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SPECTRAL APPROXIMATION OF VARIATIONALLY FORMULATED EIGENVALUE PROBLEMS ON CURVED DOMAINS*

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To the memory of Professor Jorge D. Samur

Abstract. This paper is concerned with the spectral approximation of variationally formulated eigenvalue problems posed on curved domains. As an example of the present theory, convergence and optimal error estimates are proved for the piecewise linear finite element approximation of the eigenvalues and eigenfunctions of a second order elliptic differential operator on a general curved three-dimensional domain.

Key words. spectral approximation, eigenvalue problems, curved domains

AMS subject classifications. 65N15, 65N25, 65N30

1. Introduction. In this paper we present an extension of the spectral approximation theory for non-compact operators in Hilbert spaces. In particular, we consider the numerical approximation of the eigenvalues and eigenvectors of variationally formulated problems posed over general curved domains. There are not many references about error estimates for this kind of problems. In particular, the finite element approximations of the spectrum of the Laplace operator on non-convex domains with curved boundaries have been studied only in a few papers.

The first proof of the convergence for a Laplace eigenproblem, for simple eigenvalues and Dirichlet boundary conditions, was given by Vanmaele and Ženíšek [16] by using the *min-max* characterization; see [15]. The same authors generalized their results to include multiple eigenvalues [17] and numerical integration effects [18]. Almost at the same time, Lebaud [11] analyzed a similar problem posed on two-dimensional domains by using isoparametric finite elements methods in the framework of the classical spectral approximation theory; see [1]. She also considered simple eigenvalues and Dirichlet boundary conditions but assuming exact integration. In this case, the known results (see [13]) give only an order $O(h^{k+1})$ for the eigenvalues, in contrast to $O(h^{2k})$ which would be achievable on the polygonal domains if the eigenfunctions were smooth enough. Lebaud showed how to construct "a good approximation" of the boundary in order to obtain the optimal order of convergence for eigenvalues. However, no direct extension of this method to three-dimensional domains seems to be possible.

More recently, Hernández and Rodríguez [8] considered finite element approximation of the spectral problem for the Laplace equation with Neumann boundary conditions on curved non-convex domains. By using the abstract spectral approximation theory, they proved optimal order error estimates for the eigenfunctions and a double order for eigenvalues. Later, the same authors proved convergence results and error estimates for the Raviart-Thomas approximations of the spectral acoustic problem on a curved non-convex two-dimensional domain [9].

The goal of this paper is to prove some abstract results on spectral approximation that can be applied to a wide variety of eigenvalue problems defined over curved domains. These results are obtained by introducing suitable modifications in the theory developed by Descloux,

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Nassif and Rappaz [4, 5]. Our analysis adapts the theory presented there to the fact that we are dealing with nonconforming discretizations because of the approximation of the given domain by a polyhedral one.

The remainder of the paper is organized as follows. Section 2 is devoted to introducing the notation. In Section 3, we give a precise statement of the eigenvalue problems and the approximation methods we will consider. In Section 4, we prove the abstract results. Finally, in Section 5, as an application of our results, we analyze the finite element approximation of the spectral problem for the Lamé equation with boundary conditions of Dirichlet type on a general curved three-dimensional domain. We prove convergence and optimal order error estimates for standard piecewise linear continuous elements.

Let us remark that our analysis is suitable for studying numerical approximations of operators with non-compact inverse. In particular, in a forthcoming paper we will apply this theory to investigate the finite element approximation of the Maxwell eigenproblem on curved Lipschitz polyhedral domains.

2. Notation. Throughout this paper Ω denotes a bounded open domain in $\mathbb{R}^n$, $n = 2$ or 3, in general non-convex, with a Lipschitz continuous boundary $\partial\Omega$. Let $W(\mathbb{R}^n)$ be a complex Hilbert function space with norm $\|\cdot\|_{\mathbb{R}^n}$. Given an open set $O \subset \mathbb{R}^n$, let $W(O)$ denote a generic complex Hilbert space of functions defined in O and $\|\cdot\|_O$ its norm.

First, we define the restriction operator $\check{\mathbf{S}}$ by

$$\begin{aligned} \check{\mathbf{S}} : W(\mathbb{R}^n) &\to W(O) \\ f &\mapsto f|_O. \end{aligned}$$

We restrict our attention to Hilbert spaces such that the norm $\|\cdot\|_{\mathbb{R}^n}$ satisfies

$$\|\cdot\|^2_{\mathbb{R}^n} = \|\cdot\|^2_O + \|\cdot\|^2_{\mathbb{R}^n\setminus O}.$$

Then, as an immediate consequence of this assumption, we obtain that $\check{\mathbf{S}}$ is a bounded operator.

We will need to provide extensions for functions on O to $\mathbb{R}^n$. With $u \in W(O)$, we extend it by zero from its original domain to $\mathbb{R}^n$ and we denote this extended function by $\bar{u}$. Now, let $W_0(O)$ be the space of all functions in $W(O)$ defined in such a way that the extension operator $\hat{\mathbf{S}}$, given by

$$\begin{aligned} \hat{\mathbf{S}} : W_0(O) &\to W(\mathbb{R}^n) \\ u &\mapsto \bar{u}, \end{aligned} \tag{2.1}$$

is well defined and bounded. Finally, we can define the function space $\widetilde{W}(\mathbb{R}^n) := \hat{\mathbf{S}}(W_0(O))$ endowed with the norm $\|\cdot\|_{\mathbb{R}^n}$. In what follows, to simplify notation, we will write $\|\cdot\|_{\mathbb{R}^n} = \|\cdot\|$.

3. Statement of the eigenvalue problem. Let $X(\Omega)$ be a complex Hilbert function space with norm $|\cdot|_\Omega$. Let $V(\Omega)$ be a closed subspace of $X(\Omega)$, with norm $\|\cdot\|_\Omega$, such that the inclusion $V(\Omega) \hookrightarrow X(\Omega)$ is continuous. We denote by $V_0(\Omega)$ the subspace of $V(\Omega)$ defined as in (2.1).

Consider the eigenvalue problem:

Find $\mu \in C, u \neq 0, u \in V_0(\Omega)$, such that

$$a(u, v) = \mu b(u, v), \quad \forall v \in V_0(\Omega), \tag{3.1}$$

where $a : V(\Omega) \times V(\Omega) \to C$ is a continuous and coercive sesquilinear form and $b : X(\Omega) \times X(\Omega) \to C$ is a continuous sesquilinear form.

Let $\mathbf{T}$ be the linear operator defined by

$$\begin{aligned} \mathbf{T} : X(\Omega) \quad &\rightarrow V_0(\Omega) \hookrightarrow X(\Omega) \\ x \quad &\mapsto u, \end{aligned}$$

where $u \in V_0(\Omega)$ is the solution of

$$a(u, y) = b(x, y), \quad \forall y \in V_0(\Omega). \tag{3.2}$$

Since a is elliptic, b is continuous, and $V(\Omega) \hookrightarrow X(\Omega)$, The Lax-Milgram Lemma allows us to conclude that $\mathbf{T}$ is a bounded linear operator. It is simple to show that μ is an eigenvalue of (3.1) if and only if $\lambda = 1/\mu$ is an eigenvalue of the operator $\mathbf{T}$ and the corresponding associated eigenfunctions u coincide.

Now, we define the linear operator $\mathbf{A}$ by

$$\begin{aligned} \mathbf{A} : X(\mathbb{R}^n) \quad &\rightarrow \widetilde{V}(\mathbb{R}^n) \\ x \quad &\mapsto \bar{u} = \hat{\mathbf{S}}\mathbf{T}\check{\mathbf{S}}x. \end{aligned}$$

It is clear that $\bar{u}|_\Omega = u$, where $u \in V_0(\Omega)$ is the solution of problem (3.2).

The curved domain Ω is approximated by a family of domains $\Omega_h,\ h > 0$, with polygonal boundary $\partial\Omega_h$. Let $\mathcal{T}_h$ be a standard partition of Ω_h into n-simplices such that each vertex of $\partial\Omega_h$ also lies on $\partial\Omega$. The index h denotes, as usual, the mesh size of $\mathcal{T}_h$. We assume that the family $\{\mathcal{T}_h\}$ is regular in the sense of the minimal angle condition, i.e., there is a constant C independent of the choice of $\mathcal{T}_h$ such that $\mathrm{vol}(T) \geq C\mathrm{diam}^n(T)$ for all $T \in \mathcal{T}_h$, where $\mathrm{vol}(T)$ denotes the n-dimensional volume of T; see [2], for instance.

Let $V_h(\Omega_h)$ be a finite-dimensional space on Ω_h such that $V_h(\Omega_h) \subset V(\Omega_h)$, for all h. We denote by $V_{0h}(\Omega_h)$ the space of all the functions in $V_h(\Omega_h)$ defined as in (2.1). Then, we consider the following discretization of eigenvalue problem (3.1):

Find $\mu_h \in C$, $u_h \neq 0$, $u_h \in V_{0h}(\Omega_h)$, such that

$$a_h(u_h, v) = \mu_h b_h(u_h, v), \quad \forall v \in V_{0h}(\Omega_h). \tag{3.3}$$

In what follows we shall assume that the approximate sesquilinear forms a_h and b_h are continuous on $V(\Omega_h)$ uniformly in h and that a_h is coercive on $V(\Omega_h)$ uniformly in h. We remark that, since $V_{0h}(\Omega_h) \not\subset V_0(\Omega)$, (3.3) represents a nonconforming approximation to (3.1).

Let us now define the function space $\widetilde{V}_h(\mathbb{R}^n) := \hat{\mathbf{S}}(V_{0h}(\Omega_h))$. Then, the discrete analogue of the operator $\mathbf{A}$ can be define as follows:

$$\begin{aligned} \mathbf{A}_h : X(\mathbb{R}^n) \quad &\rightarrow \widetilde{V}_h(\mathbb{R}^n) \\ x \quad &\mapsto \bar{u}_h \quad : \quad \bar{u}_h|_{\Omega_h} = u_h, \end{aligned}$$

where $u_h \in V_{0h}(\Omega_h)$ is the solution of

$$a_h(u_h, y) = b_h(x, y), \quad \forall y \in V_{0h}(\Omega_h).$$

Once again, because of the Lax-Milgram Lemma, the operator $\mathbf{A}_h$ is bounded uniformly in h. As in the continuous case, it is simple to show that μ_h is an eigenvalue of problem (3.3) if and only if $\lambda_h = 1/\mu_h$ is an eigenvalue of the operator $\mathbf{A}_h$, and the corresponding associated eigenfunctions are related by $u_h = \bar{u}_h|_{\Omega_h}$.

