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HOW SHARP IS BERNSTEIN'S INEQUALITY FOR JACOBI POLYNOMIALS?*

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Dedicated to Richard S. Varga on his 80th birthday

Abstract. Bernstein's inequality for Jacobi polynomials $P_n^{(\alpha,\beta)}$, established in 1987 by P. Baratella for the region $\mathcal{R}_{1/2} = \{|\alpha| \leq 1/2, |\beta| \leq 1/2\}$, and subsequently supplied with an improved constant by Y. Chow, L. Gatteschi, and R. Wong, is analyzed here analytically and, above all, computationally with regard to validity and sharpness, not only in the original region $\mathcal{R}_{1/2}$, but also in larger regions $\mathcal{R}_s = \{-1/2 \leq \alpha \leq s, -1/2 \leq \beta \leq s\}$, $s > 1/2$. Computation suggests that the inequality holds with new, somewhat larger, constants in any region $\mathcal{R}_s$. Best constants are provided for $s = 1 : .5 : 4$ and $s = 5 : 1 : 10$. Our work also sheds new light on the so-called Erdélyi–Magnus–Nevai conjecture for orthonormal Jacobi polynomials, adding further support for its validity and suggesting .66198126 . . . as the best constant implied in the conjecture.

Key words. Bernstein's inequality, Jacobi polynomials, sharpness, Erdélyi–Magnus–Nevai conjecture

AMS subject classifications. 33C45, 41A17

1. Introduction. Bernstein's inequality for Legendre polynomials P_n, slightly sharpened by Antonov and Holševnikov [1] and Lorch [5], states that for $n = 1, 2, 3, \ldots$,

$$(\sin\theta)^{1/2}|P_n(\cos\theta)| < \left(\frac{2}{\pi}\right)^{1/2} (n + \tfrac{1}{2})^{-1/2}, \quad 0 \leq \theta \leq \pi. \tag{1.1}$$

According to Bernstein, the constant $(2/\pi)^{1/2}$ is best possible. An extension of (1.1) to ultraspherical polynomials $P_n^{(\lambda)} = P_n^{(\lambda-1/2,\lambda-1/2)}$, $0 < \lambda < 1$, is due to Lorch [6], and a further extension to Jacobi polynomials $P_n^{(\alpha,\beta)}$ with $|\alpha| \leq 1/2$, $|\beta| \leq 1/2$ to Baratella [2]. Chow, Gatteschi, and Wong [3], by sharpening her constant, improved Baratella's result to read

$$\begin{gathered}(\sin\tfrac{1}{2}\theta)^{\alpha+1/2}(\cos\tfrac{1}{2}\theta)^{\beta+1/2}|P_n^{(\alpha,\beta)}(\cos\theta)| \leq \frac{\Gamma(q+1)}{\Gamma(1/2)}\binom{n+q}{n}N^{-q-1/2}, \\ N = n + (\alpha+\beta+1)/2, \ \ 0 \leq \theta \leq \pi,\end{gathered} \tag{1.2}$$

where $q = \max(\alpha, \beta)$ and $|\alpha| \leq 1/2$, $|\beta| \leq 1/2$. Equality sign is included in (1.2), since in the case $\alpha = \beta = \mp 1/2$ the inequality reduces to $|\cos(n\theta)| \leq 1$ resp. $|\sin((n+1)\theta)| \leq 1$, and in the case $\alpha = \pm 1/2$, $\beta = \mp 1/2$ to $|\sin(n+1/2)\theta| \leq 1$ resp. $|\cos(n+1/2)\theta| \leq 1$. It appears, though, that strict inequality holds in all other cases.

