Electronic Transactions on Numerical Analysis.
Volume 36, pp. 39-53, 2009-2010.

ISSN 1068-9613.

ETNA
Kent State University
http://etna.math.kent.edu

P-REGULAR SPLITTING ITERATIVE METHODS FOR NON-HERMITIAN POSITIVE DEFINITE LINEAR SYSTEMS*

CHENG-YI ZHANG† AND MICHELE BENZI‡

Dedicated to Richard S. Varga on the occasion of his 80th birthday

Abstract. We study the convergence of P-regular splitting iterative methods for non-Hermitian positive definite linear systems. Our main result is that P-regular splittings of the form $A = M - N$, where $N = N^*$, are convergent. Natural examples of splittings satisfying the convergence conditions are constructed, and numerical experiments are performed to illustrate the convergence results obtained.

Key words. Non-Hermitian positive definite matrices, P-regular splitting, convergence, SOR methods, preconditioned GMRES

AMS subject classifications. 65F10, 15A15, 15F10.

1. Introduction. Many problems in scientific computing give rise to a system of n linear equations in n unknowns,

$$Ax = b, \quad A = [a_{ij}] \in \mathbb{C}^{n\times n} \text{ nonsingular}, \text{ and } b, x \in \mathbb{C}^n, \tag{1.1}$$

where A is a large, sparse non-Hermitian matrix. In this paper we consider the important case where A is *non-Hermitian positive definite*; i.e., the Hermitian part $H = (A + A^*)/2$ is Hermitian positive definite, where A^* denotes the conjugate transpose of the matrix A. We note that the phrase *non-Hermitian positive definite*, while widely used, is a bit misleading since A could actually be Hermitian. The expression *possibly non-Hermitian, positive definite matrix* is more precise, but also too cumbersome. The expression *strictly accretive* is also used, but is not widely adopted. Large, sparse systems with non-Hermitian positive definite coefficient matrix arise in many applications, including discretizations of convection-diffusion problems [17], regularized weighted least-squares problems [13], real-valued formulations of certain complex symmetric systems [9], and so forth. In order to solve system (1.1) by iterative methods, it is useful to construct splittings of the coefficient matrix A. Such splittings are associated with stationary iterative methods, and are frequently used as preconditioners for Krylov subspace

*Received May 18, 2009. Accepted for publication September 6, 2009. Published online January 15, 2010. Recommended by D. Szyld.
†Department of Mathematics of School of Science, Xi'an Jiaotong University, Xi'an, Shaanxi 710049, P.R. China (zhangchengyi_2004@163.com).
‡Department of Mathematics and Computer Science, Emory University, Atlanta, GA 30322, USA (benzi@mathcs.emory.edu). The work of this author was supported in part by the National Science Foundation grant DMS-0511336.

methods or as smoothers for multigrid or Schwarz-type schemes; see, e.g., [20, 31, 38]. In general, the coefficient matrix $A \in \mathbb{C}^{n\times n}$ is split into

$$A = M - N, \tag{1.2}$$

where $M \in \mathbb{C}^{n\times n}$ is nonsingular and $N \in \mathbb{C}^{n\times n}$. Then, the general form of stationary iterative methods for (1.1) can be described as follows:

$$x^{(i+1)} = M^{-1}Nx^{(i)} + M^{-1}b, \quad i = 0, 1, 2, \dots . \tag{1.3}$$

The matrix $T = M^{-1}N$ is called the iteration matrix of the method (1.3). It is well known [34] that (1.3) converges for any given $x^{(0)}$ if and only if $\rho(T) < 1$, where $\rho(T)$ denotes the spectral radius of the matrix T. Thus, to establish convergence results for stationary iterative methods, we need to study the spectral radius of the iteration matrix in (1.3).

Next, consider the general class of alternating iterative methods for the solution of (1.1) of the form

$$\begin{cases} x^{(i+1/2)} = M^{-1}Nx^{(i)} + M^{-1}b \\ x^{(i+1)} = P^{-1}Qx^{(i+1/2)} + P^{-1}b \end{cases}, \quad i = 0, 1, 2, \dots, \tag{1.4}$$

where $A = M - N = P - Q$ are splittings of the coefficient matrix A. Many well known iterative schemes such as the symmetric Gauss-Seidel method [1], the SSOR method [33], alternating-direction and implicit (ADI) methods [26, 34, 38], the Hermitian/skew-Hermitian splitting (HSS) methods [4, 8] and several others belong to this class of methods. To analyze the convergence of the general scheme (1.4), Benzi and Szyld [14] construct a single splitting $A = B - C$ associated with the iteration matrix as follows. Eliminating $x^{(i+1/2)}$ from (1.4), we obtain the iterative process

$$x^{(i+1)} = P^{-1}QM^{-1}Nx^{(i)} + P^{-1}(QM^{-1} + I)b, \quad i = 0, 1, 2, \dots, \tag{1.5}$$

which is of the form (1.3), where now $T = P^{-1}QM^{-1}N$ is the iteration matrix. If A is nonsingular and 1 is not an eigenvalue of T, then there exists a unique splitting $A = B - C$ such that $T = B^{-1}C = I - B^{-1}A$. It is not difficult to see that $M + P - A$ is necessarily invertible and that $B = M(M + P - A)^{-1}P$. The splitting $A = B - C$ is said to be *induced* by T; see [14] for details.

There have been several studies on the convergence of splitting iterative methods for non-Hermitian positive definite linear systems. In [15, pages 190–193], some convergence conditions for the splitting of non-Hermitian positive definite matrices have been established. More recently, [35] and [36] give some conditions for the convergence of splittings for this class of linear systems.

Recently, there has been considerable interest in the Hermitian and skew-Hermitian splitting (HSS) method introduced by Bai, Golub and Ng for solving non-Hermitian positive definite linear systems, see [4]; we further note the generalizations and extensions of this basic

ETNA
Kent State University
http://etna.math.kent.edu

method proposed in [5, 7, 8, 3, 6] and [25]. Furthermore, these methods and their convergence theories have been shown to apply to (generalized) saddle point problems, either directly or indirectly (as a preconditioner); see [5, 2, 3, 7, 6, 35, 36, 25, 11, 12].

