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Contents

41 Analysis of the finite element method for transmission/mixed boundary value problems on general polygonal domains. *Hengguang Li, Anna Mazzucato, and Victor Nistor*.

Abstract.
We study theoretical and practical issues arising in the implementation of the Finite Element Method for a strongly elliptic second order equation with jump discontinuities in its coefficients on a polygonal domain Ω that may have cracks or vertices that touch the boundary. We consider in particular the equation $-\operatorname{div}(A\nabla u) = f \in H^{m-1}(\Omega)$ with mixed boundary conditions, where the matrix A has variable, piecewise smooth coefficients. We establish regularity and Fredholm results and, under some additional conditions, we also establish well-posedness in weighted Sobolev spaces. When Neumann boundary conditions are imposed on adjacent sides of the polygonal domain, we obtain the decomposition $u = u_{\mathrm{reg}} + \sigma$, into a function u_{reg} with better decay at the vertices and a function σ that is locally constant near the vertices, thus proving well-posedness in an augmented space. The theoretical analysis yields interpolation estimates that are then used to construct improved graded meshes recovering the (quasi-)optimal rate of convergence for piecewise polynomials of degree $m \geq 1$. Several numerical tests are included.

Key Words.
Neumann-Neumann vertex, transmission problem, augmented weighted Sobolev space, finite element method, graded mesh, optimal rate of convergence

AMS Subject Classifications.
65N30, 35J25, 46E35, 65N12

70 The analytic SVD: On the non-generic points on the path. *Dáša Janovská and Vladimír Janovský*.

Abstract.
A new technique for computing the Analytic SVD is proposed. The idea is to follow a branch of just one simple singular value and the corresponding left/right singular vector. Numerical computation may collapse at non-generic points; we will consider the case when the continuation gets stuck due to a nonzero multiple singular value. We interpret such a point as a singularity of the branch. We employ singularity theory in order to describe and classify this point. Since its codimension is one, we meet such a point “rarely.”

Key Words.
SVD, ASVD, continuation, singularity theory

AMS Subject Classifications.
65F15

87 Convergence rates for regularization with sparsity constraints. *Ronny Ramlau and Elena Resmerita*.

Abstract.
Tikhonov regularization with p-powers of the weighted ℓ_p norms as penalties, with $p \in (1,2)$, have been employed recently in reconstruction of sparse solutions of ill-posed inverse problems. This paper shows convergence rates for such a regularization with respect to the norm of the weighted spaces by assuming that the

solutions satisfy a certain smoothness (source) condition. The meaning of the latter is analyzed in some detail. Moreover, converse results are established: Linear convergence rates for the residual, together with convergence of the approximations to the solution, can be achieved only if the solution satisfies a source condition. Further insights for the particular case of a convolution equation are provided by analyzing the equation both theoretically and numerically.

Key Words.
ill-posed problem, regularization, Bregman distance, sparsity

AMS Subject Classifications.
47A52, 65J20

Abstract.
Since for $q > 1$, the q-Bernstein polynomials $B_{n,q}$ are not positive linear operators on $C[0, 1]$, the investigation of their convergence properties turns out to be much more difficult than that in the case $0 < q < 1$. In this paper, new results on the approximation of continuous functions by the q-Bernstein polynomials in the case $q > 1$ are presented. It is shown that if $f \in C[0, 1]$ and admits an analytic continuation $f(z)$ into $\{z : |z| < a\}$, then $B_{n,q}(f; z) \to f(z)$ as $n \to \infty$, uniformly on any compact set in $\{z : |z| < a\}$.

Key Words.
q-integers, q-binomial coefficients, q-Bernstein polynomials, uniform convergence

AMS Subject Classifications.
41A10, 30E10

Abstract.
In this paper the regularity of a special lacunary interpolation problem is investigated, where for a given r ($r \geq 2$, $r \in \mathbb{N}$) the derivatives up to the r-2nd order together with the weighted rth derivative are prescribed at the nodes. Sufficient conditions on the nodes and the weight function, for the problem to be regular, are derived. Under these conditions a method to construct the explicit formulae for the fundamental polynomials of the regular weighted lacunary interpolation is discussed. Examples are presented using the roots of the classical orthogonal polynomials.