We end this section by making other assumptions for the sesquilinear forms a and a_h. We assume that the form $a(x, y)$ can be expressed as

$$a(x, y) = a_1(x|_{\Omega\cap\Omega_h}, y|_{\Omega\cap\Omega_h}) + a_2(x|_{\Omega\setminus\Omega_h}, y|_{\Omega\setminus\Omega_h}), \tag{3.4}$$

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where a_1 and a_2 are continuous bilinear forms on $V(\Omega \cap \Omega_h)$ and $V(\Omega \setminus \Omega_h)$, respectively. We also assume that

$$a_h(x_h, y_h) = a_{1h}(x_h|_{\Omega\cap\Omega_h}, y_h|_{\Omega\cap\Omega_h}) + a_{2h}(x_h|_{\Omega_h\setminus\Omega}, y_h|_{\Omega_h\setminus\Omega}). \tag{3.5}$$

Finally, if $x, y \in V(\mathbb{R}^n)$, we assume that

$$a_1(x|_{\Omega\cap\Omega_h}, y|_{\Omega\cap\Omega_h}) = a_{1h}(x|_{\Omega\cap\Omega_h}, y|_{\Omega\cap\Omega_h}) \tag{3.6}$$

holds.

4. Spectral approximation. In this section, we present several abstract results on the approximation of eigenvalues and eigenvectors of non-compact operators defined over curved domains. These results are obtained by suitable modifications of the theory presented in [4] and [5]. As a consequence of these modifications, consistency terms arise in the error estimates.

First, we introduce some notation that will be used in the sequel. For further information on eigenvalue problems we refer the reader to [1]. We denote by $\rho(\mathbf{A})$ the resolvent set of $\mathbf{A}$ and by $\sigma(\mathbf{A})$ the spectrum of $\mathbf{A}$. For any $z \in \rho(\mathbf{A})$, $R_z(\mathbf{A}) = (z - \mathbf{A})^{-1}$ defines the resolvent operator.

Let λ be a nonzero isolated eigenvalue of $\mathbf{A}$ with algebraic multiplicity m. Let Γ be a circle in the complex plane centered at λ which lies in $\rho(\mathbf{A})$ and which encloses no other points of $\sigma(\mathbf{A})$. The continuous spectral projector, $\mathbf{E} : V(\mathbb{R}^n) \to \widetilde{V}(\mathbb{R}^n)$, relative to λ, is defined by

$$\mathbf{E} = \frac{1}{2\pi i}\int_\Gamma R_z(\mathbf{A})\, dz.$$

We assume that the following properties are satisfied:

P1:

$$\lim_{h\to 0} \|(\mathbf{A} - \mathbf{A}_h)|_{\widetilde{V}_h(\mathbb{R}^n)}\| = 0.$$

P2: For each function x of $\mathbf{E}(V(\mathbb{R}^n))$,

$$\lim_{h\to 0} \|x\|_{\Omega\setminus\Omega_h} = 0.$$

P3: For each function x of $\mathbf{E}(V(\mathbb{R}^n))$,

$$\lim_{h\to 0}\left(\inf_{x_h\in\widetilde{V}_h(\mathbb{R}^n)} \|x - x_h\|\right) = 0.$$

P4:

$$\lim_{h\to 0} \|(\mathbf{A} - \mathbf{A}_h)|_{\mathbf{E}(V(\mathbb{R}^n))}\| = 0.$$

We are going to give an extension of the theory developed in [4] to deal with curved domains. Most of the proofs of the results stated below are slight modifications of those in [4]. From now on, C denotes a constant, not necessarily the same at each occurrence, but always independent of h.

LEMMA 4.1. *Let $\mathcal{G}$ be a closed subset of $\rho(\mathbf{A})$. Under assumption* **P1**, *there exist positive constants C and h_0, independent of h, such that*

$$\|(z - \mathbf{A}_h|_{\widetilde{V}_h(\mathbb{R}^n)})^{-1}\| \le C, \quad \forall z \in \mathcal{G}, \quad \forall h < h_0.$$

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Proof. The proof is identical to that of [4, Lemma 1]. ◻

THEOREM 4.2. *Let $\mathcal{O} \in C$ be a compact set not intersecting $\sigma(\mathbf{A})$. There exist $h_0 > 0$ such that, if $h < h_0$, then $\mathcal{O}$ does not intersect $\sigma(\mathbf{A}_h|_{\widetilde{V}_h(\mathbb{R}^n)})$.*

Proof. The proof is a direct consequence of assumption $\mathbf{P1}$, as it is shown in [4, Theorem 1]. ◻

Therefore, by virtue of the previous theorem, if h is small enough, $\Gamma \subset \rho(\mathbf{A}_h|_{\widetilde{V}_h(\mathbb{R}^n)})$ and the discrete spectral projector, $\mathbf{E}_h : V(\mathbb{R}^n) \to \widetilde{V}_h(\mathbb{R}^n)$, can be defined by

$$\mathbf{E}_h = \frac{1}{2\pi i}\int_\Gamma R_z(\mathbf{A}_h|_{\widetilde{V}_h(\mathbb{R}^n)})\,dz.$$

Let us recall the definition of the gap $\widehat{\delta}$ between two closed subspaces, Y and Z, of $V(\mathbb{R}^n)$. We define

$$\widehat{\delta}(Y,Z) := \max\{\delta(Y,Z), \delta(Z,Y)\},$$

where

$$\delta(Y,Z) := \sup_{\substack{y \in Y \\ \|y\| = 1}} \left(\inf_{z\in Z} \|y - z\|\right).$$

The following theorem implies uniform convergence of $\mathbf{E}_h|_{\widetilde{V}_h(\mathbb{R}^n)}$ to $\mathbf{E}|_{\widetilde{V}_h(\mathbb{R}^n)}$ as h goes to 0.

THEOREM 4.3. *Under assumption* **P1***,*

$$\lim_{h\to 0} \|(\mathbf{E} - \mathbf{E}_h)|_{\widetilde{V}_h(\mathbb{R}^n)}\| = 0.$$

Proof. It follows combining Lemma 4.1 with assumption **P1** and it is essentially identical to that of [4, Lemma 2]. ◻

THEOREM 4.4. *Under the assumption* **P1***, for all $x \in \mathbf{E}_h(V(\mathbb{R}^n))$ there holds*

$$\lim_{h\to 0} \delta(x, \mathbf{E}(V(\mathbb{R}^n))) = 0.$$

Proof. It is a direct consequence of Theorem 4.3. ◻

THEOREM 4.5. *Under the assumptions* **P1** *and* **P3***, for all $x \in \mathbf{E}(V(\mathbb{R}^n))$ holds*

$$\lim_{h\to 0} \delta(x, \mathbf{E}_h(V(\mathbb{R}^n))) = 0.$$

Proof. The proof is identical to that of [4, Theorem 3]. ◻

THEOREM 4.6. *Under the assumptions* **P1** *and* **P3***,*

$$\lim_{h\to 0} \widehat{\delta}(\mathbf{E}(V(\mathbb{R}^n)), \mathbf{E}_h(V(\mathbb{R}^n))) = 0.$$

Proof. It is direct consequence of Theorem 4.4 and Theorem 4.5. ◻

As a consequence of the previous theorems, isolated parts of the spectrum of $\mathbf{A}$ are approximated by isolated parts of the spectrum of $\mathbf{A}_h$; see [10] and [4]. More precisely, for

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any eigenvalue λ of $\mathbf{A}$ of finite multiplicity m, there exist exactly m eigenvalues $\lambda_{1h}, \cdots, \lambda_{mh}$ of $\mathbf{A}_h$, repeated according to their respective multiplicities, converging to λ as h goes to zero.

Next we are going to give estimates which show how the eigenvalues of $\mathbf{T}$ are approximated by those of $\mathbf{T}_h$. To attain this goal, we extend the theory developed in [5] so that it can be applied to more general situations where the original and the discrete domains do not coincide. By so doing, consistency terms arise in the error estimates. These consistency terms are associated with the variational crime committed by approximating the curved boundary with a polyhedral one. We shall give general expressions for these additional consistency terms.

We begin considering the bounded operator $\mathbf{A}_*$ defined by

$$\begin{aligned} \mathbf{A}_* : X(\mathbb{R}^n) &\to \widetilde{V}(\mathbb{R}^n) \\ x &\mapsto \bar{u} \quad : \quad \bar{u}|_\Omega = u, \end{aligned}$$

where $u \in V_0(\Omega)$ is the solution of

$$a(y, u) = b(y, x), \quad \forall y \in V_0(\Omega).$$

It is known that $\bar{\lambda}$ is an eigenvalue of $\mathbf{A}_*$ with the same multiplicity m as that of λ. We also consider the bounded operator $\mathbf{A}_{*h}$ defined by

$$\begin{aligned} \mathbf{A}_{*h} : X(\mathbb{R}^n) &\to \widetilde{V}_h(\mathbb{R}^n) \\ x &\mapsto \bar{u}_h \quad : \quad \bar{u}_h|_{\Omega_h} = u_h, \end{aligned}$$

where $u_h \in V_{0h}(\Omega_h)$ is the solution of

$$a_h(y, u_h) = b_h(y, x), \quad \forall y \in V_{0h}(\Omega_h).$$

Here, $\bar{\lambda}_{1h}, \cdots, \bar{\lambda}_{mh}$ are the eigenvalues of $\mathbf{A}_{*h}$ which converge to $\bar{\lambda}$ as h goes to zero.