Continuing in this direction, in this paper we establish new results on splitting methods for solving system (1.1) iteratively, focusing on a particular class of splittings. For a given matrix $A \in \mathbb{C}^{n\times n}$, a splitting $A = M - N$ with M nonsingular is called a *P-regular splitting* if the matrix $M^* + N$ is non-Hermitian positive definite; see [29]. It is a well-known result [37, 29] that if A is Hermitian positive definite and $A = M - N$ is a P-regular splitting, then the splitting iterative method is convergent: $\rho(M^{-1}N) < 1$. In this paper, we examine the spectral properties of the iteration matrix induced by P-regular splittings of a non-Hermitian positive definite matrix. Based on these properties, we construct various SOR-type methods for non-Hermitian linear systems and prove their convergence under appropriate restrictions on the choice of the relaxation parameter. While convergence results have been known for many years for Hermitian positive definite matrices, monotone matrices and H-matrices (see, e.g., [15, 20, 31, 38, 29, 21, 34]), very little appears to be known in the non-Hermitian positive definite case. Among the few studies known to us we mention [15, pages 194–195], [27], [28], and [24]. Our results are more general than the few results found in literature, and they complete the SOR theory for non-Hermitian matrices. It is our hope that these results will prove useful in the study of convergence of more sophisticated iterative schemes, including Schwarz-type and algebraic multilevel methods; see, e.g., [19] and [10].

For convenience, some of the terminology used in this paper will be given. The symbol $\mathbb{C}^{n\times n}$ will denote the set of all $n\times n$ complex matrices. Let $A,\ B \in \mathbb{C}^{n\times n}$. We use the notation $A \succ 0$ ($A \succeq 0$) if A is Hermitian positive (semi-)definite. If A and B are both Hermitian, we write $A \succ B$ ($A \succeq B$) if and only if $A - B \succ 0$ ($A - B \succeq 0$). If A is Hermitian all the eigenvalues of A are real, and we denote by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ the smallest (i.e., leftmost) and largest (rightmost) eigenvalues, respectively. Let $A \in \mathbb{C}^{n\times n}$ with $H = (A + A^*)/2$ and $S = (A - A^*)/2$ its Hermitian and skew-Hermitian parts, respectively; then A is non-Hermitian positive (semi-)definite if and only if $H \succ 0$ ($H \succeq 0$). Throughout the paper, I will denote the $n \times n$ identity matrix.

The paper is organized as follows. Some convergence results for P-regular splittings of non-Hermitian positive definite linear systems are given in section 2. In section 3 we construct some SOR-type methods and use the general theory of section 2 to study their convergence. In section 4 a few numerical examples are given to demonstrate the convergence results obtained in this paper. Some conclusions are given in section 5.

2. General convergence results for *P*-regular splittings. In this section we establish some convergence results for P-regular splitting methods for non-Hermitian positive definite linear systems. First, some lemmas will be presented to be used in the sequel.

LEMMA 2.1. *Let $H,\ B \in \mathbb{C}^{n\times n}$ be Hermitian and let $S \in \mathbb{C}^{n\times n}$ be skew-Hermitian. If $H \succ 0$, then $\rho[(H+S)^{-1}B] \le \rho(H^{-1}B)$.*

Proof. Since $H \succ 0$, $H^{-1} \succ 0$ and $H^{-1/2} \succ 0$, it follows that $H^{-1}B$ is similar to the Hermitian matrix $H^{-1/2}BH^{-1/2}$. As a result,

$$\rho(H^{-1}B) = \rho(H^{-1/2}BH^{-1/2}) = \max_{\|x\|_2=1} |x^*H^{-1/2}BH^{-1/2}x|. \tag{2.1}$$

Similarly, $(H+S)^{-1}B$ is similar to the matrix

$$H^{1/2}(H+S)^{-1}BH^{-1/2} = (I + H^{-1/2}SH^{-1/2})^{-1}H^{-1/2}BH^{-1/2}.$$

Hence, $(H+S)^{-1}B$ and $(I + H^{-1/2}SH^{-1/2})^{-1}H^{-1/2}BH^{-1/2}$ have the same eigenvalues and therefore

$$\rho((H+S)^{-1}B) = \rho[(I + H^{-1/2}SH^{-1/2})^{-1}H^{-1/2}BH^{-1/2}].$$

Let λ be an eigenvalue of $(I + H^{-1/2}SH^{-1/2})^{-1}H^{-1/2}BH^{-1/2}$ satisfying $|\lambda| = \rho((H+S)^{-1}B)$ and let x (with $\|x\|_2 = 1$) be a corresponding eigenvector. Then, one has

$$(I + H^{-1/2}SH^{-1/2})^{-1}H^{-1/2}BH^{-1/2}x = \lambda x$$

and consequently

$$H^{-1/2}BH^{-1/2}x = \lambda(I + H^{-1/2}SH^{-1/2})x$$

and

$$\lambda = \frac{x^*H^{-1/2}BH^{-1/2}x}{x^*(I + H^{-1/2}SH^{-1/2})x} = \frac{x^*H^{-1/2}BH^{-1/2}x}{1 + x^*(H^{-1/2}SH^{-1/2})x}.$$

Since S is skew-Hermitian, so is $H^{-1/2}SH^{-1/2}$. As a result, $x^*(H^{-1/2}SH^{-1/2})x$ is either purely imaginary. Thus,

$$|1 + x^*(H^{-1/2}SH^{-1/2})x| = \sqrt{1 + |x^*(H^{-1/2}SH^{-1/2})x|^2} \ge 1.$$

Therefore,

$$\begin{aligned} \rho((H+S)^{-1}B) &= |\lambda| = \frac{|x^*H^{-1/2}BH^{-1/2}x|}{|1 + x^*(H^{-1/2}SH^{-1/2})x|} \\ &\le |x^*H^{-1/2}BH^{-1/2}x| \le \max_{\|x\|_2=1} |x^*H^{-1/2}BH^{-1/2}x| \\ &= \rho(H^{-1}B), \end{aligned} \tag{2.2}$$

ETNA
Kent State University
http://etna.math.kent.edu

which completes the proof. □

LEMMA 2.2. (See Ortega [29, page 123].) *Let $A \succ 0$, and let $A = M - N$ be a P-regular splitting. Then $\rho(M^{-1}N) < 1$.*

THEOREM 2.3. *Let $A \in \mathbb{C}^{n\times n}$ be non-Hermitian positive definite, and let $A = M - N$ be a P-regular splitting with N Hermitian. Then $\rho(M^{-1}N) < 1$.*

