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A WEAKLY OVER-PENALIZED SYMMETRIC INTERIOR PENALTY METHOD FOR THE BIHARMONIC PROBLEM*

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Abstract. We study a weakly over-penalized symmetric interior penalty method for the biharmonic problem that is intrinsically parallel. Both *a priori* error analysis and *a posteriori* error analysis are carried out. The performance of the method is illustrated by numerical experiments.

Key words. biharmonic problem, finite element, interior penalty method, weak over-penalization, Morley element, fourth order

AMS subject classifications. 65N30, 65N15

1. Introduction. Recently, it was noted in [9] that the Poisson problem can be solved by a weakly over-penalized symmetric interior penalty (WOPSIP) method [10, 12, 25] with high intrinsic parallelism. The WOPSIP method satisfies the same error estimates as the standard P_1 finite element method and also the same condition number estimates after preconditioning. Furthermore, there exist two orderings (edge-wise and element-wise) of the degrees of freedom (dofs) so that the stiffness matrix for the WOPSIP method is the sum of two matrices, each of which is block diagonal with respect to one of these two orderings. In fact, the matrix representing the piecewise Dirichlet form has 3×3 diagonal blocks with respect to the element-wise ordering of the dofs, while the matrix representing the jumps across edges has 1×1 or 2×2 diagonal blocks in the edge-wise ordering. The simple preconditioner is also block diagonal with 1×1 or 2×2 blocks in the edge-wise ordering of the dofs. These properties of the WOPSIP method make it an attractive candidate for iterative solvers for the Poisson problem.

In this paper, we extend the WOPSIP approach to fourth order problems and develop a method that also has high intrinsic parallelism. For simplicity, we consider the biharmonic problem on a bounded polygonal domain $\Omega \subset \mathbb{R}^2$.

Let $f \in L_2(\Omega)$. A weak form of the biharmonic problem is to find $u \in H_0^2(\Omega)$ such that

$$a(u, v) = (f, v) \qquad \forall\, v \in H_0^2(\Omega), \tag{1.1}$$

where

$$a(w, v) = \int_\Omega D^2 w : D^2 v \, dx \qquad \forall\, v, w \in H_0^2(\Omega),$$

$$D^2 w : D^2 v = \sum_{i,j=1}^{2} \frac{\partial^2 w}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j},$$

and $(\cdot, \cdot)$ denotes the L_2 inner product. Here and throughout the paper, we follow the standard notations for Sobolev spaces [1, 16].

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Conforming finite element methods for (1.1) require C^1 finite elements [3, 15] that involve higher order polynomials and hence are quite complicated. Alternatively, one can solve (1.1) by nonconforming finite elements that involve only low order polynomials. The WOPSIP method in this paper is based on the Morley element [23, 26]. By removing the continuity conditions of the Morley element through weak over-penalization, we obtain an intrinsically parallel finite element method for (1.1).

Our goal is to demonstrate theoretically and numerically that the performance of the WOPSIP method is similar to the performance of the Morley finite element method (in terms of the magnitudes of the discretization errors), and that an efficient adaptive algorithm is available for the WOPSIP method. This is an important step before the intrinsic parallelism of the WOPSIP method is further exploited.

The rest of the paper is organized as follows. We introduce some basic definitions in Section 2. The WOPSIP method is defined in Section 3. Section 4 contains some preliminary estimates. The *a priori* analysis and *a posteriori* analysis of the WOPSIP method in the energy norm are carried out in Sections 5 and 6. Some extensions of the WOPSIP method are discussed in Section 7. Results of numerical experiments are reported in Section 8, and we end with some concluding remarks in Section 9.

2. The set-up. Let $\mathcal{T}_h$ be a simplicial triangulation of Ω. We adopt the following notation:

h_T = diameter of T ($h = \max_{T\in\mathcal{T}_h} h_T$)
h_e = the length of the edge e
$|T|$ = the area of the triangle T
m_e = the midpoint of the edge e
$\mathcal{E}_h^i$ = the set of all the interior edges of (the triangles of) $\mathcal{T}_h$
$\mathcal{E}_h^b$ = the set of all the boundary edges of (the triangles of) $\mathcal{T}_h$
$\mathcal{E}_h = \mathcal{E}_h^i \cup \mathcal{E}_h^b$
$\mathcal{V}_h$ = the set of all the vertices of (the triangles of) $\mathcal{T}_h$
$\mathcal{V}_T$ = the set of the three vertices of T
$\mathcal{E}_T$ = the set of the three edges of T
$\mathcal{T}_e$ = the set of the triangle(s) in $\mathcal{T}_h$ such that $e \in \mathcal{E}_T$
$\mathcal{E}_p$ = the set of edges in $\mathcal{E}_h$ that share the common vertex $p \in \mathcal{V}_h$
$\mathcal{V}_e$ = the set of the two endpoints of the edge e
$v_T = v|_T$, the restriction of the function v on the triangle T

Let k be a nonnegative integer. We define the piecewise Sobolev space $H^k(\Omega,\mathcal{T}_h)$ associated with the triangulation $\mathcal{T}_h$ by

$$H^k(\Omega,\mathcal{T}_h) = \{v \in L_2(\Omega) : v_T \in H^k(T) \quad \forall\ T \in \mathcal{T}_h\},$$

and the semi-norm $|\cdot|_{H^k(\Omega,\mathcal{T}_h)}$ by

$$|v|^2_{H^k(\Omega,\mathcal{T}_h)} = \sum_{T\in\mathcal{T}_h} |v|^2_{H^k(T)}. \tag{2.1}$$

Let $e \in \mathcal{E}_h^i$ be a common edge of the triangles $T_\pm \in \mathcal{T}_h$. For $v \in H^1(\Omega,\mathcal{T}_h)$, we define the jump $[\![v]\!]_e$ of v across e (in the sense of trace) by

$$[\![v]\!]_e = v_+ - v_-,$$

where $v_\pm = v\big|_{T_\pm}$. If $v \in H^2(\Omega,\mathcal{T}_h)$ and p belongs to the closure of e, we define

$$[\![v(p)]\!]_e = [\![v]\!]_e(p) = v_+(p) - v_-(p).$$

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Let n_e be the unit normal of e pointing from T_- to T_+ and t_e be the unit tangent vector of e obtained by rotating n_e through a counterclockwise right angle (cf. Figure 2.1). For any $v \in H^2(\Omega, \mathcal{T}_h)$, we define the jumps and means of the normal and tangential derivatives of v across e by

$$\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right]_e = \frac{\partial v_+}{\partial n_e}\bigg|_e - \frac{\partial v_-}{\partial n_e}\bigg|_e \quad \text{and} \quad \left\{\!\!\left\{\frac{\partial v}{\partial n}\right\}\!\!\right\}_e = \frac{1}{2}\left(\frac{\partial v_+}{\partial n_e}\bigg|_e + \frac{\partial v_-}{\partial n_e}\bigg|_e\right),$$
$$\left[\!\left[\frac{\partial v}{\partial t}\right]\!\right]_e = \frac{\partial v_+}{\partial t_e}\bigg|_e - \frac{\partial v_-}{\partial t_e}\bigg|_e \quad \text{and} \quad \left\{\!\!\left\{\frac{\partial v}{\partial t}\right\}\!\!\right\}_e = \frac{1}{2}\left(\frac{\partial v_+}{\partial t_e}\bigg|_e + \frac{\partial v_-}{\partial t_e}\bigg|_e\right).$$

If $v \in H^3(\Omega, \mathcal{T}_h)$, the pointwise values of the jumps and means of the derivatives are well-defined. Similarly, for any $v \in H^3(\Omega, \mathcal{T}_h)$, we define the jumps and means of the second order derivatives of v across e by

$$\left[\!\left[\frac{\partial^2 v}{\partial n^2}\right]\!\right]_e = \frac{\partial^2 v_+}{\partial n_e^2}\bigg|_e - \frac{\partial^2 v_-}{\partial n_e^2}\bigg|_e \quad \text{and} \quad \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}_e = \frac{1}{2}\left(\frac{\partial^2 v_+}{\partial n_e^2}\bigg|_e + \frac{\partial^2 v_-}{\partial n_e^2}\bigg|_e\right),$$
$$\left[\!\left[\frac{\partial^2 v}{\partial n \partial t}\right]\!\right]_e = \frac{\partial^2 v_+}{\partial n_e \partial t_e}\bigg|_e - \frac{\partial^2 v_-}{\partial n_e \partial t_e}\bigg|_e \quad \text{and} \quad \left\{\!\!\left\{\frac{\partial^2 v}{\partial n \partial t}\right\}\!\!\right\}_e = \frac{1}{2}\left(\frac{\partial^2 v_+}{\partial n_e \partial t_e}\bigg|_e + \frac{\partial^2 v_-}{\partial n_e \partial t_e}\bigg|_e\right).$$