Let $\mathbf{E}_*$ be the spectral projector of $\mathbf{A}_*$ relative to $\bar{\lambda}$. We also assume the following properties for $\mathbf{A}_*$ and $\mathbf{A}_{*h}$:

P5:

$$\lim_{h\to 0} \|(\mathbf{A}_* - \mathbf{A}_{*h})|_{\widetilde{V}_h(\mathbb{R}^n)}\| = 0.$$

P6: For each function x of $\mathbf{E}_*(V(\mathbb{R}^n))$,

$$\lim_{h\to 0} \|x\|_{\Omega\setminus\Omega_h} = 0.$$

P7: For each function x of $\mathbf{E}_*(V(\mathbb{R}^n))$,

$$\lim_{h\to 0} \left(\inf_{x_h \in \widetilde{V}_h(\mathbb{R}^n)} \|x - x_h\| \right) = 0.$$

P8:

$$\lim_{h\to 0} \|(\mathbf{A}_* - \mathbf{A}_{*h})|_{\mathbf{E}(V(\mathbb{R}^n))}\| = 0.$$

We now need to introduce other operators. Let $\Pi_h : V(\mathbb{R}^n) \to V(\mathbb{R}^n)$ be the projector defined by the relations

$$\begin{aligned} a_h(x - \Pi_h x, y) &= 0, \qquad \forall y \in V_{0h}(\Omega_h) \\ (\Pi_h x)|_{\mathbb{R}^n\setminus\Omega_h} &= 0. \end{aligned} \tag{4.1}$$

Because $V_{0h}(\Omega_h)$ is a closed subset of $V(\Omega_h)$, $(\Pi_h x)|_{\Omega_h} \in V_{0h}(\Omega_h)$. Hence, we have $\Pi_h x \in \widetilde{V}_h(\mathbb{R}^n)$. Analogously, we define the projector $\Pi_{*h} : V(\mathbb{R}^n) \to V(\mathbb{R}^n)$ with range $\widetilde{V}_h(\mathbb{R}^n)$ by the relations:

$$\begin{aligned} a_h(y, x - \Pi_{*h}x) &= 0, \qquad \forall y \in V_{0h}(\Omega_h) \\ (\Pi_{*h}x)|_{\mathbb{R}^n \setminus \Omega_h} &= 0. \end{aligned} \tag{4.2}$$

Since a_h is continuous and coercive on $V(\Omega_h)$, both uniformly in h, the operators Π_h and Π_{*h} are bounded uniformly in h. Let us remark that for conforming methods $\mathbf{A}_h = \Pi_h\mathbf{A}$. This is assumed in the spectral approximation theory in [5] and used in the proofs therein. When variational crimes in the discretization of the domains are allowed, $\mathbf{A}_h$ and $\Pi_h\mathbf{A}$ do not coincide.

Let $\mathbf{B}_h := \mathbf{A}_h\Pi_h : V(\mathbb{R}^n) \to V(\mathbb{R}^n)$. Notice that $\sigma(\mathbf{A}_h) = \sigma(\mathbf{B}_h)$ and that, for any non-null eigenvalue, the corresponding invariant subspaces coincide. Let $\mathbf{F}_h : V(\mathbb{R}^n) \to V(\mathbb{R}^n)$ be the spectral projector of $\mathbf{B}_h$ relative to its eigenvalues $\lambda_{1h}, \cdots, \lambda_{mh}$. It can be proved that $\|R_z(\mathbf{B}_h)\|$ is bounded uniformly in h for $z \in \Gamma$; see [5, Lemma 1]. Consequently, the spectral projectors $\mathbf{F}_h$ are bounded uniformly on h.

Finally, let $\mathbf{B}_{*h} := \mathbf{A}_{*h}\Pi_{*h} : V(\mathbb{R}^n) \to V(\mathbb{R}^n)$ and let $\mathbf{F}_{*h}$ be the spectral projector of $\mathbf{B}_{*h}$ relative to $\bar{\lambda}_{1h}, \cdots, \bar{\lambda}_{mh}$. It is easy to show that $\mathbf{B}_{*h}$ is the actual adjoint of $\mathbf{B}_h$ with respect to a_h. In fact, for all x and $y \in V(\mathbb{R}^n)$, we have

$$a_h(\mathbf{B}_h x, y) = a_h(\mathbf{A}_h\Pi_h x, y) = a_h(\mathbf{A}_h\Pi_h x, \Pi_{*h}y) = b_h(\Pi_h x, \Pi_{*h}y).$$

Similarly, we get

$$a_h(x, \mathbf{B}_{*h}y) = b_h(\Pi_h x, \Pi_{*h}y).$$

Therefore, the spectral projector $\mathbf{F}_{*h}$ is also the adjoint of $\mathbf{F}_h$ with respect to a_h.

Let

$$\gamma_h := \delta(\mathbf{E}(V(\mathbb{R}^n)), \widetilde{V}_h(\mathbb{R}^n)) + \sup_{\substack{y \in \mathbf{E}(V(\mathbb{R}^n)) \\ \|y\| = 1}} \|y\|_{\Omega\setminus\Omega_h}.$$

Properties **P2** and **P3** imply that $\gamma_h \to 0$ as $h \to 0$. Analogously, let

$$\gamma_{*h} := \delta(\mathbf{E}_*(V(\mathbb{R}^n)), \widetilde{V}_h(\mathbb{R}^n)) + \sup_{\substack{y \in \mathbf{E}_*(V(\mathbb{R}^n)) \\ \|y\| = 1}} \|y\|_{\Omega\setminus\Omega_h}.$$

Here, because **P6** and **P7**, $\gamma_{*h} \to 0$ as $h \to 0$.

LEMMA 4.7.

$$\|(\mathbf{I} - \Pi_h)|_{\mathbf{E}(V(\mathbb{R}^n))}\| \le C\gamma_h,$$

$$\|(\mathbf{I} - \Pi_{*h})|_{\mathbf{E}_*(V(\mathbb{R}^n))}\| \le C\gamma_{*h}.$$

Proof. For a $x \in \mathbf{E}(V(\mathbb{R}^n))$, we have

$$\|(\mathbf{I} - \Pi_h)x\|^2 = \|(\mathbf{I} - \Pi_h)x\|^2_{\Omega_h} + \|x\|^2_{\Omega\setminus\Omega_h}. \tag{4.3}$$

Using that a_h is coercive on $V(\Omega_h)$ uniformly in h, we have

$$\|(\mathbf{I}-\Pi_h)x\|^2_{\Omega_h} \le Ca_h((\mathbf{I}-\Pi_h)x, (\mathbf{I}-\Pi_h)x) = Ca_h((\mathbf{I}-\Pi_h)x, x-y_h),\ \forall y_h \in V_{0h}(\Omega_h),$$

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where the last equality results from the definition of Π_h. Now, taking into account that a_h is continuous on $V(\Omega_h)$ uniformly in h, we obtain

$$\|(\mathbf{I} - \Pi_h)x\|_{\Omega_h} \le C \inf_{y_h \in \widetilde{V}_h(\mathbb{R}^n)} \|x - y_h\|,$$

which together (4.3) allows us to conclude the proof of the first estimation. An analogous proof is valid for the second one. □

LEMMA 4.8.

$$\|(\mathbf{E} - \mathbf{F}_h)|_{\mathbf{E}(V(\mathbb{R}^n))}\| \le C\|(\mathbf{A} - \mathbf{B}_h)|_{\mathbf{E}(V(\mathbb{R}^n))}\|,$$

$$\|(\mathbf{E}_* - \mathbf{F}_{*h})|_{\mathbf{E}_*(V(\mathbb{R}^n))}\| \le C\|(\mathbf{A}_* - \mathbf{B}_{*h})|_{\mathbf{E}_*(V(\mathbb{R}^n))}\|.$$

Proof. The proof is identical to that of [5, Lemma 3]. □

Let

$$\delta_h := \gamma_h + \|(\mathbf{A} - \mathbf{A}_h)|_{\mathbf{E}(V(\mathbb{R}^n))}\|.$$

From properties **P2**, **P3** and **P4** it is easily seen that $\delta_h \to 0$ as $h \to 0$. Analogously, let

$$\delta_{*h} := \gamma_{*h} + \|(\mathbf{A}_* - \mathbf{A}_{*h})|_{\mathbf{E}(V(\mathbb{R}^n))}\|.$$

Given **P6**, **P7** and **P8** $\delta_{*h} \to 0$ as $h \to 0$.

LEMMA 4.9.

$$\|(\mathbf{A} - \mathbf{B}_h)|_{\mathbf{E}(V(\mathbb{R}^n))}\| \le C\delta_h,$$

$$\|(\mathbf{A}_* - \mathbf{B}_{*h})|_{\mathbf{E}(V(\mathbb{R}^n))}\| \le C\delta_{*h}.$$

Proof. Let $x \in \mathbf{E}(V(\mathbb{R}^n))$ with $\|x\| = 1$. We have

$$\begin{aligned}
\|(\mathbf{A} - \mathbf{B}_h)x\| &\le \|(\mathbf{A} - \mathbf{A}_h)x\| + \|\mathbf{A}_h(\mathbf{I} - \Pi_h)x\| \\
&\le \|(\mathbf{A} - \mathbf{A}_h)|_{\mathbf{E}(V(\mathbb{R}^n))}\| + \|\mathbf{A}_h\|\,\|(\mathbf{I} - \Pi_h)|_{\mathbf{E}(V(\mathbb{R}^n))}\| \\
&\le (\|(\mathbf{A} - \mathbf{A}_h)|_{\mathbf{E}(V(\mathbb{R}^n))}\| + \gamma_h,
\end{aligned}$$

where the last inequality follows from Lemma 4.7 and the fact that $\|\mathbf{A}_h\|$ is uniformly bounded with respect to h. An analogous proof is valid for the second estimate of the Lemma. □

Let

$$\Lambda_h := \mathbf{F}_h|_{\mathbf{E}(V(\mathbb{R}^n))} : \mathbf{E}(V(\mathbb{R}^n)) \to \mathbf{F}_h(V(\mathbb{R}^n)).$$

LEMMA 4.10. *For h small enough, Λ_h is a bijection and $\|\Lambda_h^{-1}\|$ is bounded uniformly in h.*

Proof. See the proof of [5, Theorem 1]. □

THEOREM 4.11.

$$\widehat{\delta}(\mathbf{F}_h(V(\mathbb{R}^n)), \mathbf{E}(V(\mathbb{R}^n))) \le C\delta_h.$$

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Proof. The proof is identical to that of [5, Theorem 1]. ∎

Let us now define the operators $\hat{\mathbf{A}} := \mathbf{A}|_{\mathbf{E}(V(\mathbb{R}^n))} : \mathbf{E}(V(\mathbb{R}^n)) \to \mathbf{E}(V(\mathbb{R}^n))$ and $\hat{\mathbf{B}}_h := \Lambda_h^{-1}\mathbf{B}_h\Lambda_h : \mathbf{E}(V(\mathbb{R}^n)) \to \mathbf{E}(V(\mathbb{R}^n))$. From these definitions, it follows that $\hat{\mathbf{A}}$ has a unique eigenvalue λ of algebraic multiplicity m and that $\hat{\mathbf{B}}_h$ has the eigenvalues $\lambda_{1h}, \cdots, \lambda_{mh}$.