Key Words.
Birkhoff interpolation, lacunary interpolation, Hermite interpolation, weighted $(0, 2)$-interpolation, weighted $(0, 1, 3)$-interpolation, regularity, explicit formulae

AMS Subject Classifications.
41A05

Abstract.
An algebraic multigrid method is presented to solve large systems of linear equations. The coarsening is obtained by aggregation of the unknowns. The aggregation scheme uses two passes of a pairwise matching algorithm applied to the matrix

graph, resulting in most cases in a decrease of the number of variables by a factor slightly less than four. The matching algorithm favors the strongest negative coupling(s), inducing a problem dependent coarsening. This aggregation is combined with piecewise constant (unsmoothed) prolongation, ensuring low setup cost and memory requirements. Compared with previous aggregation-based multigrid methods, the scalability is enhanced by using a so-called K-cycle multigrid scheme, providing Krylov subspace acceleration at each level. This paper is the logical continuation of [SIAM J. Sci. Comput., 30 (2008), pp. 1082–1103], where the analysis of a anisotropic model problem shows that aggregation-based two-grid methods may have optimal order convergence, and of [Numer. Lin. Alg. Appl., 15 (2008), pp. 473–487], where it is shown that K-cycle multigrid may provide optimal or near optimal convergence under mild assumptions on the two-grid scheme. Whereas in these papers only model problems with geometric aggregation were considered, here a truly algebraic method is presented and tested on a wide range of discrete second order scalar elliptic PDEs, including nonsymmetric and unstructured problems. Numerical results indicate that the proposed method may be significantly more robust as black box solver than the classical AMG method as implemented in the code AMG1R5 by K. Stüben. The parallel implementation of the method is also discussed. Satisfactory speedups are obtained on a medium size multi-processor cluster that is typical of today computer market. A code implementing the method is freely available for download both as a FORTRAN program and a MATLAB function.

Key Words.
multigrid, linear systems, iterative methods, AMG, preconditioning, parallel computing

AMS Subject Classifications.
65F10, 65N55

147 Computing $\exp(-\tau A)b$ with Laguerre polynomials. *Bernard N. Sheehan, Yousef Saad, and Roger B. Sidje*.

Abstract.
This paper discusses a method based on Laguerre polynomials combined with a Filtered Conjugate Residual (FCR) framework to compute the product of the exponential of a matrix by a vector. The method implicitly uses an expansion of the exponential function in a series of orthogonal Laguerre polynomials, much like existing methods based on Chebyshev polynomials do. Owing to the fact that orthogonal polynomials satisfy a three-term recurrence, what these series expansion methods offer over other approaches such as Krylov subspace methods lies in the elimination of inner products and the economy in storage since there is no need to compute and keep a set of basis vectors. Compared with Chebyshev polynomials that are orthogonal within a restricted interval and need estimates of the outermost eigenvalues, Laguerre polynomials offer the added feature that they are orthogonal on the half real line, alleviating therefore the need to estimate eigenvalues.

Key Words.
conjugate residual, filtered conjugate residual, polynomial filtering, exponential propagation, orthogonal polynomials

AMS Subject Classifications.
65F50, 65L05, 41A10

Abstract.
We present a construction of a gradient recovery operator based on an oblique projection, where the basis functions of two involved spaces satisfy a condition of biorthogonality. The biorthogonality condition guarantees that the recovery operator is local.

Key Words.
gradient recovery, a posteriori error estimate, biorthogonal system

AMS Subject Classifications.
65N30, 65N15, 65N50

Abstract.
We present a preconditioner for saddle point problems which is based on an approximation of an implicit representation of the inverse of the saddle point matrix. Whereas this preconditioner does not require an approximation to the Schur complement, its theoretical analysis yields some interesting relationship to some Schur-complement-based preconditioners. Whereas the evaluation of this new preconditioner is slightly more expensive than the evaluation of standard block preconditioners from the literature, it has the advantage that, similar to constraint preconditioners, the iterates of the preconditioned system satisfy the constraint equations exactly. We will demonstrate the performance of the implicit approximate inverse preconditioner in the iterative solution of the discrete two- as well as three-dimensional Oseen equations.

