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SOME TIDBITS ON IDEAL PROJECTORS, COMMUTING MATRICES AND THEIR APPLICATIONS*

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To my teacher and friend Richard Varga on the occasion of his eightieth birthday.

Abstract. The main result of this paper is the parametrization of ideal projectors onto an arbitrary finite-dimensional linear subspace $G \subset \Bbbk[\mathbf{x}]$. This parametrization extends the previous ones by B. Mourrain and by M. Kreuzer and L. Robbiano. We also give applications of the technique developed in this paper to a question of similarity between a sequence of commuting matrices and its transpose and to the existence of real solutions to a system of polynomial equations.

Key words. Ideal projector, commuting operators, border schemes.

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1. Introduction. Throughout the paper $\Bbbk$ will stand for the field of complex numbers or the field of real numbers, $\Bbbk[\mathbf{x}] := \Bbbk[x_1, \ldots, x_d]$ will denote the space (algebra, ring) of polynomials in d indeterminants with coefficients in the field $\Bbbk$.

DEFINITION 1.1 ([1]). *A linear idempotent operator $P : \Bbbk[\mathbf{x}] \to \Bbbk[\mathbf{x}]$ is called an ideal projector if* $\ker P$ *is an ideal in* $\Bbbk[\mathbf{x}]$.

Lagrange interpolation projectors, Taylor projectors and, in one variable, Hermite interpolation projectors are all examples of ideal projectors. For this reason the study of ideal projectors holds a promise of an elegant extension of operators, traditionally used in numerical analysis, to multivariate setting. The theory was initiated by G. Birkhoff [1], C. de Boor [2], C. de Boor and A. Ron [4], H. M. Möller [9], and T. Sauer [14].

In this paper I will describe the family of ideal projectors onto a given a finite-dimensional linear subspace $G \subset \Bbbk[\mathbf{x}]$. The family of all such projectors is denoted by $\mathfrak{P}_G$. Due to Birkhoff's restriction of the domain of a projector to the ring $\Bbbk[\mathbf{x}]$, the study of ideal projectors parallels the study of ideals $J \subset \Bbbk[\mathbf{x}]$, that complement G:

$$J \oplus G = \Bbbk[\mathbf{x}]; \tag{1.1}$$

equivalently, those ideals $J \subset \Bbbk[\mathbf{x}]$ for which G spans the quotient algebra $\Bbbk[\mathbf{x}]/J$. Let $\mathfrak{J}_G$ denotes the family of all ideals satisfying (1.1). In commutative algebra the characterization of $\mathfrak{J}_G$ was previously considered by Mourrain [10] and by Kreuzer and Robbiano [7, Chapter 6.4] in connection with some questions in computer algebra. They gave a description of the bases for the ideals in $\mathfrak{J}_G$ when G satisfies some additional assumptions: G is connected to 1, in case of [10] and G is a D-invariant space spanned by monomials (order ideals) in case of [7]. Further discussion on the relationship between these assumptions can be found in [3].

The main result of the next section characterizes (parametrizes) $\mathfrak{P}_G$ (equivalently $\mathfrak{J}_G$) without any assumptions on G. In this sense Theorem 2.3 below extends the results of [10] and [7] to arbitrary G. This parametrization allows to "compute" all ideal projectors for general subspace G. The two examples in Section 3 of the paper show the difference between parametrization of $\mathfrak{P}_G$ for special G and arbitrary G.

As in [10] and [7], the indispensable tool in caring out the characterization of $\mathfrak{P}_G$ is a commuting family of multiplication operators (matrices) on G: $\mathbf{M}_P := (M_j, j = 1 : d)$

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defined by $M_j(g) := P(x_j g)$ for every $g \in G$. These operators are similar (literally and figuratively) to the multiplication maps m_j on $\Bbbk[\mathbf{x}]/J$ defined by $m_j([f]) := [x_j f] \in \Bbbk[\mathbf{x}]/J$ for every $[f] \in \Bbbk[\mathbf{x}]/J$. A relationship between ideals, multiplication maps and numerical analysis was initiated and explored by H. Stetter [17].

Unlike [10] and [7], our proofs rely on the language of ideal projectors. It is my belief that this language, as a substitute for a division algorithm, allows extensions and simplification of some of the arguments used in algebraic geometry.

In the consecutive sections we will present applications of the interrelations between ideal projectors, zero-dimensional ideals and commuting matrices to such diverse fields as linear algebra, solutions of polynomial equations and algebraic geometry.