Proof. Let $H(A) = (A + A^*)/2$ and $S(A) = (A - A^*)/2$ be the Hermitian and skew-Hermitian parts of A, respectively, and let $H(M) = (M + M^*)/2$ be the Hermitian part of M. Non-Hermitian positive definiteness of A gives that $H(A) \succ 0$. Since N is Hermitian, the skew-Hermitian part of M coincides with the skew-Hermitian part of A:

$$S(M) = \frac{1}{2}(M - M^*) = \frac{1}{2}[(M - N) - (M - N)^*] = \frac{1}{2}(A - A^*) = S(A),$$

and $H(A) = H(M) - N \succ 0$. Again, $A = M - N$ is a P-regular splitting and thus $M^* + N$ is positive definite, consequently $H(M) + N \succ 0$. Therefore, $H(M) \succ 0$ and $H(A) = H(M) - N$ is a P-regular splitting. Lemma 2.2 shows $\rho[(H(M))^{-1}N] < 1$. Since $H(M) \succ 0$, N is Hermitian and $S(M)$ is skew-Hermitian, it follows from Lemma 2.1 that

$$\rho(M^{-1}N) = \rho[(H(M) + S(M))^{-1}N] \leq \rho[(H(M))^{-1}N] < 1. \tag{2.3}$$

This completes the proof. □

COROLLARY 2.4. *Let $A \in \mathbb{C}^{n\times n}$ be non-Hermitian positive definite, and let $A = M - N$ be a splitting with $N \succeq 0$. Then $\rho(M^{-1}N) < 1$.*

REMARK 2.5. *In the last two results, the condition that N be Hermitian is essential and cannot be relaxed. An obvious example is $A = I - S$ where $S = -S^*$ and $\|S\|_2 \geq 1$. Setting $M = I$ and $N = S$ leads to a P-regular splitting where N is non-Hermitian positive semidefinite and $\rho(M^{-1}N) \geq 1$.*

REMARK 2.6. *In the Hermitian case, Lemma 2.2 has the following converse: if $A = A^* = M - N$ is a P-regular splitting and $\rho(M^{-1}N) < 1$, then A is positive definite; see [30, page 255]. It is therefore natural to ask whether the converse of Theorem 2.3 holds. That is, given a P-regular splitting $A = M - N$ with $N = N^*$ and $\rho(M^{-1}N) < 1$, is it true that $H(A) = \frac{1}{2}(A + A^*)$ is positive definite? The answer is negative, as is shown by the splitting $A = M - N$ where*

$$A = \begin{bmatrix} 0 & -4 \\ 4 & -1 \end{bmatrix}, \quad M = \begin{bmatrix} 2 & -4 \\ 4 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

This splitting is P-regular, $N = N^*$, and $\rho(M^{-1}N) < 1$; the Hermitian part of the matrix A, however, is not positive definite.

ETNA
Kent State University
http://etna.math.kent.edu

Next, we consider the convergence of the iterative scheme (1.4) or (1.5) for non-Hermitian positive definite linear systems. In [14] the following convergence result for symmetric positive definite linear systems is proved.

THEOREM 2.7. (See [14].) *Let $A \in \mathbb{R}^{n\times n}$ be symmetric positive definite, and let $A = M - N = P - Q$ be both P-regular splittings. Then $\rho(T) < 1$, where $T = P^{-1}QM^{-1}N$, and therefore the sequence $\{x^{(i)}\}$ generated by (1.4) converges to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$. Furthermore, the unique splitting $A = B - C$ induced by T is P-regular.*

In what follows, we partially generalize this result to non-Hermitian positive definite linear systems. First, some useful lemmas are introduced.

LEMMA 2.8. (See Corollary 7.6.5 in [22].) *Let $A, B \in \mathbb{C}^{n\times n}$ be Hermitian with $A \succ 0$. Then there exists a nonsingular matrix $C \in \mathbb{C}^{n\times n}$ such that $A = C^*C$ and $B = C^*DC$, where $D \in \mathbb{R}^{n\times n}$ is diagonal.*

LEMMA 2.9. *Let $B = C^*DC \in \mathbb{C}^{n\times n}$ with $C \in \mathbb{C}^{n\times n}$ nonsingular and $D = \operatorname{diag}(d_1, \dots, d_n) \in \mathbb{C}^{n\times n}$, and let $\tilde{B} = C^*|D|C \in \mathbb{C}^{n\times n}$ with $|D| = \operatorname{diag}(|d_1|, \dots, |d_n|)$. Then the Hermitian matrix $\mathscr{B} = \begin{bmatrix} \tilde{B} & B \\ B^* & \tilde{B} \end{bmatrix}$ is positive semidefinite.*

Proof. Observe that $\mathscr{B}$ can be decomposed as

$$
\begin{aligned}
\mathscr{B} &= \begin{bmatrix} \tilde{B} & B \\ B^* & \tilde{B} \end{bmatrix} = \begin{bmatrix} C^* & 0 \\ 0 & C^* \end{bmatrix} \begin{bmatrix} |D| & D \\ D^* & |D| \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \\
&= \mathscr{C}^* \begin{bmatrix} |D| & D \\ D^* & |D| \end{bmatrix} \mathscr{C},
\end{aligned}
\tag{2.4}
$$

where $\mathscr{C} = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix}$ is nonsingular since C is. Writing $\mathscr{D} = \begin{bmatrix} |D| & D \\ D^* & |D| \end{bmatrix}$, (2.4) shows that the Hermitian matrices $\mathscr{B}$ and $\mathscr{D}$ are congruent, and therefore they must have the same inertia. Hence, all we need to show is that $\mathscr{D}$ is positive semidefinite. Letting $\mathscr{P}$ denote the odd-even permutation matrix of order $2n$, it is immediate to see that

$$
\mathscr{P}^*\mathscr{D}\mathscr{P} = \begin{bmatrix} |d_1| & d_1 \\ \bar{d}_1 & |d_1| \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} |d_n| & d_n \\ \bar{d}_n & |d_n| \end{bmatrix}.
$$

Hence, $\mathscr{P}^*\mathscr{D}\mathscr{P}$ is just a direct sum of n two-by-two Hermitian matrices, each of which is obviously positive semidefinite. This shows that $\mathscr{D} \succeq 0$, and the proof is complete. □

LEMMA 2.10. *Let $A_i, B_i \in \mathbb{C}^{n\times n}$ be Hermitian and such that $A_i \succ B_i \succeq 0$ for $i = 1, 2$. Then there exist positive real numbers e_1, e_2 such that $2e_1A_1 \succ e_1B_1 + e_2B_2$ and $2e_2A_2 \succ e_1B_1 + e_2B_2$.*

Proof. Let $\mathscr{L} = \begin{bmatrix} A_1 & -B_1 \\ -B_2 & A_2 \end{bmatrix}$. Since $A_i \succ B_i \succeq 0$ for $i = 1, 2$, it follows that