Key Words.
saddle point problem, preconditioning

AMS Subject Classifications.
65F05, 65F30, 65F50, 65N22, 65N30

Abstract.
In this paper, we introduce a new family of mixed finite element spaces of higher order ($k \geq 1$) on general quadrilateral grids. A typical element has two fewer degrees of freedom than the well-known Raviart-Thomas finite element $RT_{[k]}$, yet enjoys an optimal-order approximation for the velocity in L^2-norm. The order of approximation in the divergence norm is one less than the velocity, as is common to all other known elements, except for a recent element introduced by Arnold et al. [SIAM J. Numer. Anal., 42 (2005), pp. 2429–2451]. However, we introduce a local post-processing technique to obtain an optimal order in L^2-norm of divergence. This technique can be used to enhance the result of $RT_{[k]}$ element as well, and hence, can be easily incorporated into existing codes.
Our element has one lower order of approximation in pressure than the $RT_{[k]}$ element. However, the pressure also can be locally post-processed to produce an optimal-order approximation. The greatest advantage of our finite element lies in the fact that it has the fewest degrees of freedom among all the known quadrilateral mixed finite elements and thus, together with the post-processing techniques,

provides a very efficient way of computing flow variables in mixed formulation. Numerical examples are in quite good agreement with the theory even for the case of almost degenerate quadrilateral grids.

Key Words.
mixed finite element method, quadrilateral grid, optimal velocity, post-processing

AMS Subject Classifications.
65N15, 65N30

Abstract.
In Simoncini and Szyld [Numer. Math., 109 (2008), pp. 477–487] a new non-stagnation condition for the convergence of GMRES on indefinite problems was proposed. In this paper we derive an enhanced strategy leading to a more general non-stagnation condition. Moreover, we show that the analysis also provides a good setting to derive asymptotic convergence rate estimates for indefinite problems. The analysis is then explored in the context of saddle point matrices, when these are preconditioned in a way so as to lead to nonsymmetric and indefinite systems. Our results indicate that these matrices may represent an insightful training set towards the understanding of the interaction between indefiniteness and stagnation.

Key Words.
saddle point matrices, large linear systems, GMRES, stagnation

AMS Subject Classifications.
65F10, 65N22, 65F50

Abstract.
We study a weakly over-penalized symmetric interior penalty method for the biharmonic problem that is intrinsically parallel. Both *a priori* error analysis and *a posteriori* error analysis are carried out. The performance of the method is illustrated by numerical experiments.

Key Words.
biharmonic problem, finite element, interior penalty method, weak over-penalization, Morley element, fourth order

AMS Subject Classifications.
65N30, 65N15

Abstract.
The class of Newton methods is popular for solving large sparse nonlinear algebraic systems of equations arising from the discretization of partial differential equations. The method offers superlinear or quadratic convergence when the solution is sufficiently smooth and the initial guess is close to the desired solution. However, in

many practical problems, the solution may exhibit some non-smoothness in part of the computational domain, due to, for example, the presence of a shock wave. In this situation, the convergence rate of Newton-type methods deteriorates considerably. In this paper, we introduce a two-level nonlinear elimination algorithm, in which we first identify a subset of equations that prevents Newton from having the fast convergence and then iteratively eliminate them from the global nonlinear system of equations. We show that such implicit nonlinear elimination restores the fast convergence for problems with local non-smoothness. As an example, we study a compressible transonic flow in a shocked duct.

Key Words.
nonlinear PDEs, nonlinear elimination, inexact Newton, finite difference, shock wave

AMS Subject Classifications.
65H10, 65N06, 65N55

252 Accumulation of global error in Lie group methods for linear ordinary differential equations. *Bojan Orel*.

Abstract.
In this paper we will investigate how the local errors accumulate to the global error in Lie group methods for linear ODEs. The concept of the local and global errors has to be redefined to fit in the framework of Lie groups and algebras. Formulas for tracking the global error are proposed and demonstrated on numerical examples.

Key Words.
ordinary differential equations, Lie group methods, global error, local error

AMS Subject Classifications.
65L05

263 A streaming approach for sparse matrix products and its application in Galerkin multigrid methods. *Joachim Georgii and Rüdiger Westermann*.

Abstract.
In this paper, we present a numerical algorithm for computing products of the form RKR^{T}, where R, R^{T}, and K are sparse matrices. By reformulating the problem into the simultaneous processing of a sequential data and control stream, cache miss penalties are significantly reduced. Even though the algorithm increases memory requirements, it accelerates sparse matrix products on recent processor architectures by a factor of up to 4 compared to previous approaches. We apply the algorithm to compute consistent system matrices at different resolution levels in a dynamic multigrid elasticity simulation, and we show its efficiency for nested and non-nested mesh hierarchies.

