B_{4k}(z)y_{\nu+1}^{(k+1)}(z) = 0\,. \tag{3.11}$$

Moreover, the functions B_{ik}, $i = 1,2,3,4$, are given by

(3.12)
$$\begin{cases} B_{1k}(z) = \dfrac{C_\nu}{C_{\nu-1}}\tau'_\nu\tau'_{\frac{\nu+k-1}{2}}\left[\tau^2_{\nu-1}(0)\dfrac{\sigma''}{2} - \tau'_{\nu-1}\Big(\tau_{\nu-1}(0)\sigma'(0) - \tau'_{\nu-1}\sigma(0)\Big)\right]\times \\ \qquad\qquad \Big(R(z) - 2\sigma'(z)\Big), \\ B_{2k}(z) = -(\nu-k)\tau'_\nu\tau'_{\frac{\nu}{2}-1}\tau'_{\frac{\nu+k-1}{2}}\left[R(z)\tau'_{\nu-1} + \Big(\tau_{\nu-1}(z)\sigma'' - 2\sigma'(z)\tau'_{\nu-1}\Big)\right], \\ B_{3k}(z) = \tau'_\nu\tau'_{\frac{\nu}{2}-1}\left[\tau'_{\frac{\nu+k-1}{2}}R(z) + (\nu-k)\dfrac{(\sigma'')^2}{2\tau'_\nu}\tau_\nu(z) - 2\tau'_{\nu-\frac{1}{2}}\sigma'(z)\right]\tau_{\nu-1}(z), \\ B_{4k}(z) = (\nu-k)\sigma''\dfrac{C_\nu}{C_{\nu+1}}\tau'_{\frac{\nu}{2}-1}\tau'_{\frac{\nu-1}{2}}\tau_{\nu-1}(z), \end{cases}$$

where $R(z)$ is an arbitrary function of z.

REMARK 3.4. As in Remark 3.2, when $\sigma'' = 0$ from (3.12), the following two cases apply.

1. If $\deg[\sigma(s)] = 1 \wedge \deg[\tau(s)] = 1$, then $B_{4k} = 0$.
2. If $\deg[\sigma(s)] = 0 \wedge \deg[\tau(s)] = 1$, then $B_{4k} = 0$.

Solving the corresponding systems, we find

- $\deg\sigma = 1$ and $\deg\tau = 1$, $\sigma(s) = \sigma(z) + \sigma'(s-z)$ and $\tau(s) = \tau(z) + \tau'(s-z)$

$$\begin{cases} B_{1k}(z) = \dfrac{2C_\nu}{C_{\nu-1}}\Big(\sigma'\tau_{\nu-1}(z) - \tau'\sigma(z)\Big), \\ B_{2k}(z) = (\nu-k)\tau', \\ B_{3k}(z) = -2\tau_{\frac{3\nu-k-2}{2}}, \\ B_{4k}(z) = \dfrac{C_\nu}{C_{\nu+1}}(\nu-k)\,; \end{cases}$$

- $\deg\sigma = 0$ and $\deg\tau = 1$, $\sigma(s) = \sigma(z)$ and $\tau(s) = \tau(z) + \tau'(s-z)$

$$\begin{cases} B_{3k}(z) = \dfrac{C_{\nu-1}}{\tau' C_\nu}\tau(z)A_{1k}(z), \\ B_{4k}(z) = -\dfrac{C_{\nu-1}}{\tau' C_{\nu+1}}(\nu-k)A_{1k}(z) - \dfrac{C_\nu}{\tau' C_{\nu+1}}A_{2k}(z)\,, \end{cases}$$

  where $B_{1k}(z)$ and $B_{2k}(z)$ are arbitrary polynomials in z.

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THEOREM 3.5. *Consider the functions of hypergeometric type $y^{(k)}_{\nu-1}(z)$, $y^{(k)}_{\nu}(z)$, $y^{(k+1)}_{\nu}(z)$, and $y^{(k)}_{\nu+1}(z)$. Suppose that $\rho(z)$ is a solution of (2.3), satisfying condition (3.1). Then, there exist polynomial coefficients $D_{ik}(z)$, $i=1,2,3,4$, not all identically zero, such that*

$$D_{1k}(z)y^{(k)}_{\nu-1}(z)+D_{2k}(z)y^{(k)}_{\nu}(z)+D_{3k}(z)y^{(k+1)}_{\nu}(z)+D_{4k}(z)y^{(k)}_{\nu+1}(z)=0. \tag{3.13}$$

Moreover, the functions D_{ik}, $i=1,2,3,4$ are given by

$$\begin{cases} D_{1k}=\dfrac{C_\nu}{C_{\nu-1}}\tau'_\nu\tau'_{\frac{\nu+k-1}{2}}\tau'_{\frac{\nu+k-2}{2}}\left(H(z)-\tau'_{\nu-\frac12}\right)\times \\ \qquad\qquad \left[\tau^2_{\nu-1}(0)\frac{\sigma''}{2}+\tau'_{\nu-1}\Big(\sigma(0)\tau'_{\nu-1}-\sigma'(0)\tau_{\nu-1}(0)\Big)\right], \\ D_{2k}=(\nu-k)\tau'_{\nu-\frac12}\tau'_\nu\tau'_{\frac{\nu}{2}-1}\tau'_{\frac{\nu+k-1}{2}}\times \\ \qquad\qquad \left[\Big(\tau(0)\sigma''-\sigma'(0)\tau'\Big)-\dfrac{\sigma''}{2}\tau_{\nu-1}(z)+H(z)\dfrac{\sigma'(0)\tau'_\nu-\tau_\nu(0)\sigma''}{\tau'_\nu}\right], \\ D_{3k}=-\tau'_{\nu-1}\tau'_{\nu-\frac12}\tau'_{\frac{\nu}{2}-1}\tau'_\nu\left(H(z)-\tau'_{\frac{\nu+k-1}{2}}\right)\sigma(z), \\ D_{4k}=H(z)(\nu-k)(\nu-k+1)\dfrac{\sigma''}{2}\dfrac{C_\nu}{C_{\nu+1}}\tau'_{\nu-1}\tau'_{\frac{\nu}{2}-1}\tau'_{\frac{\nu-1}{2}}, \end{cases} \tag{3.14}$$

where $H(z)$ is an arbitrary function of z.

Proof. Substituting (3.4) and (3.5) in the equation

$$S(z)=D_{1k}(z)y^{(k)}_{\nu-1}(z)+D_{2k}(z)y^{(k)}_{\nu}(z)+D_{3k}(z)y^{(k+1)}_{\nu}(z)+D_{4k}(z)y^{(k)}_{\nu+1}(z),$$

we obtain

$$S(z)=\frac{1}{\sigma^{k+1}(z)\rho(z)}\int_C\frac{\sigma^{\nu-1}(s)\rho(s)}{(s-z)^{\nu-k}}P(s)ds\,,$$

where $P(s)$ is given by

$$\begin{aligned} P(s)=D_{1k}C^{(k)}_{\nu-1}\sigma(z)+D_{2k}\frac{C^{(k)}_\nu}{\nu-k}\sigma(z)\tau_{\nu-1}(s)+D_{3k}C^{(k+1)}_\nu\sigma(s)+ \\ D_{4k}\frac{C^{(k)}_{\nu+1}}{(\nu-k-1)(\nu-k)}\sigma(z)\left(\tau'_\nu\sigma(s)+\tau_\nu(s)\tau_{\nu-1}(s)\right)\,. \end{aligned} \tag{3.15}$$

Reasoning as in the proof of Theorem 3.1, we define a polynomial $Q(z,s)$ in the variable s such that

$$\frac{\sigma^{\nu-1}(s)\rho(s)}{(s-z)^{\nu-k}}P(s)=\frac{\partial}{\partial s}\left[\frac{\sigma^\nu(s)\rho(s)}{(s-z)^{\nu-k-1}}Q(z,s)\right].$$

Therefore, if the boundary conditions (3.1) hold, $S(z)=0$ and (3.13) follows. Taking the derivative of the right-hand side we find

$$P(s)=\left[\tau_{\nu-1}(s)(s-z)-(\nu-k-1)\sigma(s)\right]Q(z,s)+\sigma(s)(s-z)\frac{\partial Q}{\partial s}(z,s)\,. \tag{3.16}$$

Hence, by comparing (3.15) with (3.16), we conclude that $\deg_s[Q(z,s)]=0$, i.e., $Q(z,s)=f(z)$, which we choose, without loss of generality, equal to 1.