ETNA
Kent State University
http://etna.math.kent.edu

$\mathscr{L}$ is a generalized M-matrix in the sense of Elsner and Mehrmann; see [18, Notation 2.3] and [23] for details. Therefore, $\mathscr{L}^*$ is also a generalized M-matrix, and consequently, [18, Notation 2.3] implies that there exist positive real numbers e_1, e_2 such that $e_1 A_1 - e_2 B_2 \succ 0$ and $e_2 A_2 - e_1 B_1 \succ 0$. Observe that $A_i \succ B_i \succeq 0$ implies $e_i A_i - e_i B_i \succ 0$ for $i = 1, 2$. Therefore, we have $2e_1 A_1 \succ e_1 B_1 + e_2 B_2$ and $2e_2 A_2 \succ e_1 B_1 + e_2 B_2$. This completes the proof. □

LEMMA 2.11. *Let $A \in \mathbb{C}^{n\times n}$ be non-Hermitian positive definite, and let $A = M - N = P - Q$ be both P-regular splittings with N and Q Hermitian. Then the matrix $\mathscr{H}_\mu = \begin{bmatrix} M & N \\ \mu Q & P \end{bmatrix}$ is nonsingular for all $\mu \in \mathbb{C}$ with $|\mu| \le 1$.*

Proof. Let $H(M)$ and $H(P)$ be the Hermitian parts of M and P, respectively. Since A is non-Hermitian positive definite and $A = M - N = P - Q$ are both P-regular splittings with N and Q Hermitian, one has

$$\begin{aligned} &H(M) + N \succ 0, \;\; H(M) - N \succ 0; \\ &H(P) + Q \succ 0, \;\; H(P) - Q \succ 0. \end{aligned} \tag{2.5}$$

Clearly, (2.5) implies that $H(M) \succ 0$ and $H(P) \succ 0$. Also, N and Q are both Hermitian. It follows from Lemma 2.8 that there exist two nonsingular matrices $C_1, C_2 \in \mathbb{C}^{n\times n}$ such that $H(M) = C_1^* C_1$, $N = C_1^* D_1 C_1$ and $H(P) = C_2^* C_2$, $Q = C_2^* D_2 C_2$, where $D_1, D_2 \in \mathbb{R}^{n\times n}$ are diagonal matrices. Following (2.5), we have

$$\begin{aligned} &C_1^*(I + D_1)C_1 \succ 0, \;\; C_1^*(I - D_1)C_1 \succ 0; \\ &C_2^*(I + D_2)C_2 \succ 0, \;\; C_2^*(I - D_2)C_2 \succ 0. \end{aligned} \tag{2.6}$$

Consequently,

$$\begin{aligned} &I + D_1 \succ 0, \;\; I - D_1 \succ 0; \\ &I + D_2 \succ 0, \;\; I - D_2 \succ 0, \end{aligned} \tag{2.7}$$

which shows that

$$I - |D_1| \succ 0, \;\; I - |D_2| \succ 0. \tag{2.8}$$

Let $\tilde{N} = C_1^* |D_1| C_1$ and $\tilde{Q}_\mu = C_2^* |\mu D_2| C_2$ for $\mu \in \mathbb{C}$ with $|\mu| \le 1$. Then $\tilde{N} \succeq 0$ and $\tilde{Q}_\mu \succeq 0$. Furthermore,

$$\begin{aligned} &H(M) - \tilde{N} = C_1^*(I - |D_1|)C_1 \succ 0; \\ &H(P) - \tilde{Q}_\mu = C_2^*(I - |\mu D_2|)C_2 \succ 0. \end{aligned} \tag{2.9}$$

This leads to $H(M) \succ \tilde{N} \succeq 0$ and $H(P) \succ \tilde{Q}_\mu \succeq 0$. It then follows from Lemma 2.10 that there exist positive real numbers e_1, e_2 such that

$$\begin{aligned} &2e_1 H(M) \succ e_1 \tilde{N} + e_2 \tilde{Q}_\mu \succeq 0; \\ &2e_2 H(P) \succ e_1 \tilde{N} + e_2 \tilde{Q}_\mu \succeq 0. \end{aligned} \tag{2.10}$$

ETNA
Kent State University
http://etna.math.kent.edu

Letting $\mathscr{E} = \mathrm{diag}(e_1 I, e_2 I)$, we have

$$\mathscr{E}\mathscr{H}_\mu + (\mathscr{E}\mathscr{H}_\mu)^* = \begin{bmatrix} 2e_1 H(M) & e_1 N + e_2\bar{\mu} Q \\ e_1 N + e_2 \mu Q & 2e_2 H(P) \end{bmatrix}. \tag{2.11}$$

Let $K_1 = 2e_1 H(M) - (e_1\tilde{N} + e_2\tilde{Q}_\mu)$, $K_2 = 2e_2 H(P) - (e_1\tilde{N} + e_2\tilde{Q}_\mu)$ and $\mathscr{K} = \mathrm{diag}(K_1, K_2)$. Then (2.10) yields $K_1 \succ 0$ and $K_2 \succ 0$. As a result, $\mathscr{K} \succ 0$. Letting $\mathscr{N} = \begin{bmatrix} \tilde{N} & N \\ N & \tilde{N} \end{bmatrix}$ and $\mathscr{Q}_\mu = \begin{bmatrix} \tilde{Q}_\mu & \bar{\mu}Q \\ \mu Q & \tilde{Q}_\mu \end{bmatrix}$, Lemma 2.9 shows that $\mathscr{N} \succeq 0$ and $\mathscr{Q}_\mu \succeq 0$. Therefore,

$$\begin{aligned} \mathscr{E}\mathscr{H}_\mu + (\mathscr{E}\mathscr{H}_\mu)^* &= \begin{bmatrix} 2e_1 H(M) & e_1 N + e_2\bar{\mu} Q \\ e_1 N + e_2 \mu Q & 2e_2 H(P) \end{bmatrix} \\ &= \mathscr{K} + e_1\mathscr{N} + e_2\mathscr{Q}_\mu \\ &\succ 0, \end{aligned} \tag{2.12}$$

i.e., $\mathscr{E}\mathscr{H}_\mu$ is non-Hermitian positive definite and thus nonsingular. Since $\mathscr{E}$ is nonsingular, $\mathscr{H}_\mu$ is nonsingular. This completes the proof. □