Squaring both sides of the inequality (1.2) and writing the result in terms of $x = \cos\theta$ and the orthonormal Jacobi polynomial $\hat{P}_n^{(\alpha,\beta)}$ yields (if $\beta \geq \alpha$; cf. (4.1))

$$\begin{aligned}&(1-x)^{\alpha+1/2}(1+x)^{\beta+1/2}[\hat{P}_n^{(\alpha,\beta)}(x)]^2 \\ &\quad \leq \frac{2\,\Gamma(n+\alpha+\beta+1)\Gamma(n+\beta+1)}{\pi\,\Gamma(n+\alpha+1)n!(n+(\alpha+\beta+1)/2)^{2\beta}}, \quad |\alpha| \leq 1/2, \ |\beta| \leq 1/2.\end{aligned} \tag{1.3}$$

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Since as $n \to \infty$ the right-hand side is $\sim 2/\pi$, it follows that the left-hand side is $O(1)$ for $|x| \le 1$, which proves the Erdélyi–Magnus–Nevai conjecture

$$(1.4) \qquad (1-x)^{\alpha+1/2}(1+x)^{\beta+1/2}[\hat{P}_n^{(\alpha,\beta)}(x)]^2 = O\left(\max[1, (\alpha^2+\beta^2)^{1/4}]\right)$$

[7, p. 604] (see also [4]) on the domain $|\alpha| \le 1/2$, $|\beta| \le 1/2$. The constant on the right of (1.3) takes on the value $2/\pi$ not only at $n = \infty$, but also at $n = 1$ when $\beta = 0$ or $|\alpha| = |\beta| = 1/2$. It is probably for $n = 1$ and $\beta = 1/2$ that the maximum is attained, near $\alpha = -.0691$, its value being .64297807.

Incidentally, if we denote the ratio of the left-hand side of (1.2) and the right-hand side (as in (3.2), (3.3)) by $c_n F_n(x)$, we have

$$(1.5) \qquad (1-x)^{\alpha+1/2}(1+x)^{\beta+1/2}[\hat{P}_n^{(\alpha,\beta)}(x)]^2 = \gamma_n c_n^2 F_n^2(x),$$

where

$$(1.6) \qquad \gamma_n = \frac{2\,\Gamma(n+\alpha+\beta+1)\Gamma(n+\beta+1)}{\pi\,\Gamma(n+\alpha+1)n!(n+(\alpha+\beta+1)/2)^{2\beta}}.$$

While the constant $\Gamma(q+1)/\Gamma(1/2)$ in (1.2), when $\alpha = \beta = 0$, is best possible, it does not follow necessarily that the same is true in the general case, although asymptotic arguments will suggest that it is. In this note, the sharpness of the inequality is determined computationally, at least for $n \le 100$, in the square $|\alpha| \le 1/2$, $|\beta| \le 1/2$. Outside thereof, it is examined to what extent the inequality is an underestimation. We will also experiment with different choices of the parameter q, which, asymptotically, is irrelevant.

All of this will be done by computing the infinity norm $\rho_n = \rho_n(\alpha,\beta,q)$ (on the interval $0 \le \theta \le \pi$) of the ratio of the left-hand side of (1.2) divided by the right-hand side. This is an important quantity inasmuch as it allows us to assess the quality of the inequality (1.2) on a domain $\mathcal{D}$ of the parameter space (n,α,β,q). In fact, let $\rho_{\mathcal{D}}^+ = \max_{\mathcal{D}} \rho_n(\alpha,\beta,q)$ and $\rho_{\mathcal{D}}^- = \min_{\mathcal{D}} \rho_n(\alpha,\beta,q)$. Then, if $\rho_{\mathcal{D}}^+ \le 1$, i.e., the inequality holds on $\mathcal{D}$, on a scale from 0 to 1, the best degree of sharpness of (1.2) on $\mathcal{D}$ is $\rho_{\mathcal{D}}^+$, and the worst degree of sharpness on $\mathcal{D}$ is $\rho_{\mathcal{D}}^-$. If $\rho_{\mathcal{D}}^+ > 1$, then the inequality on the domain $\mathcal{D}$ should be modified by multiplying the right-hand side by $\rho_{\mathcal{D}}^+$, to make it valid on $\mathcal{D}$. The best and worst degrees of sharpness, $\hat{\rho}_{\mathcal{D}}^+$, $\hat{\rho}_{\mathcal{D}}^-$ of the modified inequality are then $\hat{\rho}_{\mathcal{D}}^+ = 1$, $\hat{\rho}_{\mathcal{D}}^- = \rho_{\mathcal{D}}^-/\rho_{\mathcal{D}}^+$.