Let us consider the following consistency terms:

$$M_h = \sup_{\substack{x \in \mathbf{E}(V(\mathbb{R}^n)) \\ \|x\|=1}} \sup_{\substack{y \in \mathbf{E}_*(V(\mathbb{R}^n)) \\ \|y\|=1}} |a_h(\mathbf{A}x, \Pi_{*h}y - y) - b_h(x, \Pi_{*h}y - y)|,$$

$$M_{*h} = \sup_{\substack{x \in \mathbf{E}(V(\mathbb{R}^n)) \\ \|x\|=1}} \sup_{\substack{y \in \mathbf{E}_*(V(\mathbb{R}^n)) \\ \|y\|=1}} |a_h(\Pi_h x - x, \mathbf{A}_* y) - b_h(\Pi_h x - x, y)|,$$

$$N_h = \sup_{\substack{x \in \mathbf{E}(V(\mathbb{R}^n)) \\ \|x\|=1}} \sup_{\substack{y \in \mathbf{E}_*(V(\mathbb{R}^n)) \\ \|y\|=1}} |a_h(\mathbf{A}x, y) - b_h(x, y)|.$$

THEOREM 4.12.

$$\|\hat{\mathbf{A}} - \hat{\mathbf{B}}_h\| \leq C\Big(\delta_h\, \delta_{*h} + M_h + M_{*h} + N_h\Big).$$

Proof. We have

$$\begin{aligned}
\|\hat{\mathbf{A}} - \hat{\mathbf{B}}_h\| &= \sup_{\substack{x \in \mathbf{E}(V(\mathbb{R}^n)) \\ \|x\|=1}} \|(\hat{\mathbf{A}} - \hat{\mathbf{B}}_h)x\| = \sup_{\substack{x \in \mathbf{E}(V(\mathbb{R}^n)) \\ \|x\|=1}} \|(\hat{\mathbf{A}} - \hat{\mathbf{B}}_h)x\|_\Omega \\
&\leq C \sup_{\substack{x \in \mathbf{E}(V(\mathbb{R}^n)) \\ \|x\|=1}} \sup_{\substack{y \in \widetilde{V}(\mathbb{R}^n) \\ \|y\|=1}} a((\hat{\mathbf{A}} - \hat{\mathbf{B}}_h)x, y) \\
&= C \sup_{\substack{x \in \mathbf{E}(V(\mathbb{R}^n)) \\ \|x\|=1}} \sup_{\substack{y \in \widetilde{V}(\mathbb{R}^n) \\ \|y\|=1}} a(\mathbf{E}(\hat{\mathbf{A}} - \hat{\mathbf{B}}_h)x, y) \\
&= C \sup_{\substack{x \in \mathbf{E}(V(\mathbb{R}^n)) \\ \|x\|=1}} \sup_{\substack{y \in \widetilde{V}(\mathbb{R}^n) \\ \|y\|=1}} a((\hat{\mathbf{A}} - \hat{\mathbf{B}}_h)x, \mathbf{E}_* y) \\
&\leq C \sup_{\substack{x \in \mathbf{E}(V(\mathbb{R}^n)) \\ \|x\|=1}} \sup_{\substack{y \in \mathbf{E}_*(V(\mathbb{R}^n)) \\ \|y\|=1}} a((\hat{\mathbf{A}} - \hat{\mathbf{B}}_h)x, y).
\end{aligned} \tag{4.4}$$

Since $(\hat{\mathbf{A}} - \hat{\mathbf{B}}_h)x,\, y \in \widetilde{V}(\mathbb{R}^n)$, we can use (3.4) and (3.6) to get

$$\begin{aligned}
a((\hat{\mathbf{A}} - \hat{\mathbf{B}}_h)x, y) &= a_{1h}((\hat{\mathbf{A}} - \hat{\mathbf{B}}_h)x|_{\Omega\cap\Omega_h}, y|_{\Omega\cap\Omega_h}) + a_2((\hat{\mathbf{A}} - \hat{\mathbf{B}}_h)x|_{\Omega\setminus\Omega_h}, y|_{\Omega\setminus\Omega_h}) \\
&= a_h((\hat{\mathbf{A}} - \hat{\mathbf{B}}_h)x, y) + a_2((\hat{\mathbf{A}} - \hat{\mathbf{B}}_h)x|_{\Omega\setminus\Omega_h}, y|_{\Omega\setminus\Omega_h}).
\end{aligned} \tag{4.5}$$

Now, using that $(\Lambda_h^{-1}\mathbf{F}_h - \mathbf{I})\mathbf{A}|_{\mathbf{E}(V(\mathbb{R}^n))} = 0$ and that $\mathbf{B}_h$ commutes with its spectral projector $\mathbf{F}_h$, we obtain

$$\hat{\mathbf{A}} - \hat{\mathbf{B}}_h = (\mathbf{A} - \mathbf{B}_h)|_{\mathbf{E}(V(\mathbb{R}^n))} + (\Lambda_h^{-1}\mathbf{F}_h - \mathbf{I})(\mathbf{A} - \mathbf{B}_h)|_{\mathbf{E}(V(\mathbb{R}^n))}. \tag{4.6}$$

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Let $x \in \mathbf{E}(V(\mathbb{R}^n))$ and $y \in \mathbf{E}_*(V(\mathbb{R}^n))$, with $\|x\| = \|y\| = 1$. Since $\mathbf{F}_h(\Lambda_h^{-1}\mathbf{F}_h - \mathbf{I}) = 0$ and $\mathbf{F}_{*h}$ is the adjoint of $\mathbf{F}_h$ with respect to a_h, we have

$$\begin{aligned}
&|a_h((\Lambda_h^{-1}\mathbf{F}_h - \mathbf{I})(\mathbf{A} - \mathbf{B}_h)x, y)| \\
&\quad = |a_h((\Lambda_h^{-1}\mathbf{F}_h - \mathbf{I})(\mathbf{A} - \mathbf{B}_h)x, y)| - |a_h(\mathbf{F}_h(\Lambda_h^{-1}\mathbf{F}_h - \mathbf{I})(\mathbf{A} - \mathbf{B}_h)x, y)| \\
&\quad = |a_h((\Lambda_h^{-1}\mathbf{F}_h - \mathbf{I})(\mathbf{A} - \mathbf{B}_h)x, y)| - |a_h((\Lambda_h^{-1}\mathbf{F}_h - \mathbf{I})(\mathbf{A} - \mathbf{B}_h)x, \mathbf{F}_{*h}y)| \\
&\quad = |a_h((\Lambda_h^{-1}\mathbf{F}_h - \mathbf{I})(\mathbf{A} - \mathbf{B}_h)x, (\mathbf{I} - \mathbf{F}_{*h})y)| \\
&\quad \le C\,\|\Lambda_h^{-1}\mathbf{F}_h - \mathbf{I}\|\,\|(\mathbf{A} - \mathbf{B}_h)|_{\mathbf{E}(V(\mathbb{R}^n))}\|\,\|(\mathbf{I} - \mathbf{F}_{*h})|_{\mathbf{E}_*(V(\mathbb{R}^n))}\| \le C\delta_h\delta_{*h}.
\end{aligned} \tag{4.7}$$

The last inequality in (4.7) follows from Lemmas 4.8, 4.9, and 4.10, the fact that a_h is continuous on $V(\Omega_h)$ independently of h and that $\mathbf{F}_h$ is bounded uniformly in h. On the other hand,

$$a_h((\mathbf{A} - \mathbf{B}_h)x, y) = a_h((\mathbf{A} - \mathbf{B}_h)x, \Pi_{*h}y) + a_h((\mathbf{A} - \mathbf{B}_h)x, (\mathbf{I} - \Pi_{*h})y). \tag{4.8}$$

To bound the second term in the right-hand side of this equation, we use Lemmas 4.7 and 4.9. We thus obtain

$$\begin{aligned}
|a_h((\mathbf{A} - \mathbf{B}_h)x, (\mathbf{I} - \Pi_{*h})y)| &\le C\,\|(\mathbf{A} - \mathbf{B}_h)|_{\mathbf{E}(V(\mathbb{R}^n))}\|\,\|(\mathbf{I} - \Pi_{*h})|_{\mathbf{E}_*(V(\mathbb{R}^n))}\| \\
&\le C\,\delta_h\gamma_{*h}.
\end{aligned} \tag{4.9}$$

For the first term, we have

$$a_h((\mathbf{A} - \mathbf{B}_h)x, \Pi_{*h}y) = a_h((\mathbf{A} - \mathbf{A}_h)x, \Pi_{*h}y) + a_h((\mathbf{A}_h - \mathbf{B}_h)x, \Pi_{*h}y). \tag{4.10}$$

Now,

$$|a_h((\mathbf{A} - \mathbf{A}_h)x, \Pi_{*h}y)| = |a_h(\mathbf{A}x, \Pi_{*h}y) - b_h(x, \Pi_{*h}y)| \le M_h + N_h, \tag{4.11}$$

and

$$\begin{aligned}
a_h((\mathbf{A}_h - \mathbf{B}_h)x, \Pi_{*h}y) &= a_h(\mathbf{A}_h(\mathbf{I} - \Pi_h)x, \Pi_{*h}y) = b_h((\mathbf{I} - \Pi_h)x, \Pi_{*h}y) \\
&= b_h((\mathbf{I} - \Pi_h)x, y) - b_h((\mathbf{I} - \Pi_h)x, (\mathbf{I} - \Pi_{*h})y).
\end{aligned} \tag{4.12}$$

The first term in the right-hand side of (4.12) can be written as

$$b_h((\mathbf{I}-\Pi_h)x, y) = [a_h((\Pi_h-\mathbf{I})x, \mathbf{A}_*y) - b_h((\Pi_h-\mathbf{I})x, y)] - a_h((\Pi_h-\mathbf{I})x, (\mathbf{I}-\Pi_{*h})\mathbf{A}_*y). \tag{4.13}$$

Now, the last term of the right-hand side above can be easily bounded by

$$|a_h((\Pi_h - \mathbf{I})x, (\mathbf{I} - \Pi_{*h})\mathbf{A}_*y)| \le C\,\|(\Pi_h - \mathbf{I})|_{\mathbf{E}(V(\mathbb{R}^n))}\|\,\|(\mathbf{I} - \Pi_{*h})|_{\mathbf{E}_*(V(\mathbb{R}^n))}\|\,\|\mathbf{A}_*\|. \tag{4.14}$$

Then, Lemma 4.7, (4.13), and (4.14) immediately yield

$$|b_h((\mathbf{I} - \Pi_h)x, y)| \le C(M_{*h} + \gamma_h\gamma_{*h}). \tag{4.15}$$

Finally, we estimate the last term in (4.5). Using that $(\Lambda_h^{-1}\mathbf{F}_h - \mathbf{I})$ is bounded uniformly in h, we obtain from (4.6) and Lemma 4.9

$$\begin{aligned}
|a_2((\hat{\mathbf{A}} - \hat{\mathbf{B}}_h)x|_{\Omega\setminus\Omega_h}, y|_{\Omega\setminus\Omega_h})| &\le C\|(\mathbf{A} - \mathbf{B}_h)|_{\mathbf{E}(V(\mathbb{R}^n))}\|\,\|y\|_{\Omega\setminus\Omega_h} \\
&\le C\delta_h \sup_{\substack{y \in \mathbf{E}_*(V(\mathbb{R}^n)) \\ \|y\| = 1}} \|y\|_{\Omega\setminus\Omega_h} \le C\delta_h\gamma_{*h}.
\end{aligned} \tag{4.16}$$

Now, the theorem is a consequence of formulae (4.4) to (4.16). ∎

By using the previous theorem, we deduce the following result about the approximation of the eigenvalue λ:

THEOREM 4.13.

i)
$$\Big|\lambda - \frac{1}{m}\sum_{i=1}^{m}\lambda_{ih}\Big| \le C\Big(\delta_h\,\delta_{*h} + M_h + M_{*h} + N_h\Big)$$

ii)
$$\max_{i=1,\cdots,m}|\lambda-\lambda_{ih}| \le C\Big(\delta_h\,\delta_{*h} + M_h + M_{*h} + N_h\Big)^{1/\alpha}$$

where α is the ascent of the eigenvalue λ of $\hat{\mathbf{A}}$.