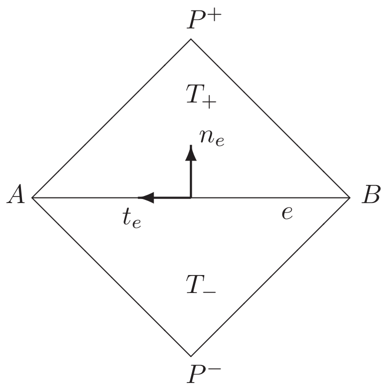

FIGURE 2.1. *Two neighboring triangles T_- and T_+ that share the edge e with the unit normal n_e pointing from T_- into T_+.*

Let $e \in \mathcal{E}_h^b$ be an edge of the triangle $T \in \mathcal{T}_h$. We define

$$[\![v]\!]_e = -v_T\big|_e \qquad \forall\, v \in H^1(\Omega, \mathcal{T}_h),$$
$$\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right]_e = -\frac{\partial v_T}{\partial n_e}\bigg|_e \quad \text{and} \quad \left[\!\left[\frac{\partial v}{\partial t}\right]\!\right]_e = -\frac{\partial v_T}{\partial t_e}\bigg|_e \qquad \forall\, v \in H^2(\Omega, \mathcal{T}_h),$$
$$\left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}_e = \frac{\partial^2 v_T}{\partial n_e^2}\bigg|_e \quad \text{and} \quad \left\{\!\!\left\{\frac{\partial^2 v}{\partial n \partial t}\right\}\!\!\right\}_e = \frac{\partial^2 v_T}{\partial n_e \partial t_e}\bigg|_e \qquad \forall\, v \in H^3(\Omega, \mathcal{T}_h),$$

where n_e is the unit normal of e pointing towards the outside of Ω and t_e is the unit tangent vector of e obtained by rotating n_e through a counterclockwise right angle.

The finite element space V_h for the WOPSIP method is the space of discontinuous piecewise quadratic polynomials associated with $\mathcal{T}_h$. The Morley finite element space $\tilde{V}_h^M$ associated with $\mathcal{T}_h$ is a subspace of V_h. A function $v \in V_h$ belongs to $\tilde{V}_h^M$ if and only if (i) v is

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continuous at the vertices in $\mathcal{V}_h$ and (ii) $\partial v/\partial n$ is continuous at the midpoints of the edges in $\mathcal{E}_h$. The dofs for $\tilde{V}_h^M$ are the values of a function at the vertices and the mean values of its normal derivative on the edges. The interpolation operator $\mathcal{I}_h : H^2(\Omega) \longrightarrow \tilde{V}_h^M$ is defined by the following conditions:

$$(\mathcal{I}_h\zeta)(p) = \zeta(p) \qquad \forall\, p \in \mathcal{V}_p, \tag{2.2}$$

$$\Pi_e \frac{\partial(\mathcal{I}_h\zeta)}{\partial n_e} = \Pi_e \frac{\partial\zeta}{\partial n_e} \qquad \forall\, e \in \mathcal{E}_h, \tag{2.3}$$

where the projection (mean value) operator $\Pi_e : L_2(e) \longrightarrow P_0(e)$ is defined by

$$\Pi_e v = \frac{1}{h_e}\int_e v\, ds.$$

It follows from (2.2)–(2.3) and integration by parts that

$$\int_T D^2\zeta : D^2 v\, dx = \int_T D^2(\mathcal{I}_h\zeta) : D^2 v\, dx \qquad \forall\, \zeta \in H^2(\Omega),\ v \in V_h,\ T \in \mathcal{T}_h. \tag{2.4}$$

Moreover, the Morley interpolation operator satisfies the standard [16] error estimate

$$\|\zeta - \mathcal{I}_h\zeta\|_{L_2(T)} + h_T|\zeta - \mathcal{I}_h\zeta|_{H^1(T)} + h_T^2|\zeta - \mathcal{I}_h\zeta|_{H^2(T)} \le C h_T^s |\zeta|_{H^s(T)} \tag{2.5}$$

for all $\zeta \in H^s(T)$, $2 \le s \le 3$ and $T \in \mathcal{T}_h$.

REMARK 2.1. Throughout this paper we use C (with or without subscripts) to denote a generic positive constant that depends only on Ω and/or the shape regularity of $\mathcal{T}_h$. To avoid the proliferation of constants, we also use the notation $A \lesssim B$ to represent the inequality $A \le (\text{constant}) \cdot B$, where the constant only depends on Ω and/or the shape regularity of $\mathcal{T}_h$.

Note that $\mathcal{I}_h$ maps $H_0^2(\Omega)$ into $V_h^M = \{v \in \tilde{V}_h^M : v$ vanishes at the vertices of $\mathcal{T}_h$ along $\partial\Omega$ and $\partial v/\partial n$ vanishes at the midpoints on the boundary edges$\}$.

Finally, we recall the definition of the Hsieh-Clough-Tocher finite element space V_h^{HCT} associated with a triangulation $\mathcal{T}_h$ [13, 16]. A function v belongs to V_h^{HCT} if and only if (i) $v \in C^1(\bar{\Omega}) \cap H_0^2(\Omega)$ and (ii) v is piecewise cubic on each $T \in \mathcal{T}_h$ with respect to the partition generated by connecting the three vertices of T to the center of T. The degrees of freedom of a function in V_h^{HCT} are the values of the function and its first order derivatives at the interior vertices and the mean values of its normal derivative on the interior edges.

3. A Weakly Over-Penalized Interior Penalty Method.

The WOPSIP method for (1.1) is to find $u_h \in V_h$ such that

$$a_h(u_h, v) = (f, v) \qquad \forall\, v \in V_h, \tag{3.1}$$

where

$$\begin{aligned} a_h(w,v) = \sum_{T\in\mathcal{T}_h}\int_T D^2 w : D^2 v\, dx &+ \sum_{e\in\mathcal{E}_h} h_e^{-2}\left(\Pi_e[\![\partial w/\partial n]\!]_e\right)\left(\Pi_e[\![\partial v/\partial n]\!]_e\right) \\ &+ \sum_{p\in\mathcal{V}_h}\sum_{e\in\mathcal{E}_p} h_e^{-4}[\![w(p)]\!]_e[\![v(p)]\!]_e. \end{aligned} \tag{3.2}$$

REMARK 3.1. The bilinear form $a_h(\cdot,\cdot)$ is independent of the choices of $T_\pm$ in the definitions of the jumps.

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REMARK 3.2. Note that, by the midpoint rule, we have

$$\Pi_e \left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right]_e = \left[\!\!\left[\frac{\partial v}{\partial n}(m_e)\right]\!\!\right]_e \qquad \forall\, v \in V_h,\ e \in \mathcal{E}_h.$$

REMARK 3.3. We refer to this method as a weakly over-penalized method because the over-penalized terms are well-defined on $H^2(\Omega)$, the Sobolev space where the weak form (1.1) of the biharmonic problem is posed.

REMARK 3.4. For $v \in V_h$ and $w \in V_h^M + H_0^2(\Omega)$, we have

$$a_h(v, w) = \sum_{T\in\mathcal{T}_h} \int_T D^2v : D^2w\,dx.$$

Thus, the WOPSIP method becomes the Morley nonconforming method when restricted to the Morley finite element space.

We will use the function values at the vertices and the values of the normal derivatives at the midpoints of the edges as dofs for the finite element space V_h. There are two natural orderings for the dofs. In the first ordering, where the dofs associated with a triangle $T \in \mathcal{T}_h$ are always consecutive, the bilinear form

$$\sum_{T\in\mathcal{T}_h} \int_T D^2w : D^2v\,dx$$

is represented by a block-diagonal matrix with 6×6 diagonal blocks. We will refer to this ordering of the dofs as the element-wise ordering.

In the second ordering, where the dofs associated with a vertex are always consecutive and the dofs associated with a midpoint are always consecutive, the bilinear form

$$\sum_{e\in\mathcal{E}_h} h_e^{-2} \left(\Pi_e[\![\partial w/\partial n]\!]_e\right)\left(\Pi_e[\![\partial v/\partial n]\!]_e\right) + \sum_{p\in\mathcal{V}_h}\sum_{e\in\mathcal{E}_p} h_e^{-4}[\![w(p)]\!]_e[\![v(p)]\!]_e$$

is represented by a block-diagonal matrix. The diagonal block corresponding to a midpoint is either 1×1 (boundary midpoint) or 2×2 (interior midpoint), while the diagonal block corresponding to a vertex is $m \times m$, where m is the number of triangles in $\mathcal{T}_h$ that share the vertex as a common vertex. We will refer to this ordering of the dofs as the vertex-edge-wise ordering.

In view of this splitting of the stiffness matrix, the operation of multiplying a vector representing the dofs of a finite element function by the stiffness matrix can be easily parallelized. Thus the WOPSIP method is intrinsically parallel.