THEOREM 2.12. *Let $A \in \mathbb{C}^{n\times n}$ be non-Hermitian positive definite, and let $A = M - N = P - Q$ be both P-regular splittings with N and Q Hermitian. Then $\rho(T) < 1$, where $T = P^{-1}QM^{-1}N$, and therefore the sequence $\{x^{(i)}\}$ generated by* (1.4) *converges to the unique solution of* (1.1) *for any choice of the initial guess $x^{(0)}$.*

Proof. The proof is by contradiction. We assume that λ is an eigenvalue of T with $|\lambda| \geq 1$. Then $\lambda I - T = \lambda I - P^{-1}QM^{-1}N$ is singular. As a result, $P - (\lambda^{-1}Q)M^{-1}N$ is singular. Let $\mu = \lambda^{-1}$, then $|\mu| \leq 1$ and $P - (\mu Q)M^{-1}N = P - (\lambda^{-1}Q)M^{-1}N$ is singular. Observe that $S = P - (\mu Q)M^{-1}N = \mathscr{H}_\mu / M$, the Schur complement of the matrix $\mathscr{H}_\mu = \begin{bmatrix} M & N \\ \mu Q & P \end{bmatrix}$ with respect to the matrix M. It follows from the block LU decomposition [39]

$$\mathscr{H}_\mu = \begin{bmatrix} I & 0 \\ \mu Q M^{-1} & I \end{bmatrix} \begin{bmatrix} M & N \\ 0 & S \end{bmatrix}$$

that $\mathscr{H}_\mu$ must be singular. This contradicts Lemma 2.11, according to which matrix $\mathscr{H}_\mu$ is nonsingular for $|\mu| \leq 1$. Therefore T has no eigenvalue λ with $|\lambda| \geq 1$; that is, $\rho(T) < 1$ and $T = P^{-1}QM^{-1}N$ is convergent. This completes the proof. □

REMARK 2.13. *It remains an open question whether the unique splitting $A = B - C$ induced by T in Theorem 2.12 is P-regular.*

3. SOR methods for non-Hermitian positive definite systems. In this section we apply the general theory developed in the previous section to study the convergence of SOR-like methods applied to non-Hermitian positive definite systems.

Without loss of generality, we write

$$A = I - L - U = (I - L + U^*) - (U + U^*) = (I - U + L^*) - (L + L^*), \tag{3.1}$$

where L and U are strictly lower and strictly upper triangular matrices, respectively. The successive over-relaxation method (SOR method) is defined by the iteration matrix

$$\mathscr{L}_\omega = [I - \omega(L - U^*)]^{-1}[\omega(U + U^*) + (1 - \omega)I], \tag{3.2}$$

while the unsymmetric SOR method (USSOR method) is given by the iteration matrix

$$\mathscr{I}_{\omega,\bar{\omega}} = \mathscr{U}_{\bar{\omega}}\mathscr{L}_\omega, \tag{3.3}$$

where

$$\mathscr{U}_{\bar{\omega}} = [I - \bar{\omega}(U - L^*)]^{-1}[\bar{\omega}(L + L^*) + (1 - \bar{\omega})I]. \tag{3.4}$$

As a special case, when $\omega = \bar{\omega}$ we have the symmetric SOR method (SSOR method), defined by the iteration matrix

$$\mathscr{J}_\omega = \mathscr{U}_\omega\mathscr{L}_\omega. \tag{3.5}$$

THEOREM 3.1. *Let $A \in \mathbb{C}^{n\times n}$ be non-Hermitian positive definite with $H = (A + A^*)/2$ its Hermitian part, and let $A = I - L - U$ be defined by (3.1). Also, let $\eta = \lambda_{\min}(B)$ be the smallest eigenvalue of $B := H + 2(U + U^*)$.*

(i) *If $\eta \geq 0$, then the SOR method is convergent for $\omega \in (0, 1)$;*

(ii) *If $\eta < 0$, then the SOR method is convergent for $\omega \in (0, \frac{2}{2-\eta})$.*

Proof. Let $M = \frac{1}{\omega}I - (L - U^*)$ and $N = \left(\frac{1}{\omega} - 1\right) I + (U + U^*)$. Then $\mathscr{L}_\omega = M^{-1}N$ and $A = M - N$ is a splitting of A since M is nonsingular. Let $H(M) = (M + M^*)/2$. Since N is Hermitian,

$$\begin{aligned} H(M) + N &= H + 2N = \frac{2 - 2\omega}{\omega}I + H + 2(U + U^*) \\ &= \frac{2 - 2\omega}{\omega}I + B. \end{aligned} \tag{3.6}$$

(i) If $\eta \geq 0$ and $\omega \in (0, 1)$, then we have $B \succeq 0$ and $\frac{2 - 2\omega}{\omega} > 0$. Identity (3.6) shows $H(M) + N = \frac{2 - 2\omega}{\omega}I + B \succ 0$; that is, $M + N$ is positive definite. Therefore, $A = M - N$ is a *P*-regular splitting of A. Hence, Theorem 2.3 yields that $\rho(\mathscr{L}_\omega) = \rho(M^{-1}N) < 1$, i.e., the SOR method is convergent.

(ii) If $\eta < 0$ and $\omega \in (0, \frac{2}{2-\eta})$, then we have with (3.6) that

$$
\begin{aligned}
H(M) + N &= \frac{2-2\omega}{\omega} I + B \\
&\succ -\eta I + B \succeq 0,
\end{aligned} \tag{3.7}
$$

which shows that $M+N$ is positive definite. As a result, $A = M - N$ is a *P*-regular splitting of A. It follows again from Theorem 2.3 that $\rho(\mathscr{L}_\omega) = \rho(M^{-1}N) < 1$, i.e., the SOR method is convergent. This completes the proof. □

REMARK 3.2. *Theorem 3.1 becomes Theorem 1 in [27] if $A = I - L + L^T \in \mathbb{R}^{n\times n}$; hence, Theorem 3.1 generalizes the convergence result of Niethammer and Schade.*

THEOREM 3.3. *Let $A \in \mathbb{C}^{n\times n}$ be non-Hermitian positive definite with $H = (A + A^*)/2$ its Hermitian part, and let $A = I - L - U$ be defined by (3.1) and $\eta = \lambda_{\min}(B)$ and $\mu = \lambda_{\min}(C)$ be the smallest eigenvalues of $B := H + 2(U + U^*)$ and $C := H + 2(L + L^*)$, respectively.*

(i) *If $\eta \geq 0$ and $\mu \geq 0$, then the USSOR method is convergent for $\omega, \bar{\omega} \in (0, 1)$;*

