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ON AN UNSYMMETRIC EIGENVALUE PROBLEM GOVERNING FREE VIBRATIONS OF FLUID-SOLID STRUCTURES*

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Abstract. In this paper we consider an unsymmetric eigenvalue problem occurring in fluid-solid vibrations. We present some properties of this eigenvalue problem and a Rayleigh functional which allows for a min-max-characterization. With this Rayleigh functional the one-sided Rayleigh functional iteration converges cubically, and a Jacobi-Davidson-type method improves the local and global convergence properties.

Key words. eigenvalue, variational characterization, minmax principle, fluid-solid interaction, Rayleigh quotient iteration, Jacobi-Davidson method

AMS subject classification. 65F15

1. Introduction. For a wide class of linear selfadjoint operators $A : \mathcal{H} \to \mathcal{H}$, the eigenvalues of the linear eigenvalue problem $Ax = \lambda x$ can be characterized by three fundamental variational principles, namely the Rayleigh's principle [13], the Poincaré's minmax characterization [12], and the maxmin principle of Courant [4], Fischer [5], and Weyl [20]. These variational characterizations of eigenvalues are known to be very powerful tools when studying selfadjoint linear operators on a Hilbert space $\mathcal{H}$. Bounds for eigenvalues, comparison theorems, interlacing results, and monotonicity of eigenvalues can be proved easily with these characterizations, to name just a few.

In this paper we discuss the unsymmetric eigenvalue problem

$$\begin{bmatrix} K_s & C \\ 0 & K_f \end{bmatrix} \begin{bmatrix} x_s \\ x_f \end{bmatrix} = \lambda \begin{bmatrix} M_s & 0 \\ -C^T & M_f \end{bmatrix} \begin{bmatrix} x_s \\ x_f \end{bmatrix}, \tag{1.1}$$

which governs free vibrations of a fluid–solid structure. Here $K_s \in \mathbb{R}^{s\times s}$ and $K_f \in \mathbb{R}^{f\times f}$ are the stiffness matrices, $M_s \in \mathbb{R}^{s\times s}$ and $M_f \in \mathbb{R}^{f\times f}$ are the mass matrices of the structure and the fluid, respectively, and $C \in \mathbb{R}^{s\times f}$ describes the coupling of structure and fluid. x_s is the structure displacement vector and x_f the fluid pressure vector.

Problem (1.1) can be symmetrized easily. Hence, all eigenvalues are real, and the variational principles mentioned in the first paragraph hold for the symmetrized problem. However, the transformed problem incorporates the inverse of the mass matrix M_s, which is usually obtained from a finite element discretization of a partial differential operator and is therefore very large and sparse. Hence, to evaluate the Rayleigh quotient of the symmetrized problem is quite costly.

In this paper we introduce a Rayleigh functional p of the original problem (1.1) which can be evaluated easily since it is the positive solution of a quadratic equation involving only bilinear forms of the matrices K_s, M_s, K_f, M_f, and C. We prove that right eigenvectors of (1.1) are stationary points of p, and that all eigenvalues satisfy Rayleigh's principle with respect to p and are minimum-maximum and maximum-minimum values of p.

For symmetric eigenvalue problems the Rayleigh quotient iteration is known to converge cubically to simple eigenvalues, but for unsymmetric problems its convergence is only quadratic. Replacing the Rayleigh quotient by p, the resulting Rayleigh functional iteration

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converges also cubically. This property suggests an iterative projection method of Jacobi-Davidson type for (1.1).

The paper is organized as follows. In Section 2 we discuss the symmetrized version of the eigenvalue problem (1.1), and prove some useful properties of the eigenvalues and eigenvectors. In particular, if $[x_s^T, x_f^T]^T$ is a right eigenvector of (1.1) corresponding to λ, then $[\lambda x_s^T, x_f^T]^T$ is a left eigenvector corresponding to λ. This property suggests the definition of the Rayleigh functional p for which we prove in Section 3 variational characterizations of the eigenvalues of (1.1). Section 4 proves the cubic convergence of the Rayleigh functional iteration. In Section 5 we consider structure-preserving iterative projection methods of nonlinear Arnoldi and of Jacobi-Davidson type based on the Rayleigh functional, the efficiency of which is evaluated by a numerical example in Section 6.

2. Fluid-solid vibrations. Vibrations of fluid-solid structures are governed by the linear eigenvalue problem [1, 7, 8]

$$Kx := \begin{bmatrix} K_s & C \\ 0 & K_f \end{bmatrix} \begin{bmatrix} x_s \\ x_f \end{bmatrix} = \lambda \begin{bmatrix} M_s & 0 \\ -C^T & M_f \end{bmatrix} \begin{bmatrix} x_s \\ x_f \end{bmatrix} =: \lambda Mx, \tag{2.1}$$

where the matrices $K_s, M_s \in \mathbb{R}^{s\times s}$, and $K_f, M_f \in \mathbb{R}^{f\times f}$ are assumed to be symmetric and positive definite.

It is important that problem (2.1) can be symmetrized, i.e., it is equivalent to a symmetric and definite eigenvalue problem.

PROPOSITION 2.1. *Let*

$$T := \begin{bmatrix} M_s^{-1}K_s & M_s^{-1}C \\ 0 & I \end{bmatrix}. \tag{2.2}$$

Then it holds that

$$T^TK = \begin{bmatrix} K_sM_s^{-1}K_s & K_sM_s^{-1}C \\ C^TM_s^{-1}K_s & K_f + C^TM_s^{-1}C \end{bmatrix} \quad \textit{and} \quad T^TM = \begin{bmatrix} K_s & 0 \\ 0 & M_f \end{bmatrix}. \tag{2.3}$$

This result yields at once the following properties of problem (2.1), a part of which was proved directly in [9].