The ill-conditioning of the WOPSIP method due to over-penalization can be remedied by a simple preconditioner. Let the bilinear form $b_h(\cdot,\cdot)$ on $V_h \times V_h$ be defined by

$$\begin{aligned} b_h(w, v) = \sum_{T\in\mathcal{T}_h} \Bigg[\sum_{p\in\mathcal{V}_T} w_T(p)v_T(p) + h_T^2 \sum_{e\in\mathcal{E}_T} \left(\Pi_e \frac{\partial w_T}{\partial n}\right)\left(\Pi_e \frac{\partial v_T}{\partial n}\right)\Bigg] \\ + \sum_{e\in\mathcal{E}_h} \left(\Pi_e[\![\partial w/\partial n]\!]_e\right)\left(\Pi_e[\![\partial v/\partial n]\!]_e\right) \\ + \sum_{p\in\mathcal{V}_h}\sum_{e\in\mathcal{E}_p} h_e^{-2}[\![w(p)]\!]_e[\![v(p)]\!]_e. \end{aligned}$$

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The following lemma shows that the discrete problem resulting from the WOPSIP method behaves like a typical fourth order problem after preconditioning by the operator associated with $b_h(\cdot,\cdot)$.

LEMMA 3.5. *Let the operators $A_h, B_h : V_h \longrightarrow V_h'$ be defined by*

$$\langle A_h w, v\rangle = a_h(w,v) \quad \textit{and} \quad \langle B_h w, v\rangle = b_h(w,v) \qquad \forall\, v,w \in V_h,$$

where $\langle\cdot,\cdot\rangle$ is the canonical bilinear form on $V_h' \times V_h$. Then the following condition number estimate holds for a quasi-uniform triangulation $\mathcal{T}_h$:

$$\frac{\lambda_{\max}(B_h^{-1}A_h)}{\lambda_{\min}(B_h^{-1}A_h)} \le Ch^{-4}, \tag{3.3}$$

where $\lambda_{\max}(B_h^{-1}A_h)$ (resp. $\lambda_{\min}(B_h^{-1}A_h)$) is the maximum (resp. minimum) eigenvalue of $B_h^{-1}A_h$.

Proof. First we note that all the eigenvalues of $B_h^{-1}A_h$ are positive since both $a_h(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$ are symmetric positive definite bilinear forms on V_h. From scaling, we have

$$|v|^2_{H^2(T)} \lesssim h_T^{-2}\left[\sum_{p\in\mathcal{V}_T} v^2(p) + h_T^2 \sum_{e\in\mathcal{E}_T}\left(\Pi_e \frac{\partial v}{\partial n}\right)^2\right] \qquad \forall\, v \in P_2(T),$$

and hence

$$\langle A_h v, v\rangle = a_h(v,v) \lesssim h^{-2} b_h(v,v) = h^{-2}\langle B_h v, v\rangle \qquad \forall\, v \in V_h,$$

which together with the Rayleigh quotient formula implies

$$\lambda_{\max}(B_h^{-1}A_h) = \max_{v\in V_h\setminus\{0\}} \frac{\langle A_h v, v\rangle}{\langle B_h v, v\rangle} \le Ch^{-2}. \tag{3.4}$$

In the other direction, it follows from a Poincaré-Friedrichs inequality for piecewise H^2 functions [14] that

$$\|v\|^2_{L_2(\Omega)} \lesssim \left(|v|^2_{H^2(\Omega,\mathcal{T}_h)} + \sum_{e\in\mathcal{E}_h}\int_e \left[h_e^{-3}[\![v]\!]_e^2 + h_e^{-1}\left(\Pi_e[\![\partial v/\partial n]\!]_e\right)^2\right] ds\right) \tag{3.5}$$

for all $v \in V_h$. Furthermore, we have, by scaling and a standard interpolation error estimate [13, 16],

$$\begin{aligned} \sum_{e\in\mathcal{E}_h}\int_e h_e^{-3}[\![v]\!]_e^2 ds &\le 2\sum_{e\in\mathcal{E}_h}\int_e h_e^{-3}\left([\![v-v^I]\!]_e^2 + [\![v^I]\!]_e^2\right) ds \\ &\lesssim \left(\sum_{T\in\mathcal{T}_h}\int_{\partial T} h_T^{-3}(v-v^I)^2 ds + \sum_{p\in\mathcal{V}_h}\sum_{e\in\mathcal{E}_p} h_e^{-2}[\![v(p)]\!]_e^2\right) \\ &\lesssim \left(|v|^2_{H^2(\Omega,\mathcal{T}_h)} + \sum_{p\in\mathcal{V}_h}\sum_{e\in\mathcal{E}_p} h_e^{-2}[\![v(p)]\!]_e^2\right) \qquad \forall\, v \in V_h, \end{aligned} \tag{3.6}$$

where v^I is the piecewise linear polynomial such that v_T^I agrees with v_T at the vertices of T for all $T \in \mathcal{T}_h$. Note also that (again from scaling)

$$h_T^2\left[\sum_{p\in\mathcal{V}_T} v^2(p) + h_T^2\sum_{e\in\mathcal{E}_T}\left(\Pi_e\frac{\partial v}{\partial n}\right)^2\right] \lesssim \|v\|^2_{L_2(T)} \qquad \forall\, v \in P_2(T),\ T \in \mathcal{T}_h. \tag{3.7}$$

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Combining the estimates (3.5)–(3.7), we find

$$h^2 \sum_{T\in\mathcal{T}_h}\Big[\sum_{p\in\mathcal{V}_T} v_T^2(p) + h_T^2 \sum_{e\in\mathcal{E}_T}\Big(\Pi_e \frac{\partial v_T}{\partial n}\Big)^2\Big] \lesssim \sum_{T\in\mathcal{T}_h} \|v\|^2_{L_2(T)}$$
$$\lesssim \Big[|v|^2_{H^2(\Omega,\mathcal{T}_h)} + \sum_{p\in\mathcal{V}_h}\sum_{e\in\mathcal{E}_p} h_e^{-2}[\![v(p)]\!]_e^2 + \sum_{e\in\mathcal{E}_h}(\Pi_e[\![\partial v/\partial n]\!]_e)^2\Big] \qquad \forall\, v\in V_h,$$

which implies

$$h^2\langle B_h v, v\rangle = h^2 b_h(v,v) \lesssim a_h(v,v) = \langle A_h v, v\rangle \qquad \forall\, v\in V_h,$$

and hence by the Rayleigh quotient formula

$$\lambda_{\min}(B_h^{-1}A_h) = \min_{v\in V_h\setminus\{0\}} \frac{\langle A_h v, v\rangle}{\langle B_h v, v\rangle} \geq Ch^2. \tag{3.8}$$

The estimate (3.3) follows from (3.4) and (3.8). ∎

REMARK 3.6. Note that the matrix representing the bilinear form $b_h(\cdot,\cdot)$ is block-diagonal (with small diagonal blocks) in the vertex-edge-wise ordering of the dofs. Therefore the preconditioning can also be easily performed in parallel.

4. Preliminary estimates. In this section we establish several results that are useful for the error analysis of the WOPSIP method.

We begin by constructing a linear operator $E_h : V_h \longrightarrow V_h^{HCT}$ by averaging. Let N be any (global) degree of freedom of V_h^{HCT}, i.e., $N(w)$ is either the value of a function w or its first order derivatives at an interior vertex of $\mathcal{T}_h$ or the mean value of the normal derivative of w on an interior edge. For $v \in V_h$, we define

$$N(E_h v) = \frac{1}{|\mathcal{T}_N|}\sum_{T\in\mathcal{T}_N} N(v_T), \tag{4.1}$$

where $\mathcal{T}_N$ is the set of triangles in $\mathcal{T}_h$ that share the degree of freedom N and $|\mathcal{T}_N|$ is the number of elements of $\mathcal{T}_N$.

LEMMA 4.1. *The operator E_h has the following properties*:

$$\sum_{T\in\mathcal{T}_h} h_T^{-4}\|v - E_h v\|^2_{L_2(T)} \leq C \sum_{e\in\mathcal{E}_h}\int_e \big(h_e^{-3}[\![v]\!]_e^2 + h_e^{-1}[\![\partial v/\partial n]\!]_e^2\big)\,ds, \tag{4.2}$$

$$|v - E_h v|^2_{H^2(\Omega,\mathcal{T}_h)} \leq C \sum_{e\in\mathcal{E}_h}\int_e \big(h_e^{-3}[\![v]\!]_e^2 + h_e^{-1}[\![\partial v/\partial n]\!]_e^2\big)\,ds, \tag{4.3}$$

for all $v \in V_h$.