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Substituting the expansions (3.9) of $\tau_{\nu-1}(s)$, $\tau_\nu(s)$, and $\sigma(s)$ in powers of $s-z$ in (3.15) and (3.16), we obtain

$$\begin{cases} D_{1k}C_{\nu-1}^{(k)} + D_{2k}\dfrac{C_\nu^{(k)}}{\nu-k}\tau_{\nu-1}(z) + D_{3k}(z)C_\nu^{(k+1)} + \\ \qquad D_{4k}\dfrac{C_{\nu+1}^{(k)}}{(\nu-k)(\nu-k+1)}\left[\tau_\nu'\sigma(z)+\tau_\nu(z)\tau_{\nu-1}(z)\right] = -(\nu-k-1), \\ D_{2k}\dfrac{C_\nu^{(k)}}{\nu-k}\sigma(z)\tau_{\nu-1}' + D_{3k}C_\nu^{(k+1)}\sigma'(z) + D_{4k}\dfrac{C_{\nu+1}^{(k)}}{(\nu-k)(\nu-k+1)}\sigma(z)2\tau_\nu(z)\tau_{\nu-\frac12}' \\ \qquad = \tau_k(z), \\ D_{3k}C_\nu^{(k+1)}\dfrac{\sigma''}{2} + D_{4k}\dfrac{C_{\nu+1}^{(k)}}{(\nu-k)(\nu-k+1)}\sigma(z)\tau_\nu'\tau_{\nu-\frac12}' = \tau_{\frac{\nu+k-1}{2}}' . \end{cases}$$

Assuming $\sigma''\neq 0$, from last equation we get

$$D_{3k}(z)C_\nu^{(k+1)} = \frac{2}{\sigma''}\left[\tau_{\frac{\nu+k-1}{2}}' - D_{4k}(z)\frac{C_{\nu+1}^{(k)}}{(\nu-k)(\nu-k+1)}\sigma(z)\tau_\nu'\tau_{\nu-\frac12}'\right].$$

Choosing now

$$D_{4k}(z) = R(z)\tau_{\frac{\nu+k-1}{2}}'\frac{(\nu-k)(\nu-k+1)}{C_{\nu+1}^{(k)}\sigma(z)\tau_\nu'\tau_{\nu-\frac12}'},$$

where $R(z)$ is an arbitrary function of z, we obtain

$$D_{3k}(z)C_\nu^{(k+1)} = \frac{2}{\sigma''}\tau_{\frac{\nu+k-1}{2}}'\left(1-R(z)\right),$$

and therefore

$$\begin{cases} D_{4k}(z) = R(z)\tau_{\frac{\nu+k-1}{2}}'\dfrac{(\nu-k)(\nu-k+1)}{C_{\nu+1}^{(k)}\tau_\nu'\tau_{\nu-\frac12}'}, \\ D_{3k}(z)C_\nu^{(k+1)} = \dfrac{2}{\sigma''}\tau_{\frac{\nu+k-1}{2}}'\Big(1-R(z)\Big)\sigma(z), \\ D_{2k}(z)C_\nu^{(k)} = \dfrac{\nu-k}{\tau_{\nu-1}'}\left[2\left(\tau(0)-\dfrac{\sigma'(0)}{\sigma''}\tau'\right)-\tau_{\nu-1}(z)+2R(z)\tau_{\frac{\nu+k-1}{2}}'\left(\dfrac{\sigma'(0)}{\sigma''}-\dfrac{\tau_\nu(0)}{\tau_\nu'}\right)\right], \\ D_{1k}(z)C_{\nu-1}^{(k)} = -\dfrac{\tau_{\nu-1}(z)}{\tau_{\nu-1}'}\left[\tau_{\nu-1}(z)-\dfrac{2}{\sigma''}\tau_{\nu-1}'+2R(z)\tau_{\frac{\nu+k-1}{2}}'\left(\dfrac{\sigma'(0)}{\sigma''}-\dfrac{\tau_\nu(0)}{\tau_\nu'}\right)\right] - \\ \qquad (\nu-k-1)\sigma(z) - R(z)\tau_{\frac{\nu+k-1}{2}}'\dfrac{\tau_\nu'\sigma(z)+\tau_\nu(z)\tau_{\nu-1}(z)}{\tau_\nu'\tau_{\nu-\frac12}'} - \dfrac{2}{\sigma''}\sigma(z)\tau_{\frac{\nu+k-1}{2}}'(1-R(z)). \end{cases}$$

If we now substitute the above values D_{ik}, $i=1,2,3,4$, in (3.13), put $R(z)=H(z)/\tau_{\frac{\nu+k-1}{2}}'$, where $H(z)$ is a function of z, and simplify the resulting expressions, we obtain the values (3.14). ∎

Notice that if we choose $H(z)$ to be a polynomial in z, then the corresponding coefficients A_{ik}, $i=1,2,3,4$, will be polynomials in z, too. Formulae (3.14) are still valid, by analytic continuation, when $\sigma''=0$.

REMARK 3.6. In the case when $\sigma''=0$, from (3.14), the following two cases apply.

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1. If $\deg[\sigma(s)] = 1 \wedge \deg[\tau(s)] = 1$, then $D_{4k}(z) = 0$.
2. If $\deg[\sigma(s)] = 0 \wedge \deg[\tau(s)] = 1$, then $D_{2k}(z) = 0 = A_{4k}(z)$.

Thus, a similar analysis yields

- in the first case, $\sigma(s) = \sigma(z) + \sigma(s - z)$ and $\tau(s) = \tau(z) + \tau'(s - z)$

$$\begin{cases} D_{1k}(z) = \dfrac{2C_\nu}{C_{\nu-1}}\tau_\nu(z)\Big(\tau'\sigma(z) - \sigma'\tau_{\nu-1}(z)\Big), \\ D_{2k}(z) = (\nu - k)\sigma'\tau_\nu(z), \\ D_{3k}(z) = -2\tau_{\frac{3\nu-k}{2}}\sigma(z), \\ D_{4k}(z) = \dfrac{C_\nu}{C_{\nu+1}}(\nu - k - 1)(\nu - k)\sigma' \, ; \end{cases}$$

- in the second case $\sigma(s) = \sigma(z)$ and $\tau(s) = \tau(z) + \tau'(s - z)$

$$\begin{cases} D_{1k}(z) = -\dfrac{C_\nu}{C_{\nu-1}}(\nu - k - 1)\tau'\sigma(z), \\ D_{2k}(z) = -(\nu - k)\tau(z), \\ D_{3k}(z) = \sigma(z), \\ D_{4k}(z) = \dfrac{C_\nu}{C_{\nu+1}}(\nu - k - 1)(\nu - k). \end{cases}$$

3.2. Three-term recurrence relations. In general, in order to obtain three-term recurrence relations involving functions of hypergeometric type and their derivatives of any order, one could follow the technique described in the previous section; see, e.g., [25]. Here we will obtain several three-term recurrence relations that follow from Theorem 3.1 (Corollaries 3.7–3.9), Theorem 3.3 (Corollary 3.10) and Theorem 3.5 (Corollaries 3.11–3.14), when one of the coefficients $A_{ik}(z)$, $i = 1, 2, 3, 4$, is chosen to be identically zero. Since the proofs of all Corollaries are quite similar we will include here only the first one.

COROLLARY 3.7.

$$E_{1k}(z)y_\nu^{(k)}(z) + E_{2k}(z)y_\nu^{(k+1)}(z) + E_{3k}(z)y_{\nu+1}^{(k+1)}(z) = 0, \tag{3.17}$$

$$\begin{cases} E_{1k}(z) = -\tau'_{\frac{\nu+k-1}{2}}\tau'_\nu, \\ E_{2k}(z) = -\tau'_\nu\sigma'(0) + \frac{\sigma''}{2}\left(\tau_\nu(0) - \tau'_\nu z\right), \\ E_{3k}(z) = \dfrac{C_\nu}{C_{\nu+1}}\tau'_{\frac{\nu-1}{2}}. \end{cases} \tag{3.18}$$

Proof. Using the fact that $R(z)$ in Theorem 3.1 is an arbitrary polynomial of z and putting $R(z) = 2\sigma'(z)$, we get $A_{1k} = 0$. Thus, relation (3.2) becomes

$$E_{1k}(z)y_\nu^{(k)}(z) + E_{2k}(z)y_\nu^{(k+1)}(z) + E_{3k}(z)y_{\nu+1}^{(k+1)}(z) = 0\,,$$

where the coefficients $E_{1k} = A_{2k}$, $E_{2k} = A_{3k}$ and $E_{3k} = A_{4k}$ are given by

$$\begin{cases} E_{1k}(z) = -(\nu - k)\tau'_{\frac{\nu+k-1}{2}}\sigma''\sigma(z), \\ E_{2k}(z) = \dfrac{\nu - k}{\tau'_\nu}\sigma''\sigma(z)\left[-\dfrac{1}{2}\left(2\tau'_\nu\sigma'(0) + \tau'_\nu\sigma''z - \tau_\nu(0)\sigma''\right)\right], \\ E_{3k}(z) = (\nu - k)\dfrac{C_\nu}{C_{\nu+1}}\dfrac{\tau'_{\frac{\nu-1}{2}}}{\tau'_\nu}\sigma''\sigma(z). \end{cases}$$

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Hence, after some simplifications, we obtain (3.18). □

The previous three-term recurrence relation was published in [9, relations (6)–(7), p. 663].