PROPOSITION 2.2. *(i) All eigenvalues of the fluid-solid eigenvalue problem* (2.1) *are real.*

(ii) Right eigenvectors of (2.1) *can be chosen orthonormal with respect to*

$$\tilde{M} := \begin{bmatrix} K_s & 0 \\ 0 & M_f \end{bmatrix}, \tag{2.4}$$

and left eigenvectors can be chosen orthonormal with respect to

$$\bar{M} := \begin{bmatrix} M_s & 0 \\ 0 & K_f \end{bmatrix}. \tag{2.5}$$

(iii) If $x := \begin{bmatrix} x_s \\ x_f \end{bmatrix}$ *is a right eigenvector of* (2.1) *corresponding to the eigenvalue* λ*, then* $\tilde{x} := \begin{bmatrix} \lambda x_s \\ x_f \end{bmatrix}$ *is a left eigenvector also corresponding to* λ*.*

(iv) Let y *be a left eigenvector and* x *be a right eigenvector belonging to distinct eigenvalues. Then it holds that*

$$y^TKx = 0, \quad \textit{and} \quad y^TMx = 0. \tag{2.6}$$

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Proof. (i) It follows immediately from the equivalence of (2.1) and (2.3).

(ii) The $\tilde{M}$-orthogonality of right eigenvectors is a consequence of the equivalence of (2.1) and (2.3). The $\bar{M}$-orthogonality of left eigenvectors can be derived similarly by postmultiplying the left eigenvector equation by

$$\begin{bmatrix} K_s^{-1}M_s & -K_s^{-1}C \\ 0 & I \end{bmatrix}.$$

(iii) By the symmetry of (2.3) x is also a left eigenvalue of (2.3), and therefore

$$Tx = \begin{bmatrix} M_s^{-1}K_s & M_s^{-1}C \\ 0 & I \end{bmatrix} \begin{bmatrix} x_s \\ x_f \end{bmatrix} = \begin{bmatrix} M_s^{-1}(K_s x_s + C x_f) \\ x_f \end{bmatrix} = \begin{bmatrix} \lambda x_s \\ x_f \end{bmatrix}$$

is a left eigenvector of (2.1) corresponding to λ.

(iv) x and $T^{-1}y$ are eigenvectors of the symmetric eigenproblem (2.3) corresponding to distinct eigenvalues, and therefore they are orthogonal with respect to $\tilde{M}$. Hence,

$$0 = y^T T^{-T} \tilde{M} x = y^T T^{-T} T^T M x = y^T M x \quad \text{and} \quad y^T K x = \lambda y^T M x = 0. \qquad \square$$

A further consequence of the equivalence of problems (2.1) and (2.3) is that the eigenvalues of (2.1) can be characterized by the variational principles mentionend in the introduction.

PROPOSITION 2.3. *Let $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_{s+f}$ be the eigenvalues of problem (2.1) ordered by magnitude, and let $x_1, x_2, \ldots$ corresponding right eigenvectors which are orthogonal with respect to $\tilde{M}$. Then it holds that*

(i) (Rayleigh's principle)

$$\begin{aligned} \lambda_i &= \min\left\{\frac{x^T T^T K x}{x^T T^T M x} : x^T T^T M x_j = 0,\ j = 1, \ldots, i-1\right\} \\ &= \max\left\{\frac{x^T T^T K x}{x^T T^T M x} : x^T T^T M x_j = 0,\ j = i+1, \ldots, s+f\right\}. \end{aligned} \tag{2.7}$$

(ii) (minmax characterization)

$$\lambda_i = \min_{\dim V = i} \max_{x \in V,\, x \ne 0} \frac{x^T T^T K x}{x^T T^T M x} = \max_{\dim V = s+f+1-i} \min_{x \in V,\, x \ne 0} \frac{x^T T^T K x}{x^T T^T M x}. \tag{2.8}$$

The minmax characterization allows for comparing the eigenvalues of the s dimensional solid eigenproblem and of the f dimensional fluid eigenproblem with the $s+f$ dimensional coupled fluid-solid eigenvalue problem.

PROPOSITION 2.4. *Let $\lambda_j(A, B)$ denote the j smallest eigenvalue of the eigenproblem $Ax = \lambda Bx$. Then it holds that*

$$\begin{aligned} \lambda_j(K, M) &\le \lambda_j(K_s, M_s),\ j = 1, \ldots, s, \\ \lambda_{s+f+1-j}(K, M) &\ge \lambda_{s+1-j}(K_s, M_s),\ j = 1, \ldots, s, \\ \lambda_j(K, M) &\le \lambda_j(K_f, M_f),\ j = 1, \ldots, f, \\ \lambda_{s+f+1-j}(K, M) &\ge \lambda_{f+1-j}(K_f, M_f),\ j = 1, \ldots, f. \end{aligned}$$

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Proof. Let $E_s := \text{span}\{e_1, \dots, e_s\}$, where $e_j \in \mathbb{R}^{s+f}$ denotes the jth unit vector containing a 1 in its jth component and zeros elsewhere. Then it holds that

$$\begin{aligned}
\lambda_j(K, M) &= \min_{\dim V = j} \max_{x \in V, x \neq 0} \frac{x^T T^T K x}{x^T T^T M x} \\
&\leq \min_{\dim V = j, V \subset E_s} \max_{x \in V, x \neq 0} \frac{x^T T^T K x}{x^T T^T M x} \\
&= \min_{\dim W = j, W \subset \mathbb{R}^s} \max_{y \in W, x \neq 0} \frac{y^T K_s M_s^{-1} K_s y}{y^T K_s y} \\
&= \lambda_j(K_s, M_s).
\end{aligned}$$

The second inequality is obtained analogously from the maxmin version of (2.7), and the third and fourth inequalities follow in the same way exchanging the roles of the structure and the fluid. □

3. An inverse-free Rayleigh functional.

The minmax characterization of eigenvalues in Proposition 2.3 suffers the disadvantage that one has to solve two linear systems with system matrix M_s to evaluate the Rayleigh quotient. Since the fluid-solid eigenvalue problem usually is obtained as a finite element discretization of partial differential operators the dimension s is usually very large, and the evaluation of the Rayleigh quotient is very costly. In this chapter we prove a minmax characterization using a Rayleigh functional which does not require the solution of large linear systems.