We will make use of the following observation due to Carl de Boor [2].

THEOREM 1.2. *A linear operator $P : \Bbbk[\mathbf{x}] \to \Bbbk[\mathbf{x}]$ is an ideal projector if and only if*

$$P(fg) = P(f \cdot P(g)) \tag{1.2}$$

for all $f, g \in \Bbbk[\mathbf{x}]$.

In terms of the quotient algebra $\Bbbk[\mathbf{x}]/J$, (1.2) says that $[f[g]] = [fg] \in \Bbbk[\mathbf{x}]/J$, for all $f, g \in \Bbbk[\mathbf{x}]$.

For every $J \in \mathfrak{I}_G$, we use $\mathbf{M}_J = (M_1, \ldots, M_d)$ to denote the sequence of multiplication operators on G defined by $M_i(g) = P_J(x_i g)$. It is easy to see (cf. [2]) that this is a sequence of pairwise commuting operators which is cyclic, with the cyclic vector $P_J 1 \in G$:

$$\{p(\mathbf{M}_J)(P_J 1), p \in \Bbbk[\mathbf{x}]\} = G. \tag{1.3}$$

2. Border schemes. Let $\mathfrak{g} = (g_1, \ldots, g_N)$ be a linear basis for G. We define the border of $\mathfrak{g}$ as

$$\partial\mathfrak{g} := \{1, x_i g_k, i = 1, \ldots, d, k = 1, \ldots, N\} \setminus G.$$

For every ideal $J \in \mathfrak{I}_G$, the decomposition (1.1) induces an *ideal projector* P_J onto G with $\ker P_J = J$. From (1.1) it follows that for every ideal $J \in \mathfrak{I}_G$ and for every $b \in \partial\mathfrak{g}$ there exists a unique (!) polynomial $p_b = P_J b \in G$ such that $b - p_b \in J$. As it turns out, the set $\{b - p_b, b \in \partial\mathfrak{g}\}$ forms an ideal basis for J, called a (generalized) border basis.

PROPOSITION 2.1. *Let $J \in \mathfrak{I}_G$ and for every $b \in \partial\mathfrak{g}$ let $p_b := P_J b$ be the unique polynomial in G such that $b - p_b \in J$. Then $\{b - p_b, b \in \partial\mathfrak{g}\}$ forms an ideal basis for J.*

This proposition is not new; cf. [2], [16]. The proof below is essentially the same as in [2] and presented purely for convenience.

Proof. We wish to prove that for every element of $f \in J$ there are $f_{i,k} \in \Bbbk[\mathbf{x}]$ such that

$$f = \sum_{i,k} f_{i,k} \cdot (x_i g_k - P_J(x_i g_k)),$$

since $x_i g_k - P_J(x_i g_k) = 0$ if $x_i g_k \in G$.

Since J is the range of $I - P_J$, and by linearity of P_J, it suffices to show that for every monomial $\mathbf{x}^\alpha$ there are $f_{i,k} \in \Bbbk[\mathbf{x}]$ such that

$$\mathbf{x}^\alpha - P_J \mathbf{x}^\alpha = \sum f_{i,k} \cdot (x_i g_k - P_J(x_i g_k)).$$

The rest of the proof is by induction on the degree $|\alpha| := \sum \alpha_j$. If $|\alpha| = 0$ then $1 - P_J 1 = 1 - p_1 \in \{b - p_b, b \in \partial\mathfrak{g}\}$ and there is nothing to prove. Next, assume that $\mathbf{x}^\alpha - P_J \mathbf{x}^\alpha \in \langle b - g_b, b \in \partial\mathfrak{g}\rangle$. Then, for every $i = 1, \ldots, d$,

$$x_i \mathbf{x}^\alpha - P_J(x_i \mathbf{x}^\alpha) = x_i \mathbf{x}^\alpha - P_J(x_i P_J \mathbf{x}^\alpha) = x_i(\mathbf{x}^\alpha - P_J \mathbf{x}^\alpha) + x_i P_J \mathbf{x}^\alpha - P_J(x_i P_J \mathbf{x}^\alpha).$$