2. The constant in (1.2) is sharp. An elementary computation, using Stirling's formula, will show that the right-hand side of (1.2), as $n \to \infty$, is asymptotically equivalent to $(\pi n)^{-1/2}$, regardless of the values of the parameters α, β, and q. The inequality (1.2) thus says that the function on the left, multiplied by $(\pi n)^{1/2}$, is less than, or equal to, a constant that tends to 1 as $n \to \infty$. But Darboux's formula [8, Theorem 8.21.8] tells us that this same expression, at least on a compact subinterval of $0 < \theta < \pi$, but for arbitrary real α and β, is $\le 1 + O(1/n)$, where the constant 1 is best possible (bounding, as it does, a cosine function). This not only shows that the constant $\Gamma(q+1)/\Gamma(1/2)$ in (1.2) is indeed best possible, but also that the inequality, with the constant somewhat enlarged, may well hold in larger domains of the (α,β)-plane. The purpose of this note is to explore this computationally in some detail.

3. Bernstein's inequality for monic Jacobi polynomials. In what follows, we prefer to use the monic Jacobi polynomial $\pi_n^{(\alpha,\beta)}$, i. e.,

$$P_n^{(\alpha,\beta)}(x) = k_n \pi_n^{(\alpha,\beta)}(x), \quad k_n = 2^{-n}\binom{2n+\alpha+\beta}{n},$$

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and we shall write it as

$$\pi_n^{(\alpha,\beta)}(x) = \prod_{r=1}^{n}(x - x_r) \tag{3.1}$$

in terms of the zeros $x_r = x_{n,r}^{(\alpha,\beta)}$ (in ascending order) of the Jacobi polynomial $P_n^{(\alpha,\beta)}$. If we divide both sides of (1.2) by the expression on its right-hand side, and let $x = \cos\theta$, Bernstein's inequality takes the form

$$c_n|F_n(x)| \le 1, \quad -1 \le x \le 1, \tag{3.2}$$

where

$$\begin{aligned} c_n = c_n(\alpha,\beta,q) &= \frac{\sqrt{\pi}(n+(\alpha+\beta+1)/2)^{q+1/2}\binom{2n+\alpha+\beta}{n}}{\Gamma(q+1)2^{n+(\alpha+\beta+1)/2}\binom{n+q}{n}}\,, \\ F_n(x) = F_n^{(\alpha,\beta)}(x) &= (1-x)^{(2\alpha+1)/4}(1+x)^{(2\beta+1)/4}\pi_n^{(\alpha,\beta)}(x), \end{aligned} \tag{3.3}$$

where $q = \max(\alpha,\beta)$. Since we later consider q to be an independent parameter, we include it in the constant c_n as one of three parameters. Notice that

$$c_n(\alpha,\beta,q) = c_n(\beta,\alpha,q),$$

regardless of how $q = q(\alpha,\beta)$ is defined so long as $q(\alpha,\beta) = q(\beta,\alpha)$.

4. The infinity norm $\|F_n\|_\infty$ of F_n.

We now wish to compute $\|F_n\|_\infty = \max_{-1\le x\le 1} |F_n(x)|$. Since by the reflection formula for Jacobi polynomials,

$$\|F_n^{(\alpha,\beta)}\|_\infty = \|F_n^{(\beta,\alpha)}\|_\infty,$$

it suffices to consider $\beta \ge \alpha$, and since $\|F_n^{(\alpha,\beta)}\|_\infty = \infty$ if $2\alpha + 1 < 0$, we may assume

$$\beta \ge \alpha \ge -1/2. \tag{4.1}$$

Computing $\|F_n\|_\infty$ amounts to computing the local extrema of F_n in the interior of the interval $[-1,1]$ along with $|F_n(\pm 1)|$. With regard to the former, we have