Let $\begin{bmatrix} x_s \\ x_f \end{bmatrix}$ be a right eigenvector of problem (2.1) corresponding to the eigenvalue λ, and $\begin{bmatrix} \lambda x_s \\ x_f \end{bmatrix}$ be a left eigenvector. Then it holds that

$$\lambda = \frac{[\lambda x_s^T, x_f^T] \begin{bmatrix} K_s & C \\ 0 & K_f \end{bmatrix} \begin{bmatrix} x_s \\ x_f \end{bmatrix}}{[\lambda x_s^T, x_f^T] \begin{bmatrix} M_s & 0 \\ -C^T & M_f \end{bmatrix} \begin{bmatrix} x_s \\ x_f \end{bmatrix}} = \frac{\lambda x_s^T K_s x_s + \lambda x_s^T C x_f + x_f^T K_f x_f}{\lambda x_s^T M_s x_s - x_f^T C^T x_s + x_f^T M_f x_f}.$$

This equation suggests to define a Rayleigh functional for a general vector $[x_s^T, x_f^T] \in \mathbb{R}^{s+f}$ by the requirement

$$p(x_s, x_f) = \frac{p(x_s, x_f) x_s^T K_s x_s + p(x_s, x_f) x_s^T C x_f + x_f^T K_f x_f}{p(x_s, x_f) x_s^T M_s x_s - x_f^T C^T x_s + x_f^T M_f x_f}, \tag{3.1}$$

which is equivalent to the quadratic equation

$$p(x_s, x_f)^2 x_s^T M_s x_s + p(x_s, x_f)(x_f^T M_f x_f - x_s^T K_s x_s - 2 x_s^T C x_f) - x_f^T K_f x_f = 0. \tag{3.2}$$

The smaller root of (3.2) is negative, and hence physically meaningless. We therefore choose the unique positive root of this equation as Rayleigh functional.

DEFINITION 3.1.

$$p(x_s, x_f) := \begin{cases} q(x_s, x_f) + \sqrt{q(x_s, x_f)^2 + \frac{x_f^T K_f x_f}{x_s^T M_s x_s}} & \textit{if} \quad x_s \neq 0, \\ \frac{x_f^T K_f x_f}{x_f^T M_f x_f} & \textit{if} \quad x_s = 0, \end{cases} \tag{3.3}$$

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where

$$q(x_s, x_f) := \frac{x_s^T K_s x_s - x_f^T M_f x_f + 2x_s^T C x_f}{2x_s^T M_s x_s} \tag{3.4}$$

is called Rayleigh functional of the fluid-solid vibration eigenvalue problem (2.1).

We denote by $g : \mathbb{R} \times \mathbb{R}^{s+f} \to \mathbb{R}$ the function which is used to define the Rayleigh functional, i.e.,

$$g(p, (x_s, x_f)) = p^2 x_s^T M_s x_s + p(x_f^T M_f x_f - x_s^T K_s x_s - 2x_s^T C x_f) - x_f^T K_f x_f.$$

As for the linear symmetric eigenproblem and nonlinear eigenvalue problems the following result holds.

PROPOSITION 3.2. *Every right eigenvector* $(x_s^T, x_f^T)^T$ *of* (2.1) *is a stationary point of the Rayleigh functional, i.e.,*

$$\nabla p(x_s, x_f) = 0. \tag{3.5}$$

Proof. Differentiating the defining equation (3.2) of the Rayleigh functional yields

$$\nabla p(x_s, x_f)\big(2p(x_s, x_f) x_s^T M_s x_s + x_f^T M_f x_f - x_s^T K_s x_s - 2x_s^T C x_f\big) \\ + \begin{bmatrix} 2p(x_s, x_f)^2 M_s x_s - 2p(x_s, x_f)(C x_f + K_s x_s) \\ 2p(x_s, x_f)(-C^T x_s + M_f x_f) - 2K_f x_f \end{bmatrix} = 0.$$

If $[x_s^T, x_f^T]^T$ is a right eigenvector and $\lambda := p(x_s, x_f)$ then it follows from

$$x_s^T K_s x_s + x_s^T C x_f = \lambda x_s^T M_s x_s$$

that

$$2\lambda x_s^T M_s x_s + x_f^T M_f x_f - x_s^T K_s x_s - 2x_s^T C x_f = x_s^T K_s x_s + x_f^T M_f x_f > 0,$$

and from (2.1)

$$2\lambda^2 M_s x_s - 2\lambda(K_s x_s + C x_f) = 0 \text{ and } 2\lambda(M_f x_f - C^T x_s) - 2K_f x_f = 0.$$

Hence, $\nabla p(x_s, x_f) = 0$. ☐

The following lemma prepares the proof of the variational characterizations of the eigenvalues of problem (2.1) with respect to the Rayleigh functional p.