Proof. Taking into account that $\sigma(\hat{\mathbf{A}}) = \lambda$ and that $\lambda_{1h}, \cdots, \lambda_{mh}$ are the eigenvalues of $\hat{\mathbf{B}}_h$, we have $tr(\hat{\mathbf{A}}) = m\lambda$ and $tr(\hat{\mathbf{B}}_h) = \sum_{i=1}^{m}\lambda_{ih}$. Then, from the continuity of the traces

$$\Big|\lambda - \frac{1}{m}\sum_{i=1}^{m}\lambda_{ih}\Big| = \frac{1}{m}|tr(\hat{\mathbf{A}}) - tr(\hat{\mathbf{B}}_h)| \le C\|\hat{\mathbf{A}} - \hat{\mathbf{B}}_h\|.$$

On the other hand, it is known that,

$$|\lambda - \lambda_{ih}|^{\alpha} \le C\|\hat{\mathbf{A}} - \hat{\mathbf{B}}_h\|,$$

for any $1 \le i \le m$. Therefore, we can conclude *i)* and *ii)* directly from Theorem 4.12. ∎

REMARK 4.14. In many applications, the operator $\mathbf{A}$ is self-adjoint. In this case, if μ is a nonzero eigenvalue of $\mathbf{A}$, the ascent α of $(\mu - \mathbf{A})$ is one. So, the space of generalized eigenvectors $E(\mathbb{R}^n)$ coincide with the space of the actual eigenvectors corresponding to μ; see [1].

5. Example. Let Ω be a bounded three-dimensional domain with a Lipschitz continuous boundary $\partial\Omega$. We assume that $\partial\Omega$ is piecewise smooth, more precisely, is piecewise of class C^2. To avoid additional technical difficulties, we will assume that the set of points where the condition of C^2- smoothness of $\partial\Omega$ is not satisfied consists of a finite number of straight lines and single points.

Let $(\cdot,\cdot)$ be the scalar product in $L^2(\Omega)$ and let $|\cdot|$ denote the corresponding L^2 norm. Further, $H^\sigma(\Omega)$ denotes the standard Sobolev spaces with the usual norms $\|\cdot\|_\sigma$ and $H_0^1(\Omega)$ denotes the subspace of functions in $H^1(\Omega)$ satisfying a zero Dirichlet boundary conditions.

We consider the spectral problem:

Given $s > 0$, find $\lambda \in \mathbb{R}$ and $\mathbf{u} \neq 0$ such that

$$\left\{\begin{array}{rl} s\,\mathbf{grad}\,(\operatorname{div}\mathbf{u}) - \mathbf{curl}\,\mathbf{curl}\,\mathbf{u} = \lambda\mathbf{u} & \text{in } \Omega,\\ \mathbf{u} = 0 & \text{on } \partial\Omega. \end{array}\right. \tag{5.1}$$

Let $X(\Omega) := (L^2(\Omega))^3$, $V(\Omega) := (H^1(\Omega))^3$ and $V_0(\Omega) := (H_0^1(\Omega))^3$. Let a_0 and b be the symmetric bilinear forms defined by

$$a_0(\mathbf{u},\mathbf{v}) := \int_\Omega \mathbf{curl}\,\mathbf{u}\cdot\,\mathbf{curl}\,\mathbf{v}\; + s\operatorname{div}\mathbf{u}\operatorname{div}\mathbf{v},\quad \forall\mathbf{u},\mathbf{v}\in V(\Omega),$$

$$b(\mathbf{u},\mathbf{v}) := \int_\Omega \mathbf{u}\cdot\mathbf{v},\quad \forall\mathbf{u},\mathbf{v}\in X(\Omega).$$

The bilinear form a_0 is coercive on $V_0(\Omega)$ but is not coercive on $V(\Omega)$. However, $a := a_0 + b$ can be used in our problem and it turns out to be coercive on $V(\Omega)$. Furthermore, a is continuous on $V(\Omega)$.

REMARK 5.1. When $s = \dfrac{\lambda_s + 2\mu_s}{\mu_s}$, the bilinear form $a_0(\mathbf{u}, \mathbf{v})$ is associated to the elasticity system for a material of Lamé coefficients λ_s and μ_s. Denoting the material density by ρ_s, problem (5.1) gives the vibration eigenfrequencies $\omega = \sqrt{\dfrac{\lambda \mu_s}{\rho_s}}$ of an elastic, homogeneous and isotropic three-dimensional body fixed along its boundary.

The variational formulation of problem (5.1) associated with a is given by:

Find $\lambda \in \mathbb{R}$ *and* $\mathbf{u} \in V_0(\Omega)$, $\mathbf{u} \neq 0$, *such that*

$$a(\mathbf{u}, \mathbf{v}) = (\lambda + 1)\, b(\mathbf{u}, \mathbf{v}), \quad \forall \mathbf{v} \in V_0(\Omega). \tag{5.2}$$

It is well known that problem (5.2) has an increasing sequence of finite multiplicity eigenvalues $\lambda_n > 0$, $n \in N$. There is no finite accumulation point. The corresponding $L^2(\Omega)$-orthonormal eigenfunctions $\mathbf{u}_n$ belong to $V_0(\Omega)$. Now, as in Section 2, we consider the bounded linear operator $\mathbf{T} : X(\Omega) \to X(\Omega)$ defined by $\mathbf{Tf} = \mathbf{u} \in V_0(\Omega)$ and

$$a(\mathbf{u}, \mathbf{y}) = b(\mathbf{f}, \mathbf{y}), \quad \forall \mathbf{y} \in V_0(\Omega). \tag{5.3}$$

By virtue of the Lax-Milgram Lemma, we have

$$\|\mathbf{u}\|_{\Omega} \leq C\, \|\mathbf{f}\|_{0,\Omega}.$$

As a consequence of the classical a priori estimates, for any $\mathbf{f} \in X(\Omega)$, $\mathbf{u} = \mathbf{Tf}$ is known to satisfy some further regularity. In fact, $\mathbf{u} \in (H^{1+r}(\Omega))^3$ for $r \in (1/2, 1]$ (see [3]) and there holds

$$\|\mathbf{u}\|_{1+r,\Omega} \leq C\, \|\mathbf{f}\|_{0,\Omega}. \tag{5.4}$$

Now, we consider the bounded linear operator $\mathbf{A} : X(\mathbb{R}^3) \to \widetilde{V}(\mathbb{R}^3)$ defined by $\mathbf{Af} = \hat{\mathbf{S}}\mathbf{T}\check{\mathbf{S}}\mathbf{f}$, where $\hat{\mathbf{S}}$ and $\check{\mathbf{S}}$ are the extension and the restriction operators, respectively, defined in Section 2. Since a and b are symmetric, $\mathbf{T}$ is self-adjoint with respect to a. Clearly, $\mathbf{A}$ is also self-adjoint with respect to a. Notice that $(\lambda, \mathbf{u})$ is a solution of problem (5.2) if and only if $(\frac{1}{\lambda+1}, \mathbf{u})$ is an eigenpair of $\mathbf{T}$ which, in its turn, is equivalent to $(\frac{1}{\lambda+1}, \bar{\mathbf{u}})$ being an eigenpair of $\mathbf{A}$, where $\bar{\mathbf{u}} = \hat{\mathbf{S}}(\mathbf{u})$.

Let the curved domain Ω be approximated by a polyhedron Ω_h with vertices on $\partial\Omega$. Let $\mathcal{T}_h$ be a partition of Ω_h, i.e., a set of a finite number of closed tetrahedra T, which has the following properties:

- each vertex of Ω_h is a vertex of a $T \in \mathcal{T}_h$,
- each $T \in \mathcal{T}_h$ has at least one vertex in the interior of Ω_h,
- any two tetrahedra, $T, T' \in \mathcal{T}_h$ share at most a vertex, a whole side, or a whole face.

Let $\mathcal{N}_h$ and $\mathcal{E}_h$ denote the set of all vertices and the set of all edges in $\mathcal{T}_h$, respectively. We assume that

- $\mathcal{N}_h \subset \bar{\Omega}$,
- $\mathcal{N}_h \cap \partial\Omega_h \subset \partial\Omega$,
- $\mathcal{E}_h$ contains all the points where the boundary $\partial\Omega$ is not C^2,
- for all $T \in \mathcal{T}_h$, at most one face of T lies on $\partial\Omega_h$.

We also assume that the family $\{\mathcal{T}_h\}$ is regular.

In what follows we will use some notation and definitions introduced in [6]. Consider a $T \in \mathcal{T}_h$ which has a face $\mathcal{S}_h^T \subset \partial\Omega_h$, called a boundary tetrahedra. We enumerate the vertices of T such that the vertices of $\mathcal{S}_h^T$ are numbered first and we denote them by P_1^T, P_2^T, P_3^T, and P_4^T, in local notation. Let Σ_h^T be the part of $\partial\Omega$ which is approximated by the face $\mathcal{S}_h^T$. We denote by T^{id} the closed tetrahedra with three plane sides, having P_4 as a common vertex, and with one curved side, coinciding with Σ_h^T, and we call it the ideal tetrahedra associated with $T \in \mathcal{T}_h$. For the sake of simplicity, we assume that the partitions $\mathcal{T}_h$ are such that for each boundary tetrahedra T, either $T \subset T^{id}$ or $T \supset T^{id}$. If we replace all boundary tetrahedra in $\mathcal{T}_h$ by their associated ideal tetrahedra T^{id}, we obtain the so-called ideal partition $\mathcal{T}_h^{id}$ of the domain Ω.