$$F_n'(x) = \tfrac{1}{2}(1-x)^{(2\alpha-3)/4}(1+x)^{(2\beta-3)/4}\{[\beta-\alpha-(\alpha+\beta+1)x]\pi_n^{(\alpha,\beta)}(x)+2(1-x^2)\pi_n^{(\alpha,\beta)\prime}(x)\},$$

so that the local extrema occur at those roots of the equation $[\beta-\alpha-(\alpha+\beta+1)x]\pi_n^{(\alpha,\beta)}(x)+2(1-x^2)\pi_n^{(\alpha,\beta)\prime}(x) = 0$ that are inside $(-1,1)$, that is, dividing by $\pi_n^{(\alpha,\beta)}$ and noting (3.1), at the respective roots of

$$f(x) = 0, \quad f(x) = \beta-\alpha-(\alpha+\beta+1)x + 2(1-x^2)\sum_{r=1}^{n}\frac{1}{x-x_r}\,. \tag{4.2}$$

There can be at most $n+1$ real roots. To discuss their location, we first observe that

$$f(-1) = 2\beta+1, \quad f(1) = -(2\alpha+1).$$

It is clear from from (4.2) that

$$f(x_r+0) = +\infty, \quad f(x_r-0) = -\infty, \quad r = 1,2,\ldots,n,$$

and on each interval (x_r, x_{r+1}), $r = 1, 2, \dots n-1$, the function f descends monotonically (cf. Section 5) from $+\infty$ to $-\infty$. It therefore crosses the real line exactly once, accounting for $n-1$ internal extrema. We distinguish three cases with regard to the parameter α. If, first, $2\alpha+1>0$, and hence by (4.1) also $2\beta+1>0$, then $f(-1)>0$ and $f(1)<0$, so that there are two more roots, one each in $(-1, x_1)$ and $(x_n, 1)$, accounting for two more internal extrema, and thus for a complete set of $n+1$ extrema. If, secondly, $2\alpha+1=0$, there are two subcases: $2\beta+1>0$ and $2\beta+1=0$. In the former, there is still a local extremum in $(-1, x_1)$, but none in $(x_n, 1)$; in the latter, both these lateral intervals are devoid of local extrema (in fact, this is one of the trivial cases noted in Section 1, in which $c_n\|F_n\|_\infty = 1$.) Finally, in the third case, $2\alpha+1<0$, as was already mentioned, $\|F_n\|_\infty = \infty$.

5. Computing $\|F_n\|_\infty$ in terms of local extrema. To compute a local extremum of F_n, say in the interval (a, b), $-1 \le a < b \le 1$, we use Newton's method applied to the equation (4.2), with the midpoint of the interval (a, b) as the initial approximation,

$$x^{(i+1)} = x^{(i)} - \frac{f(x^{(i)})}{f'(x^{(i)})}, \quad i = 0, 1, 2, \dots, \quad x^{(0)} = (a+b)/2. \tag{5.1}$$

Since the interval (a, b) in our application is small and f rapidly descending from $+\infty$ to $-\infty$ (i.e., f' is large negative), Newton's iteration (5.1) converges very quickly. The derivative of f is easily computed from (4.2),

$$f'(x) = -(\alpha+\beta+1) - 2\sum_{r=1}^{n} \frac{x^2 - 2x_r x + 1}{(x - x_r)^2}.$$

Since $\alpha+\beta+1 \ge 0$ by (4.1), and the discriminants of the quadratics in the numerator on the right are $-4(1-x_r^2) < 0$, each term of the sum is positive and $f'(x) < 0$ on (a, b), as already noted in the previous section. Thus we arrive at the following

Computational procedure.
If $\alpha > -1/2$, apply (5.1) to the intervals $(a, b) = (x_r, x_{r+1})$, $r = 0, 1, 2, \dots, n$ (where $x_0 = -1$, $x_{n+1} = 1$), giving $\xi_r = x^{(\infty)}$. Then, since $F_n(\pm 1) = 0$,

$$\|F_n\|_\infty = \max_{0\le r\le n} |F_n(\xi_r)|, \quad 2\beta+1 \ge 2\alpha+1 > 0. \tag{5.2}$$

If $\alpha = -1/2$ and $\beta > -1/2$, do the same, but in (5.2) let r run only up to $n-1$, and compute

$$\|F_n\|_\infty = \max\left\{ F_n(1), \max_{0\le r\le n-1} |F_n(\xi_r)| \right\}, \quad 2\beta+1 > 0 = 2\alpha+1. \tag{5.3}$$

If $\alpha = \beta = -1/2$, put $c_n\|F_n\|_\infty = 1$.