LEMMA 3.3. *Assume that* $x_1, \ldots, x_m$ *are eigenvectors corresponding to the pairwise distinct eigenvalues* $\lambda_1 < \ldots < \lambda_m$. *Then it holds that*

(i)

$$g(\lambda_j, \sum_{i=1}^m x_i) = g(\lambda_j, \sum_{i=1}^m x_i - x_j), \quad \textit{for } j = 1, \ldots, m.$$

(ii)

$$\lambda_1 \le p\left(\sum_{i=1}^m x_i\right) \le \lambda_m. \tag{3.6}$$

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Proof. (i) Let $x = \sum_{i=1}^m x_i$ and denote by x_s, x_f and x_{si}, x_{fi} the solid and fluid components of x and x_i, respectively. Then it holds that

$$\begin{aligned}
&g(\lambda_j, \sum_{i=1}^m x_i)\\
=&\lambda_j^2 x_s^T M_s x_s + \lambda_j(-x_f^T C^T x_s + x_f^T M_f x_f - x_s^T K_s x_s - x_s^T C x_f) - x_f^T K_f x_f\\
=&\sum_{k,l} \lambda_j^2 x_{sk}^T M_s x_{sl} + \lambda_j(-x_{fk}^T C^T x_{sl} + x_{fk}^T M_f x_{fl} - x_{sk}^T K_s x_{sl} - x_{sk}^T C x_{fl}) - x_{fk}^T K_f x_{fl}\\
=&\sum_{k,l} \lambda_j(\lambda_j - \lambda_l) x_{sk}^T M_s x_{sl} + (\lambda_j - \lambda_l)(-x_{fk}^T C^T x_{sl} + x_{fk}^T M_f x_{fl})\\
=&\sum_{k,l\neq j} \lambda_j(\lambda_j - \lambda_l) x_{sk}^T M_s x_{sl} + (\lambda_j - \lambda_l)(-x_{fk}^T C^T x_{sl} + x_{fk}^T M_f x_{fl})\\
=&g(\lambda_j, \sum_{i=1}^m x_i - x_j),
\end{aligned}$$

where we used the M-orthogonality of left and right eigenvectors.

(ii) If $m = 1$, we have $p(x_1) = \lambda_1$ by construction of p. Assume that (3.6) is true for some $m \in \mathbb{N}$. Then, by the non-positivity of $g(\cdot, x_2 + \ldots + x_{m+1})$ in $[0, \lambda_2]$,

$$g(\lambda_1, x_1 + x_2 + \ldots + x_{m+1}) = g(\lambda_1, x_2 + \ldots + x_{m+1}) \leq 0,$$

and by the non-negativity of $g(\cdot, x_1 + \ldots + x_m)$ in $[\lambda_m, \infty)$,

$$g(\lambda_{m+1}, x_1 + \ldots + x_m + x_{m+1}) = g(\lambda_{m+1}, x_1 + \ldots + x_m) \geq 0.$$

This implies

$$\lambda_1 \leq p\left(\sum_{i=1}^{m+1} x_i\right) \leq \lambda_{m+1}. \qquad \square$$

THEOREM 3.4. *Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{s+f}$ be the eigenvalues of problem (2.1), and let $x_1, x_2, \ldots$ corresponding right eigenvectors. Then it holds that*

(i) (Rayleigh's principle)

$$\begin{aligned}
\lambda_k &= \min\{p(x) \,:\, x^T \tilde{M} x_j = 0,\ j = 1, \ldots, k-1\}\\
&= \max\{p(x) \,:\, x^T \tilde{M} x_j = 0,\ j = k+1, \ldots, s+f\}.
\end{aligned} \tag{3.7}$$

(ii) (minmax characterization)

$$\lambda_k = \min_{\dim S_k = k} \max_{0\neq x \in S_k} p(x) = \max_{\dim S_k = s+f+1-k} \min_{0 \neq x \in S_k} p(x).$$

Proof. (i) The proof of Rayleigh's principle follows directly from Lemma 3.3.

(ii) Let $S' = \text{span}\{x_k, \ldots, x_{s+f}\}$. Due to $S' \cap S_k \neq \{0\}$, for any k-dimensional subspace S_k, there exists $x \in S' \cap S_k$, such that

$$p(x) \geq \lambda_k, \text{ for any } k\text{-dimensional subspace } S_k.$$

Hence,

$$\min_{\dim S_k = k} \max_{x \in S_k} p(x) \geq \lambda_k.$$

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Choosing in particular $S_k = \text{span}\{x_1, \dots, x_k\}$, we obtain

$$\min_{\dim S_k = k} \max_{0 \neq x \in S_k} p(x) = \lambda_k,$$

and similarly we have

$$\lambda_k = \max_{\dim S_k = n+1-k} \min_{0 \neq x \in S_k} p(x). \qquad \square$$

From the minmax characterization we obtain that projection methods which preserve the structure of the eigenvalue problem (2.1) yield upper bounds of the eigenvalues of (2.1) as follows.

PROPOSITION 3.5. *Assume that* $V = \begin{bmatrix} V_s & 0 \\ 0 & V_f \end{bmatrix} \in \mathbb{R}^{s+f \times k}$ *has rank* k. *Let*

$$K_V := V^T K V = \begin{bmatrix} V_s^T K_s V_s & V_s^T C V_f \\ 0 & V_f^T K_f V_f \end{bmatrix}, \; M_V := V^T M V = \begin{bmatrix} V_s^T M_s V_s & 0 \\ -V_f^T C^T V_s & V_f^T M_f V_f \end{bmatrix}, \tag{3.8}$$

and let $\tilde\lambda_1 \le \tilde\lambda_2 \dots \le \tilde\lambda_k$ *be the eigenvalues of the projected eigenvalue problem*

$$K_V y = \tilde\lambda M_V y. \tag{3.9}$$

Then it holds that

$$\lambda_j \le \tilde\lambda_j, \quad j = 1, 2, \dots, k. \tag{3.10}$$

Proof. Let

$$z = \begin{bmatrix} z_s \\ z_f \end{bmatrix} \quad \text{and} \quad x := Vz = \begin{bmatrix} V_s z_s \\ V_f z_f \end{bmatrix}.$$