COROLLARY 3.8.

$$F_{1k}(z)y_{\nu-1}^{(k)}(z) + F_{2k}(z)y_{\nu}^{(k)}(z) + F_{3k}(z)y_{\nu+1}^{(k+1)}(z) = 0, \tag{3.19}$$

$$\begin{cases} F_{1k}(z) = \dfrac{C_\nu}{C_{\nu-1}}\tau'_{\frac{\nu+k-2}{2}}\left[\tau_{\nu-1}(0)\left(\tau'_{\nu-1}\sigma'(0) - \dfrac{\sigma''}{2}\tau_{\nu-1}(0)\right) - \left(\tau'_{\nu-1}\right)^2\sigma(0)\right]\times \\ \qquad\left(\dfrac{\sigma''}{2}\tau'_\nu z + \sigma'(0)\tau'_\nu - \dfrac{\sigma''}{2}\tau_\nu(0)\right), \\ F_{2k}(z) = (\nu-k)\tau'_{\frac{\nu}{2}-1}\left(\dfrac{\sigma''}{2}\tau'_\nu z + \sigma'(0)\tau'_\nu - \dfrac{\sigma''}{2}\tau_\nu(0)\right)\times \\ \qquad\left(\dfrac{\sigma''}{2}\tau_{\nu-1}(0) - \sigma'(0)\tau'_{\nu-1} - \dfrac{\sigma''}{2}\tau'_{\nu-1}z\right) - \tau'_{\frac{\nu}{2}-1}\tau'_{\frac{\nu+k-1}{2}}\tau'_{\nu-1}\tau'_\nu\sigma(z), \\ F_{3k}(z) = \dfrac{C_\nu}{C_{\nu+1}}\tau'_{\frac{\nu}{2}-1}\tau'_{\frac{\nu-1}{2}}\tau'_{\nu-1}\sigma(z). \end{cases} \tag{3.20}$$

COROLLARY 3.9.

$$G_{1k}(z)y_{\nu-1}^{(k)}(z) + G_{2k}(z)y_{\nu}^{(k+1)}(z) + G_{3k}(z)y_{\nu+1}^{(k+1)}(z) = 0,$$

$$\begin{cases} G_{1k}(z) = -\dfrac{C_\nu}{C_{\nu-1}}\tau'_{\frac{\nu+k-1}{2}}\tau'_{\frac{\nu+k-2}{2}}\tau'_\nu\left[\tau^2_{\nu-1}(0)\sigma'' - 2\tau'_{\nu-1}\left(\tau_{\nu-1}(0)\sigma'(0) - \tau'_{\nu-1}\sigma(0)\right)\right], \\ G_{2k}(z) = \dfrac{\nu-k}{2}\tau'_{\frac{\nu}{2}-1}\left(\tau_\nu(0)\sigma'' - 2\sigma'(0)\tau'_\nu - \sigma''\tau'_\nu z\right)\times \\ \qquad\left(\tau_{\nu-1}(0)\sigma'' - 2\sigma'(0)\tau'_{\nu-1} - \sigma''\tau'_{\nu-1}z\right) + 2\tau'_{\frac{\nu}{2}-1}\tau'_{\frac{\nu+k-1}{2}}\tau'_{\nu-1}\tau'_\nu\sigma(z), \\ G_{3k}(z) = (\nu-k)\dfrac{C_\nu}{C_{\nu+1}}\tau'_{\frac{\nu}{2}-1}\tau'_{\frac{\nu-1}{2}}\left(\tau_{\nu-1}(0)\sigma'' - 2\sigma'(0)\tau'_{\nu-1} - \sigma''\tau'_{\nu-1}z\right). \end{cases}$$

COROLLARY 3.10.

$$I_{1k}(z)y_{\nu-1}^{(k+1)}(z) + I_{2k}(z)y_{\nu}^{(k)}(z) + I_{3k}(z)y_{\nu+1}^{(k+1)}(z) = 0,$$

$$\begin{cases} I_{1k}(z) = \dfrac{C_\nu}{C_{\nu-1}}\left[\tau_{\nu-1}(0)\left(\sigma'(0)\tau'_{\nu-1} - \dfrac{\sigma''}{2}\tau_{\nu-1}(0)\right) - \left(\tau'_{\nu-1}\right)^2\sigma(0)\right] \\ \qquad\times\left[\sigma'(0)\tau'_\nu - \frac{\sigma''}{2}\left(\tau_\nu(0) - \tau'_\nu z\right)\right], \\ I_{2k}(z) = (\nu-k-1)\tau'_{\frac{\nu}{2}-1}\tau'_\nu\left(\sigma'(0)\tau'_{\nu-1} - \frac{\sigma''}{2}\tau_{\nu-1}(0)\right) + \tau_\nu(0)\tau'_{\frac{\nu}{2}-1}\tau'_{\frac{\nu+k}{2}}\tau'_{\nu-1} \\ \qquad+\tau'_{\frac{\nu}{2}-1}\tau'_{\nu-1}\tau'_{\nu-\frac{1}{2}}\tau'_\nu z, \\ I_{3k}(z) = -\dfrac{C_\nu}{C_{\nu+1}}\tau'_{\frac{\nu}{2}-1}\tau'_{\frac{\nu-1}{2}}\tau_{\nu-1}(z)\,. \end{cases}$$

COROLLARY 3.11.

$$J_{1k}(z)y^{(k)}_{\nu-1}(z) + J_{2k}(z)y^{(k)}_{\nu}(z) + J_{3k}(z)y^{(k)}_{\nu+1}(z) = 0,$$

$$\begin{cases} J_{1k}(z) = \dfrac{C_\nu}{C_{\nu-1}}\tau'_{\frac{\nu+k-2}{2}}\tau'_\nu\left[\tau'_{\nu-1}\left(\sigma'(0)\tau_{\nu-1}(0)-\sigma(0)\tau'_{\nu-1}\right)-\tau^2_{\nu-1}(0)\dfrac{\sigma''}{2}\right], \\ J_{2k}(z) = -\tau'_{\frac{\nu}{2}-1}\tau'_{\nu-\frac{1}{2}}\left[\tau'_\nu\tau'_{\nu-1}z+\tau'\tau_{2\nu-k}(0)+\sigma''\left(k\tau(0)-\tau_{\nu(1-\nu)}(0)\right)\right], \\ J_{3k}(z) = (\nu-k+1)\dfrac{C_\nu}{C_{\nu+1}}\tau'_{\frac{\nu}{2}-1}\tau'_{\frac{\nu-1}{2}}\tau'_{\nu-1}. \end{cases}$$

This Corollary was first published in [25, relations (5)–(8), p. 713] and it is nothing else than the *standard* three-term recurrence relation for the derivative of any order of the hypergeometric function.

COROLLARY 3.12.

$$L_{1k}(z)y^{(k)}_{\nu}(z) + L_{2k}(z)y^{(k+1)}_{\nu}(z) + L_{3k}(z)y^{(k)}_{\nu+1}(z) = 0,$$

$$\begin{cases} L_{1k}(z) = \tau'_{\frac{\nu+k-1}{2}}\tau_\nu(z), \\ L_{2k}(z) = \tau'_\nu\sigma(z), \\ L_{3k}(z) = -(\nu-k+1)\dfrac{C_\nu}{C_{\nu+1}}\tau'_{\frac{\nu-1}{2}}\,. \end{cases}$$

COROLLARY 3.13.

$$M_{1k}(z)y^{(k)}_{\nu-1}(z) + M_{2k}(z)y^{(k)}_{\nu}(z) + M_{3k}(z)y^{(k+1)}_{\nu}(z) = 0,$$

$$\begin{cases} M_{1k}(z) = -\dfrac{C_\nu}{C_{\nu-1}}\tau'_{\frac{\nu+k-2}{2}}\left[\tau^2_{\nu-1}(0)\dfrac{\sigma''}{2}+\tau'_{\nu-1}\Big(\sigma(0)\tau'_{\nu-1}-\sigma'(0)\tau_{\nu-1}(0)\Big)\right], \\ M_{2k}(z) = -\dfrac{\nu-k}{2}\tau'_{\frac{\nu}{2}-1}\tau'_{\nu-1}\left(\sigma''z+2\sigma'(0)-\sigma''\dfrac{\tau_{\nu-1}(0)}{\tau'_{\nu-1}}\right), \\ M_{3k}(z) = \tau'_{\frac{\nu}{2}-1}\tau'_{\nu-1}\sigma(z). \end{cases}$$

The above relation was first obtained in [7, relations (30)–(31), p. 717].