Proof. Let $v \in V_h$ be arbitrary and $w = v - E_h v$. From scaling, we have

$$\sum_{T\in\mathcal{T}_h} h_T^{-4}\|v - E_h v\|^2_{L_2(T)} \tag{4.4}$$
$$\lesssim \sum_{T\in\mathcal{T}_h} h_T^{-2}\Big(\sum_{p\in\mathcal{V}_T}\big(w_T^2(p) + h_T^2|\nabla w_T(p)|^2\big) + \sum_{e\in\mathcal{E}_T} h_T^2|(\partial w_T/\partial n)(m_e)|^2\Big).$$

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It follows from (4.1), scaling, and a standard inverse estimate [13, 16] that

$$\sum_{T\in\mathcal{T}_h} h_T^{-2} \sum_{p\in\mathcal{V}_T} w_T^2(p) \lesssim \sum_{T\in\mathcal{T}_h} h_T^{-2} \sum_{p\in\mathcal{V}_T} \sum_{e\in\mathcal{E}_p} [\![v(p)]\!]_e^2 \lesssim \sum_{e\in\mathcal{E}_h} \int_e h_e^{-3} [\![v]\!]_e^2 ds, \tag{4.5}$$

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} \sum_{p\in\mathcal{V}_T} |\nabla w_T(p)|^2 &\lesssim \sum_{T\in\mathcal{T}_h} \sum_{p\in\mathcal{V}_T} \sum_{e\in\mathcal{E}_p} \Big(\Big[\!\!\Big[\frac{\partial v}{\partial t}(p)\Big]\!\!\Big]_e^2 + \Big[\!\!\Big[\frac{\partial v}{\partial n}(p)\Big]\!\!\Big]_e^2 \Big)\\ &\lesssim \sum_{e\in\mathcal{E}_h} h_e^{-1} \int_e \Big([\![\partial v/\partial t]\!]_e^2 + [\![\partial v/\partial n]\!]_e^2 \Big) ds\\ &\lesssim \sum_{e\in\mathcal{E}_h} \int_e \Big(h_e^{-3} [\![v]\!]_e^2 + h_e^{-1} [\![\partial v/\partial n]\!]_e^2 \Big) ds,\end{aligned} \tag{4.6}$$

$$\sum_{T\in\mathcal{T}_h} \sum_{e\in\mathcal{E}_T} |(\partial w_T/\partial n)(m_e)|^2 \lesssim \sum_{T\in\mathcal{T}_h} \sum_{e\in\mathcal{E}_T} \Big[\!\!\Big[\frac{\partial v}{\partial n}(m_e)\Big]\!\!\Big]_e^2 \lesssim \sum_{e\in\mathcal{E}_h} h_e^{-1} \int_e [\![\partial v/\partial n]\!]_e^2 ds. \tag{4.7}$$

The estimate (4.2) follows from (4.4)–(4.7). The estimate (4.3) in turn follows from (2.1), (4.2), and an inverse estimate. ∎

COROLLARY 4.2. *We have*

$$\begin{aligned}&\sum_{T\in\mathcal{T}_h} \big(h_T^{-4} \|v - E_h v\|_{L_2(T)}^2 + h_T^{-2} |v - E_h v|_{H^1(T)}^2 \big)\\ &\qquad + |v - E_h v|_{H^2(\Omega,\mathcal{T}_h)}^2 + |E_h v|_{H^2(\Omega)}^2\\ &\leq C\Big(|v|_{H^2(\Omega,\mathcal{T}_h)}^2 + \sum_{e\in\mathcal{E}_h} \big(\Pi_e [\![\partial v/\partial n]\!]_e\big)^2 + \sum_{p\in\mathcal{V}_h} \sum_{e\in\mathcal{E}_p} h_e^{-2} [\![v(p)]\!]_e^2 \Big),\end{aligned} \tag{4.8}$$

for all $v \in V_h$.

Proof. Let $v \in V_h$ be arbitrary and v^I be the piecewise linear polynomial such that v_T and v_T^I agrees at the vertices of T for all $T \in \mathcal{T}_h$. It follows from scaling, a standard interpolation error estimate and the trace theorem that

$$\begin{aligned}&\sum_{e\in\mathcal{E}_h} \int_e h_e^{-1} [\![\partial v/\partial n]\!]_e^2 ds\\ &\quad\lesssim \sum_{e\in\mathcal{E}_h} \int_e h_e^{-1} \big(\Pi_e [\![\partial v^I/\partial n]\!]_e\big)^2 ds + \sum_{e\in\mathcal{E}_h} \int_e h_e^{-1} [\![\partial(v - v^I)/\partial n]\!]_e^2 ds\\ &\quad\lesssim \sum_{e\in\mathcal{E}_h} \int_e h_e^{-1} \big(\Pi_e [\![\partial v/\partial n]\!]_e\big)^2 ds + \sum_{e\in\mathcal{E}_h} \int_e h_e^{-1} [\![\partial(v - v^I)/\partial n]\!]_e^2 ds\\ &\quad\lesssim \sum_{e\in\mathcal{E}_h} \big(\Pi_e [\![\partial v/\partial n]\!]_e\big)^2 + \sum_{T\in\mathcal{T}_h} h_T^{-1} \int_{\partial T} \big(\partial(v - v^I)/\partial n\big)^2 ds\\ &\quad\lesssim \sum_{e\in\mathcal{E}_h} \big(\Pi_e [\![\partial v/\partial n]\!]_e\big)^2 + \sum_{T\in\mathcal{T}_h} |v|_{H^2(T)}^2.\end{aligned} \tag{4.9}$$

The estimate for $\sum_{T\in\mathcal{T}_h} h_T^{-4} \|v - E_h v\|_{L_2(T)}^2$ follows from (3.6), (4.2) and (4.9). The rest of

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the estimates then follow from inverse estimates and the triangle inequality. ◻

COROLLARY 4.3. *We have*

$$\sum_{T\in\mathcal{T}_h} h_T^{-1}\int_{\partial T}|\nabla(v-E_hv)|^2ds \lesssim \Big(|v|^2_{H^2(\Omega,\mathcal{T}_h)}+\sum_{e\in\mathcal{E}_h}\big(\Pi_e[\![\partial v/\partial n]\!]_e\big)^2+\sum_{p\in\mathcal{V}_h}\sum_{e\in\mathcal{E}_p}h_e^{-2}[\![v(p)]\!]_e^2\Big) \tag{4.10}$$

for all $v\in V_h$.

Proof. From scaling, we have

$$\sum_{T\in\mathcal{T}_h} h_T^{-1}\int_{\partial T}|\nabla(v-E_hv)|^2ds \lesssim \sum_{T\in\mathcal{T}_h} h_T^{-4}\|v-E_hv\|^2_{L_2(T)},$$

which together with (4.8) implies (4.10). ◻

The following result shows that $E_h\mathcal{I}_h$ can be treated as a quasi-interpolation operator.

LEMMA 4.4. *Let* $\zeta\in H^s(\Omega)$ *for* $2\le s\le 3$ *and* $\mathcal{I}_h\zeta\in\tilde{V}_h^M$ *be the Morley interpolant of* ζ. *We have*

$$\|\zeta-E_h\mathcal{I}_h\zeta\|_{L_2(\Omega)}+h|\zeta-E_h\mathcal{I}_h\zeta|_{H^1(\Omega)}+h^2|\zeta-E_h\mathcal{I}_h\zeta|_{H^2(\Omega)}\le Ch^s|\zeta|_{H^s(\Omega)}. \tag{4.11}$$

Proof. Since $\zeta-E_h\mathcal{I}_h\zeta=0$ if ζ is a quadratic polynomial, the estimate (4.11) follows from the Bramble-Hilbert lemma [6, 19] applied to element patches. (Details for similar results can be found in [8]). ◻

REMARK 4.5. The construction of E_h and the derivation of its properties exploit the fact that the Hsieh-Clough-Tocher element is a C^1 relative of the Morley element. Such *enriching* operators appeared in the analysis of domain decomposition methods and multigrid methods for nonconforming finite elements [7, 8] and in the a posteriori error analysis of a C^0 interior penalty method for the biharmonic problem [11].

Next, we recall two estimates from [11, 22] that generalize the local efficiency estimates in the *a posteriori* analysis. They are derived by the technique of bubble functions [2, 29].

LEMMA 4.6. *Let* u *be the solution of* (1.1). *We have*

$$\sum_{T\in\mathcal{T}_h} h_T^4\|f\|^2_{L_2(T)}\le C\big(|u-v|_{H^2(\Omega,\mathcal{T}_h)}+\mathrm{Osc}(f,\mathcal{T}_h)\big)^2 \qquad \forall\, v\in V_h, \tag{4.12}$$

$$\sum_{e\in\mathcal{E}_h} h_e\int_e\Big[\!\!\Big[\frac{\partial^2 v}{\partial n^2}\Big]\!\!\Big]_e^2 ds\le C\big(|u-v|_{H^2(\Omega,\mathcal{T}_h)}+\mathrm{Osc}(f,\mathcal{T}_h)\big)^2 \qquad \forall\, v\in V_h, \tag{4.13}$$

where

$$\mathrm{Osc}(f,\mathcal{T}_h)=\Big(\sum_{T\in\mathcal{T}_h} h_T^4\|f-\bar f\|^2_{L_2(T)}\Big)^{1/2}, \tag{4.14}$$

and $\bar f$ *is the piecewise constant function that takes the mean value of* f *on each* $T\in\mathcal{T}_h$, *i.e.*,

$$\bar f\big|_T=\frac{1}{|T|}\int_T f\,dx.$$

Finally, we observe that, by replacing v with $u-v$ in (3.6) and (4.9), we have

$$\sum_{e\in\mathcal{E}_h} h_e^{-3}\int_e [\![v]\!]_e^2 ds = \sum_{e\in\mathcal{E}_h} h_e^{-3}\int_e [\![u-v]\!]_e^2 ds \le C\Big(|u-v|^2_{H^2(\Omega,\mathcal{T}_h)} + \sum_{p\in\mathcal{V}_h}\sum_{e\in\mathcal{E}_p} h_e^{-2}[\![(u-v)(p)]\!]_e^2\Big) \qquad \forall\, v\in V_h, \tag{4.15}$$

$$\sum_{e\in\mathcal{E}_h} h_e^{-1}\int_e \left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right]_e^2 ds = \sum_{e\in\mathcal{E}_h} h_e^{-1}\int_e \left[\!\!\left[\frac{\partial (u-v)}{\partial n}\right]\!\!\right]_e^2 ds \le C\Big(|u-v|^2_{H^2(\Omega,\mathcal{T}_h)} + \sum_{e\in\mathcal{E}_h}\big(\Pi_e[\![\partial(u-v)/\partial n]\!]_e\big)^2\Big) \qquad \forall\, v\in V_h. \tag{4.16}$$