Key Words.
sparse matrix products, cache-awareness, multigrid, Galerkin update

AMS Subject Classifications.
65F50, 65M55, 65M60, 65Y20, 68W01, 74B99, 74H15

276 Adaptive reduction-based multigrid for nearly singular and highly disordered physical systems. *J. Brannick, A. Frommer, K. Kahl, S. MacLachlan, and L. Zikatanov*.

Abstract.
Classical multigrid solution of linear systems with matrices that have highly variable entries and are nearly singular is made difficult by the compounding difficulties introduced by these two model features. Efficient multigrid solution of nearly singular matrices is known to be possible, provided the so-called Brandt-McCormick (or eigenvector approximation) criterion is satisfied, which requires building interpolation to fit the near-null-space modes with high accuracy. When these modes are known, traditional multigrid approaches may be very effective. In this paper, we consider the case of matrices describing highly disordered systems, such as those that arise in lattice quantum chromodynamics (QCD), where the near-null modes cannot be easily expressed in closed form. We develop a variational adaptive reduction-based algebraic multigrid preconditioner for such systems and present a two-level convergence theory for the approach for Hermitian and positive-definite systems. The proposed method is applied to a two-dimensional model known as the Gauge Laplacian, a common test problem for development of solvers in quantum dynamics applications, showing promising numerical results. The proposed reduction-based setup uses compatible relaxation coarsening together with a sparse approximation to the so-called ideal interpolation operator to recursively construct the coarse spaces.

Key Words.
algebraic multigrid, lattice QCD, gauge Laplacian, iterative methods, adaptive multigrid

AMS Subject Classifications.
65N55, 65N22, 65F10

296 A robust spectral method for finding lumpings and meta stable states of non-reversible Markov chains. *Martin Nilsson Jacobi*.

Abstract.
A spectral method for identifying lumping in large Markov chains is presented. The identification of meta stable states is treated as a special case. The method is based on the spectral analysis of a self-adjoint matrix that is a function of the original transition matrix. It is demonstrated that the technique is more robust than existing methods when applied to noisy non-reversible Markov chains.

Key Words.
Markov chain, stochastic matrix, metastable states, lumping, aggregation, modularity, block diagonal dominance, block stochastic

AMS Subject Classifications.
15A18, 15A51, 60J10, 65F15

307 An analysis of low-rank modifications of preconditioners for saddle point systems. *Chen Greif and Michael L. Overton*.

Abstract.
We characterize the spectral behavior of a primal Schur-complement-based block diagonal preconditioner for saddle point systems, subject to low-rank modifications.

This is motivated by a desire to reduce as much as possible the computational cost of matrix-vector products with the (1,1) block, while keeping the eigenvalues of the preconditioned matrix reasonably clustered. The formulation leads to a perturbed hyperbolic quadratic eigenvalue problem. We derive interlacing results, highlighting the differences between this problem and perturbed linear eigenvalue problems. As an example, we consider primal-dual interior point methods for semidefinite programs, and express the eigenvalues of the preconditioned matrix in terms of the centering parameter.

Key Words.
saddle point systems, preconditioners, Schur complement, semidefinite programming

AMS Subject Classifications.
65F08, 65F10, 90C22

Abstract.
This paper is concerned with the Simultaneous Iterative Reconstruction Technique (SIRT) class of iterative methods for solving inverse problems. Based on a careful analysis of the semi-convergence behavior of these methods, we propose two new techniques to specify the relaxation parameters adaptively during the iterations, so as to control the propagated noise component of the error. The advantage of using this strategy for the choice of relaxation parameters on noisy and ill-conditioned problems is demonstrated with an example from tomography (image reconstruction from projections).