COROLLARY 3.14.

$$N_{1k}(z)y^{(k)}_{\nu-1}(z) + N_{2k}(z)y^{(k+1)}_{\nu}(z) + N_{3k}(z)y^{(k)}_{\nu+1}(z) = 0,$$

$$\begin{cases} N_{1k}(z) = \dfrac{C_\nu}{C_{\nu-1}}\tau'_{\frac{\nu+k-2}{2}}\tau'_{\frac{\nu+k-1}{2}} \\ \qquad\qquad \times\left[\tau_{\nu-1}(0)\left(2\sigma'(0)\tau'_{\frac{\nu-1}{2}}-\tau(0)\sigma''\right)-2\left(\tau'_{\nu-1}\right)^2\sigma(0)\right]\tau_\nu(z), \\ N_{2k}(z) = 2\tau'_{\frac{\nu}{2}-1}\tau'_{\nu-\frac{1}{2}}\left[\tau'_\nu\tau'_{\nu-1}z+\tau_\nu(0)\tau'_{k-1}+\tau'_\nu\sigma'(0)(\nu-k)\right]\sigma(z), \\ N_{3k}(z) = -(\nu-k)(\nu-k+1)\dfrac{C_\nu}{C_{\nu+1}}\tau'_{\frac{\nu}{2}-1}\tau'_{\frac{\nu-1}{2}}\left[\sigma''\tau'_{\nu-1}z+2\sigma'(0)\tau'_{\frac{\nu-1}{2}}-\sigma''\tau(0)\right]. \end{cases}$$

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REMARK 3.15. To conclude this section let us mention that, in spite of the fact that the Nikiforov and Uvarov method used here allows us to obtain the recurrence relations in Theorems 3.1 and 3.3 directly, in principle, Theorem 3.1 can be obtained also by combining appropriately the known Corollaries 3.7 and 3.13. In a similar way, it is possible to obtain Theorem 3.3 by a suitable combination of Corollaries 3.7 and 3.10. Finally, we point out that the Nikiforov and Uvarov technique is required in order to obtain the new Corollary 3.10.

4. Applications.

4.1. Recurrence relations for hypergeometric-type functions. We can reduce equation (1.1) to a canonical form by a linear change of independent variable. According to [15, 16], there exist three different cases, corresponding to the different possibilities for the degree of σ:

- $\deg\big(\sigma(z)\big) = 2$:

$$z(1-z)\,u'' + \big[\gamma - \big(\alpha+\beta+1\big)z\big]\,u' - \alpha\beta\,u = 0. \tag{4.1}$$

  This corresponds to equation (1.1) with

$$\sigma(z) = z(1-z), \qquad \tau(z) = \gamma - \big(\alpha+\beta+1\big)z, \qquad \lambda = -\alpha\beta\,. \tag{4.2}$$

- $\deg\big(\sigma(z)\big) = 1$:

$$z\,u'' + \big(\gamma - z\big)\,u' - \alpha\,u = 0\,. \tag{4.3}$$

  This is equation (1.1) with

$$\sigma(z) = z, \qquad \tau(z) = \gamma - z, \qquad \lambda = -\alpha\,. \tag{4.4}$$

- $\deg\big(\sigma(z)\big) = 0$:

$$u'' - 2z\,u' + 2\nu\,u = 0\,. \tag{4.5}$$

  This corresponds to equation (1.1) with

$$\sigma(z) = 1, \qquad \tau(z) = -2z, \qquad \lambda = 2\nu\,. \tag{4.6}$$

Equations (4.1), (4.3), and (4.5) are known as the *hypergeometric*, *confluent hypergeometric*, and *Hermite* equations, respectively. Explicit solutions of these three different equations are well known; see, e.g., [15, 16]. In [15, §20, Section 2, p. 258] particular solutions were found using the corresponding integral representations: the hypergeometric $F(\alpha,\beta,\gamma,z)$, confluent hypergeometric $F(\alpha,\gamma,z)$, and Hermite $H_\nu(z)$ functions. In terms of the generalized hypergeometric notation [4], the first two correspond to ${}_2F_1(\alpha,\beta;\gamma;z)$ and ${}_1F_1(\alpha;\gamma;z)$, respectively. We will use, for these functions, the normalization considered in [15, §20, Section 2, p. 255].

We remark that if $\sigma(z)$ has a double root, then the generalized hypergeometric-type equation can be reduced to an equation where the corresponding $\sigma(z)$ is of degree one [15, pp. 3,4].

Here we will present recurrence relations that follow from Theorems 3.1, 3.3, and 3.5, and some particular examples from the corresponding Corollaries. In the sequel, $R(z)$ represents an arbitrary function of z.

4.1.1. Relations derived from Theorem 3.1.

- *Hypergeometric equation*; see (4.1) and (4.2). From (4.2) and (2.6) it follows that $\nu = -\alpha$ or $\nu = -\beta$. Choosing $\nu = -\alpha$, relation (3.2) is fulfilled with

$$\begin{cases} A_{1k}(z) = -\alpha(\alpha-1)(\beta+k)(\beta+k-1)(\beta-\gamma)(\beta-\alpha+1)\big(R(z)-1+2z\big), \\ A_{2k}(z) = (\alpha-1)(\alpha+k)(\beta-1)(\beta+k)(\beta-\alpha+1)\times \\ \qquad \Big\{\Big(R(z)-1+2z\Big)\Big[(1-2z)(\beta-\alpha-1)-\Big((\gamma-\alpha-1)-(\beta-\alpha+1)z\Big)\Big]- \\ \qquad (\beta-\alpha-1)z(1-z)\Big\}, \\ A_{3k}(z) = -(\alpha-1)(\beta-1)(\beta-\alpha-1)z(1-z)\times \\ \Big[(\beta-\alpha+1)(\beta+k)R(z)-(\alpha+k)\Big((\gamma-\alpha)-(\beta-\alpha+1)z\Big)-(\beta-\alpha)(\beta-\alpha+1)(1-2z)\Big], \\ A_{4k}(z) = (\alpha+k)\beta(\beta-1)(\alpha-\gamma)(\beta-\alpha-1)z(1-z). \end{cases}$$

- *Confluent hypergeometric equation*; see (4.3) and (4.4). Using (4.4) and (2.6) we find $\nu = -\alpha$, being relation (3.2) fulfilled with

$$A_{1k}(z) = -\alpha, \quad A_{2k}(z) = \alpha+k, \quad A_{3k}(z) = z, \quad A_{4k}(z) = 0.$$

- *Hermite equation*; see (4.5) and (4.6). Using now (4.6) and (2.6) we conclude that ν may be an arbitrary complex number and relation (3.2) is fulfilled with

$$A_{1k}(z) = -2\nu, \quad A_{2k}(z) = 0, \quad A_{3k}(z) = 1, \quad A_{4k}(z) = 0.$$

4.1.2. Relations derived from Theorem 3.3.

- *Hypergeometric equation*; see (4.1) and (4.2). This corresponds to $\nu = -\alpha$ (or $\nu = -\beta$) and relation (3.11) is fulfilled with

$$\begin{cases} B_{1k}(z) = \alpha(\alpha-1)\beta(\beta-\alpha+1)(\beta-\alpha-2)\big(R(z)-1+2z\big), \\ B_{2k}(z) = -(\alpha-1)(\alpha+k)(\beta-1)(\beta+k!-1)(\beta-\alpha+1)\Big[(\beta-\gamma)-(\beta-\alpha-1)\Big(R(z)+z\Big)\Big], \\ B_{3k}(z) = (\alpha-1)(\beta-1)\Big[(\gamma-\alpha-1)-(\beta-\alpha-1)z\Big]\times \\ \Big[(\beta-\alpha)(\beta-\alpha+1)(1-2z)-R(z)(\beta+k-1)(\beta-\alpha+1)-(\alpha+k)\Big((\gamma-\alpha)-(\beta-\alpha+1)z\Big)\Big], \\ B_{4k}(z) = -(\alpha+k)\beta(\beta-1)(\gamma-\alpha)\Big[(\gamma-\alpha-1)-(\beta-\alpha-1)z\Big]. \end{cases}$$

- *Confluent hypergeometric equation*; see (4.3) and (4.4). This corresponds to $\nu = -\alpha$ and relation (3.11) is fulfilled with

$$B_{1k}(z) = \alpha, \quad B_{2k}(z) = -(\alpha+k), \quad B_{3k}(z) = (\gamma-\alpha-1)-z, \quad B_{4k}(z) = 0.$$

- *Hermite equation*; see (4.5) and (4.6). Relation (3.11) is fulfilled, for an arbitrary complex number ν, with

$$B_{1k}(z) = \nu, \quad B_{2k}(z) = \nu-k, \quad B_{3k}(z) = -z, \quad B_{4k}(z) = 0.$$