A standard inverse estimate and (4.16) immediately yield the following estimate:

$$\sum_{e\in\mathcal{E}_h^i} h_e\int_e \left[\!\!\left[\frac{\partial^2 v}{\partial n\partial t}\right]\!\!\right]_e^2 ds \le C\Big(|u-v|^2_{H^2(\Omega,\mathcal{T}_h)} + \sum_{e\in\mathcal{E}_h}\big(\Pi_e[\![\partial(u-v)/\partial n]\!]_e\big)^2\Big) \tag{4.17}$$

for all $v\in V_h$.

5. A priori error analysis.

We measure the error in the energy norm

$$\|v\|_h = \sqrt{a_h(v,v)} = \Big(|v|^2_{H^2(\Omega,\mathcal{T}_h)} + \sum_{e\in\mathcal{E}_h} h_e^{-2}\big(\Pi_e[\![\partial v/\partial n]\!]_e\big)^2 + \sum_{p\in\mathcal{V}_h}\sum_{e\in\mathcal{E}_p} h_e^{-4}[\![v(p)]\!]_e^2\Big)^{1/2}. \tag{5.1}$$

Following the ideas in [22], we will show that the WOPSIP method is quasi-optimal in the energy norm up to terms that are of order $O(h)$, using only the weak problem (1.1) and the tools developed in Section 4. Thus, the proof of the theorem below does not rely on any elliptic regularity theory for the biharmonic problem.

THEOREM 5.1. *Let u and u_h be the solution of (1.1) and (3.1) respectively. We have*

$$\|u-u_h\|_h \le C\bigg[\inf_{v\in V_h}\|u-v\|_h + \Big(\sum_{T\in\mathcal{T}_h} h_T^2|u|^2_{H^2(T)}\Big)^{1/2} + \mathrm{Osc}(f,\mathcal{T}_h)\bigg]. \tag{5.2}$$

Proof. Let $v\in V_h$ be arbitrary. First, by duality, we have

$$\|u-u_h\|_h \le \|u-v\|_h + \|v-u_h\|_h \le \|u-v\|_h + \max_{w\in V_h\setminus\{0\}}\frac{a_h(v-u_h,w)}{\|w\|_h}. \tag{5.3}$$

Next we write, using (1.1) and (3.1),

$$\begin{aligned} a_h(v-u_h,w) &= a_h(v,w)-(f,w)\\ &= a_h(v,w-E_hw)+a_h(v,E_hw)-(f,w)\\ &= a_h(v,w-E_hw)-a_h(u-v,E_hw)-(f,w-E_hw)\end{aligned} \tag{5.4}$$

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and record the obvious estimates

$$
\begin{aligned}
(5.5)\qquad |a_h(u-v,E_hw)| &= \Big|\sum_{T\in\mathcal{T}_h}\int_T D^2(u-v):D^2(E_hw)\,dx\Big| \\
&\le |u-v|_{H^2(\Omega,\mathcal{T}_h)}|E_hw|_{H^2(\Omega)} \lesssim \|u-v\|_h\|w\|_h, \\
(5.6)\qquad |(f,w-E_hw)| &\lesssim \Big(\sum_{T\in\mathcal{T}_h} h_T^4\|f\|^2_{L_2(T)}\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}_h} h_T^{-4}\|w-E_hw\|^2_{L_2(T)}\Big)^{1/2} \\
&\lesssim \Big(|u-v|_{H^2(\Omega,\mathcal{T}_h)}+\mathrm{Osc}(f,\mathcal{T}_h)\Big)\|w\|_h
\end{aligned}
$$

that follow immediately from (4.8), (4.12), and (5.1).

It remains to estimate the term $a_h(v,w-E_hw)$. We have

$$
\begin{aligned}
a_h(v,w-E_hw) &= \sum_{T\in\mathcal{T}_h}\int_T D^2v:D^2(w-E_hw)\,dx \\
(5.7)\qquad &\quad + \sum_{e\in\mathcal{E}_h} h_e^{-2}(\Pi_e[\![\partial v/\partial n]\!]_e)(\Pi_e[\![\partial(w-E_hw)/\partial n]\!]_e) \\
&\quad + \sum_{p\in\mathcal{V}_h}\sum_{e\in\mathcal{E}_p} h_e^{-4}[\![v(p)]\!]_e[\![(w-E_hw)(p)]\!]_e.
\end{aligned}
$$

Using (5.1), the two last terms on the right-hand side of (5.7) can be easily estimated:

$$
\begin{aligned}
&\sum_{e\in\mathcal{E}_h} h_e^{-2}(\Pi_e[\![\partial v/\partial n]\!]_e)(\Pi_e[\![\partial(w-E_hw)/\partial n]\!]_e) \\
&\qquad + \sum_{p\in\mathcal{V}_h}\sum_{e\in\mathcal{E}_p} h_e^{-4}[\![v(p)]\!]_e[\![(w-E_hw)(p)]\!]_e \\
(5.8)\qquad &\qquad = \sum_{e\in\mathcal{E}_h} h_e^{-2}(\Pi_e[\![\partial(v-u)/\partial n]\!]_e)(\Pi_e[\![\partial w/\partial n]\!]_e) \\
&\qquad\qquad + \sum_{p\in\mathcal{V}_h}\sum_{e\in\mathcal{E}_p} h_e^{-4}[\![(v-u)(p)]\!]_e[\![w(p)]\!]_e \\
&\qquad \le \|u-v\|_h\|w\|_h.
\end{aligned}
$$

For the first term on the right-hand side of (5.7), we find from integration by parts that

$$
\begin{aligned}
&\sum_{T\in\mathcal{T}_h}\int_T D^2v:D^2(w-E_hw)\,dx \\
&\qquad = -\sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}_e \left[\!\!\left[\frac{\partial(w-E_hw)}{\partial n}\right]\!\!\right]_e ds \\
(5.9)\qquad &\qquad\quad - \sum_{e\in\mathcal{E}_h^i}\int_e \left[\!\!\left[\frac{\partial^2 v}{\partial n^2}\right]\!\!\right]_e \left\{\!\!\left\{\frac{\partial(w-E_hw)}{\partial n}\right\}\!\!\right\}_e ds \\
&\qquad\quad - \sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 v}{\partial n\partial t}\right\}\!\!\right\}_e \left[\!\!\left[\frac{\partial(w-E_hw)}{\partial t}\right]\!\!\right]_e ds \\
&\qquad\quad - \sum_{e\in\mathcal{E}_h^i}\int_e \left[\!\!\left[\frac{\partial^2 v}{\partial n\partial t}\right]\!\!\right]_e \left\{\!\!\left\{\frac{\partial(w-E_hw)}{\partial t}\right\}\!\!\right\}_e ds \\
&\qquad = S_1+S_2+S_3+S_4,
\end{aligned}
$$

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and we can estimate the four sums as follows.

From direct calculations, scaling, and (5.1), we have

$$
\begin{aligned}
S_1 &= -\sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}_e \Pi_e \left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right]_e ds \\
&= -\sum_{e\in\mathcal{E}_h} h_e \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}_e \Pi_e \left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right]_e \\
&\le \Big(\sum_{e\in\mathcal{E}_h} h_e^4 \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}_e^2\Big)^{1/2}\Big(\sum_{e\in\mathcal{E}_h} h_e^{-2}\Big(\Pi_e\left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right]_e\Big)^2\Big)^{1/2} \qquad (5.10)\\
&\lesssim \Big(\sum_{T\in\mathcal{T}_h} h_T^2 |v|^2_{H^2(T)}\Big)^{1/2}\|w\|_h \\
&\lesssim \Big(\sum_{T\in\mathcal{T}_h} h_T^2 \big[|u-v|^2_{H^2(T)} + |u|^2_{H^2(T)}\big]\Big)^{1/2}\|w\|_h,
\end{aligned}
$$

and

$$
\begin{aligned}
S_3 &= -\sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 v}{\partial n\partial t}\right\}\!\!\right\}_e \left[\!\!\left[\frac{\partial w}{\partial t}\right]\!\!\right]_e ds \\
&\le \sum_{e\in\mathcal{E}_h} h_e^2 \left\{\!\!\left\{\frac{\partial^2 v}{\partial n\partial t}\right\}\!\!\right\}_e h_e^{-2}\sum_{p\in\mathcal{V}_e} |[\![w(p)]\!]_e| \\
&\le \Big(\sum_{e\in\mathcal{E}_h} h_e^4 \left\{\!\!\left\{\frac{\partial^2 v}{\partial n\partial t}\right\}\!\!\right\}_e^2\Big)^{1/2}\Big(\sum_{e\in\mathcal{E}_h} 2h_e^{-4}\sum_{p\in\mathcal{V}_e}[\![w(p)]\!]_e^2\Big)^{1/2} \qquad (5.11)\\
&\lesssim \Big(\sum_{T\in\mathcal{T}_h} h_T^2 |v|^2_{H^2(T)}\Big)^{1/2}\|w\|_h \\
&\lesssim \Big(\sum_{T\in\mathcal{T}_h} h_T^2 \big[|u-v|^2_{H^2(T)} + |u|^2_{H^2(T)}\big]\Big)^{1/2}\|w\|_h.
\end{aligned}
$$