The Matlab script `bernstein.m` listed in the Appendix implements this procedure and for any given n, α, β, q outputs $\rho_n(\alpha, \beta, q) = c_n(\alpha, \beta, q)\|F_n^{(\alpha,\beta)}\|_\infty$.

6. Numerical results. In this section we present numerical results for the square $|\alpha| \le 1/2$, $|\beta| \le 1/2$. We determine $\rho_{\mathcal{D}}^+$ and $\rho_{\mathcal{D}}^-$ (cf. Section 1) on the domain $\mathcal{D} = \{n = [5\ 10\ 20\ 50\ 100], \alpha = -.5\,{:}\,.01\,{:}\,.5, \beta = \alpha\,{:}\,.01\,{:}\,.5,\ q\}$, where in turn $q = q^+ = \max(\alpha, \beta) = \beta$, $q = q^- = \min(\alpha, \beta) = \alpha$, $q = -.75\,{:}\,.25\,{:}\,1$. The results are shown in Table 6.1.

It was observed that the sequence $\{\rho_n(\alpha, \beta, q)\}$ is monotone, either increasing or decreasing. Therefore, if $n_0 \le n \le n_1$, it would suffice to compute ρ_n for $n = n_0$ and

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TABLE 6.1
Sharpness of (1.2) on the square $|\alpha| \leq 1/2$, $|\beta| \leq 1/2$, with selected values of q.

$q \rightarrow$	q^+	q^-	0	.25	.5	.75	1
$\rho^+_{\mathcal{D}}$	1.0000	1.0000	1.0230	1.0169	1.0000	.9997	.9988
$\rho^-_{\mathcal{D}}$	.9978	.9978	.9754	.9468	.9091	.8639	.8128

$q \rightarrow$	$-.25$	$-.5$	$-.75$
$\rho^+_{\mathcal{D}}$	1.0174	1.0000	1.0000
$\rho^-_{\mathcal{D}}$	.9532	.9167	.8707

TABLE 6.2
Sharpness of (the modified) Bernstein's inequality (1.2) with the right-hand side multiplied by $\rho^+_{\mathcal{D}_s}$ on the square $-1/2 \leq \alpha \leq s$, $-1/2 \leq \beta \leq s$.

s	$\rho^+_{\mathcal{D}_s}$	$\hat{\rho}^-_{\mathcal{D}_s}$
1.0	1.038670463288	.960631920975
1.5	1.077936370739	.925639053930
2.0	1.119905216638	.890950401502
2.5	1.166112996124	.855646070084
3.0	1.217697600829	.819398840672
3.5	1.275581233437	.782215962616
4.0	1.340588974513	.744284804200
5.0	1.495211643984	.667316902208
6.0	1.688484850743	.590932161440
7.0	1.928648600010	.517346707121
8.0	2.225950341336	.448248994544
9.0	2.593289070919	.384754639811
10.0	3.046949165887	.327468542495

$n = n_1$, since $\max_{n_0 \leq n \leq n_1} \rho_n = \max(\rho_{n_0}, \rho_{n_1})$ and $\min_{n_0 \leq n \leq n_1} \rho_n = \min(\rho_{n_0}, \rho_{n_1})$. Consequently, $\rho^+_{\mathcal{D}} = \max(\max_{\mathcal{D}} \rho_{n_0}, \max_{\mathcal{D}} \rho_{n_1})$ and $\rho^-_{\mathcal{D}} = \min(\min_{\mathcal{D}} \rho_{n_0}, \min_{\mathcal{D}} \rho_{n_1})$. In other words, if monotonicity in fact holds true, $\mathcal{D} = \{n_0 \leq n \leq n_1, \dots\}$ may be replaced by $\mathcal{D} = \{n = \{n_0, n_1\}, \dots\}$. In all our experiments we have verified that indeed the results for $\rho^+_{\mathcal{D}}$ and $\rho^-_{\mathcal{D}}$ are the same whether we restrict n to the smallest and largest value, or include intermediate values as well.