4.1.3. Relations derived from Theorem 3.5.

- *Hypergeometric equation*; see (4.1) and (4.2). This corresponds to $\nu = -\alpha$ (or $\nu = -\beta$) and relation (3.13) is fulfilled with

$$\begin{cases} D_{1k}(z) = -\alpha(\alpha-1)(\beta+k)(\beta+k-1)(\beta-\gamma)(\beta-\alpha+1)\big[R(z)-(\beta-\alpha)\big], \\ D_{2k}(z) = (\alpha-1)(\alpha+k)(\beta-1)(\beta+k)(\beta-\alpha)\times \\ \qquad \Big\{\big[(\beta-\gamma)-(\beta-\alpha-1)z\big](\beta-\alpha+1)+R(z)(\beta-3\alpha+2\gamma+1)\Big)\Big\}, \\ D_{3k}(z) = (\alpha-1)(\beta-1)(\beta-\alpha-1)(\beta-\alpha)(\beta-\alpha+1)\Big(R(z)-(\beta+k)\Big)z(1-z), \\ D_{4k}(z) = R(z)(\alpha+k)(\alpha+k-1)\beta(\beta-1)(\gamma-\alpha)(\beta-\alpha-1). \end{cases}$$

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- *Confluent hypergeometric equation*; see (4.3) and (4.4). This corresponds to $\nu = -\alpha$ and relation (3.13) is fulfilled with

$$D_{1k}(z) = -\alpha, \quad D_{2k}(z) = \alpha + k, \quad D_{3k}(z) = z, \quad D_{4k}(z) = 0.$$

- *Hermite equation*; see (4.5) and (4.6). This corresponds to $\nu = -\alpha$ and relation (3.13) is fulfilled with

$$D_{1k}(z) = 2\nu, \quad D_{2k}(z) = 0, \quad D_{3k}(z) = -1, \quad D_{4k}(z) = 0.$$

In the following we will put $k = 1$ and use the identities [15, p. 261]

$$ {}_2F_1{}'(\alpha,\beta;\gamma;z) = \frac{\alpha\beta}{\gamma}\, {}_2F_1(\alpha{+}1,\beta{+}1;\gamma{+}1;z),\ \ {}_1F_1{}'(\alpha;\gamma;z) = \frac{\alpha}{\gamma}\, {}_1F_1(\alpha{+}1;\gamma{+}1;z).$$

4.1.4. Relations derived from Corollary 3.7.

- *Hypergeometric function*. This corresponds to $\nu = -\alpha$ (or $\nu = -\beta$). Substituting the quantities (4.2) in (3.17)–(3.18), we find the following recurrence relation for the hypergeometric function:

$$\gamma\big(\alpha-\beta-1\big)\, {}_2F_1(\alpha,\beta;\gamma;z) + \beta\big(\alpha-\gamma\big)\, {}_2F_1(\alpha,\beta+1;\gamma+1;z) + \\ \alpha\big[\big(\beta-\gamma+1\big) + \big(\beta-\alpha+1\big)z\big]\, {}_2F_1(\alpha+1,\beta+1;\gamma+1;z) = 0\,.$$

- *Confluent hypergeometric function*. This corresponds to $\nu = -\alpha$. Therefore, (3.17)–(3.18) yield, for the hypergeometric confluent function, the following recurrence relation

$$\gamma\, {}_1F_1(\alpha;\gamma;z) + \big(\gamma-\alpha\big)\, {}_1F_1(\alpha;\gamma+1;z) + \alpha\, {}_1F_1(\alpha+1;\gamma+1;z) = 0.$$

- *Hermite function*. Let ν be an arbitrary complex number. Substituting (4.4) in (3.17)–(3.18), we find the following very well known relation for the Hermite function:

$$H'_{\nu+1}(z) = 2\big(\nu+1\big)H_\nu(z).$$

Other recurrences relations can be obtained from the other Corollaries 3.8–3.14. Since the technique is similar we just present here the resulting relations.

4.1.5. Relations derived from Corollary 3.8.

- *Hypergeometric function.*

$$\gamma\Big\{\alpha\big[\big(\beta-\gamma+1\big) - \big(\beta-\alpha+1\big)z\big]\big[\big(\beta-\gamma\big) - \big(\beta-\alpha-1\big)z\big] - \\ \beta\big[(\beta-\alpha)^2-1\big]z(1-z)\Big\}\, {}_2F_1(\alpha,\beta;\gamma;z) + \\ \alpha\gamma\big(\beta-\gamma\big)\big[\big(\beta-\alpha+1\big)z - \big(\beta-\gamma+1\big)\big]\, {}_2F_1(\alpha+1,\beta;\gamma;z) + \\ \beta^2\big(\alpha-\gamma\big)\big(\beta-\alpha-1\big)z(1-z)\, {}_2F_1(\alpha,\beta+1;\gamma+1;z) = 0\,.$$

- *Confluent hypergeometric function.*

$$-\gamma\big(\alpha+z\big)\, {}_1F_1(\alpha;\gamma;z) + \big(\gamma-\alpha\big)z\, {}_1F_1(\alpha;\gamma+1;z) + \alpha\gamma\, {}_1F_1(\alpha+1;\gamma;z) = 0.$$

- *Hermite function.*

$$H'_{\nu+1}(z) = 2\big(\nu+1\big)H_\nu(z).$$

4.1.6. Relations derived from Corollary 3.9.

- *Hypergeometric function.*

$$\begin{aligned}&\beta\Big\{\alpha\big[(\beta-\gamma+1)-(\beta-\alpha+1)z\big]\big[(\beta-\gamma)-(\beta-\alpha-1)z\big]+\\&\qquad\beta\big[(\beta-\alpha)^2-1\big]z(1-z)\Big\}\,{}_2F_1(\alpha+1,\beta+1;\gamma+1;z)+\\&\qquad+\beta\gamma\big(\beta-\gamma\big)\big(\beta-\alpha+1\big)\,{}_2F_1(\alpha+1,\beta;\gamma;z)-\\&\qquad\beta^2\big(\alpha-\gamma\big)\Big[\big(\beta-\gamma\big)-\big(\beta-\alpha-1\big)z\Big]\,{}_2F_1(\alpha,\beta+1;\gamma+1;z)=0\,.\end{aligned}$$

- *Confluent hypergeometric function.*

$$-\big(\alpha+z\big)\,{}_1F_1(\alpha+1;\gamma+1;z)+\big(\alpha-\gamma\big)z\,{}_1F_1(\alpha;\gamma+1;z)+\gamma\,{}_1F_1(\alpha+1;\gamma;z)=0.$$

- *Hermite function.*

$$H_\nu'(z)=2\nu\,H_{\nu-1}(z).$$

4.1.7. Relations derived from Corollary 3.10.

- *Hypergeometric function.*

$$\begin{aligned}&\gamma\big(\beta-1\big)\Big[\big(\alpha+1\big)\big(\beta-\alpha+1\big)\big(\gamma-\beta\big)+\big(\gamma-\alpha\big)\big(\beta-\alpha-1\big)\big(\beta+1\big)-\\&\big(\beta-\alpha+1\big)\big(\beta-\alpha\big)\big(\beta-\alpha-1\big)z\Big]\,{}_2F_1(\alpha,\beta;\gamma;z)+\\&\beta^2\big(\alpha-\gamma\big)\Big[\big(\gamma-\alpha-1\big)-\big(\beta-\alpha-1\big)z\Big]\,{}_2F_1(\alpha,\beta+1;\gamma+1;z)+\\&\alpha\beta\big(\alpha+1\big)\big(\gamma-\beta\big)\Big[\big(\gamma-\beta-1\big)+\big(\beta-\alpha+1\big)z\Big]\,{}_2F_1(\alpha+2,\beta+1;\gamma+1;z)=0\,.\end{aligned}$$

- *Confluent hypergeometric function.*

$$\begin{aligned}&\gamma\Big[\big(\gamma-2\alpha-1\big)-z\Big]\,{}_1F_1(\alpha;\gamma;z)+\big(\alpha-\gamma\big)\Big[\big(\gamma-\alpha-1\big)-z\Big]\,{}_1F_1(\alpha;\gamma+1;z)+\\&\qquad\alpha\big(\alpha+1\big)\,{}_1F_1(\alpha+2;\gamma+1;z)=0\,.\end{aligned}$$

- *Hermite function.*

$$H_{\nu+1}'(z)=2\big(\nu+1\big)\,H_\nu(z).$$

4.1.8. Relations derived from Corollary 3.11.

- *Hypergeometric function.*

$$\begin{aligned}&\big(\beta-1\big)\big(\gamma-\alpha\big)\big(\beta-\alpha-1\big)\,{}_2F_1(\alpha-1,\beta;\gamma;z)+\\&\big(\beta-\alpha\big)\Big\{\big[(\beta-\alpha)^2-1\big]z-\big(\alpha+\beta+1\big)\big(\gamma-2\alpha\big)+2\big(\gamma-\alpha(\alpha+1)\big)\Big\}\,{}_2F_1(\alpha,\beta;\gamma;z)+\\&\alpha\big(\beta-\alpha+1\big)\big(\gamma-\beta\big)\,{}_2F_1(\alpha+1,\beta;\gamma;z)=0\,.\end{aligned}$$

- *Confluent hypergeometric function.*

$$\big(\gamma-\alpha\big)\,{}_1F_1(\alpha-1;\gamma;z)+\Big[z-\big(\gamma-2\alpha\big)\Big]\,{}_1F_1(\alpha;\gamma;z)-\alpha\,{}_1F_1(\alpha+1;\gamma;z)=0.$$

- *Hermite function.*

$$H_{\nu+1}(z)-2z\,H_\nu(z)+2\nu\,H_{\nu-1}(z)=0.$$

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4.1.9. Relations derived from Corollary 3.12.