From Corollary 4.3, (4.13), (4.17), and (5.1), we have

$$
\begin{aligned}
S_2 &\le \Big(\sum_{e\in\mathcal{E}_h} h_e\int_e \left[\!\!\left[\frac{\partial^2 v}{\partial n^2}\right]\!\!\right]_e^2 ds\Big)^{1/2}\Big(\sum_{e\in\mathcal{E}_h} h_e^{-1}\int_e \left\{\!\!\left\{\frac{\partial(w-E_hw)}{\partial n}\right\}\!\!\right\}_e^2 ds\Big)^{1/2} \qquad (5.12)\\
&\lesssim \big(|u-v|_{H^2(\Omega,\mathcal{T}_h)} + \mathrm{Osc}(f,\mathcal{T}_h)\big)\|w\|_h,
\end{aligned}
$$

and

$$
\begin{aligned}
S_4 &\le \Big(\sum_{e\in\mathcal{E}_h} h_e\int_e \left[\!\!\left[\frac{\partial^2 v}{\partial n\partial t}\right]\!\!\right]_e^2 ds\Big)^{1/2}\Big(\sum_{e\in\mathcal{E}_h} h_e^{-1}\int_e \left\{\!\!\left\{\frac{\partial(w-E_hw)}{\partial t}\right\}\!\!\right\}_e^2 ds\Big)^{1/2} \qquad (5.13)\\
&\lesssim \|u-v\|_h\|w\|_h.
\end{aligned}
$$

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It follows from (5.9)–(5.13) that

$$\sum_{T\in\mathcal{T}_h}\int_T D^2v : D^2(w-E_hw)\,dx$$
$$\lesssim\left[\|u-v\|_h+\Big(\sum_{T\in\mathcal{T}_h}h_T^2|u|^2_{H^2(T)}\Big)^{1/2}+\mathrm{Osc}(f,\mathcal{T}_h)\right]\|w\|_h,$$

which together with (5.4)–(5.8) implies

$$\max_{w\in V_h\setminus\{0\}}\frac{a_h(v-u_h,w)}{\|w\|_h}\lesssim\left[\|u-v\|_h+\Big(\sum_{T\in\mathcal{T}_h}h_T^2|u|^2_{H^2(T)}\Big)^{1/2}+\mathrm{Osc}(f,\mathcal{T}_h)\right]$$

and, in view of (5.3), the estimate (5.2). ◻

We now invoke the elliptic regularity theory for nonsmooth domains [5, 17, 20, 21] to obtain a concrete error estimate. According to the regularity theory there exists a number $\alpha\in(1/2,1]$ such that the solution of (1.1) satisfies

$$\|u\|_{H^{2+\alpha}(\Omega)}\le C_\Omega\|f\|_{H^{-2+\alpha}(\Omega)} \tag{5.14}$$

when the right-hand side $f\in H^{-2+\alpha}(\Omega)$ $(\hookleftarrow L_2(\Omega))$. We shall refer to α as the index of elliptic regularity for the biharmonic problem.

THEOREM 5.2. *Let α be the index of elliptic regularity for the biharmonic problem. We have*

$$\|u-u_h\|_h\le Ch^\alpha\|f\|_{L_2(\Omega)}. \tag{5.15}$$

Proof. Let $\mathcal{I}_hu$ be the Morley interpolant of u. From (2.5), (4.14), (5.2), and (5.14), we deduce

$$\|u-u_h\|_h^2\le C\Big(\|u-\mathcal{I}_hu\|_h^2+\sum_{T\in\mathcal{T}_h}h_T^4\big[|u|^2_{H^2(T)}+\|f-\bar{f}\|^2_{L_2(T)}\big]\Big)$$
$$\le C\big(h^{2\alpha}|u|^2_{H^{2+\alpha}(\Omega)}+h^4|u|^2_{H^2(\Omega)}+h^4\|f\|^2_{L_2(\Omega)}\big)\le Ch^{2\alpha}\|f\|^2_{L_2(\Omega)}.$$ ◻

6. A posteriori error analysis.

We will use the following residual-based error estimator in our analysis:

$$\eta_h=\Big(\sum_{j=0}^4\eta_j^2\Big)^{1/2}, \tag{6.1}$$

where

$$\eta_0=\Big(\sum_{T\in\mathcal{T}_h}h_T^4\|f\|^2_{L_2(T)}\Big)^{1/2},$$
$$\eta_1=\Big(\sum_{e\in\mathcal{E}_h}h_e^{-3}\int_e[\![u_h]\!]_e^2\,ds\Big)^{1/2},$$
$$\eta_2=\Big(\sum_{e\in\mathcal{E}_h}h_e^{-1}\int_e[\![\partial u_h/\partial n]\!]_e^2\,ds\Big)^{1/2},$$
$$\eta_3=\Big(\sum_{e\in\mathcal{E}_h}h_e^{-2}\big(\Pi_e[\![\partial u_h/\partial n]\!]_e\big)^2\Big)^{1/2},$$
$$\eta_4=\Big(\sum_{p\in\mathcal{V}_h}\sum_{e\in\mathcal{E}_p}h_e^{-4}[\![u_h(p)]\!]_e^2\Big)^{1/2}.$$

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THEOREM 6.1. *Let u and u_h be the solution of* (1.1) *and* (3.1) *respectively. We have*

$$\|u - u_h\|_h \le C\eta_h. \tag{6.2}$$

Proof. From the definition (5.1), we see that

$$\|u - u_h\|_h^2 = |u - u_h|^2_{H^2(\Omega,\mathcal{T}_h)} + \eta_3^2 + \eta_4^2. \tag{6.3}$$

Furthermore we have, from Lemma 4.1,

$$\begin{aligned}|u - u_h|_{H^2(\Omega,\mathcal{T}_h)} &\le |u - E_h u_h|_{H^2(\Omega)} + |E_h u_h - u_h|_{H^2(\Omega,\mathcal{T}_h)}\\ &\lesssim |u - E_h u_h|_{H^2(\Omega)} + \eta_1 + \eta_2,\end{aligned} \tag{6.4}$$

and by duality,

$$|u - E_h u_h|_{H^2(\Omega)} = \max_{\phi\in H_0^2(\Omega)} \frac{a(u - E_h u_h, \phi)}{|\phi|_{H^2(\Omega)}}. \tag{6.5}$$

Let $\phi \in H_0^2(\Omega)$ be arbitrary. We write, using (1.1) and (3.1),

$$\begin{aligned}a(u - E_h u_h, \phi) &= (f, \phi) - a_h(u_h, \phi) + a_h(u_h - E_h u_h, \phi)\\ &= (f, \phi - \mathcal{I}_h\phi) - a_h(u_h, \phi - \mathcal{I}_h\phi) + a_h(u_h - E_h u_h, \phi).\end{aligned} \tag{6.6}$$

From (2.4), (2.5), (3.2), and Lemma 4.1, we find

$$\begin{aligned}a_h(u_h, \phi - \mathcal{I}_h\phi) &= \sum_{T\in\mathcal{T}_h} \int_T D^2 u_h : D^2(\phi - \mathcal{I}_h\phi)\, dx = 0\\
(f, \phi - \mathcal{I}_h\phi) &\le \sum_{T\in\mathcal{T}_h} \|f\|_{L_2(T)} \|\phi - \mathcal{I}_h\phi\|_{L_2(T)}\\
&\le \Big(\sum_{T\in\mathcal{T}_h} h_T^4 \|f\|^2_{L_2(T)}\Big)^{1/2} \Big(\sum_{T\in\mathcal{T}_h} h_T^{-4} \|\phi - \mathcal{I}_h\phi\|^2_{L_2(T)}\Big)^{1/2}\\
&\lesssim \eta_0 |\phi|_{H^2(\Omega)},\\
a_h(u_h - E_h u_h, \phi) &= \sum_{T\in\mathcal{T}_h} \int_T D^2(u_h - E_h u_h) : D^2\phi\, dx\\
&\le |u_h - E_h u_h|_{H^2(\Omega,\mathcal{T}_h)} |\phi|_{H^2(\Omega)} \lesssim (\eta_1 + \eta_2)|\phi|_{H^2(\Omega)},\end{aligned}$$

which together with (6.5)–(6.6) implies

$$|u - E_h u_h|_{H^2(\Omega,\mathcal{T}_h)} \lesssim \eta_0 + \eta_1 + \eta_2. \tag{6.7}$$

The estimate (6.2) follows from (6.1), (6.3)–(6.4), and (6.7). □

Theorem 6.1 shows that η_h is a reliable error estimator. The next theorem, which states that η_h is also an efficient error estimator, follows immediately from (4.12), (4.15), (4.16), (5.1), and (6.1).