Then it is obvious that $\tilde p(z) = p(x)$ where $\tilde p$ denotes the Rayleigh functional of the projected problem (3.9). Hence, for $j = 1, \dots, k$, it holds that

$$\begin{aligned} \lambda_j &= \min_{\dim W = j, W \subset \mathbb{R}^{s+f}} \max_{x \in W, x \neq 0} p(x) \le \min_{\dim Z = j, Z \subset \mathbb{R}^k} \max_{x \in VZ, x \neq 0} p(x) \\ &= \min_{\dim Z = j, Z \subset \mathbb{R}^k} \max_{z \in Z, z \neq 0} p(Vz) = \min_{\dim Z = j, Z \subset \mathbb{R}^k} \max_{z \in Z, z \neq 0} \tilde p(z) = \tilde\lambda_j. \qquad \square \end{aligned}$$

4. Rayleigh functional iteration. For symmetric eigenvalue problems the Rayleigh quotient iteration converges cubically to simple eigenvalue. For unsymmetric problems the convergence is only quadratic, but a two-sided version was introduced by Ostrowski [10] which was shown to be also cubically convergent [11].

In this section we consider a one-sided Rayleigh functional iteration for the unsymmetric fluid-solid interaction eigenvalue problem and prove its local cubical convergence.

THEOREM 4.1 (Convergence of Rayleigh Functional iteration). *Consider the Rayleigh functional iteration given in Algorithm 1. Then* ρ_k *and* $x^{(k)}$ *converge locally and cubically towards an eigenvalue* λ *and a corresponding eigenvector* x.

Proof. The iteration formula can be rewritten as

$$x^{(k)} = M^{-1}(K - \rho_k M) x^{(k+1)}.$$

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Algorithm 1 Rayleigh Functional iteration for fluid-solid eigenvalue problems.

Require: Initial vector $x^{(1)}$.
for $k = 1, 2, \ldots,$ until convergence **do**
  Evaluate Rayleigh functional $\rho_k = p(x^{(k)})$.
  Solve $(K - \rho_k M)x^{(k+1)} = Mx^{(k)}$ for $x^{(k)}$.
  Normalize $x^{(k)}$.
end for

Let the columns of X form a normalized basis of right eigenvectors of (2.1), such that $X^T\tilde{M}X = I$ and $x = Xu$. Then it holds that

$$\begin{aligned} u^{(k)} =& X^{-1}M^{-1}(K - \rho_k M)Xu^{(k+1)} \\ =& X^T \begin{bmatrix} K_s & 0 \\ 0 & M_f \end{bmatrix} \begin{bmatrix} M_s^{-1} & 0 \\ M_f^{-1}C^T M_s^{-1} & M_f^{-1} \end{bmatrix} \begin{bmatrix} K_s - \rho_k M_s & C \\ \rho_k C^T & K_f - \rho_k M_f \end{bmatrix} Xu^{(k+1)} \\ =& X^T \left\{ \begin{bmatrix} K_s M_s^{-1} K_s & K_s M_s^{-1} C \\ C^T M_s^{-1} K_s & C^T M_s^{-1} C + K_f \end{bmatrix} - \rho_k \begin{bmatrix} K_s & 0 \\ 0 & M_f \end{bmatrix} \right\} Xu^{(k+1)}, \end{aligned}$$

and it follows from (2.3)

$$u^{(k)} = X^T T^T (K - \rho_k M) X u^{(k+1)} = (\Lambda - \rho_k I) u^{(k+1)}. \tag{4.1}$$

Assume that $x^{(k)}$ approximates an eigenvector $\hat{x}$ of (2.1) corresponding to an eigenvalue $\hat{\lambda}$. Denote by m the multiplicity of $\hat{\lambda}$, and let $\Lambda = \text{diag}\{\hat{\lambda} I_m, \tilde{\Lambda}\}$. Then, the eigenvector basis X can be chosen such that

$$u^{(k)} = \begin{bmatrix} e_1 \\ \varepsilon d \end{bmatrix},$$

where e_1 is the first unit vector of dimension m, $\|d\| = 1$ and ε is small.

Due to the stationarity of the nonlinear Rayleigh functional at eigenvectors, it holds that

$$\rho_k = \lambda + O(\varepsilon^2),$$

and the iteration procedure (4.1) yields in the eigenvector basis X

$$u^{(k+1)} = (\Lambda - \rho_k I)^{-1} u^{(k)} = \begin{bmatrix} \frac{e_1}{\lambda - \rho_k} \\ \varepsilon(\tilde{\Lambda} - \rho_k I)^{-1} d \end{bmatrix} = \alpha \begin{bmatrix} e_1 \\ O(\varepsilon^3)(\tilde{\Lambda} - \rho_k I)^{-1} d \end{bmatrix},$$

where α is a scaling factor. For sufficiently small ε the diagonal elements of $\tilde{\Lambda} - \rho_k I$ are bounded away from 0. Therefore,

$$\frac{\|\hat{x} - x^{(k+1)}\|_{\tilde{M}}}{\|\hat{x} - x^{(k)}\|_{\tilde{M}}^3} = \frac{\|\hat{u} - u^{(k+1)}\|_2}{\|\hat{u} - u^{(k)}\|_2^3} = \frac{O(\varepsilon^3)}{\|\varepsilon d\|_2^3} = O(1),$$

where $\tilde{M} = \text{diag}(K_s, M_f)$ and the eigenvector iterates converge locally cubically to $\hat{x}$.

The eigenvalue iterates satisfy

$$\frac{|\hat{\lambda} - \rho_{k+1}|}{|\hat{\lambda} - \rho_k|^3} = \frac{|O(\varepsilon^6)|}{|O(\varepsilon^2)|^3} = O(1),$$

and they also converge cubically. $\square$

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5. Iterative projection methods for fluid-solid vibrations. Rayleigh functional iteration converges fast, but often it is highly sensitive with respect to initial vectors. The basin of attraction can be very small, and an erratic behaviour of the iteration can be observed. To avoid the possible failure of the Rayleigh functional iteration one combines it with an iterative projection method.