With the triangulation $\mathcal{T}_h$, we consider the finite element spaces

$$X(\Omega_h) := (L^2(\Omega_h))^3, \qquad V(\Omega_h) := (H^1(\Omega_h))^3,$$

$$V_h(\Omega_h) := \{\mathbf{v}_h \in V(\Omega_h) \, : \, \mathbf{v}_h|_T \in (\mathcal{P}_1(T))^3 \quad \forall T \in \mathcal{T}_h\},$$

and

$$V_{0h}(\Omega_h) := \{\mathbf{v}_h \in V_h(\Omega_h) \, : \, \mathbf{v}_h|_{\partial\Omega_h} = 0\}.$$

Let a_h and b_h be the symmetric bilinear forms defined by

$$a_h(\mathbf{u}, \mathbf{v}) := \int_{\Omega_h} \mathbf{curl}\,\mathbf{u} \cdot \mathbf{curl}\,\mathbf{v} + s \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v} + \int_{\Omega_h} \mathbf{u} \cdot \mathbf{v}, \quad \forall \mathbf{u}, \mathbf{v} \in V(\Omega_h),$$

$$b_h(\mathbf{u}, \mathbf{v}) := \int_{\Omega_h} \mathbf{u} \cdot \mathbf{v}, \quad \forall \mathbf{u}, \mathbf{v} \in X(\Omega_h).$$

Notice that the bilinear form a_h is coercive and continuous on $V(\Omega_h)$ uniformly in h. Then, the discretization of the spectral problem (5.2) is given by

Find $\lambda_h \in \mathbb{R}$ and $\mathbf{u}_h \in V_{0h}(\Omega_h)$, $\mathbf{u}_h \neq 0$, such that

$$a_h(\mathbf{u}_h, \mathbf{v}_h) = (\lambda_h + 1)\, b_h(\mathbf{u}_h, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in V_{0h}(\Omega_h).$$

Now, we can define a discrete analogue of $\mathbf{A}$. Let $\mathbf{A}_h : X(\mathbb{R}^3) \to \widetilde{V}_h(\mathbb{R}^3)$ be the bounded linear operator defined by $\mathbf{A}_h\mathbf{f} \in \widetilde{V}_h(\mathbb{R}^3)$ and

$$a_h(\mathbf{A}_h\mathbf{f}, \mathbf{v}_h) = b_h(\mathbf{f}, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in V_{0h}(\Omega_h).$$

It remains to show that the bilinear forms a and a_h satisfy the assumptions (3.4), (3.5), and (3.6). To that end, let ω be a closed subset of $\Omega \cup \Omega_h$ and consider the bilinear form

$$a_\omega(\mathbf{u}, \mathbf{v}) := \int_\omega \mathbf{curl}\,\mathbf{u} \cdot \mathbf{curl}\,\mathbf{v} + s \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v} + \int_\omega \mathbf{u} \cdot \mathbf{v}, \quad \forall \mathbf{u}, \mathbf{v} \in (H^1(\omega))^3,$$

Thus, noting that a_ω is continuous on $(H^1(\omega))^3$ uniformly in h, it suffices to take

$$a_1(\mathbf{u}|_{\Omega\cap\Omega_h}, \mathbf{v}|_{\Omega\cap\Omega_h}) = a_{1h}(\mathbf{u}|_{\Omega\cap\Omega_h}, \mathbf{v}|_{\Omega\cap\Omega_h}) = a_{\Omega\cap\Omega_h}(\mathbf{u}, \mathbf{v}),$$

$$a_2(\mathbf{u}|_{\Omega\setminus\Omega_h}, \mathbf{v}|_{\Omega\setminus\Omega_h}) = a_{\Omega\setminus\Omega_h}(\mathbf{u}, \mathbf{v}),$$

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$$a_{2h}(\mathbf{u}|_{\Omega_h\setminus\Omega},\mathbf{v}|_{\Omega_h\setminus\Omega}) = a_{\Omega_h\setminus\Omega}(\mathbf{u},\mathbf{v}).$$

In order to prove properties **P1**, **P2**, **P3**, and **P4** for this problem, we establish the following lemmas and definitions.

LEMMA 5.2. *There exists a positive constant C such that:*

$$\|\mathbf{v}\|_{0,\Omega\setminus\bar{\Omega}_h} \leq Ch^\sigma\|\mathbf{v}\|_{\sigma,\Omega} \quad \forall \mathbf{v}\in (H^\sigma(\Omega))^3, \quad 0\leq\sigma\leq 1,$$

$$\|\mathbf{v}\|_{0,\Omega_h\setminus\bar{\Omega}} \leq Ch^\sigma\|\mathbf{v}\|_{\sigma,\Omega_h} \quad \forall \mathbf{v}\in (H^\sigma(\Omega_h))^3, \quad 0\leq\sigma\leq 1.$$

Proof. By adapting the arguments used in the proof of [6, Lemma 3.3.11] for the three-dimensional case, the inequalities can be proved for $\sigma = 1$. Since the two inequalities are clearly true for $\sigma = 0$, they follow for $0 < \sigma < 1$ from standard results on interpolation in Sobolev spaces. □

DEFINITION 5.3. *Let $w_h \in V_{0h}(\Omega_h)$. A function $\hat{w}\in V_0(\Omega)$ is called associated with w_h if it has the following properties:*

- $\hat{w}\in C^0(\bar{\Omega})$,
- $\hat{w}(P_i) = w_h(P_i), \quad \forall P_i\in\mathcal{N}_h$,
- *$\hat{w}$ is linear on each tetrahedra $T\in\mathcal{T}_h\cap\mathcal{T}_h^{id}$,*
- *if $T\subset T^{id}$, $\hat{w}=0$ on $T^{id}\setminus T$ and $\hat{w}=w_h$ on T,*
- *if $T^{id}\subset T$, $\hat{w}|_{\partial T^{id}\subset\partial\Omega}=0$.*

The definition above is due to Feistauer and Ženíšek; see [6]. The construction of such a function follows basically from the interpolation theory developed to Zlámal [19] for two-dimensional curved finite elements. The extension of his ideas to the three-dimensional case is relatively straightforward so we do not include the details here.

LEMMA 5.4. *Let $\hat{w}\in V_0(\Omega)$ be associated with $w_h\in V_{0h}(\Omega_h)$. Let $T^{id}\in\mathcal{T}_h^{id}$ lie along $\partial\Omega$ and let $T\in\mathcal{T}_h$ be its approximation. If $T^{id}\subset T$, then*

$$\|\hat{w}-w_h\|_{T^{id}} \leq C\,h\|w_h\|_T,$$

where C is a constant independent of h.

Proof. The proof is a consequence of Definition 5.3 and a suitable extension of [19, Theorem 2]. □

In what follows, we will use smooth extensions of functions originally defined in Ω. We denote by φ^e an extension of φ from $H^\sigma(\Omega)$, $\sigma>0$, into $H^\sigma(\mathbb{R}^3)$ satisfying $\varphi^e\in H^\sigma(\mathbb{R}^3)$ and

$$\|\varphi^e\|_{\sigma,\mathbb{R}^3} \leq C\|\varphi\|_{\sigma,\Omega}; \tag{5.5}$$

see [7], for instance.

Let $\mathbf{f}\in\widetilde{V}_h(\mathbb{R}^3)$ and define $\bar{\mathbf{u}} := \mathbf{A}\mathbf{f}$ and $\bar{\mathbf{u}}_h := \mathbf{A}_h\mathbf{f}$.

LEMMA 5.5. *There exists a positive constant C such that*

$$\begin{aligned}\|\bar{\mathbf{u}}-\bar{\mathbf{u}}_h\| \leq\ & C\Big(\inf_{\mathbf{v}_h\in V_{0h}(\Omega_h)}\|\mathbf{v}_h-\mathbf{u}^e\|_{\Omega_h} \\ &+\sup_{\mathbf{w}_h\in V_{0h}(\Omega_h)}\frac{|a_h(\mathbf{u}^e-\mathbf{u}_h,\mathbf{w}_h)|}{\|\mathbf{w}_h\|_{\Omega_h}} + \|\mathbf{u}\|_{\Omega\setminus\Omega_h} + \|\mathbf{u}^e\|_{\Omega_h\setminus\Omega}\Big).\end{aligned}$$

Proof. We have

$$\|\bar{\mathbf{u}}-\bar{\mathbf{u}}_h\|^2 = \|\bar{\mathbf{u}}-\bar{\mathbf{u}}_h\|^2_{\Omega\cup\Omega_h} = \|\mathbf{u}-\mathbf{u}_h\|^2_{\Omega\cap\Omega_h} + \|\mathbf{u}\|^2_{\Omega\setminus\Omega_h} + \|\mathbf{u}_h\|^2_{\Omega_h\setminus\Omega}.$$

Now, let $\mathbf{v}_h$ be an arbitrary element in the space $V_{0h}(\Omega_h)$. We can write

$$\|\mathbf{u}-\mathbf{u}_h\|^2_{\Omega\cap\Omega_h} \leq 2\,(\|\mathbf{u}-\mathbf{v}_h\|^2_{\Omega\cap\Omega_h} + \|\mathbf{v}_h-\mathbf{u}_h\|^2_{\Omega\cap\Omega_h}),$$

and

$$\|\mathbf{u}_h\|_{\Omega_h\setminus\Omega} \leq \|\mathbf{v}_h-\mathbf{u}_h\|_{\Omega_h\setminus\Omega} + \|\mathbf{v}_h\|_{\Omega_h\setminus\Omega}.$$

By using the uniform coerciveness and continuity of the bilinear form a_h, we obtain

$$\begin{aligned}
\alpha\|\mathbf{v}_h-\mathbf{u}_h\|^2_{\Omega_h} \quad &\leq \int_{\Omega_h} |\,\mathbf{curl}\,(\mathbf{v}_h-\mathbf{u}_h)|^2 + s|\operatorname{div}(\mathbf{v}_h-\mathbf{u}_h)|^2 + \int_{\Omega_h} |\mathbf{v}_h-\mathbf{u}_h|^2\\
&\leq \int_{\Omega_h} \mathbf{curl}\,(\mathbf{v}_h-\mathbf{u}^e)\cdot\mathbf{curl}\,(\mathbf{v}_h-\mathbf{u}_h) + s\operatorname{div}(\mathbf{v}_h-\mathbf{u}^e)\operatorname{div}(\mathbf{v}_h-\mathbf{u}_h)\\
&+\int_{\Omega_h} \mathbf{curl}\,(\mathbf{u}^e-\mathbf{u}_h)\cdot\mathbf{curl}\,(\mathbf{v}_h-\mathbf{u}_h) + s\operatorname{div}(\mathbf{u}^e-\mathbf{u}_h)\operatorname{div}(\mathbf{v}_h-\mathbf{u}_h)\\
&+\int_{\Omega_h} (\mathbf{v}_h-\mathbf{u}^e)(\mathbf{v}_h-\mathbf{u}_h) + \int_{\Omega_h} (\mathbf{u}^e-\mathbf{u}_h)(\mathbf{v}_h-\mathbf{u}_h)\\
&\leq C\Big(\|\mathbf{v}_h-\mathbf{u}^e\|_{\Omega_h}\|\mathbf{v}_h-\mathbf{u}_h\|_{\Omega_h} + a_h((\mathbf{u}^e-\mathbf{u}_h),(\mathbf{v}_h-\mathbf{u}_h))\Big),
\end{aligned}$$

from which we deduce

$$\|\mathbf{v}_h-\mathbf{u}_h\|_{\Omega_h} \leq C\left(\|\mathbf{v}_h-\mathbf{u}^e\|_{\Omega_h} + \sup_{\mathbf{w}_h\in V_{0h}(\Omega_h)} \frac{|a_h(\mathbf{u}^e-\mathbf{u}_h,\mathbf{w}_h)|}{\|\mathbf{w}_h\|_{\Omega_h}}\right).$$