THEOREM 6.2. *Let u and u_h be the solution of* (1.1) *and* (3.1) *respectively. We have*

$$\eta_h \le C\big(\|u - u_h\|_h + \mathrm{Osc}(f, \mathcal{T}_h)\big).$$

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7. Extensions. Let $u_h \in V_h$ be the solution of (3.1). We can view $E_h u_h \in V_h^{HCT}$ as a C^1 solution of (1.1) obtained by post-processing.

THEOREM 7.1. *The following error estimate holds for the post-processed solution* $E_h u_h$:

$$|u - E_h u_h|_{H^2(\Omega)} \le Ch^\alpha \|f\|_{L_2(\Omega)}.$$

Proof. This is a direct consequence of (2.5), (4.8), (4.11), (5.14), and (5.15).

$$\begin{aligned}
|u - E_h u_h|_{H^2(\Omega)} &\le |u - E_h \mathcal{I}_h u|_{H^2(\Omega)} + |E_h(\mathcal{I}_h u - u_h)|_{H^2(\Omega)} \\
&\lesssim |u - E_h \mathcal{I}_h u|_{H^2(\Omega)} + \|\mathcal{I}_h u - u_h\|_h \\
&\lesssim |u - E_h \mathcal{I}_h u|_{H^2(\Omega)} + \|\mathcal{I}_h u - u\|_h + \|u - u_h\|_h \\
&\lesssim h^\alpha \|f\|_{L_2(\Omega)} \qquad \square
\end{aligned}$$

Following the ideas in [8], we can also derive other error estimates for $E_h u_h$. The key is to understand the adjoint operator $E_h^* : H_0^2(\Omega) \longrightarrow V_h$ defined by

$$a_h(E_h^* \phi, v) = a(\phi, E_h v) \qquad \forall\, \phi \in H_0^2(\Omega),\ v \in V_h. \tag{7.1}$$

REMARK 7.2. It follows from (1.1) and (7.1) that $E_h^* u \in V_h$ satisfies

$$a_h(E_h^* u, v) = (f, E_h v) \qquad \forall\, v \in V_h.$$

Therefore $E_h^* u$ is the solution of a modified version of the WOPSIP method that can be applied to (1.1) for a general right-hand side $f \in H^{-s}(\Omega)$, where $-2 \le s \le 0$.

We begin with a technical lemma.

LEMMA 7.3. *Let α be the index of elliptic regularity for the biharmonic problem. We have*

$$\Big| \sum_{T \in \mathcal{T}_h} \int_T D^2\zeta : D^2(w - E_h w)\, dx \Big| \le Ch^\alpha \|\zeta\|_{H^{2+\alpha}(\Omega)} \|w\|_h, \tag{7.2}$$

$$\Big| \sum_{T \in \mathcal{T}_h} \int_T D^2\zeta : D^2(\mathcal{I}_h\phi - E_h \mathcal{I}_h \phi)\, dx \Big| \le Ch^{2\alpha} \|\zeta\|_{H^{2+\alpha}(\Omega)} \|\phi\|_{H^{2+\alpha}(\Omega)}, \tag{7.3}$$

for all $\zeta \in H^{2+\alpha}(\Omega)$, $w \in V_h$ and $\phi \in H^{2+\alpha}(\Omega) \cap H_0^2(\Omega)$.

Proof. Let $w \in V_h$ be arbitrary. We have, by Corollary 4.2 and (5.1),

$$\begin{aligned}
\Big| \sum_{T \in \mathcal{T}_h} \int_T D^2\zeta : D^2(w - E_h w)\, dx \Big| &\le |\zeta|_{H^2(\Omega)} |w - E_h w|_{H^2(\Omega, \mathcal{T}_h)} \\
&\le C|\zeta|_{H^2(\Omega)} \|w\|_h \qquad \forall\, \zeta \in H^2(\Omega).
\end{aligned} \tag{7.4}$$

For $\zeta \in H^3(\Omega)$, it follows from integration by parts that

$$\begin{aligned}
&\sum_{T \in \mathcal{T}_h} \int_T D^2\zeta : D^2(w - E_h w)\, dx \\
&= - \sum_{T \in \mathcal{T}_h} \int_T \nabla(\Delta\zeta) \cdot \nabla(w - E_h w)\, dx - \sum_{e \in \mathcal{E}_h} \int_e \frac{\partial^2 \zeta}{\partial n_e^2} \Big[\!\Big[\frac{\partial w}{\partial n} \Big]\!\Big]_e ds \\
&\quad - \sum_{e \in \mathcal{E}_h} \int_e \frac{\partial^2 \zeta}{\partial n_e \partial t_e} \Big[\!\Big[\frac{\partial w}{\partial t} \Big]\!\Big]_e ds,
\end{aligned} \tag{7.5}$$

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and the three terms on the right-hand side can be estimated as follows.

We apply Corollary 4.2 and (5.1) to bound the first term:

$$\Big|\sum_{T\in\mathcal{T}_h}\int_T \nabla(\Delta\zeta)\cdot\nabla(w-E_hw)\,dx\Big| \le |\zeta|_{H^3(\Omega)}|w-E_hw|_{H^1(\Omega,\mathcal{T}_h)} \le Ch|\zeta|_{H^3(\Omega)}\|w\|_h. \tag{7.6}$$

For the second term, we write

$$-\sum_{e\in\mathcal{E}_h}\int_e \frac{\partial^2\zeta}{\partial n_e^2}\left[\!\left[\frac{\partial w}{\partial n}\right]\!\right]_e ds = -\sum_{e\in\mathcal{E}_h}\int_e\Big(\frac{\partial^2\zeta}{\partial n_e^2}-\omega_e\Big)\left[\!\left[\frac{\partial w}{\partial n}\right]\!\right]_e ds - \sum_{e\in\mathcal{E}_h}\int_e \omega_e\Pi_e\left[\!\left[\frac{\partial w}{\partial n}\right]\!\right]_e ds,$$

where

$$\omega_e = \frac{1}{|T|}\int_T \frac{\partial^2\zeta}{\partial n_e^2}dx$$

is the mean value of $\partial^2\zeta/\partial n_e^2$ over a triangle $T\in\mathcal{T}_e$. We then find by using (4.9), (5.1), the trace theorem, and a standard interpolation error estimate that

$$\begin{aligned}\Big|\sum_{e\in\mathcal{E}_h}\int_e \frac{\partial^2\zeta}{\partial n_e^2}\left[\!\left[\frac{\partial w}{\partial n}\right]\!\right]_e ds\Big| &\lesssim h|\zeta|_{H^3(\Omega)}\Big(\sum_{e\in\mathcal{E}_h}h_e^{-1}\int_e\left[\!\left[\frac{\partial w}{\partial n}\right]\!\right]_e^2 ds\Big)^{1/2} + \sum_{e\in\mathcal{E}_h}h_e|\omega_e|\cdot|\Pi_e[\![\partial w/\partial n]\!]_e| \\ &\lesssim h|\zeta|_{H^3(\Omega)}\|w\|_h + \Big(\sum_{e\in\mathcal{E}_h}h_e^4\omega_e^2\Big)^{1/2}\Big(\sum_{e\in\mathcal{E}_h}h_e^{-2}\big[\Pi_e[\![\partial w/\partial n]\!]_e\big]^2\Big)^{1/2} \\ &\lesssim h|\zeta|_{H^3(\Omega)}\|w\|_h + h|\zeta|_{H^2(\Omega)}\|w\|_h \lesssim h\|\zeta\|_{H^3(\Omega)}\|w\|_h.\end{aligned} \tag{7.7}$$

For the third term, we write

$$-\sum_{e\in\mathcal{E}_h}\int_e \frac{\partial^2\zeta}{\partial n_e\partial t_e}\left[\!\left[\frac{\partial w}{\partial t}\right]\!\right]_e ds = -\sum_{e\in\mathcal{E}_h}\int_e\Big(\frac{\partial^2\zeta}{\partial n_e\partial t_e}-\tau_e\Big)\left[\!\left[\frac{\partial w}{\partial t}\right]\!\right]_e ds - \sum_{e\in\mathcal{E}_h}\int_e \tau_e\left[\!\left[\frac{\partial w}{\partial t}\right]\!\right]_e ds,$$

where

$$\tau_e = \frac{1}{|T|}\int_T \frac{\partial^2\zeta}{\partial n_e\partial t_e}dx$$

is the mean value of $\partial^2\zeta/\partial n_e\partial t_e$ over a triangle $T\in\mathcal{T}_e$. We then obtain by using (3.6), (5.1),