Iterative projection methods have proven to be very efficient if a small number of eigenvalues and eigenvectors are desired. Here the eigenproblem is projected to a subspace of small dimension which yields approximate eigenpairs. If an error tolerance is not met then the search space is expanded in an iterative way with the aim that some of the eigenvalues of the reduced matrix become good approximations of some of the wanted eigenvalues of the given large matrix.

An expansion with high approximation potential is given by the Rayleigh functional iteration, i.e., if (θ, x), where $x = Vu$, is a Ritz pair of the current projected problem

$$V^T(K - \theta M)Vu = 0, \tag{5.1}$$

then a reasonable expansion of the search space $\mathcal{V} = \text{span}\{V\}$ is the solution v of the linear system

$$(K - \theta M)v = Mx. \tag{5.2}$$

At least close to an eigenpair the expansion v is very sensitive to inexact solves of the linear system (5.2). In [19] it was shown that the most robust expansion of $\mathcal{V}$ which contains the direction v of inverse iteration is $t := x + \alpha v$ where α is chosen such that $x^T\tilde{M}t = 0$, i.e.,

$$t = x - \frac{x^T\tilde{M}x}{x^T\tilde{M}(K - \theta M)^{-1}Mx}(K - \theta M)^{-1}Mx. \tag{5.3}$$

It is easily seen that t solves the correction equations

$$\Big(I - \frac{Mxx^T}{x^TMx}\Big)(K - \theta M)\Big(I - \frac{xx^T\tilde{M}}{x^T\tilde{M}x}\Big)t = (K - \theta M)x,\ x^T\tilde{M}t = 0, \tag{5.4}$$

which demonstrates that the resulting iterative projection method is a Jacobi-Davidson-type method [6, 14, 15].

Expanding $\mathcal{V}$ by the solution t of (5.4), the structure of the eigenvalue problem (2.1) is destroyed and eigenvalues of the projected problem (5.1) can become not real. We therefore expand in every iteration step the search space by two vectors

$$\begin{bmatrix} v_s & 0 \\ 0 & v_f \end{bmatrix} := \begin{bmatrix} t_s - V_sV_s^TK_st_s & 0 \\ 0 & t_f - V_fV_f^TM_ft_f \end{bmatrix},$$

where $t = [t_s^T, t_f^T]^T$ is an approximate solution of (5.3). Then, reordering the columns of the projection matrix V, we obtain the form $V = \begin{bmatrix} V_s & 0 \\ 0 & V_f \end{bmatrix}$, and by Proposition 3.5, the eigenvalues of the projected problems (5.1) are upper bounds of the corresponding eigenvalues of (2.1) of better accuracy.

Moreover, if θ is an eigenvalue of (5.1) with corresponding eigenvector u then it is also the value of the Rayleigh functional at the Ritz vector Vu, i.e., $\theta = p(Vu)$, which is no longer true in the non-structure-preserving Jacobi-Davidson method, where V is expanded by the (approximate) solution t of (5.4). A template of the resulting Jacobi-Davidson-type method is contained in Algorithm 2.

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Algorithm 2 Jacobi-Davidson-type method.

Require: Initial basis $V = \begin{bmatrix} V_s & O \\ O & V_f \end{bmatrix}$, $V_s^T K_s V_s = I$, $V_f^T M_f V_f = I$, $m = 1$.

1: Determine preconditioner $L \approx (K - \sigma M)^{-1}$, for σ close to first wanted eigenvalue.
2: **while** $m \leq$ number of wanted eigenvalues **do**
3: Compute the $m-$th smallest eigenvalue θ_m and the corresponding eigenvector $y = [y_s^T, y_f^T]^T$ of the projected problem

$$\begin{bmatrix} V_s^T K_s V_s & V_s^T C V_f \\ O & V_f^T K_f V_f \end{bmatrix} \begin{bmatrix} y_s \\ y_f \end{bmatrix} = \theta \begin{bmatrix} V_s^T M_s V_s & O \\ -V_f^T C^T V_s & V_f^T M_f V_f \end{bmatrix} \begin{bmatrix} y_s \\ y_f \end{bmatrix}. \tag{5.5}$$

4: Determine Ritz vector $x = \begin{bmatrix} V_s y_s \\ V_f y_f \end{bmatrix}$ and the residual $r = (K - \theta_m M)x$.
5: **if** $\|r\|/\|x\| < \epsilon$ **then**
6: Accept approximate mth eigenpair (θ_m, x), and increase $m \leftarrow m + 1$.
7: Reduce search space V if indicated.
8: Determine new preconditioner $L \approx (K - \theta M)^{-1}$ if necessary.
9: Choose approximation (θ_m, x) to next eigenpair.
10: Compute residual $r = (K - \theta_m M)x$.
11: **end if**
12: Find approximate solution $t = [t_s^T, t_f^T]^T$ of correction equation

$$\Big(I - \frac{Mxx^T}{x^T M x}\Big)(K - \theta_m M)\Big(I - \frac{xx^T \tilde{M}}{x^T \tilde{M} x}\Big)t = r, \; x^T \tilde{M} t = 0,$$

(e.g., by a preconditioned Krylov solver).
13: Orthogonalize $v_s = t_s - V_s V_s^T K_s t_s$, $v_f = t_f - V_f V_f^T M_f t_f$.
14: If $\|v_s\|_{K_s} >$ tol, then expand $V_s \leftarrow [V_s, v_s/\|v_s\|_{K_s}]$.
15: If $\|v_f\|_{M_f} >$ tol, then expand $V_f \leftarrow [V_f, v_f/\|v_f\|_{M_f}]$.
16: Update projected problem (5.5).
17: **end while**

Some comments are as follows:

(i) If the dimension of the search space has become too large, then we reduce the matrices V_s and V_f in step 7 such that the columns of V_s (and V_f) form a K_s- (and M_f-) orthogonal basis of the space spanned by the structure (and the fluid) part of the eigenvectors found so far. Notice, that the search space is reduced only after an eigenpair has converged because the reduction spoils too much information and the convergence can be retarded.