By the inductive assumption, $x_i\left(\mathbf{x}^\alpha - P_J\mathbf{x}^\alpha\right) \in \langle b - g_b, b \in \partial\mathfrak{g}\rangle$.. Also $P_J\mathbf{x}^\alpha \in G$ hence $P_J\mathbf{x}^\alpha = \sum a_k g_k$ and

$$\begin{aligned} x_i P_J \mathbf{x}^\alpha - P_J\left(x_i P_J \mathbf{x}^\alpha\right) &= \sum_k a_k x_i g_k - \sum_k a_k P_J\left(x_i g_k\right) \\ &= \sum a_k\left(x_i g_k - P_J\left(x_i g_k\right)\right) \in \langle b - g_b, b \in \partial\mathfrak{g}\rangle\,, \end{aligned}$$

since $x_i g_k - P_J\left(x_i g_k\right) = 0$ if $x_i g_k \in G$. □

REMARK 2.2. This proposition is a direct generalization of Proposition 6.4.15 in [7], where it is proved for D-invariant subspaces G spanned by monomials. The argument in [7] uses the division algorithm with remainder coming from the space G. This, once again, shows that the language of ideal projectors is an alternative to that of division algorithm: for every $f \in \Bbbk[\mathbf{x}]$,

$$f = \sum f_b \cdot p_b + P_J f$$

where $P_J f$ is the unique "remainder" in G of the division of f by the ideal J.

What about a converse? That is, what polynomials $(p_b, b \in \partial\mathfrak{g})$ have the property that the ideal $\langle b - p_b, b \in \partial\mathfrak{g}\rangle$ is in $\mathfrak{I}_G$? This is the question first dealt with in [10] with some additional assumptions on G; cf. also [3] and [7].

Mimicking the terminology of [7, 6.4B], we will characterize those border prebases that are border bases. We will present necessary and sufficient conditions on polynomials $\{p_b, b \in \partial\mathfrak{g}\}$ for $\{b - p_b, b \in \partial\mathfrak{g}\}$ to be a basis for an ideal in $\mathfrak{I}_G$. As in [7, 6.4B], the criterion involves formal multiplication operators $M_j : G \to G$ defined by

$$M_i g_k = \begin{cases} x_i g_k & \text{if} \quad x_i g_k \in G, \\ p_{x_i g_k} & \text{if} \quad x_i g_k \notin G. \end{cases}$$

Here is the main theorem of this section.

THEOREM 2.3. *Let $(p_b, b \in \partial\mathfrak{g})$ be a sequence of polynomials in G. Then the ideal $\langle f - p_f, f \in \partial\mathfrak{g}\rangle \in \mathfrak{I}_G$ if and only if*

(i) *$M_i M_k = M_k M_i$ for all $i, k = 1, \ldots, d$,*

(ii) *$g\left(M_1, \ldots, M_d\right) p_1 = g$ for all $g \in G$.*

Proof. First assume that $J = \langle b - p_b, f \in \partial\mathfrak{g}\rangle \in \mathfrak{I}_G$ and let P_J be the ideal projector onto G with $\ker P_J = J$.. Then $M_i g = P_J\left(x_i g\right)$ for all $i = 1, \ldots, d$. It follows from (1.2) that

$$M_j M_k(g) = P_J\left(x_i P_J\left(x_k g\right)\right) = P_J\left(x_i x_k g\right) = P_J\left(x_k x_i g\right) = M_k M_i(g),$$

which proves (i). Also observe that if $g = \sum a_\alpha \mathbf{x}^\alpha$, then, for $\mathbf{M} := (M_1, \ldots, M_d)$ and $g_0 := P1$, we have

$$\begin{aligned} g\left(M_1, \ldots, M_d\right) g_0 = g &= \sum a_\alpha \mathbf{M}^\alpha\left(P_J 1\right) = \sum a_\alpha P_J\left(x^\alpha\left(P_J 1\right)\right) \\ &= \sum a_\alpha P_J\left(x^\alpha\right) = P_J\left(\sum a_\alpha x^\alpha\right) = P_J g = g, \end{aligned}$$

which proves (ii).

Now, suppose that (i) and (ii) hold. Then the mapping $\varphi : \Bbbk[\mathbf{x}] \to \Bbbk[\mathbf{x}]$ defined by

$$\varphi f = f(\mathbf{M}) p_1$$

is a ring homomorphism, hence its kernel

$$K := \ker\varphi = \{f \in \Bbbk[\mathbf{x}] : f(\mathbf{M})p_1 = 0\} \overset{\text{by (ii)}}{=} \{f \in \Bbbk[\mathbf{x}] : f(\mathbf{M}) = 0\}$$

is an ideal in $\Bbbk[\mathbf{x}]$. By (ii) the range of φ is G and $K \cap G = 0$. By the fundamental theorem of homomorphisms $\Bbbk[\mathbf{x}]/K$ is isomorphic to G. In particular codimension of K is equal to $\dim G$ and K complements G.