(ii) *If $\eta < 0$ and $\mu \geq 0$, then the USSOR method is convergent for $\omega \in (0, \frac{2}{2-\eta})$ and $\bar{\omega} \in (0, 1)$;*

(iii) *If $\eta \geq 0$ and $\mu < 0$, then the USSOR method is convergent for $\omega \in (0, 1)$ and $\bar{\omega} \in (0, \frac{2}{2-\mu})$;*

(iv) *If $\eta < 0$ and $\mu < 0$, then the USSOR method is convergent for $\omega \in (0, \frac{2}{2-\eta})$ and $\bar{\omega} \in (0, \frac{2}{2-\mu})$.*

Proof. Let $M = \dfrac{1}{\omega} I - (L - U^*)$, $N = \left(\dfrac{1}{\omega} - 1\right) I + (U + U^*)$ and $P = \dfrac{1}{\bar{\omega}} I - (U - L^*)$, $Q = \left(\dfrac{1}{\bar{\omega}} - 1\right) I + (L + L^*)$. Then M and P are nonsingular, N and Q are Hermitian, $\mathscr{L}_\omega = M^{-1}N$, $\mathscr{M}_{\bar{\omega}} = P^{-1}Q$, and $A = M - N = P - Q$ are splittings of A. Let $H(M) = (M + M^*)/2$ and $H(P) = (P + P^*)/2$. Since N and Q are Hermitian, (3.6) holds. Furthermore,

$$
\begin{aligned}
H(P) + Q &= H + 2Q = \frac{2 - 2\bar{\omega}}{\bar{\omega}} I + H + 2(L + L^*) \\
&= \frac{2 - 2\bar{\omega}}{\bar{\omega}} I + C.
\end{aligned} \tag{3.8}
$$

It is easy to prove that both $H(M) + N \succ 0$ and $H(P) + Q \succ 0$ when (i) $\eta \geq 0$, $\mu \geq 0$ and $\omega, \bar{\omega} \in (0, 1)$; (ii) $\eta < 0$, $\mu \geq 0$ and $\omega \in (0, \frac{2}{2-\eta})$, $\bar{\omega} \in (0, 1)$; (iii) $\eta \geq 0$, $\mu < 0$ and $\omega \in (0, 1)$, $\bar{\omega} \in (0, \frac{2}{2-\mu})$; and (iv) $\eta < 0$, $\mu < 0$ and $\omega \in (0, \frac{2}{2-\eta})$, $\bar{\omega} \in (0, \frac{2}{2-\mu})$. Therefore, both $M + N$ and $P + Q$ are positive definite and consequently $A = M - N = P - Q$ are

ETNA
Kent State University
http://etna.math.kent.edu

P-regular splittings with N and Q Hermitian. Then Theorem 2.12 shows that

$$\rho(\mathscr{I}_{\omega,\bar{\omega}}) = \rho(\mathscr{U}_{\bar{\omega}}\mathscr{L}_{\omega}) = \rho(P^{-1}QM^{-1}N) = \rho(T) < 1,$$

i.e., the USSOR method is convergent. This completes the proof. □

THEOREM 3.4. *Let $A \in \mathbb{C}^{n\times n}$ be non-Hermitian positive definite with $H = (A + A^*)/2$ its Hermitian part, and let $A = I - L - U$ be defined by (3.1) and $\eta = \lambda_{\min}(B)$ and $\mu = \lambda_{\min}(C)$ be the smallest eigenvalues of $B := H + 2(U + U^*)$ and $C := H + 2(L + L^*)$, respectively.*

(i) *If $\eta \geq 0$ and $\mu \geq 0$, then the SSOR method is convergent for $\omega \in (0, 1)$;*

(ii) *If either $\eta \leq \mu < 0$ or $\eta < 0 \leq \mu$, then the SSOR method is convergent for $\omega \in (0, \frac{2}{2-\eta})$;*

(iii) *If either $\mu \leq \eta < 0$ or $\mu < 0 \leq \eta$, then the SSOR method is convergent for $\omega \in (0, \frac{2}{2-\mu})$.*

Proof. The proof can be immediately obtained from Theorem 3.3. □

4. Numerical experiments. In this section we describe the results of some numerical experiments with the SOR method on a set of linear systems arising from a finite element discretization of a convection-diffusion equation in two dimensions. The purpose of these experiments is not to advocate the use of SOR as a solver for this particular type of problem, but to illustrate the theory developed in this paper, in particular Theorem 3.1.

The model problem is the partial differential equation

$$-\,\varepsilon\Delta u + \mathbf{w}\cdot\nabla u = f\,, \tag{4.1}$$

where $\varepsilon > 0$, Δ is the 2D Laplacian, ∇ is the gradient, $\mathbf{w}$ is a prescribed vector field (the 'wind'), and f is a given scalar field (the 'source'). The solution u is sought on the unit square $\Omega = [0,1] \times [0,1]$, and is subject to suitable boundary conditions. Here we consider the problem given as Example 3.1.3 in [17]: zero source ($f \equiv 0$), constant wind at a 30^o angle to the left of vertical ($\mathbf{w} = (-\sin\frac{\pi}{6}, \cos\frac{\pi}{6})$), and boundary conditions such that the solution exhibits a downstream boundary layer and an interior layer; see [17, page 118] for details.

Equation (4.1) is discretized on a uniform square grid of size 32×32 using Q1 Galerkin finite elements with SUPG stabilization. The resulting matrix A is nonsymmetric and has complex eigenvalues. Its symmetric part H is positive definite, for all $\varepsilon > 0$. We note that A has some positive off-diagonal entries and therefore it is not an M-matrix. Prior to forming the SOR splitting, the coefficient matrix A is diagonally scaled so that its diagonal entries are all equal to 1, hence $A = I - L - U$ with L strictly lower and U strictly upper triangular.

We consider three problem instances, corresponding to $\varepsilon = 10^{-1}$, 10^{-2} and 10^{-3}, respectively. The problems becomes increasingly convection-dominated as ε decreases. In Table

ETNA
Kent State University
http://etna.math.kent.edu

TABLE 4.1
Values of η, $2/(2-\eta)$ and GMRES iterations for different values of ε.