(ii) The preconditioner is updated in step 8 if the solver of (5.5) has become too slow.

(iii) Since the dimension of the projected eigenproblem is quite small it is solved by a dense solver and therefore an approximation to the next eigenpair is at hand without additional cost.

(iv) The correction equation is solved by a few steps of an iterative solver, e.g., GMRES, where the preconditioner takes into account the projectors occurring in (5.4), i.e., if L is a preconditioner of $K - \theta M$, then the solver of (5.4) is preconditioned by

$$\Big(I - \frac{Mxx^T}{x^T M x}\Big) L \Big(I - \frac{xx^T \tilde{M}}{x^T \tilde{M} x}\Big). \tag{5.6}$$

It may seem complicated to include the projectors into the preconditioner, but it was pointed out already by Sleijpen and van der Vorst [15], that an implementation of a Krylov solver with this preconditioner requires only one solve of a linear system $Lw = z$ in every iteration step, and one additional solve to initialize the method.

(v) Replacing the approximate solution t of the correction equation (5.4) in step 12 of Algorithm 2 by an approximation

$$t = L^{-1}(K - \theta M)x, \quad L \approx (K - \sigma M) \tag{5.7}$$

to the Cayley transform $(K - \sigma M)^{-1}(K - \theta M)x$ (σ close to the desired eigenvalue) one obtains an iterative projection method which was introduced in [17] for nonlinear eigenvalue problems and which was called nonlinear Arnoldi method. Although problem (1.1) is linear, no Krylov space is constructed and no Arnoldi recursion holds the resulting iterative projection method again is called nonlinear Arnoldi method.

(vi) It may happen that an eigenvector is mainly concentrated to the fluid and the solid, respectively, and then close to this eigenvector the component of t with respect to the complementary structure is very small. In this case we do not expand V_s in step 14 and V_f in step 15, respectively.

6. Numerical experiment. In order to compute the Jacobi-Davidson-type method and the nonlinear Arnoldi method we consider a model which describes free vibrations of a tube bundle immersed in a slightly compressible fluid, cf., [2, 3, 18]. We consider the same finite element model with 143082 degrees of freedom that was considered in [18].

We compare the structure-preserving iterative projection methods with Jacobi-Davidson method for general nonsymmetric eigenvalue problems in [14], which is based on the correction equation

$$\Big(I - \frac{pp^T}{p^Tp}\Big)(K - \theta M)\Big(I - \frac{xx^T}{x^Tx}\Big)t = r, \; x^Tt = 0,$$

where $p = (K - \sigma M)x$, and which expands the search space in every iteration step by one vector not accounting for structure preservation. We considered the orthogonal projection method (5.1) since the Petrov–Galerkin method suggested in [16] immediately generated complex eigenvectors of projected problems.

We computed all 18 eigenvalues in the interval $[0, 1]$, where we used as preconditioner an LU and incomplete LU factorization of $L := K - 0.5M$, respectively, and we did not update the preconditioner and did not reduce the subspace $\mathcal{V}$ in the course of the algorithm. We accepted an eigenpair if the the residual norm was less than 10^{-7} and we solved the correction equation with GMRES preconditioned by (5.6), where we allowed at most N iteration steps, and we terminated GMRES if the initial residual was reduced at least by the factor τ.

Table 6.1 shows the CPU times (on a Pentium R4 computer with 3.4 GHz and 8 GB RAM under MATLAB R2008b) and the number of iterations necessary to determine all eigenvalues in $[0, 1]$ for $(\tau, N) = (10^{-3}, 10)$ and $(\tau, N) = (10^{-1}, 6)$, respectively.

The nonlinear Arnoldi method is much faster than both versions of the Jacobi-Davidson method if a relatively precise preconditioner such as LU factorization of $L = (K - 0.5M)$ is used. Conversely, both versions of the Jacobi-Davidson method are much more robust if only coarse preconditioners such as incomplete LU factorization of L are available. In any case the structure-preserving variant of Jacobi-Davidson is faster than the standard one, although it uses much larger search spaces (recall that in the structure-preserving variant the search space

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TABLE 6.1
Comparison of structure-preserving iterative projection methods with the standard Jacobi-Davidson method.

Method	Precond.	(τ, N)	CPU time	dimension
nonlin. Arnoldi	lu		23.53	62
	luinc(0.001)		298.46	257
struc. pres. JD	lu	$(10^{-3}, 10)$	74.80	46
	lu	$(10^{-1}, 6)$	44.13	57
	luinc(0.001)	$(10^{-3}, 10)$	122.14	83
	luinc(0.001)	$(10^{-1}, 6)$	96.91	91
	luinc(0.01)	$(10^{-3}, 10)$	169.48	122
	luinc(0.01)	$(10^{-1}, 6)$	203.94	170
standard JD	lu	$(10^{-3}, 10)$	107.16	47
	lu	$(10^{-1}, 6)$	52.43	48
	luinc(0.001)	$(10^{-3}, 10)$	170.20	88
	luinc(0.001)	$(10^{-1}, 6)$	156.64	100
	luinc(0.01)	$(10^{-3}, 10)$	269.79	130
	luinc(0.01)	$(10^{-1}, 6)$	400.11	183

TABLE 6.2
Structure-preserving methods and standard JD method with restarts.