On the other hand,

$$\|\mathbf{u}-\mathbf{v}_h\|_{\Omega\cap\Omega_h} = \|\mathbf{u}^e-\mathbf{v}_h\|_{\Omega\cap\Omega_h},$$

and

$$\|\mathbf{v}_h\|_{\Omega_h\setminus\Omega} \leq \|\mathbf{v}_h-\mathbf{u}^e\|_{\Omega_h\setminus\Omega} + \|\mathbf{u}^e\|_{\Omega_h\setminus\Omega}.$$

Combining the above inequalities, we conclude the proof. □

We now estimate the terms appearing in the right-hand side of the inequality in Lemma 5.5. In the sequel, we shall assume that r is the constant appearing in equation (5.4).

LEMMA 5.6. *There exists a positive constant C such that*

$$\inf_{\mathbf{v}_h\in V_{0h}(\Omega_h)} \|\mathbf{u}^e-\mathbf{v}_h\|_{\Omega_h} \leq C\,h^r\|\mathbf{u}\|_{1+r,\Omega}.$$

Proof. Since $\mathbf{u}^e\in(H^{1+r}(\mathbb{R}^3))^3$, $\mathbf{u}^e\in(C^0(\mathbb{R}^3))^3$. Therefore, $\mathbf{L}\mathbf{u}^e$, the Lagrange linear interpolant of $\mathbf{u}^e|_{\Omega_h}$, is well defined; see [2], for instance. By using standard interpolation results, we have

$$\|\mathbf{u}^e-\mathbf{L}\mathbf{u}^e\|_{\Omega_h} \leq Ch^r\|\mathbf{u}^e\|_{1+r,\Omega_h}.$$

Observe that $\mathbf{L}\mathbf{u}^e\in V_{0h}(\Omega_h)$ although $\mathbf{u}^e|_{\partial\Omega_h}\neq 0$. Then, using the estimate (5.5), we conclude the proof. □

LEMMA 5.7. *There exists a positive constant C such that*

$$\sup_{\mathbf{w}_h\in V_{0h}(\Omega_h)} \frac{|a_h(\mathbf{u}^e-\mathbf{u}_h,\mathbf{w}_h)|}{\|\mathbf{w}_h\|_{\Omega_h}} \leq C\,h^r\|\mathbf{f}\|.$$

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Proof. For any function $\mathbf{w}_h \in V_{0h}(\Omega_h)$, we have

$$\begin{aligned} a_h(\mathbf{u}^e - \mathbf{u}_h, \mathbf{w}_h) &= \int_{\Omega_h} \mathbf{curl}\,(\mathbf{u}^e - \mathbf{u}_h)\cdot \mathbf{curl}\,\mathbf{w}_h + s\,\mathrm{div}\,(\mathbf{u}^e - \mathbf{u}_h)\,\mathrm{div}\,\mathbf{w}_h \\ &\quad + \int_{\Omega_h} (\mathbf{u}^e - \mathbf{u}_h)\cdot \mathbf{w}_h \\ &= \int_{\Omega\cup\Omega_h} \mathbf{curl}\,\mathbf{u}^e \cdot \mathbf{curl}\,\bar{\mathbf{w}}_h + s\,\mathrm{div}\,\mathbf{u}^e\,\mathrm{div}\,\bar{\mathbf{w}}_h + \mathbf{u}^e\cdot\bar{\mathbf{w}}_h - \int_{\Omega_h} \mathbf{f}\cdot\mathbf{w}_h \\ &= \int_{\Omega} \mathbf{curl}\,\mathbf{u}\cdot \mathbf{curl}\,\bar{\mathbf{w}}_h + s\,\mathrm{div}\,\mathbf{u}\,\mathrm{div}\,\bar{\mathbf{w}}_h + \mathbf{u}\cdot\bar{\mathbf{w}}_h \\ &\quad + \int_{\Omega_h\setminus\Omega} \mathbf{curl}\,\mathbf{u}^e\cdot \mathbf{curl}\,\mathbf{w}_h + s\,\mathrm{div}\,\mathbf{u}^e\,\mathrm{div}\,\mathbf{w}_h + \mathbf{u}^e\cdot\mathbf{w}_h - \int_{\Omega_h} \mathbf{f}\cdot\mathbf{w}_h. \end{aligned}$$

The last three terms can be easily bounded. In fact, by using the Cauchy-Schwarz inequality, Lemma 5.2 and estimate (5.5), we obtain

$$\begin{aligned} \left|\int_{\Omega_h\setminus\Omega} \mathbf{curl}\,\mathbf{u}^e\cdot \mathbf{curl}\,\mathbf{w}_h\right| &\leq C\,\|\,\mathbf{curl}\,\mathbf{u}^e\|_{0,\Omega_h\setminus\Omega}\|\,\mathbf{curl}\,\mathbf{w}_h\|_{0,\Omega_h\setminus\Omega} \\ &\leq C\,h^r\|\mathbf{u}^e\|_{1+r,\Omega_h}\|\mathbf{w}_h\|_{\Omega_h} \leq C\,h^r\|\mathbf{u}\|_{1+r,\Omega}\|\mathbf{w}_h\|_{\Omega_h}, \end{aligned} \tag{5.6}$$

$$\begin{aligned} \left|\int_{\Omega_h\setminus\Omega} \mathrm{div}\,\mathbf{u}^e\,\mathrm{div}\,\mathbf{w}_h\right| &\leq C\,\|\,\mathrm{div}\,\mathbf{u}^e\|_{0,\Omega_h\setminus\Omega}\|\,\mathrm{div}\,\mathbf{w}_h\|_{0,\Omega_h\setminus\Omega} \leq C\,h^r\|\mathbf{u}^e\|_{1+r,\Omega_h}\|\mathbf{w}_h\|_{\Omega_h} \\ &\leq C\,h^r\|\mathbf{u}\|_{1+r,\Omega}\|\mathbf{w}_h\|_{\Omega_h}, \end{aligned} \tag{5.7}$$

$$\begin{aligned} \left|\int_{\Omega_h\setminus\Omega} \mathbf{u}^e\cdot\mathbf{w}_h\right| &\leq C\,\|\mathbf{u}^e\|_{0,\Omega_h\setminus\Omega}\|\mathbf{w}_h\|_{0,\Omega_h\setminus\Omega} \leq C\,h^2\|\mathbf{u}^e\|_{1+r,\Omega_h}\|\mathbf{w}_h\|_{\Omega_h} \\ &\leq C\,h^2\|\mathbf{u}\|_{1+r,\Omega}\|\mathbf{w}_h\|_{\Omega_h}. \end{aligned} \tag{5.8}$$

We are going to estimate the remainder terms. To this end, we need to introduce some notation. We denote by ω_T the domain bounded by Σ_h^T and $\mathcal{S}_h^T$, with $\Sigma_h^T \subset \partial\Omega$ being the curved side of an ideal tetrahedra and with $\mathcal{S}_h^T \subset \partial\Omega_h$ being the corresponding side of the associated tetrahedra $T \in \mathcal{T}_h$. Now, we consider a function $\hat{\mathbf{w}} = (\hat{w}_1, \hat{w}_2, \hat{w}_3)$, with $\hat{w}_i, i = 1, 2, 3$, as defined in Definition 5.3. Since $\hat{\mathbf{w}} \in V_0(\Omega)$, we may take it as a test function in (5.3). Then, we can obtain

$$\begin{aligned} &\Big|\int_\Omega \mathbf{f}\hat{\mathbf{w}} - \int_\Omega \mathbf{curl}\,\mathbf{u}\cdot \mathbf{curl}\,(\hat{\mathbf{w}} - \bar{\mathbf{w}}_h) + s\,\mathrm{div}\,\mathbf{u}\,\mathrm{div}\,(\hat{\mathbf{w}} - \bar{\mathbf{w}}_h) + \mathbf{u}\cdot(\hat{\mathbf{w}} - \bar{\mathbf{w}}_h) \\ &\quad - \int_{\Omega_h} \mathbf{f}\cdot\mathbf{w}_h\Big| \\ &= \Big|\sum_{\substack{T\,\in\,\mathcal{T}_h^{id} \\ \partial T\cap\partial\Omega\neq\emptyset}} \int_T \mathbf{f}\hat{\mathbf{w}} - \int_T \mathbf{curl}\,\mathbf{u}\cdot \mathbf{curl}\,(\hat{\mathbf{w}} - \bar{\mathbf{w}}_h) + s\,\mathrm{div}\,\mathbf{u}\,\mathrm{div}\,(\hat{\mathbf{w}} - \bar{\mathbf{w}}_h) \\ &\quad + \int_T \mathbf{u}\cdot(\hat{\mathbf{w}} - \bar{\mathbf{w}}_h) - \sum_{\substack{T\,\in\,\mathcal{T}_h \\ \partial T\cap\partial\Omega_h\neq\emptyset}} \int_T \mathbf{f}\cdot\mathbf{w}_h\Big| \end{aligned}$$