- *Hypergeometric function.*

$$\beta\gamma\big(\alpha-\gamma\big)\,{}_2F_1(\alpha-1,\beta;\gamma;z)+\gamma\big(\beta-1\big)\Big[\big(\gamma-\alpha\big)-\big(\beta-\alpha+1\big)z\Big]\,{}_2F_1(\alpha,\beta;\gamma;z)+ \\ \alpha\beta\big(\beta-\alpha+1\big)z(1-z)\,{}_2F_1(\alpha+1,\beta+1;\gamma+1;z)=0\,.$$

- *Confluent hypergeometric function.*

$$\gamma\big(\alpha-\gamma\big)\,{}_1F_1(\alpha-1;\gamma;z)+\gamma\Big[\big(\gamma-\alpha\big)-z\Big]\,{}_1F_1(\alpha;\gamma;z)+\alpha z\,{}_1F_1(\alpha+1;\gamma+1;z)=0.$$

- *Hermite function.*

$$H_{\nu+1}(z)-2z\,H_\nu(z)+H_\nu'(z)=0.$$

4.1.10. Relations derived from Corollary 3.13.

- *Hypergeometric function.*

$$\gamma\Big[\big(\gamma-\beta\big)+\big(\beta-\alpha-1\big)z\Big]\,{}_2F_1(\alpha,\beta;\gamma;z)+\gamma\big(\beta-\gamma\big)\,{}_2F_1(\alpha+1,\beta;\gamma;z)- \\ \beta\big(\beta-\alpha-1\big)z(1-z)\,{}_2F_1(\alpha+1,\beta+1;\gamma+1;z)=0\,.$$

- *Confluent hypergeometric function.*

$$\gamma\,{}_1F_1(\alpha+1;\gamma;z)-\gamma\,{}_1F_1(\alpha;\gamma;z)-z\,{}_1F_1(\alpha+1;\gamma+1;z)=0.$$

- *Hermite function.*

$$H_\nu'(z)=2\nu\,H_{\nu-1}(z).$$

4.1.11. Relations derived from Corollary 3.14.

- *Hypergeometric function.*

$$\beta\big(\alpha-\gamma\big)\Big[\big(\gamma-\beta\big)+\big(\beta-\alpha-1\big)z\Big]\,{}_2F_1(\alpha-1,\beta;\gamma;z)+ \\ \gamma\big(\gamma-\beta\big)\big(\beta-\alpha-1\big)\Big[\big(\gamma-\alpha\big)-\big(\beta-\alpha+1\big)z\Big]\,{}_2F_1(\alpha+1,\beta;\gamma;z)+ \\ \beta\big(\beta-\alpha\big)\Big\{\Big[\big(\beta-\alpha\big)^2-1\Big]z+2\alpha\beta-\gamma\big(\alpha+\beta-1\big)\Big\}\,{}_2F_1(\alpha+1,\beta+1;\gamma+1;z)=0\,.$$

- *Confluent hypergeometric function.*

$$\gamma\big(\gamma-\alpha\big)\,{}_1F_1(\alpha-1;\gamma;z)-\gamma\Big(\big(\gamma-\alpha\big)-z\Big)\,{}_1F_1(\alpha+1;\gamma;z)+ \\ \beta z\big(\gamma-z\big)\,{}_1F_1(\alpha+1;\gamma+1;z)=0\,.$$

- *Hermite function.*

$$H_\nu'(z)=2\nu\,H_{\nu-1}(z).$$

REMARK 4.1. Notice that we can interchange α and β in (4.1). Therefore, several other recurrence relations can be obtained by interchanging α and β in all relations obtained from Corollaries 3.7–3.14 corresponding to the hypergeometric equation (4.1). Notice also that the relation derived in Section 4.1.8 for the confluent hypergeometric function was first published in [15, p. 267].

REMARK 4.2. Notice that the Bessel equation

$$z^2u''+zu'+\big(z^2-\nu^2\big)\,u=0\,,$$

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whose solutions $u_\nu \equiv \mathcal{Z}_\nu$ are called the *Bessel functions of order* ν, can be transformed, by the change of the dependent variable $u = \phi(z)y$, where $\phi(z) = z^\nu e^{iz}$ [15, p. 202] into the equation

$$z^2y'' + (2iz + 2\nu + 1)y' + i(2\nu + 1)y = 0\,,$$

i.e., into the hypergeometric confluent differential equation (4.3) with $\gamma = 2\nu + 1$ and $\alpha = \nu + \frac{1}{2}$; see [15, p. 254]. Then, the recurrence relations for the solutions of the hypergeometric confluent differential equation can be used for generating several recurrence relations for the Bessel functions.

REMARK 4.3. The Airy function is a solution of the equation

$$u'' + zu = 0\,,$$

which is a particular case of the Lommel equation

$$v'' + \frac{1-2\alpha}{z}v' + \left[\left(\beta\gamma z^{\gamma-1}\right)^2 + \frac{\alpha^2 - \nu^2\gamma^2}{z^2}\right] v = 0.$$

Its solutions can be expressed in terms of the Bessel function $\mathcal{Z}_\nu$ of order ν by $v(z) = z^\alpha \mathcal{Z}_\nu\left(\beta z^\gamma\right)$, with $\alpha = \frac{1}{2}$, $\nu = \frac{1}{3}$, $\beta = \frac{2}{3}$, and $\gamma = \frac{3}{2}$, and therefore, the recurrence relations for the Airy functions follow from the recurrences of the Bessel functions; see Remark 4.2.

4.2. Recurrences for polynomials of hypergeometric type. The polynomials of hypergeometric type $p_n(z) := y_n(z)$ are particular cases of the functions of hypergeometric type $y_\nu(z)$ when the parameter $\nu = n$ is a non-negative integer, being $p_n(z) := y_n(z)$ a particular solution of the equation (1.1) where λ is given by (2.1). They can be represented by the Rodrigues formula (2.2), where B_n are normalizing constants and $\rho(z)$ satisfies the Pearson equation (2.3), or by their integral representation (2.4), where

$$C_n = \frac{n!B_n}{2\pi i}\,. \tag{4.7}$$

If a_n denotes the leading coefficient of the polynomial $y_n(z)$, then (see [15])

$$a_n = B_n \prod_{m=0}^{n-1} \tau'_{\frac{n+m-1}{2}}\,, \qquad a_0 = B_0\,. \tag{4.8}$$

If $a_n = 1$, then $y_n(z)$ is said to be a *monic* polynomial.

The polynomials of hypergeometric type are the classical polynomials, i.e., the Hermite $H_n(z)$, Laguerre $L_n^\alpha(z)$, and Jacobi $P_n^{\alpha,\beta}(z)$ polynomials.

A very important property of the orthogonal polynomials is the three-term recurrence relation

$$zp_n(z) = \alpha_n p_{n+1}(z) + \beta_n p_n(z) + \gamma_n p_{n-1}(z).$$

For computing the coefficients α_n, β_n, and γ_n, we can use Corollary 3.11 with $k = 0$ as it has been done in [25]. Other important properties of these polynomials are the so-called raising and lowering operators (see, e.g., [1]) that can be obtained from Corollaries 3.12 and 3.13, respectively. Since they were studied using this method in [20], we will omit them here.

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TABLE 4.1
The classical orthogonal polynomials

$P_n(z)$	$H_n(z)$	$L_n^{\alpha}(z)$	$P_n^{\alpha,\beta}(z)$
$\sigma(z)$	1	z	$1-z^2$
$\tau(z)$	$-2z$	$-z+\alpha+1$	$-(\alpha+\beta+2)z+\beta-\alpha$
λ_n	$2n$	n	$n(n+\alpha+\beta+1)$
$\rho(z)$	e^{-z^2}	$z^{\alpha}e^{-z}$ $\alpha>-1$	$(1-z)^{\alpha}(1+z)^{\beta}$ $\alpha,\beta>-1$
B_n	$\dfrac{(-1)^n}{2^n}$	$(-1)^n$	$\dfrac{(-1)^n}{(n+\alpha+\beta+1)_n}$

Here we will study another recurrence relation. Namely, the so-called structure relation by Marcellán et. al [14],

$$P_n(z)=\frac{P'_{n+1}(z)}{n+1}+r_n\frac{P'_n(z)}{n}+s_n\frac{P'_{n-1}(z)}{n-1}\,,\quad n\geq 2\,, \tag{4.9}$$

where r_n and s_n are some constants. This relation constitutes another characterization theorem for the classical orthogonal polynomials. A complete study of such structure relations was done in [1].