It can be seen from Table 6.1 that the choices $q = q^+$ and $q = q^-$ yield by far the best degrees of sharpness, both choices being essentially identical in quality. Naturally, if we lower $n_0 = 5$ to $n_0 = 1$, the sharpness deteriorates (to $\rho^-_{\mathcal{D}} = .9406$ for both choices of q), while increasing n_0 to, say, $n_0 = 10$ improves sharpness (to $\rho^-_{\mathcal{D}} = .9994$ for both choices of q).

7. Bernstein's inequality on larger domains. We now explore the sharpness resp. validity of (1.2) in the larger regions $\mathcal{R}_s = \{-1/2 \leq \alpha \leq s, -1/2 \leq \beta \leq s\}$, where, to begin with, $s = 1, 2, 5$, and 10. We define $\mathcal{D} = \mathcal{D}_s = \{n = [5\ 10\ 20\ 50\ 100], (\alpha, \beta) \in \mathcal{R}_s\}$, $s \geq 1/2$. We found that $\rho^-_{\mathcal{D}_s} = \rho^-_{\mathcal{D}_{1/2}}$ for all $s > 1/2$, and computations based on successively finer screenings near the minimum point $(\alpha^-, \beta^-) \in \mathcal{R}_{1/2}$ for $\rho^-_{\mathcal{D}_{1/2}}$ yielded

$$\begin{aligned} \rho^-_{\mathcal{D}_{1/2}} &= .997780002408 \quad (\text{where } q = q^+), \\ \rho^-_{\mathcal{D}_{1/2}} &= .997804307519 \quad (\text{where } q = q^-). \end{aligned} \tag{7.1}$$

For $\rho^+_{\mathcal{D}_s}$ we found, when $s = 1, 2, 5, 10$, regardless of whether $q = q^+$ or $q = q^-$, that the maximum $\rho^+_{\mathcal{D}_s} = \max_{\mathcal{D}_s} \rho_n$ is always attained at the upper right-hand corner $(\alpha, \beta) = (s, s)$ of the square $\mathcal{R}_s$. Assuming this to be true in general, we computed Table 6.2 for $\rho^+_{\mathcal{D}_s}$ and (cf. Section 1) $\hat{\rho}^-_{\mathcal{D}_s} = \rho^-_{\mathcal{D}_s}/\rho^+_{\mathcal{D}_s}$, where we used the first of the two values for $\rho^-_{\mathcal{D}_s} = \rho^-_{\mathcal{D}_{1/2}}$ in (7.1). (The other value, of course, gives very similar results.)

It can be seen that the sharpness of the inequality, even for $s = 10$, is still well within one order of magnitude. What is remarkable is also that the results are exacly the same if we let n go up to 200, so that the results are likely to be valid for all $n \geq 5$.

As a final experiment, we recomputed the second column of Table 6.2 with $\mathcal{D}_s = \{n = [5\ 24\ 43\ \ 62\ 81\ 100],\ \{(\alpha, \beta)\} \subset \mathcal{R}_s\}$, where $\{(\alpha, \beta)\}$ is a set of 1,000 randomly generated pairs (α, β) in $\mathcal{R}_s$. We verified that the results are all strictly smaller than those in Table 6.2, the smallest and largest deviations being 3.0770×10^{-5} (for $s = 3.5$) resp. 6.2961×10^{-3} (for $s = 9$).