$$
\begin{aligned}
&= \Big| \sum_{\substack{T \in \mathcal{T}_h^{id} \\ \partial T \cap \partial\Omega \neq \emptyset}} \left(\int_T \mathbf{f} \cdot (\hat{\mathbf{w}} - \mathbf{w}_h) - \int_T \mathbf{curl}\, \mathbf{u} \cdot \mathbf{curl}\, (\hat{\mathbf{w}} - \bar{\mathbf{w}}_h) \right. \\
&\quad \left. + \int_T s \operatorname{div} \mathbf{u} \operatorname{div} (\hat{\mathbf{w}} - \bar{\mathbf{w}}_h) - \int_T \mathbf{u} \cdot (\hat{\mathbf{w}} - \bar{\mathbf{w}}_h) \right) - \sum_{\substack{T \in \mathcal{T}_h \\ \partial T \cap \partial\Omega_h \neq \emptyset}} \int_{\omega_T} \mathbf{f} \cdot \mathbf{w}_h \Big| \\
&\leq C \Bigg[\sum_{\substack{T \in \mathcal{T}_h^{id} \\ \partial T \cap \partial\Omega \neq \emptyset}} \Big(\|\mathbf{f}\|_{0,T} \|\hat{\mathbf{w}} - \bar{\mathbf{w}}_h\|_{0,T} + \|\mathbf{u}\|_T \|\hat{\mathbf{w}} - \bar{\mathbf{w}}_h\|_T \Big) + \|\mathbf{f}\|_{0,\Omega_h \setminus \Omega} \|\mathbf{w}_h\|_{0,\Omega_h \setminus \Omega} \Bigg] \\
&\leq C \Bigg[\sum_{\substack{T \in \mathcal{T}_h \\ \partial T \cap \partial\Omega_h \neq \emptyset}} \Big(h \|\mathbf{f}\|_{0,T} \|\mathbf{w}_h\|_T + h \|\mathbf{u}\|_T \|\mathbf{w}_h\|_T \Big) + h^2 \|\mathbf{f}\|_{1,\Omega_h} \|\mathbf{w}_h\|_{1,\Omega_h} \Bigg] \\
&\leq C \Big[h \Big(\|\mathbf{f}\| + \|\mathbf{u}\|_\Omega \Big) + h^2 \|\mathbf{f}\| \Big] \|\mathbf{w}_h\|_{\Omega_h},
\end{aligned}
$$

where we have used Lemmas 5.2 and 5.4. We conclude the proof combining the previous inequality with (5.6), (5.7), (5.8), and the estimate (5.4). ◻

THEOREM 5.8. **(P1)** *There exists a positive constant C such that*

$$\|(\mathbf{A} - \mathbf{A}_h)\mathbf{f}\| \leq C\, h^r \|\mathbf{f}\| \quad \forall \mathbf{f} \in \widetilde{V}_h(\mathbb{R}^3).$$

Proof. It is an immediate consequence of the previous lemmas. ◻

THEOREM 5.9. **(P2)** *For each eigenfunction $\bar{\mathbf{u}}$ of $\mathbf{A}$ associated to λ, there exists a strictly positive constant C such that*

$$\|\bar{\mathbf{u}}\|_{\Omega \setminus \Omega_h} = \|\mathbf{u}\|_{\Omega \setminus \Omega_h} \leq C\, h^r \|\mathbf{u}\|_{1+r,\Omega}.$$

Proof. It is an immediate consequence of the regularity of the eigenfunctions $\mathbf{u}$ of the operator $\mathbf{T}$ and Lemma 5.2. ◻

THEOREM 5.10. **(P3)** *For each eigenfunction $\bar{\mathbf{u}}$ of $\mathbf{A}$ associated to λ, there exists a strictly positive constant C such that*

$$\inf_{\mathbf{v}_h \in V_{0h}(\Omega_h)} \|\bar{\mathbf{u}} - \bar{\mathbf{v}}_h\| \leq C\, h^r \|\mathbf{u}\|_{1+r,\Omega}.$$

Proof. We have

$$
\begin{aligned}
\|\bar{\mathbf{u}} - \bar{\mathbf{v}}_h\|^2 &= \|\bar{\mathbf{u}} - \mathbf{v}_h\|^2_{\Omega_h} + \|u\|^2_{\Omega \setminus \Omega_h} \leq \|\bar{\mathbf{u}} - \mathbf{u}^e\|^2_{\Omega_h} + \|\mathbf{u}^e - \mathbf{v}_h\|^2_{\Omega_h} + \|\mathbf{u}\|^2_{\Omega \setminus \Omega_h} \\
&\leq \|\mathbf{u}^e\|^2_{\Omega_h \setminus \Omega} + \|\mathbf{u}^e - \mathbf{v}_h\|^2_{\Omega_h} + \|\mathbf{u}\|^2_{\Omega \setminus \Omega_h} \leq h^{2r} \|\mathbf{u}\|^2_{1+r,\Omega},
\end{aligned}
$$

which follows because the regularity of the eigenfunctions $\mathbf{u}$ of the operator $\mathbf{T}$, estimate (5.5) and Lemma 5.6. Then, taking the infimum with respect to $\mathbf{v}_h \in V_{0h}(\Omega_h)$, we can conclude the proof. ◻

By virtue of the previous theorems, the spectrum of $\mathbf{A}_h$ furnishes the approximations of the spectrum of $\mathbf{A}$ as we stated in Section 3.

THEOREM 5.11. **(P4)** *There exists a positive constant C such that*

$$\|(\mathbf{A} - \mathbf{A}_h)|_{\mathbf{E}(V(\mathbb{R}^3))}\| \leq C\, h^r.$$

Proof. For $\mathbf{x} \in \mathbf{E}(V(\mathbb{R}^3))$, $\bar{\mathbf{u}} = \mathbf{A}\mathbf{x} \in H^{1+r}(\mathbb{R}^3)$. Then, the proof runs identically to that of Theorem 5.8 . ◻

Observe that, since $\mathbf{A}$ and $\mathbf{A}_h$ are self-adjoint, properties **P5**, **P6**, **P7** and **P8** are equally valid.

Now, we are going to estimate the consistency terms appearing in Theorem 4.13. Notice that M_h and M_{*h} also coincide because of the symmetry of a_h and b_h.

LEMMA 5.12. *There exists a positive constant C such that*

$$M_h = \sup_{\substack{\mathbf{x}\in\mathbf{E}(V(\mathbb{R}^3))\\ \|\mathbf{x}\|=1}} \sup_{\substack{\mathbf{y}\in\mathbf{E}(V(\mathbb{R}^3))\\ \|\mathbf{y}\|=1}} |a_h(\mathbf{A}\mathbf{x},\Pi_h\mathbf{y}-\mathbf{y}) - b_h(\mathbf{x},\Pi_h\mathbf{y}-\mathbf{y})| \le C\,h^{2r},$$

with Π_h being the projection onto $\widetilde{V}_h(\mathbb{R}^3)$ with respect to a_h, defined by equation (4.1).

Proof. Let $\mathbf{x}\in\mathbf{E}(V(\mathbb{R}^3))$, with $\|\mathbf{x}\|=1$, and put $\mathbf{w}=\check{\mathbf{S}}\mathbf{A}\mathbf{x}$. According to the definition of $\mathbf{A}$, we have $\mathbf{w}=\check{\mathbf{S}}\hat{\mathbf{S}}\mathbf{T}\check{\mathbf{S}}\mathbf{x}=\mathbf{T}\check{\mathbf{S}}\mathbf{x}$. From (5.4) we know that $\mathbf{w}\in(H^{1+r}(\Omega))^3$ and

$$\|\mathbf{w}\|_{1+r,\Omega} \le C\,\|\mathbf{x}\| = C.$$

Now, taking $\mathbf{v}\in\mathcal{D}(\Omega)$ as test functions in (5.3), it can be shown that $\mathbf{w}$ is the solution of the following strong problem

$$\begin{cases} s\,\mathbf{grad}\,(\operatorname{div}\mathbf{w}) - \mathbf{curl}\,\mathbf{curl}\,\mathbf{w} + \mathbf{w} = \mathbf{x} & \text{in } \Omega,\\ \mathbf{w} = 0 & \text{on } \partial\Omega.\end{cases}$$

Let us denote by $\bar{\mathbf{w}}$ the extension of $\mathbf{w}$ by zero from Ω to $\mathbb{R}^3$. Let $\mathbf{y}\in\mathbf{E}(V(\mathbb{R}^3))$ with $\|\mathbf{y}\|=1$, and take $\mathbf{v}_h=\Pi_h\mathbf{y}-\mathbf{y}$. Integrating by parts, we obtain

$$\begin{aligned}
|a_h(\bar{\mathbf{w}},\mathbf{v}_h) - b_h(\mathbf{x},\mathbf{v}_h)| &= \Big|\int_{\Omega\setminus\Omega_h} \mathbf{curl}\,\mathbf{w}\cdot\mathbf{curl}\,\mathbf{v}_h + s\operatorname{div}\mathbf{w}\operatorname{div}\mathbf{v}_h + \mathbf{w}\cdot\mathbf{v}_h \\
&\quad + \int_{\Omega\setminus\Omega_h} \mathbf{x}\cdot\mathbf{v}_h\Big| \\
&\le C\Big(|\mathbf{w}|_{1,\Omega\setminus\Omega_h}|\mathbf{v}_h|_{1,\Omega\setminus\Omega_h} + \|\mathbf{w}\|_{0,\Omega\setminus\Omega_h}\|\mathbf{v}_h\|_{0,\Omega\setminus\Omega_h} \\
&\quad + \|\mathbf{x}\|_{0,\Omega\setminus\Omega_h}\|\mathbf{v}_h\|_{0,\Omega\setminus\Omega_h}\Big) \\
&\le C\Big(h^{2r}\,\|\mathbf{w}\|_{1+r,\Omega}\|\mathbf{y}\|_{1+r,\Omega} + h^{2r}\|\mathbf{w}\|_{r,\Omega}\|\mathbf{y}\|_{r,\Omega} \\
&\quad + h^{2r}\|\mathbf{x}\|_{r,\Omega}\|\mathbf{y}\|_{r,\Omega}\Big).
\end{aligned}$$

In the last inequality, we have used that $\mathbf{v}_h|_{\Omega\setminus\Omega_h} = -\mathbf{y}|_{\Omega\setminus\Omega_h}$ and the estimate in Lemma 5.2. Finally, we can conclude the proof using estimate (5.4). ☐

LEMMA 5.13. *There exists a positive constant C such that*

$$N_h = \sup_{\substack{\mathbf{x}\in\mathbf{E}(V(\mathbb{R}^3))\\ \|\mathbf{x}\|=1}} \sup_{\substack{\mathbf{y}\in\mathbf{E}(V(\mathbb{R}^3))\\ \|\mathbf{y}\|=1}} |a_h(\mathbf{A}\mathbf{x},\mathbf{y}) - b_h(\mathbf{x},\mathbf{y})| \le C\,h^{2r}.$$

Proof. It is identical to that of the previous lemma by substituting $\mathbf{v}_h$ by $\mathbf{y}$. ☐

THEOREM 5.14. *There exists a positive constant C such that*

$$\max_{i=1,\cdots,m} |\lambda - \lambda_{ih}| \le Ch^{2r}.$$

Proof. It is a immediate consequence of the properties **P2**, **P3**, **P4**, and the previous lemmas. ☐

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