Key Words.
SIRT methods, Cimmino and DROP iteration, semi-convergence, relaxation parameters, tomographic imaging

AMS Subject Classifications.
65F10, 65R32

Abstract.
We develop and test a new mapping that can be applied to directed unweighted networks. Although not a "matrix function" in the classical matrix theory sense, this mapping converts an unsymmetric matrix with entries of zero or one into a symmetric real-valued matrix of the same dimension that generally has both positive and negative entries. The mapping is designed to reveal approximate directed bipartite communities within a complex directed network; each such community is formed by two set of nodes S_1 and S_2 such that the connections involving these nodes are predominantly *from* a node in S_1 and *to* a node in S_2. The new mapping is motivated via the concept of *alternating walks* that successively respect and then violate the orientations of the links. Considering the combinatorics of these walks leads us to a matrix that can be neatly expressed via the singular value decomposition of the original adjacency matrix and hyperbolic functions. We argue that this new matrix mapping has advantages over other, exponential-based measures. Its performance is

illustrated on synthetic data, and we then show that it is able to reveal meaningful directed bipartite substructure in a network from neuroscience.

Key Words.
bipartivity, clustering, communities, exponential, networks, neuroscience, stickiness

AMS Subject Classifications.
65F60, 05C50

351 Benchmarking aggregation AMG for linear systems in CFD simulations of compressible internal flows. *Maximilian Emans*.

Abstract.
The performance of parallel implementations of three fundamentally different aggregation AMG (algebraic multigrid) solvers, including novel k-cycle methods, for systems of linear equations appearing in industrial CFD simulations are examined. The results show that the k-cycle methods are a good choice for cases with less than 20000 unknowns per process if the cost of the setup tends to become critical; for most other applications, however, established methods proved to be equally efficient or superior.

Key Words.
algebraic multigrid, fluid dynamics, finite volumes, pressure-velocity coupling

AMS Subject Classifications.
15A06, 65F08, 76G25

367 Coarsening invariance and bucket-sorted independent sets for algebraic multigrid. *David M. Alber and Luke N. Olson*.

Abstract.
Independent set-based coarse-grid selection algorithms for algebraic multigrid are defined by their policies for weight initialization, independent set selection, and weight update. In this paper, we develop theory demonstrating that algorithms employing the same policies produce identical coarse grids, regardless of the implementation. The coarse-grid invariance motivates a new coarse-grid selection algorithm, called Bucket-Sorted Independent Sets (BSIS), that is more efficient than an existing algorithm (CLJP-c) using the same policies. Experimental results highlighting the efficiency of two versions of the new algorithm are presented, followed by a discussion of BSIS in a parallel setting.

Key Words.
algebraic multigrid, parallel, coarse-grid selection

AMS Subject Classifications.
65Y05, 65Y20, 65F10

386 A spectral method for the eigenvalue problem for elliptic equations. *Kendall Atkinson and Olaf Hansen*.

Abstract.
Let Ω be an open, simply connected, and bounded region in $\mathbb{R}^d$, $d \geq 2$, and assume its boundary $\partial\Omega$ is smooth. Consider solving the eigenvalue problem $Lu = \lambda u$ for

an elliptic partial differential operator L over Ω with zero values for either Dirichlet or Neumann boundary conditions. We propose, analyze, and illustrate a 'spectral method' for solving numerically such an eigenvalue problem. This is an extension of the methods presented earlier by Atkinson, Chien, and Hansen [Adv. Comput. Math, 33 (2010), pp. 169–189, and to appear].

Key Words.
elliptic equations, eigenvalue problem, spectral method, multivariable approximation

AMS Subject Classifications.
65M70

413 Analysis of a non-standard finite element method based on boundary integral operators. *Clemens Hofreither, Ulrich Langer, and Clemens Pechstein*.

Abstract.
We present and analyze a non-standard finite element method based on element-local boundary integral operators that permits polyhedral element shapes as well as meshes with hanging nodes. The method employs elementwise PDE-harmonic trial functions and can thus be interpreted as a local Trefftz method. The construction principle requires the explicit knowledge of the fundamental solution of the partial differential operator, but only locally, i.e., in every polyhedral element. This allows us to solve PDEs with elementwise constant coefficients. In this paper we consider the diffusion equation as a model problem, but the method can be generalized to convection-diffusion-reaction problems and to systems of PDEs such as the linear elasticity system and the time-harmonic Maxwell equations with elementwise constant coefficients. We provide a rigorous error analysis of the method under quite general assumptions on the geometric properties of the elements. Numerical results confirm our theoretical estimates.

Key Words.
finite elements, boundary elements, BEM-based FEM, Trefftz methods, error estimates, polyhedral meshes

AMS Subject Classifications.
65N30, 65N38