COROLLARY 4.4. *For the monic hypergeometric-type polynomials, the following recurrence relation holds*

$$\hat{y}_n(z)=\hat{A}_1(z)\hat{y}'_{n+1}(z)+\hat{A}_2(z)\hat{y}'_n(z)+\hat{A}_3(z)\hat{y}'_{n-1}(z), \tag{4.10}$$

where the coefficients, $\hat{A}_i$, $i=1,2,3$, are given by

$$\begin{cases} \hat{A}_1(z)=\dfrac{1}{n+1}, \\ \hat{A}_2(z)=\dfrac{1}{2}\frac{\left(\tau'_{n-1}\tau_n(0)+\tau_{n-1}(0)\tau'_n\right)\sigma''-2\sigma'(0)\tau'_n\tau'_{n-1}}{\tau'_n\tau'_{n-\frac{1}{2}}\tau'_{n-1}}, \\ \hat{A}_3(z)=\left(1-\dfrac{\tau'_{\frac{n-1}{2}}}{\tau'_{n-\frac{1}{2}}}\right)\dfrac{2\tau'_{n-1}\left(\tau'_{n-1}\sigma(0)-\tau_{n-1}(0)\sigma'(0)\right)+\tau^2_{n-1}(0)\sigma''}{2\left(\tau'_{n-1}\right)^2\tau'_{n-\frac{3}{2}}}. \end{cases} \tag{4.11}$$

Proof. Since $R(z)$ in Theorem 3.3 is an arbitrary polynomial in z, we will define the function $Q(z)$, such that

$$R(z)=2\sigma'(z)-\sigma''\frac{\tau_{\nu-1}(z)}{\tau'_{\nu-1}}\frac{2+Q(z)}{2}\,.$$

Then (3.11) holds and the corresponding coefficients, $A_{ik}(z)$, $i = 1,2,3,4$, become

$$\begin{cases} A_{1k}(z) = \dfrac{C_\nu}{C_{\nu-1}} \dfrac{2+Q(z)}{Q(z)} \dfrac{2\tau'_{\nu-1}\left(\tau_{\nu-1}(0)\sigma'(0) - \tau'_{\nu-1}\sigma(0)\right) - \tau^2_{\nu-1}(0)\sigma''}{2(\nu-k)\tau'_{\nu-1}\tau'_{\frac{\nu}{2}-1}}, \\ A_{2k}(z) = 1, \\ A_{3k}(z) = \dfrac{1}{(\nu-k)Q(z)} \dfrac{(\nu-k)\tau'_{\nu-1}\left(\tau_\nu(z)\sigma'' - 2\sigma'(z)\tau'_\nu\right) - \tau_{\nu-1}(z)(2+Q(z))\tau'_\nu\tau'_{\frac{\nu+k-1}{2}}}{\tau'_\nu\tau'_{\nu-1}\tau'_{\frac{\nu+k-1}{2}}}, \\ A_{4k}(z) = \dfrac{1}{Q(z)} \dfrac{C_\nu}{C_{\nu+1}} \dfrac{2\tau'_{\frac{\nu-1}{2}}}{\tau'_\nu\tau'_{\frac{\nu+k-1}{2}}}. \end{cases}$$

Since the polynomials are monic, by (4.8), $B_n = \left(\prod_{m=0}^{n-1}\tau'_{\frac{n+m-1}{2}}\right)^{-1}$. Then choosing $Q(z) = -2\tau'_{n-\frac{1}{2}}/\tau'_{\frac{n-1}{2}}$ and setting $k = 0$, the equations (3.11) transform into (4.10), whereas (3.12) leads to (4.11). Notice that in (4.9), $r_n = n\hat{A}_2$ and $s_n = (n-1)\hat{A}_3$. $\square$

We obtain from formulas (4.10)–(4.11) of Corollary 4.4 the following identities,

$$\begin{aligned} H_n(z) &= \frac{1}{n+1}H'_{n+1}(z), \qquad L_n^\alpha(z) = \frac{1}{n+1}(L_{n+1}^\alpha)'(z) + (L_n^\alpha)'(z), \\ P_n^{\alpha,\beta}(z) &= \frac{1}{n+1}(P_{n+1}^{\alpha,\beta})'(z) + \frac{2(\alpha-\beta)}{(2n+\alpha+\beta)(2n+2+\alpha+\beta)}(P_n^{\alpha,\beta})'(z) \\ &\quad - \frac{4n(n+\alpha)(n+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}(P_{n-1}^{\alpha,\beta})'(z), \end{aligned}$$

for the Hermite, Laguerre, and Jacobi polynomials, respectively.

4.3. Further examples. In this section we will present several relations for the classical polynomials that follow from the Theorems 3.1, 3.3, and 3.5. In order to obtain the following relations, where $R(z)$ represents an arbitrary function of z; see Section 4.2 and Table 4.1.

4.3.1. Relations derived from Theorem 3.1.

- *Jacobi polynomials.*

$$\begin{cases} A_{1k}(z) = 4n(n+1)(\alpha+n)(\beta+n)(\alpha+\beta+n)(\alpha+\beta+n+k)(\alpha+\beta+n+k+1)\times \\ \qquad (\alpha+\beta+2n+2)\big(R(z)+4z\big), \\ A_{2k}(z) = (n+1)(n-k)(\alpha+\beta+n)(\alpha+\beta+n+k+1)(\alpha+\beta+2n-1)(\alpha+\beta+2n)\times \\ \qquad (\alpha+\beta+2n+2)\Big[\Big(R(z)+4z\Big)\Big((\beta-\alpha)+(\alpha+\beta+2n)z\Big)+(\alpha+\beta+2n)\big(1-z^2\big)\Big], \\ A_{3k}(z) = (n+1)\Big[(\alpha+\beta+n+k+1)(\alpha+\beta+2n+2)R(z)+ \\ \qquad 2(n-k)\Big((\beta-\alpha)-(\alpha+\beta+2n+2)z\Big)+4(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)z\Big]\times \\ \qquad (\alpha+\beta+n)(\alpha+\beta+2n-1)(\alpha+\beta+2n)^2\big(1-z^2\big), \\ A_{4k}(z) = -2(n-k)(\alpha+\beta+n)(\alpha+\beta+2n-1)(\alpha+\beta+2n)^2(\alpha+\beta+2n+1)\times \\ \qquad (\alpha+\beta+2n+2)\big(1-z^2\big). \end{cases}$$

- *Laguerre polynomials.*

$$A_{1k}(z) = -n(\alpha+n), \quad A_{2k}(z) = -(n-k), \quad A_{3k}(z) = z, \quad A_{4k}(z) = 0.$$

- *Hermite polynomials.*

$$A_{1k}(z) = -n, \quad A_{2k}(z) = 0, \quad A_{3k}(z) = 1, \quad A_{4k}(z) = 0.$$

4.3.2. Relations derived from Theorem 3.3.

- *Jacobi polynomials.*

$$\begin{cases} A_{1k}(z) = 4n(n+1)(\alpha+n)(\beta+n)(\alpha+\beta+n)(\alpha+\beta+n+k+1)\times \\ \qquad (\alpha+\beta+2n+2)\big(R(z)+4z\big), \\ A_{2k}(z) = -(n-k)(\alpha+\beta+n)(\alpha+\beta+n+k+1)(\alpha+\beta+2n-1)(\alpha+\beta+2n)\times \\ \qquad (\alpha+\beta+2n+2)\Big[R(z)(\alpha+\beta+2n)+2\Big((\beta-\alpha)+(\alpha+\beta+2n-2)z\Big)\Big], \\ A_{3k}(z) = (\alpha+\beta+n)(\alpha+\beta+2n-1)(\alpha+\beta+2n)\Big((\beta-\alpha)-(\alpha+\beta+2n)z\Big)\times \\ \qquad \Big[2(n-k)\Big((\beta-\alpha)-(\alpha+\beta+2n+2)z\Big)-(\alpha+\beta+2n+2)\times \\ \qquad \Big((\alpha+\beta+n+k+1)R(z)-4(\alpha+\beta+2n)z\Big)\Big], \\ A_{4k}(z) = -(n-k)(\alpha+\beta+n)(\alpha+\beta+2n-1)(\alpha+\beta+2n)(\alpha+\beta+2n+1)\times \\ \qquad (\alpha+\beta+2n+2)\Big((\beta-\alpha)-(\alpha+\beta+2n)z\Big). \end{cases}$$