Let h_b be the unique element in G such that $b - h_b \in K$. We need to show that $J = K$ or, alternatively that $h_b = g_b$ for every $b \in \partial\mathfrak{g}$. Since $b - h_b \in K$ we have

$$0 = (b(\mathbf{M}) - h_b(\mathbf{M}))p_1 \overset{\text{by (ii)}}{=} b(\mathbf{M})p_1 - h_b.$$

On the other hand, by definition of $\mathbf{M}$, we have $b(\mathbf{M})p_1 = p_b$ which implies that $p_b = h_b$ for all $b \in \partial\mathfrak{g}$. ☐

REMARK 2.4. If G is a D-invariant subspaces of $\Bbbk[\mathbf{x}]$ spanned by monomials, then $1 \in G$ and, by the D-invariance, condition (ii) of Theorem 2.3 is automatically satisfied (see Example 3.1 below). Hence Theorem 2.3 generalizes Theorem 6.4.30 of [7] with, what seems to be, a shorter, simpler proof, courtesy of the language of ideal projectors.

The operators $M_1, \dots, M_k$ can be written as $N \times N$ matrices in the basis $\mathfrak{g}$ and, the polynomial $p_1 \in G$ generates an $N \times 1$ matrix of its coefficients.

DEFINITION 2.5. *The affine scheme $\mathcal{B}_\mathfrak{g}$ defined by the ideal $I_\mathfrak{g}$ generated by the entries of the matrices $M_jM_i - M_iM_j$, $i, j = 1, \dots, d$, and the coordinates of the vector p_1,*

$$\langle g_k (M_1, \dots, M_d)\, p_1 - g_k \rangle, \quad k = 1, \dots, N,$$

is called the generalized border scheme for $\mathfrak{g}$ or $\mathfrak{g}$-border scheme. It parametrizes the family of ideals $\mathfrak{I}_G$ or, equivalently, the family of ideal projectors $\mathfrak{P}_G$.

Unlike the border schemes for monomial D-invariant subspaces of $\Bbbk[\mathbf{x}]$, the $\mathfrak{g}$-border scheme is defined by, possibly, N extra parameters (if $1 \notin G$): coefficients of p_1, and N extra equations: $g_k (M_1, \dots, M_d)\, p_1 - g_k = 0$; cf. Example 3.2 below.

Note that, for the ideal $J \in \mathfrak{I}_G$, the operators M_j depend only on the space G and not on its basis $\mathfrak{g}$. The entries of the matrices M_j depend on the basis, hence the $\mathfrak{g}$-border scheme depends on the particular choice of basis $\mathfrak{g}$ for G.

3. Two examples.

The first example is standard; cf. [8, Example 18.23], [13], [15].

EXAMPLE 3.1. Let $\mathfrak{g} = (1, x, y) \subset \Bbbk[\mathbf{x}]$ and G be the space spanned by $\mathfrak{g}$. Thus $\partial\mathfrak{g} = \left\{x^2, xy, y^2\right\}$ and every $P \in \mathfrak{P}_G$ is determine by its values

$$\begin{aligned} Px^2 &= a_0 + b_0x + c_0y, \\ Pxy &= a_1 + b_1x + c_1y, \\ Py^2 &= a_2 + b_2x + c_2y. \end{aligned}$$

or, equivalently, by nine coefficients

$$(a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1, c_2)\,. \tag{3.1}$$

So what are the condition on the coefficients (3.1) that guarantee that the ideal

$$\left\langle x^2 - Px^2, xy - Pxy, y^2 - Py^2 \right\rangle$$

complements G? To answer this question we form formal multiplication matrices

$$M_1 = \begin{bmatrix} 0 & a_0 & a_1 \\ 1 & b_0 & b_1 \\ 0 & c_0 & c_1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & a_1 & a_2 \\ 1 & b_1 & b_2 \\ 0 & c_1 & c_2 \end{bmatrix}.$$