	$\varepsilon = 10^{-1}$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$
η	-2.033	-2.406	-2.646
$\frac{2}{2-\eta}$	0.496	0.454	0.430
ω_{best}	0.57	0.55	0.51
GMRES	53	48	52

4.1 we report the value of $\eta = \lambda_{\min}(B)$, with $B = H + 2(U + U^T)$, together with the corresponding value of $2/(2-\eta)$ for the three values of ε considered. Recall that according to Theorem 3.1, when $\eta < 0$ (as is the case here) the SOR method is guaranteed to converge for all $\omega \in (0, 2/(2-\eta))$. This is, however, a sufficient condition only. In practice, we found that SOR converges for $\omega \in (0, \bar{\omega})$ where $\bar{\omega}$ is typically somewhat larger than $2/(2-\eta)$. In all three cases, the Gauss–Seidel method ($\omega = 1$) was found to diverge. Since $0 < \omega < 1$, the SOR method used here is actually an under-relaxation procedure rather than an over-relaxation one. In Table 4.1 we also report the optimal value ω_{best} of the relaxation parameter ω in the SOR method, determined experimentally (to two digits of accuracy). Finally, as a baseline method we report in Table 4.1 the number of (unpreconditioned) full GMRES [32] iterations. In all our experiments, we report the number of iterations required to reduce the initial residual by five orders of magnitude, starting from a zero initial guess.

In Table 4.2 we report (under 'its') the number of SOR iterations required to solve the three linear systems with the SOR method for two distinct choices of the relaxation parameter, namely, for $\omega = 2/(2-\eta)$ and $\omega = \omega_{best}$. We also include (under 'G-its') the number of iterations required by preconditioned GMRES, where the preconditioner is the SOR method with the corresponding value of ω. We note that GMRES acceleration is generally not very effective, and sometimes counterproductive. For a discussion of the use of SOR as a preconditoner for Krylov subspace methods; see [16].

Finally, in Table 4.3 we show iteration counts for SOR and SOR-preconditioned GMRES for several values of ω. We note that for $\omega \geq 0.7$, SOR diverges for all three problems. (For $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-3}$, the SOR iteration is already divergent for $\omega \geq 0.6$.) The results show that the rate of convergence suffers some deterioration as ε decreases. The results also show that GMRES acceleration with suboptimal values can be beneficial; however, the reduction in the number of iterations compared to unpreconditioned GMRES (see Table 4.1) is rather disappointing. In practice, using SOR (with the optimal ω) without GMRES acceleration is more effective, in terms of total costs, than using either SOR-preconditioned GMRES or unpreconditioned GMRES; the exception is the case $\varepsilon = 0.1$, where GMRES preconditioned with the Gauss–Seidel method converges very rapidly. This method, however, behaves poorly for smaller values of ε.

ETNA
Kent State University
http://etna.math.kent.edu

TABLE 4.2
Results for $\omega = \frac{2}{2-\eta}$ and for $\omega = \omega_{best}$.

	$\varepsilon = 10^{-1}$			$\varepsilon = 10^{-2}$			$\varepsilon = 10^{-3}$		
ω	$\rho(\mathscr{L}_\omega)$	its	G-its	$\rho(\mathscr{L}_\omega)$	its	G-its	$\rho(\mathscr{L}_\omega)$	its	G-its
$\omega = \frac{2}{2-\eta}$	0.622	32	35	0.735	48	40	0.776	58	44
$\omega = \omega_{best}$	0.581	27	33	0.705	43	39	0.757	52	47

TABLE 4.3
Results for different values of ω .

	$\varepsilon = 10^{-1}$			$\varepsilon = 10^{-2}$			$\varepsilon = 10^{-3}$		
ω	$\rho(\mathscr{L}_\omega)$	its	G-its	$\rho(\mathscr{L}_\omega)$	its	G-its	$\rho(\mathscr{L}_\omega)$	its	G-its
0.1	0.906	176	44	0.913	176	44	0.918	182	45
0.2	0.822	83	43	0.848	92	43	0.860	100	44
0.3	0.748	53	41	0.796	66	42	0.818	74	43
0.4	0.681	39	38	0.754	52	40	0.785	61	43
0.5	0.620	31	35	0.720	48	39	0.759	53	47
0.6	0.565	32	32	>1	∞	39	>1	∞	58
1.0	>1	∞	19	>1	∞	131	>1	∞	>300

We mention in passing an interesting experimental observation. In all the numerical tests reported above, the iteration matrix of the SOR method,

$$\mathscr{L}_\omega = [I - \omega(L - U^*)]^{-1}[\omega(U + U^*) + (1 - \omega)I],$$

was found to have purely real spectrum. This means that instead of GMRES acceleration, standard Chebyshev acceleration could be used instead. Moreover, for ω small enough all the eigenvalues of $\mathscr{L}_\omega$ are positive.

Our numerical experiments provide an illustration of the convergence result in Theorem 3.1, case (ii). Similar experimental tables could be used to illustrate the other convergence results in this paper, for example for the SSOR method. In practice, of course, it is difficult to use SOR-type methods for solving this type of problem, since it is generally difficult to estimate η and therefore the SOR convergence interval $(0, 2/(2 - \eta))$. Also, estimating ω_{best} is even more difficult. Of course, more practical methods exist for the solution of problem (4.1), such as Krylov subspace methods with more effective preconditioners or multigrid methods. In light of our results, it is possible that SOR with a small value of ω may prove an effective smoother for multigrid applied to problems like the ones considered here.

5. Conclusions. In this paper we have studied the convergence of P-regular splitting methods for the solution of non-Hermitian positive definite linear systems. Some of our results can be regarded as generalizations of analogous results for the Hermitian positive definite case.

As an application of our theory, we obtain new convergence conditions for SOR-like methods in the non-Hermitian case.

Acknowledgments. The first author would like to acknowledge the hospitality of the Department of Mathematics and Computer Science at Emory University, where this work was completed.