We remark that the property of the maximum $\rho^+_{\mathcal{D}_s}$ being attained at $\alpha = \beta = s$ has been verified also if $n =$[1:10, 20, 25, 50, 75, 100] in the definition of $\mathcal{D}_s$ and also for $\max_n \sqrt{\gamma_n} c_n \times \|F_n\|_\infty$ in (1.5). The property, therefore, is likely to hold for any $n \geq 1$ and any $s \geq 1/2$; if so, it would allow to extend the upper bound for

$$\max_{-1 \leq x \leq 1} (1-x)^{\alpha+1/2}(1+x)^{\beta+1/2}[\hat{P}_n^{(\alpha,\beta)}(x)]^2,$$

proved for $\alpha = \beta \geq (1 + \sqrt{2})/4$ in [4, Equation (4)] to arbitrary $\alpha > -1/2$, $\beta > -1/2$, lending added support for the validity of the Erdélyi–Magnus–Nevai conjecture. Indeed, further calculations along the lines reported on in Table 6.2, but for $n \geq 1$, in particular the computation for $\alpha = \beta = s$ of the quantity

$$\max_n \gamma_n c_n^2 \|F_n\|_\infty^2 / \max\left(1, (2s^2)^{1/4}\right)$$

for $s = [.5\!:\!.01\!:\!1\ \ 2\!:\!10\ \ 20\ \ \ 50]$ and $s = .706\!:\!.0001\!:\!.708$ reveals that it attains a global maximum $.66198126\ldots$ at $s = 1/\sqrt{2}$. This suggests that the best constant implied in the Erdélyi–Magnus–Nevai conjecture (1.4) is $.66198126\ldots$.

Appendix. In the following Matlab script, the routines `r_jacobi` and `gauss` are part of a software package `OPQ`, which can be downloaded, along with the routine below, auxiliary routines, and a driver, from

http://www.cs.purdue.edu/archives/2002/wxg/codes/BIJ.html

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```
% BERNSTEIN Sharpness of Bernstein's inequality for
%      Jacobi polynomials P_n(a,b;·) with b>=a>=-1/2.
%      The output is c_n || F_n ||.
%
function rho=bernstein(n,a,b,q)
if a<-1/2 | b<-1/2 | b<a
   disp('parameters a and/or b not in range')
   return
end tol=1e2*eps;
pnum=1; pden=1; p2=1;
for nu=1:n
   pnum=(1+(n+a+b)/nu)*pnum;
   pden=(1+q/nu)*pden;
   p2=(1-1/(2*nu))*p2;
end
c0=pnum/2^(n+(a+b+1)/2);
c1=(n+(a+b+1)/2)^(q+1/2)/(gamma(1+q)*pden);
c2=sqrt(pi)*c1*p2;
c=sqrt(pi)*c0*c1;
extr=zeros(n+1,1);
%
% When applying this routine for the same values
% of a and b, but many different values of n, the
% following command, for better efficiency, should
% be called outside the n-loop with n set equal to
% the largest n-value in the loop and the array ab
% included among the input parameters of this routine.
%
ab=r_jacobi(n,a,b);
xw=gauss(n,ab);
x=xw(:,1);
x1=[-1 x' 1]';
k0=1; k1=n+1;
if a==-1/2
   if b>-1/2
     k1=n;
   else
     rho=1;
     return
   end
end
for k=k0:k1
   t0=0; t1=(x1(k)+x1(k+1))/2;
   while abs(t1-t0)>tol
     t0=t1;
     t1=t0-fbern(t0,a,b,x)/f1bern(t0,a,b,x);
   end
   p=prod(t1-x);
   extr(k)=(1-t1)^(a/2+1/4)*(1+t1)^(b/2+1/4)*abs(p);
end
rho=c*max(extr);
if a==-1/2
   if c2>rho
     rho=c2;
   end
end
```

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```
% FBERN A function f needed in Bernstein's inequality
%       for Jacobi polynomials
%
function y=fbern(t,a,b,x)
y=b-a-(a+b+1)*t+2*(1-t^2)*sum(1./(t-x));


% F1BERN The function f' needed in Bernstein's inequality
%       for Jacobi polynomials
%
function y=f1bern(t,a,b,x)
y=-(a+b+1)-2*sum((t^2-2*t*x+1)./(t-x).^2);
```

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