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This is the case when G is a monomial D-invariant space and the conditions (ii) of Theorem 2.3 is automatically satisfied. All that is left is to enforce the commutativity. The six quadratic equations obtained from $M_1M_2 - M_2M_1 = 0$ are

$$\begin{cases} (a_0b_1 + a_1c_1) - (a_1b_0 + a_2c_0) = 0, \\ (a_1 + b_0b_1 + b_1c_1) - (b_0b_1 + b_2c_0) = 0, \\ \left(c_1^2 + b_1c_0\right) - (a_0 + b_0c_1 + c_0c_2) = 0, \\ (a_0b_2 + a_1c_2) - (a_1b_1 + a_2c_1) = 0, \\ (a_2 + b_0b_2 + b_1c_2) - \left(b_1^2 + b_2c_1\right) = 0, \\ (b_2c_0 + c_1c_2) - (a_1 + b_1c_1 + c_1c_2) = 0. \end{cases}$$

A close examination reveals that there is a lot of redundancy in these equations. The solutions to these equations are given by

$$\begin{aligned} a_0 &= -b_0c_1 + c_1^2 + b_1c_0 - c_0c_2, \\ a_1 &= b_2c_0 - b_1c_1, \\ a_2 &= b_1^2 - c_2b_1 - b_0b_2 + b_2c_1. \end{aligned} \tag{3.2}$$

The border scheme $\mathcal{B}_{\mathfrak{g}}$ is a six-dimensional affine variety in $\mathbb{k}^9$ that consists of all nine-tuples $(a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1, c_2)$ satisfying (3.2).

By checking (3.2) we see that the following four projectors defined by

$$\begin{aligned} T &: Tx^2 = T(xy) = Ty^2 = 0, \\ P_* &: P_*x^2 = y, P_*(xy) = P_*y^2 = 0, \\ L &: Lx^2 = x, L(xy) = 0, Ly^2 = y, \\ H &: Hx^2 = Hxy = Hy^2 = y, \end{aligned} \tag{3.3}$$

are in fact ideal projectors onto G. The first, T, is the Taylor projector onto G, it interpolates the function and its first partial derivatives at 0. The second, P_*, also interpolates various derivatives at zero, namely

$$\delta_0, \quad \delta_0 \circ D_x, \quad \delta_0 \circ \left(D_x^2 + 2D_y\right),$$

and is a different projector. Hence, unlike the case in one variable, there are two (infinitely many) ideal projectors onto G such that the zero locus $\mathcal{Z}(\ker P)$ of the ideal $\ker P$ is $\{0\}$.. The projector L is a Lagrange projector interpolating at sites $(0,0)$, $(1,0)$ and $(0,1)$. Finally the last projector H interpolates the value of a function and its derivative with respect to x at zero and the value of the function at $(1,1)$.

EXAMPLE 3.2. We want to determine and parametrize the family of ideals that complement the two-dimensional space G spanned by $\mathfrak{g} = \{x, y\}$. In this case, G is not D-invariant and $1 \notin G$, hence in addition to commutativity condition we need to enforce condition (ii) of Theorem 2.3: Let

$$p_1 = ux + vy = \begin{bmatrix} u \\ v \end{bmatrix}$$

and matrices

$$M_1 = \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \quad M_2 = \begin{bmatrix} A & C \\ B & D \end{bmatrix}.$$

The commutativity conditions give four equations:

$$\begin{cases} Bc - Cb = 0, \\ Ab - bD - Ba + Bd = 0, \\ cD - Ac + Ca - Cd = 0, \\ Cb - Bc = 0. \end{cases}$$

The two additional equations $M_1 p_1 = x$, $M_2 p_1 = y$ give four more equations:

$$\begin{cases} au + cv - 1 = 0, \\ bu + dv = 0, \\ Au + Cv = 0, \\ vD + Bu - 1 = 0. \end{cases}$$

Together these two sets of equations define a four-dimensional affine algebraic set (the $\mathfrak{g}$-border scheme) in $\mathbb{k}^{10}$. Clearly (u, v) can not be zero. If $u, v \neq 0$, then the solutions to these equations are given in terms of four free parameters, u, v, C, d:

$$b = -d\frac{v}{u}, \quad A = -C\frac{v}{u}, \quad c = -C\frac{v}{u}, \quad D = -\frac{-1 + du}{v}, \quad B = d, \quad a = \frac{Cv^2 + u}{u^2},$$

and all ideal projectors onto G are given by

$$P1 = ux + vy, \quad Px^2 = ax - d\frac{v}{u}y, \quad Pxy = -C\frac{v}{u}x + dy, \quad Py^2 = Cx - \frac{-1 + du}{v}y.$$

The remaining cases are listed below:

2) $C = 0,\ d = -\dfrac{vD - 1}{u},\ B = -\dfrac{vD - 1}{u},\ A = 0,\ c = 0,\ b = \dfrac{vD - 1}{u^2}v,\ a = \dfrac{1}{u}.$

3) $d = 0,\ C = 0,\ u = 0,\ A = D,\ v = \dfrac{1}{D},\ B = 0,\ c = D.$

4) $v = 0,\ d = \dfrac{1}{u},\ A = 0,\ c = 0,\ B = \dfrac{1}{u},\ b = 0,\ a = \dfrac{1}{u}.$

4. Applications.

4.1. To linear algebra. It is well-known and easy to see (cf. [18]) that every square matrix M is similar to its transpose. This is not the case for sequences of commuting matrices. For instance, the pair of matrices (M_1, M_2) associated with the Taylor projector T in (3.3) has the form

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

with the cyclic vector $e_1 = (1, 0, 0)$. The adjoint pair of matrices (M_1^t, M_2^t) is not cyclic. Hence there cannot be an invertible transformation S such that $M_i^t = SM_iS^{-1}$ for $i = 1, 2$. As an easy application of ideal projectors we obtain the following non-trivial result.

THEOREM 4.1. *A cyclic sequence $\mathbf{L} = (L_1, \ldots, L_d)$ is similar to its transpose $\mathbf{L}^t = (L_1^t, \ldots, L_d^t)$ if and only if $\mathbf{L}^t$ is cyclic.*

The starting point is the observation (cf. [5], [11, Theorem 1.9], [12]) that not only an ideal projector generates a cyclic sequence of commuting operators (and therefore matrices) but any cyclic sequence of commuting matrices is similar to a sequence of multiplication operators for some ideal projector.

PROPOSITION 4.2. *Let* $\mathbf{L} = (L_1, \ldots, L_d)$ *be a cyclic sequence of commuting* $N \times N$ *matrices. Then the ideal*

$$J_{\mathbf{L}} := \{f \in \Bbbk[\mathbf{x}] : f(\mathbf{L}) = 0\}$$

has codimension N *and* $\mathbf{L}$ *is similar to the matrices of multiplication operators* $\mathbf{M}_P$ *of any ideal projector* P *with* $\ker P = J_{\mathbf{L}}$.

Proof. Let v_0 be a cyclic vector for $\mathbf{L}$. Define

$$\begin{array}{rccc} \varphi : & \Bbbk[\mathbf{x}] & \to & \Bbbk^N, \\ & f & \to & f(\mathbf{L})v_0. \end{array}$$

Since $\mathbf{L}$ is cyclic, φ is onto and by the fundamental theorem of homomorphisms $\Bbbk[\mathbf{x}]/\ker\varphi$ is isomorphic to $\Bbbk^N$. But

$$\ker\varphi = \{f \in \Bbbk[\mathbf{x}] : f(\mathbf{L})v_0 = 0\} = J_{\mathbf{L}},$$

which shows that codimension of $J_{\mathbf{L}}$ is N.

Now, let G be a subspace that complements $J_{\mathbf{L}}$ and let P be an ideal projector onto G with $\ker P = J_{\mathbf{L}}$. Since G complements $J_{\mathbf{L}}$, it follows that the restriction $\varphi_{|G}$ of φ to G is invertible and, by direct computation,

$$P = \left(\varphi_{|G}\right)^{-1} \circ \varphi.$$

To show that $\mathbf{L}$ is similar to the sequence of multiplication operators $\mathbf{M}_P$ we will verify the identity: $\left(\varphi_{|G}\right)^{-1} \circ L_j \circ \left(\varphi_{|G}\right) = M_j$, i.e.,

$$\left(\varphi_{|G}\right)^{-1} \circ L_j \circ \left(\varphi_{|G}\right) g = P_{\mathbf{L}}\left(x_j g\right),$$

for all $g \in G$. Indeed,

$$\begin{aligned} P\left(x_j g\right) &= \left(\varphi_{|G}\right)^{-1} \varphi\left(x_j g\right) = \left(\varphi_{|G}\right)^{-1} \left(L_j g(\mathbf{L})v_0\right) = \left(\varphi_{|G}\right)^{-1} \circ L_j\left(g(\mathbf{L})v_0\right) \\ &= \left(\left(\varphi_{|G}\right)^{-1} \circ L_j \circ \varphi\right)(g) = \left(\varphi_{|G}\right)^{-1} \circ L_j \circ \left(\varphi_{|G}\right)(g), \end{aligned}$$

since $g \in G$. □

Proof of Theorem 4.1. Suppose that $\mathbf{L}^t$ is cyclic. Since $\ker\mathbf{L}^t = \ker\mathbf{L}$, by Proposition 4.2, $\mathbf{L}^t$ is similar to matrices for multiplication operators $\mathbf{M}_P$ for the projector P just as $\mathbf{L}$ is. By transitivity, $\mathbf{L}^t$ is similar to $\mathbf{L}$. The converse is obvious. That is, if $\mathbf{L}^t = S\mathbf{L}S^{-1}$, then $\mathbf{L}^t$ is also cyclic. □