- *Laguerre polynomials.*

$$A_{1k}(z) = -n(\alpha+n), \quad A_{2k}(z) = n-k, \quad A_{3k}(z) = -(\alpha+n-z), A_{4k}(z) = 0.$$

- *Hermite polynomials.*

$$A_{1k}(z) = -n, \quad A_{2k}(z) = -2(n-k), \quad A_{3k}(z) = 2, \quad A_{4k}(z) = 0.$$

4.3.3. Relations derived from Theorem 3.5.

- *Jacobi polynomials.*

$$\begin{cases} A_{1k}(z) = 4n(n+1)(\alpha+n)(\beta+n)(\alpha+\beta+n)(\alpha+\beta+n+k)(\alpha+\beta+n+k+1)\times \\ \qquad (\alpha+\beta+2n+2)\Big(R(z)+(\alpha+\beta+2n+1)\Big), \\ A_{2k}(z) = (n-k)(n+1)(\alpha+\beta+n)(\alpha+\beta+n+k+1)(\alpha+\beta+2n-1)(\alpha+\beta+2n)\times \\ \qquad (\alpha+\beta+2n+1))\Big[\Big((\beta-\alpha)+(\alpha+\beta+2n)z\Big)(\alpha+\beta+2n+2)-2(\beta-\alpha)R(z)\Big], \\ A_{3k}(z) = -(n+1)(\alpha+\beta+n)(\alpha+\beta+2n-1)(\alpha+\beta+2n)^2(\alpha+\beta+2n+1)\times \\ \qquad (\alpha+\beta+2n+2)\Big(R(z)+(\alpha+\beta+n+k+1)\Big)\big(1-z^2\big), \\ A_{4k}(z) = -2R(z)(n-k)(n-k+1)(\alpha+\beta+n)(\alpha+\beta+2n-1)(\alpha+\beta+2n)^2\times \\ \qquad (\alpha+\beta+2n+1)(\alpha+\beta+2n+2). \end{cases}$$

- *Laguerre polynomials.*

$$A_{1k}(z) = n(\alpha+n), \quad A_{2k}(z) = n-k, \quad A_{3k}(z) = -z, \quad A_{4k}(z) = 0.$$

- *Hermite polynomials.*

$$A_{1k}(z) = n, \quad A_{2k}(z) = 0, \quad A_{3k}(z) = -1, A_{4k}(z) = 0.$$

4.3.4. A known identity for the Laguerre polynomials. To conclude this paper, we derive a very well-known formula for the Laguerre polynomials using the method described here. Putting $k=0$ in relations (3.19)–(3.20), for the Laguerre polynomials, we obtain

$$A_1(x)L_{n-1}^{(\alpha)}(x) + A_2(x)L_n^{(\alpha)}(x) + A_3(x)\left(L_{n+1}^{(\alpha)}(x)\right)' = 0\,, \tag{4.12}$$

where, by (4.7), the coefficients, A_i, $i=1,2,3$, are given by

$$A_1(x) = \alpha+n, \quad A_2(x) = x-n, \quad A_3(x) = x.$$

Then, (4.12) leads to the well-known formula for the Laguerre polynomials,

$$x\left(L_{n+1}^{(\alpha)}(x)\right)' = (n-x)L_n^{(\alpha)}(x) - (\alpha+n)L_{n-1}^{(\alpha)}(x)\,.$$

Let us also point out that for Jacobi polynomials, if one considers, in Theorem 3.5, $k=0$, $\nu=n$, and $C_n = \frac{n!B_n}{2\pi i}$, then the corresponding coefficients solution (3.13) gives the four-term recurrence relation stated in [26, Corollary 1.1, p. 729].

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REFERENCES

[1] R. Álvarez-Nodarse, *On characterizations of classical polynomials*, J. Comput. Appl. Math., 196 (2006), pp. 320–337.

[2] R. Álvarez-Nodarse, J. L. Cardoso, and N. R. Quintero, *On recurrence relations for radial wave functions for the N-th dimensional oscillators and hydrogenlike atoms: analytical and numerical study*, Electron. Trans. Numer. Anal., 24 (2006), pp. 7–23. http://etna.math.kent.edu/vol.24.2006/pp7-23.dir/.

[3] V. G. Bagrov and D. M. Gitman, *Exact Solutions of Relativistic Wave Equations*, Kluwer Press, Dordrecht, 1990.

[4] W. N. Bailey, *Generalized Hypergeometric Series*, Cambridge Tracts in Mathematics and Mathematical Physics, Vol. 32, London, Cambridge University Press, VII, 108 S., 1935.

[5] J. L. Cardoso and R. Álvarez-Nodarse, *Recurrence relations for radial wave functions for the N-th dimensional oscillators and hydrogenlike atoms*, J. Phys. A: Math. Gen., 36 (2003), pp. 2055–2068.

[6] J. S. Dehesa, W. Van Assche, and R. J. Yañez, *Information entropy of classical orthogonal polynomials and their application to the harmonic oscillator and Coulomb potentials*, Methods Appl. Anal., 4 (1997), pp. 91–110.

[7] J. S. Dehesa and R. J. Yañez, *Fundamental recurrence relations of functions of hypergeometric type and their derivatives of any order*, Nuovo Cimento B (11), 109 (1994), pp. 711–723.

[8] J. S. Dehesa, R. J. Yañez, M. Perez-Victoria, and A. Sarsa, *Non-linear characterizations for functions of hypergeometric type and their derivatives of any order*, J. Math. Anal. Appl., 184 (1994), pp. 35–43.

[9] J. S. Dehesa, R. J. Yañez, A. Zarso, and J. A. Aguilar, *New linear relationships of hypergeometric-type functions with applications to orthogonal polynomial*, Rend. Mat. Appl., VII. Ser. 13 (1993), pp. 661–671.

[10] J. W. Dettman, *Applied Complex Variables,* Dover Publications Inc., New York, 1984.

[11] W. Gautschi, *Computational aspects of three-term recurrence relations*, SIAM Rev., 9 (1967), pp. 24–82.

[12] W. Koepf, *Hypergeometric Summation. An Algorithmic Approach to Summation and Special Function Identities*, Vieweg, Braunschweig/Wiesbaden, 1998.

[13] G. Levai, *A class of exactly solvable potentials related to the Jacobi polynomials*, J. Phys. A: Math. Gen., 24 (1991), pp. 131–146.

[14] F. Marcellán, A. Branquinho, and J. Petronilho, *Classical orthogonal polynomials: a functional approach*, Acta Appl. Math., 34 (1994), pp. 283–303.

[15] A. F. Nikiforov and V. B. Uvarov, *Special Functions of Mathematical Physics,* Birkhäuser, Basel, 1988.

[16] E. D. Rainville, *Special Functions,* The MacMillan Company, New York, 1960.

[17] A. Gil, J. Segura, and N. M. Temme, *The ABC of hyper recursions*, J. Comput. Appl. Math., 190 (2006), pp. 270–286.

[18] A. Gil, J. Segura, and N. M. Temme, *Numerically satisfactory solutions of hypergeometric recursions*, Math. Comp., 76 (2007), pp. 1449–1468.

[19] C. A. Singh and T. H. Devi, *Exactly solvable non-shape invariant potentials*, Phys. Lett. A, 171 (1992), pp. 249–252.

[20] S. K. Suslov, *On the theory of difference analogues of special functions of hypergeometric type (Russian)*, Uspekhi Mat. Nauk, 44 (1989), pp. 185–226; translation in Russian Math. Surveys, 44 (1989), pp. 227–278.

[21] A. G. USHVERIDZE, *Quasi-Exactly Solvable Models in Quantum Machanics*, Institute of Physics, Bristol, 1994.

[22] J. WU AND Y. ALHASSID, *The potential groups and hypergeometric differential equations*, J. Math. Phys., 31 (1990), pp. 557–562.

[23] B. W. WILLIAMS, *A second class of solvable potentials related to the Jacobi polynomials*, J. Phys. A: Math. Gen., 24 (1991), pp. L667–L670.

[24] J. WIMP, *Computation with Recurrence Relations*, Applicable Mathematics Series, Pitman Advanced Publishing Program, Boston, MA, 1984.

[25] R. J. YAÑEZ, J. S. DEHESA, AND A. F. NIKIFOROV, *The three-term recurrence relations and the differentiation formulas for hypergeometric-type functions*, J. Math. Anal. Appl., 188 (1994), pp. 855–866.

[26] R. J. YAÑEZ, J. S. DEHESA, AND A. ZARZO, *Four-term recurrence relations of hypergeometric-type polynomials*, Nuovo Cimento B (11), 109 (1994), pp. 725–733.

[27] A. ZARZO, R. J. YAÑEZ, AND J. S. DEHESA, *General recurrence and ladder relations of hypergeometric-type functions*, J. Comput. Appl. Math., 207 (2007), pp. 166–179.