Method	CPU time	restarts	max. dim.
nonlin. Arnoldi	282	5	131
struc. pres. JD	357	2	116
standard JD	505	2	126

is usually expanded by two vectors in every iteration step). Notice, however, that the vectors in the structure-preserving methods occupy mutually exclusive vector coordinates such that the required storage is even smaller than for the standard JD method.

To test the restarted version of the method, we computed all 66 eigenvalues of our problem in $[0, 5]$. We restarted whenever the dimension of the search subspace exceeded the number of already converged eigenvalues plus some prescribed threshold. Since a restart destroys information on the eigenvectors and particularly on the one the method is just aiming at, we restarted only if an eigenvector had just converged. The reduced search space was chosen as $V = \text{span}\{x_1, \dots, x_k\}$ for the standard JD method, where x_j are the eigenvectors computed so far, and as $V_s = \text{span}\{x_1^{(s)}, \dots, x_k^{(s)}\}$ and $V_f = \text{span}\{x_1^{(f)}, \dots, x_k^{(f)}\}$ for the structure-preserving methods, where $x_j^{(s)}$ and $x_j^{(f)}$ are the structure and fluid part of x_j, respectively.

Table 6.2 shows the CPU times, number of restarts, and maximal dimensions of the search spaces. The nonlinear Arnoldi method requires more restarts since the individual expansion of the search space for this method is less accurate than for the JD method. However, the overall cost becomes smaller than for the JD variants since the expansion is less costly than the approximate solution of the correction equation in every step of the JD methods. Again the structure-preserving JD method is superior to the standard version.

7. Conclusions. For an unsymmetric eigenvalue problem governing free vibrations of fluid-solid structures, we introduced a Rayleigh functional p, and we proved variational characterizations of its eigenvalues. The corresponding Rayleigh functional iteration converges cubically. Structure-preserving iterative projection methods yield upper bounds of the eigenvalues of increasing accuracy. The nonlinear Arnoldi method is superior to Jacobi-Davidson-type methods if an accurate preconditioner is available, but it is much more sensitive to coarse preconditioners than the latter ones.

REFERENCES

[1] *ANSYS, Inc., Theory Reference for ANSYS and ANSYS Workbench, Release 11.0*, ANSYS, Inc., Canonsburg, PA, 2007.

[2] C. Conca, J. Planchard, and M. Vanninathan, *Existence and location of eigenvalues for fluid-solid structures*, Comput. Meth. Appl. Mech. Engrg., 77 (1989), pp. 253–291.

[3] ———, *Fluid and Periodic Structures*, vol. 24 of Research in Applied Mathematics, Masson, Paris, 1995.

[4] R. Courant, *Über die eigenwerte bei den differentialgleichungen der mathematischen physik*, Math. Z., 7 (1920), pp. 1–57.

[5] E. Fischer, *Über quadratische Formen mit reellen Koeffizienten*, Monatsh. Math. Phys., 16 (1905), pp. 234–249.

[6] D. Fokkema, G. Sleijpen, and H. van der Vorst, *Jacobi-Davidson style QR and QZ algorithms for the partial reduction of matrix pencils*, SIAM J. Sci. Comput., 20 (1998), pp. 94–125.

[7] C. Harris and A. Piersol, eds., *Harris Shock and Vibration Handbook*, McGraw-Hill, New York, 5th ed., 2002.

[8] A. Kropp and D. Heiserer, *Efficient broadband vibro–acoustic analysis of passenger car bodies using an FE–based component mode synthesis approach*, J. Comput. Acoustics, 11 (2003), pp. 139–157.

[9] Z. D. Ma and I. Hagiwara, *Improved mode–superposition technique for modal frequency response analysis of coupled acoustic–structural systems*, AIAA J., 29 (1991), pp. 1720–1726.

[10] A. Ostrowski, *On the convergence of the Rayleigh quotient iteration for the computation of the characteristic roots and vectors, I-VI*, Arch. Ration. Mech. Anal., 1-4 (1958/59), pp. 233–241, 423–428, 325–340, 341–347, 472–481, 153–165.

[11] B. Parlett, *The Rayleigh quotient iteration and some generalizations for nonnormal matrices*, Math. Comp., 28 (1974), pp. 679–693.

[12] H. Poincaré, *Sur les equations aux dérivées partielles de la physique mathématique*, Amer. J. Math., 12 (1890), pp. 211–294.

[13] Rayleigh, *Some general theorems relating to vibrations*, Proc. London Math. Soc., 4 (1873), pp. 357–368.

[14] G. Sleijpen, G. Booten, D. Fokkema, and H. van der Vorst, *Jacobi-Davidson type methods for generalized eigenproblems and polynomial eigenproblems*, BIT, 36 (1996), pp. 595–633.

[15] G. Sleijpen and H. van der Vorst, *A Jacobi-Davidson iteration method for linear eigenvalue problems*, SIAM J. Matrix Anal. Appl., 17 (1996), pp. 401–425.

[16] ———, *Jacobi–Davidson method (Section 8.4)*, in Templates for the Solution of Algebraic Eigenvalue Problems: A Practical Guide, Z. Bai, J. Demmel, J. Dongarra, A. Ruhe, and H. van der Vorst, eds., SIAM, Philadelphia, 2000, pp. 238–246.

[17] H. Voss, *An Arnoldi method for nonlinear eigenvalue problems*, BIT, 44 (2004), pp. 387–401.

[18] ———, *Projection methods for nonlinear sparse eigenvalue problems*, in Annals of the European Academy of Sciences, 2005, pp. 152–183.

[19] ———, *A new justification of the Jacobi–Davidson method for large eigenproblems*, Linear Algebra Appl., 424 (2007), pp. 448–455.

[20] H. Weyl, *Das asymptotische verteilungsgesetz der eigenwerte linearer partieller differentialgleichungen (mit einer anwendung auf die theorie der hohlraumstrahlung)*, Math. Ann., 71 (1912), pp. 441–479.