In general the problem of (simultaneous) similarity of two d-tuple of matrices seems to be quite difficult; cf. [6]. The following generalization of Theorem 4.1 is straightforward, but may be new.

THEOREM 4.3. *Let* $\mathbf{L}$ *be a cyclic* d*-tuple of commuting* $N \times N$ *matrices and let* $\mathbf{B} = (B_1, \ldots, B_d)$ *be an arbitrary* d*-tuple of* $N \times N$ *matrices. Then* $\mathbf{B}$ *is similar to* $\mathbf{L}$ *if and only if* $\mathbf{B}$ *is a cyclic* d*-tuple of commuting matrices.*

PROBLEM 4.4. *What are the necessary and sufficient conditions for a general commuting sequence of matrices to be similar to its transpose?*

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4.2. To real solutions of polynomial systems. A standard exercise in Calculus uses the intermediate value theorem to prove that every real polynomial of odd degree has at least one real zero. In this section we give a simple proof of a (hopefully original) observation that generalizes this statement to the systems of polynomial equations in several variables. Namely, we prove that every ideal of real polynomials of odd codimension has a common zero.

In one variable every polynomial $p \in \mathbb{R}[\mathbf{x}]$ of odd degree defines the ideal

$$J := \langle p \rangle := \{fp, f \in \mathbb{R}[\mathbf{x}]\},$$

that complements the space $\mathbb{R}_{<2n-1}[\mathbf{x}]$ of polynomials of degree less than $2n-1$; thus J is of odd codimension. The existence of a solution is equivalent to

$$\mathcal{Z}(I) := \{x \in \mathbb{R} : q(x) = 0 \text{ for all } q \in J\} \neq \emptyset.$$

The existence of a real solution for ideals of odd codimension is intuitively obvious from the principle of conjugation. If $\mathbf{z}_1, \dots, \mathbf{z}_m \in \mathbb{C}^d$ are all the solutions of the generators of J, then they must be invariant under the conjugation. If they are all complex one has to have an even number of them. This translates to even codimension. But the technical proof of this needs to take into account multiplicities of solutions and is more involved than the one, presented below, using multiplication operators.

THEOREM 4.5. *Let $J \subset \mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \dots, x_d]$ be an ideal of odd codimension. Then*

$$\mathcal{Z}(J) := \{\mathbf{x} \in \mathbb{R}^d : q(\mathbf{x}) = 0 \textit{ for all } q \in J\} \neq \emptyset.$$

Proof. Let P be an ideal projector from $\mathbb{R}[\mathbf{x}]$ onto a subspace $G \subset \mathbb{R}[\mathbf{x}]$ with $\ker P = J$. To prove the theorem it suffices to show that the sequence $\mathbf{M}_J := (M_1, \dots, M_d)$ of commuting operators on an odd-dimensional space G has a common eigenvector. Indeed, if $M_j g = \lambda_j g$ for some $g \neq 0$, then for every $p \in J$, $p(M_1, \dots, M_d) = 0$. Hence

$$0 = p(M_1, \dots, M_d)\, g = p(\lambda_1, \dots, \lambda_d) \cdot g,$$

and the eigentuple $(\lambda_1, \dots, \lambda_d)$ is a zero of p.