REFERENCES

[1] A. C. AITKEN, *On the iterative solution of a system of linear equations*, Proc. Roy. Soc. Edinburgh Sect. A, 63, (1950). pp. 52–60.
[2] Z.-Z. BAI AND G. H. GOLUB, *Accelerated Hermitian and skew-Hermitian splitting iteration methods for saddle-point problems*, IMA J. Numer. Anal., 27 (2007), pp. 1–23.
[3] Z.-Z. BAI, G. H. GOLUB, L.-Z. LU, AND J.-F. YIN, *Block triangular and skew-Hermitian splitting methods for positive-definite linear systems*, SIAM J. Sci. Comput., 26 (2005), pp. 844–863.
[4] Z.-Z. BAI, G. H. GOLUB, AND M. K. NG, *Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems*, SIAM J. Matrix Anal. Appl., 24 (2003), pp. 603–626.
[5] Z.-Z. BAI, G. H. GOLUB, AND M. K. NG, *On successive-overrelaxation acceleration of the Hermitian and skew-Hermitian splitting iterations*, Numer. Linear Algebra Appl., 17 (2007), pp. 319–335.
[6] ———*On inexact Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems*, Linear Algebra Appl., 428 (2008), pp. 413–440.
[7] Z.-Z. BAI, G. H. GOLUB, AND J.-Y. PAN, *Preconditioned Hermitian and skew-Hermitian splitting methods for non-Hermitian positive semidefinite linear systems*, Numer. Math., 98 (2004), pp. 1–32.
[8] M. BENZI, *A generalization of the Hermitian and skew-Hermitian splitting iteration*, SIAM J. Matrix. Anal. Appl., 31 (2009), pp. 360–374.
[9] M. BENZI AND D. BERTACCINI, *Block preconditioning of real-valued iterative algorithms for complex linear systems*, IMA J. Numer. Anal., 28 (2008), pp. 598–618.
[10] M. BENZI, A. FROMMER, R. NABBEN AND D. B. SZYLD, *Algebraic theory of multiplicative Schwarz methods*, Numer. Math., 89 (2001), pp. 605–639.
[11] M. BENZI, M. GANDER, AND G. H. GOLUB, *Optimization of the Hermitian and skew-Hermitian splitting iteration for saddle-point problems*, BIT, 43 (2003), pp. 881–900.
[12] M. BENZI AND G. H. GOLUB, *A preconditioner for generalized saddle point problems*, SIAM J. Matrix Anal. Appl., 26 (2004), pp. 20–41.
[13] M. BENZI AND M. K. NG, *Preconditioned iterative methods for weighted Toeplitz least squares problems*, SIAM J. Matrix Anal. Appl., 27 (2006), pp. 1106–1124.
[14] M. BENZI AND D. B. SZYLD, *Existence and uniqueness of splittings for stationary iterative methods with applications to alternating methods*, Numer. Math., 76 (1997), pp. 309–321.
[15] A. BERMAN AND R. J. PLEMMONS, *Nonnegative Matrices in the Mathematical Sciences*, Academic Press, New York, NY, 1979. Reprinted by SIAM, Philadelphia, 1994.
[16] M. A. DELONG AND J. M. ORTEGA, *SOR as a preconditioner*, Appl. Numer. Math., 18 (1995), pp. 431–440.
[17] H. ELMAN, D. SILVESTER, AND A. WATHEN, *Finite Elements and Fast Iterative Solvers with Applications in Incompressible Fluid Dynamics*, Numerical Mathematics and Scientific Computation, Oxford University Press, Oxford, 2005.
[18] L. ELSNER AND V. MEHRMANN, *Convergence of block iterative methods for linear systems arising in the numerical solution of Euler equations*, Numer. Math., 59 (1991), pp. 541–559.
[19] A. FROMMER AND D. B. SZYLD, *Weighted max norms, splittings, and overlapping Schwarz iterations*, Numer. Math., 83 (1999), pp. 259–278.
[20] G. H. GOLUB AND C. F. VAN LOAN, *Matrix Computations*, third edition, Johns Hopkins University Press, Baltimore, MD, 1996.

[21] A. HADJIDIMOS, *Accelerated overrelaxation method*, Math. Comp., 32 (1978), pp. 149–157.

[22] R. A. HORN AND C. R. JOHNSON, *Matrix Analysis*, Cambridge University Press, New York, 1985.

[23] T.-Z. HUANG, S.-Q. SHEN AND H.-B. LI, *On generalized H-matrices*, Linear Algebra Appl., 396 (2005), pp. 81–90.

[24] L. A. KRUKIER AND T. S. MARTYNOVA, *Point SOR and SSOR methods for the numerical solution of the steady convection-diffusion equation with dominant convection*, in Iterative Methods in Scientific Computation IV, D. R. Kincaid and A. C. Elster, Eds., IMACS Series in Computational and Applied Mathematics, 5, New Brunswick, NJ, 1999, pp. 399–404.

[25] L. LI, T.-Z. HUANG, AND X.-P. LIU, *Modified Hermitian and skew-Hermitian splitting methods for non-Hermitian positive-definite linear systems*, Numer. Linear Algebra Appl., 14 (2007), pp. 217–235.

[26] G. I. MARCHUK, Splitting and alternating direction methods, in , Handbook of Numerical Analysis, Vol. I, P. G. Ciarlet and J. L. Lions, Eds., North Holland, New York, NY, 1990, pp. 197–462.

[27] W. NIETHAMMER AND J. SCHADE, *On a relaxed SOR-method applied to nonsymmetric linear systems*, J. Comput. Appl. Math., 1 (1975), pp. 133–136.

[28] W. NIETHAMMER AND R. S. VARGA, *Relaxation methods for non-Hermitian linear systems*, Results in Mathematics, 16 (1989), pp. 308–320.

[29] J. M. ORTEGA, *Numerical Analysis. A Second Course*, Academic Press, New York, NY, 1972. Reprinted by SIAM, Philadelphia, 1990.

[30] ———*Introduction to Parallel and Vector Solution of Linear Systems*, Plenum Press, New York, NY, 1988.

[31] Y. SAAD, *Iterative Methods for Sparse Linear Systems*, second edition, SIAM, Philadelphia, 2003.

[32] Y. SAAD AND M. H. SCHULTZ, *GMRES: A generalized minimal residual algorithm for solving nonsymmetric linear systems*, SIAM J. Sci. Stat. Comput., 7 (1986), pp. 856–869.

[33] J. W. SHELDON, *On the numerical solution of elliptic difference equations*, Mathematical Tables and Other Aids to Computation, 9 (1955), pp. 1101–112.

[34] R. S. VARGA, *Matrix Iterative Analysis*, second edition, Springer-Verlag, Berlin/Heidelberg, 2000.

[35] C.-L. WANG AND Z.-Z. BAI, *Sufficient conditions for the convergent splitting of non-Hermitian positive definite matrices*, Linear Algebra Appl., 330 (2001), pp. 215–218.

[36] ———*Convergence conditions for splitting iteration methods for non-Hermitian linear systems*, Linear Algebra Appl., 428 (2008), pp. 453–468.

[37] J. WEISSINGER, *Verallgemainerungen des Seidelschen Iterationsverfahrens*, Z. Angew. Math. Mech., 33 (1953), pp. 155–162.

[38] D. M. YOUNG, *Iterative Solution of Large Linear Systems*, Academic Press, New York, NY, 1971.

[39] F. ZHANG, *The Schur Complement and Its Applications*, Springer, New York, 2005.