The proof that $\mathbf{M}_J$ has a common eigenvector is by induction on d. If $d = 1$, then the characteristic polynomial of M_1 has an odd degree and thus has a real root that corresponds to an eigenvector of M_1. Assume that the statement is true for any sequence of $d-1$ commuting operators. Let H be a subspace of G of minimal odd dimension, invariant with respect to $M_1, \dots, M_d$. Let $\tilde{M}_1, \dots, \tilde{M}_d$ be the restrictions of $M_1, \dots, M_d$ to H. It is clearly enough to prove that $\tilde{M}_1, \dots, \tilde{M}_d$ have a common eigenvector in H. Let $h \in H$ be an eigenvector for $\tilde{M}_d$ corresponding to an eigenvalue λ. Consider the spaces

$$H_1 := \ker\left(\tilde{M}_d - \lambda I\right), \quad H_2 := \operatorname{ran}\left(\tilde{M}_d - \lambda I\right).$$

These two spaces are invariant with respect to $M_1, \dots, M_d$, and one of the two has an odd dimension since $\dim H_1 + \dim H_2 = \dim H$. Since $H_1 \neq \{0\}$, $\dim H_2 < \dim H$, and from minimality of H it follows that H_1 has and odd dimension and hence $H_1 = H$. Thus M_d is a multiple of the identity on H and any eigenvector in H, common to $M_1, \dots, M_{d-1}$, is also an eigenvector of M_d. ∎

For instance, any system of three quadratic equations

$$\begin{aligned}
x^2 - (a_0 + b_0 x + c_0 y) &= 0,\\
xy - (a_1 + b_1 x + c_1 y) &= 0,\\
y^2 - (a_2 + b_2 x + c_2 y) &= 0,
\end{aligned}$$

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with coefficients satisfying (3.2) has a real solution since, by Example 3.1, the corresponding ideal complements the space of linear polynomials, hence has codimension 3.

4.3. To the geometry of border schemes. For a monomial D-invariant space $G \subset \Bbbk[\mathbf{x}]$ the following problem was posed in [13]:

PROBLEM 4.6. *Is the border scheme $\mathcal{B}_{\mathfrak{g}}$, where $\mathfrak{g}$ is the monomials basis for G, connected?*

In other words, given two ideal projectors, P_0 and $P_1 \in \mathfrak{P}_G$, does there exist a continuous family of ideal projectors $P(t) \in \mathfrak{P}_G$ such that $P(0) = P_0$ and $P(1) = P_1$? The answer is affirmative for those D-invariant monomial spaces G where $\deg f \geq \max\{\deg g : g \in G\}$ for every $f \in \partial\mathfrak{g}$. It is interesting to note that it follows from the general theory of Gröbner basis (cf. [8, Remark 18.3], [13]), that "the Hilbert schemes" are connected. That is, for a given pair of ideal projectors $P_0, P_1 \in \mathfrak{P}_G$ there exists a continuous family $P(t)$ of ideal projectors, such that $P(0) = P_0$, $P(1) = P_1$, and $\dim \operatorname{ran} P = \dim G$ for all t. The rub is: for some t the ideal projector $P(t)$ may project onto a subspace of dimension N that is different from G.

Are generalized border schemes connected? The answer in $\mathbb{R}[x]$, real polynomials in one variable, is clearly negative. Indeed, let G be the one-dimensional space spanned by x. Then the family $\mathfrak{J}_G$ consists of maximal ideals, missing the one supported at zero. And since $\mathbb{R}/\{0\}$ is not connected, neither is the border scheme $\mathcal{B}_{(x)}$. Of course, this based on the peculiarity of one-dimensional real space, where a point separates the space. A more subtle example can be obtained by the projectors onto the two dimensional subspace $G = \operatorname{span}\left\{1, x^2\right\}$ in $\mathbb{R}[x]$. If $P \in \mathfrak{P}_G$ is given by

$$Px = a + bx^2, \quad Px^3 = c + dx^2, \tag{4.1}$$

then the multiplication matrix for P is

$$M = \begin{bmatrix} a & c \\ b & d \end{bmatrix},$$

and by Theorem 2.3, the requirement on the coefficients is

$$M^2(1,0)^t = \left(a^2 + bc, ab + bd\right)^t = (0,1)^t,$$

which implies that $b \neq 0$. Thus, the ideal projectors P_0 and P_1 given by

$$\begin{aligned} P_0 x &= 1 + x^2, \quad & P_0 x^3 &= -1 - x^2, \\ P_1 x &= 1 - x^2, \quad & P_1 x^3 &= 1 - x^2, \end{aligned}$$

satisfy (i) and (ii) of Theorem 2.3, yet cannot be connected by a continuous family of ideal projectors given by (4.1) with $b \neq 0$. Hence $\mathcal{B}_{(1,x^2)}$ is not connected.

PROBLEM 4.7. *Are the generalized border schemes in $\mathbb{C}[\mathbf{x}]$ connected? Are generalized border schemes in $\mathbb{R}[\mathbf{x}]$ connected for $d \geq 2$?*

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