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the trace theorem, and standard interpolation error and inverse estimates that

$$
\begin{aligned}
&\Big|\sum_{e\in\mathcal{E}_h}\int_e \frac{\partial^2\zeta}{\partial n_e\partial t_e}\left[\!\!\left[\frac{\partial w}{\partial t}\right]\!\!\right]_e ds\Big| \\
&\quad\lesssim h|\zeta|_{H^3(\Omega)}\Big(\sum_{e\in\mathcal{E}_h} h_e^{-1}\int_e \left[\!\!\left[\frac{\partial w}{\partial t}\right]\!\!\right]_e^2 ds\Big)^{1/2} + \sum_{e\in\mathcal{E}_h}|\tau_e|\sum_{p\in\mathcal{V}_e}|[\![w(p)]\!]_e| \\
&\quad\lesssim h|\zeta|_{H^3(\Omega)}\Big(\sum_{e\in\mathcal{E}_h} h_e^{-3}\int_e [\![w]\!]_e^2 ds\Big)^{1/2} \\
&\qquad+\Big(\sum_{e\in\mathcal{E}_h} h_e^4\tau_e^2\Big)^{1/2}\Big(\sum_{e\in\mathcal{E}_h} h_e^{-4}\sum_{p\in\mathcal{V}_e}[\![w(p)]\!]_e^2\Big)^{1/2} \\
&\quad\lesssim h|\zeta|_{H^3(\Omega)}\|w\|_h + h|\zeta|_{H^2(\Omega)}\|w\|_h \lesssim h\|\zeta\|_{H^3(\Omega)}\|w\|_h.
\end{aligned}\tag{7.8}
$$

Combining (7.5)–(7.8), we have

$$
\Big|\sum_{T\in\mathcal{T}_h}\int_T D^2\zeta : D^2(w-E_hw)\,dx\Big| \le Ch\|\zeta\|_{H^3(\Omega)}\|w\|_h \qquad \forall\,\zeta\in H^3(\Omega).\tag{7.9}
$$

The estimate (7.2) follows from (7.4), (7.9), and interpolation between Sobolev spaces [1, 4, 27, 28].

Next, we derive (7.3) by a similar approach. Let $\phi\in H^{2+\alpha}(\Omega)\cap H_0^2(\Omega)$ be arbitrary. We have, by (2.5) and (4.11),

$$
\begin{aligned}
&\Big|\sum_{T\in\mathcal{T}_h}\int_T D^2\zeta : D^2(\mathcal{I}_h\phi - E_h\mathcal{I}_h\phi)\,dx\Big| \\
&\quad\le |\zeta|_{H^2(\Omega)}|\mathcal{I}_h\phi - E_h\mathcal{I}_h\phi|_{H^2(\Omega,\mathcal{T}_h)} \\
&\quad\le |\zeta|_{H^2(\Omega)}\big(|\mathcal{I}_h\phi-\phi|_{H^2(\Omega,\mathcal{T}_h)} + |\phi - E_h\mathcal{I}_h\phi|_{H^2(\Omega)}\big) \\
&\quad\le Ch^\alpha|\zeta|_{H^2(\Omega)}\|\phi\|_{H^{2+\alpha}(\Omega)} \qquad\qquad \forall\,\zeta\in H^2(\Omega).
\end{aligned}\tag{7.10}
$$

For $\zeta\in H^3(\Omega)$, it follows from integration by parts that

$$
\begin{aligned}
&\sum_{T\in\mathcal{T}_h}\int_T D^2\zeta : D^2(\mathcal{I}_h\phi - E_h\mathcal{I}_h\phi)\,dx \\
&\quad= -\sum_{T\in\mathcal{T}_h}\int_T \nabla(\Delta\zeta)\cdot\nabla(\mathcal{I}_h\phi - E_h\mathcal{I}_h\phi)\,dx \\
&\qquad -\sum_{e\in\mathcal{E}_h}\int_e\Big(\frac{\partial^2\zeta}{\partial n_e^2}\left[\!\!\left[\frac{\partial\mathcal{I}_h\phi}{\partial n}\right]\!\!\right]_e + \frac{\partial^2\zeta}{\partial n_e\partial t_e}\left[\!\!\left[\frac{\partial\mathcal{I}_h\phi}{\partial t}\right]\!\!\right]_e\Big)ds \\
&\quad= -\sum_{T\in\mathcal{T}_h}\int_T \nabla(\Delta\zeta)\cdot\nabla(\mathcal{I}_h\phi - E_h\mathcal{I}_h\phi)\,dx \\
&\qquad -\sum_{e\in\mathcal{E}_h}\int_e\Big(\frac{\partial^2\zeta}{\partial n_e^2}-\omega_e\Big)\left[\!\!\left[\frac{\partial(\mathcal{I}_h\phi-\phi)}{\partial n}\right]\!\!\right]_e ds \\
&\qquad\quad -\sum_{e\in\mathcal{E}_h}\int_e\Big(\frac{\partial^2\zeta}{\partial n_e\partial t_e}-\tau_e\Big)\left[\!\!\left[\frac{\partial(\mathcal{I}_h\phi-\phi)}{\partial t}\right]\!\!\right]_e ds.
\end{aligned}\tag{7.11}
$$

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Using (2.5), (4.11), the trace theorem, and a standard interpolation error estimate, we obtain from (7.11)

$$\begin{aligned}(7.12)\qquad &\Big|\sum_{T\in\mathcal{T}_h}\int_T D^2\zeta : D^2(\mathcal{I}_h\phi - E_h\mathcal{I}_h\phi)\,dx\Big| \\ &\qquad \le Ch^{1+\alpha}|\zeta|_{H^3(\Omega)}\|\phi\|_{H^{2+\alpha}(\Omega)} \qquad \forall\,\zeta\in H^3(\Omega).\end{aligned}$$

The estimate (7.3) now follows from (7.10), (7.12), and interpolation between Sobolev spaces. □

We are now ready to establish the properties of E_h^*.

LEMMA 7.4. *Let α be the index of elliptic regularity for the biharmonic problem. We have*

$$(7.13)\qquad \|\zeta - E_h^*\zeta\|_h \le Ch^\alpha\|\zeta\|_{H^{2+\alpha}(\Omega)} \qquad \forall\,\zeta\in H^{2+\alpha}(\Omega)\cap H_0^2(\Omega).$$

Proof. We have the following standard estimate for nonconforming methods:

$$\begin{aligned}\|\zeta - E_h^*\zeta\|_h &\le \|\zeta - v\|_h + \|v - E_h^*\zeta\|_h \\ (7.14)\qquad &\le \|\zeta - v\|_h + \sup_{w\in V_h\setminus\{0\}}\frac{a_h(v - E_h^*\zeta, w)}{\|w\|_h} \\ &\le 2\|\zeta - v\|_h + \sup_{w\in V_h\setminus\{0\}}\frac{a_h(\zeta - E_h^*\zeta, w)}{\|w\|_h} \qquad \forall\,v\in V_h.\end{aligned}$$

Let $w\in V_h$ be arbitrary. Since $\zeta\in H_0^2(\Omega)$, we have

$$a_h(\zeta - E_h^*\zeta, w) = a_h(\zeta, w - E_h w) = \sum_{T\in\mathcal{T}_h}\int_T D^2\zeta : D^2(w - E_h w)\,dx.$$

It then follows from (7.2) that

$$a_h(\zeta - E_h^*\zeta, w) \le Ch^\alpha\|\zeta\|_{H^{2+\alpha}(\Omega)}\|w\|_h,$$

which implies

$$\sup_{w\in V_h\setminus\{0\}}\frac{a_h(\zeta - E_h^*\zeta, w)}{\|w\|_h} \le Ch^\alpha\|\zeta\|_{H^{2+\alpha}(\Omega)}.$$

Finally, by taking $v = \mathcal{I}_h\zeta$, we obtain from (2.5)

$$(7.15)\qquad \|\zeta - v\|_h = \|\zeta - \mathcal{I}_h\zeta\|_h = |\zeta - \mathcal{I}_h\zeta|_{H^2(\Omega,\mathcal{T}_h)} \le Ch^\alpha\|\zeta\|_{H^{2+\alpha}(\Omega)}.$$

The estimate (7.13) follows from (7.14)–(7.15). □

LEMMA 7.5. *Let α be the index of elliptic regularity for the biharmonic problem. We have*

$$|a_h(\zeta - E_h^*\zeta, \mathcal{I}_h\phi)| \le Ch^{2\alpha}\|\zeta\|_{H^{2+\alpha}(\Omega)}\|\phi\|_{H^{2+\alpha}(\Omega)}$$

for all $\zeta,\phi\in H^{2+\alpha}(\Omega)\cap H_0^2(\Omega)$.

Proof. In view of the fact that

$$a_h(\zeta - E_h^*\zeta, \mathcal{I}_h\phi) = a_h(\zeta, \mathcal{I}_h\phi - E_h\mathcal{I}_h\phi) = \sum_{T\in\mathcal{T}_h}\int_T D^2\zeta : D^2(\mathcal{I}_h\phi - E_h\mathcal{I}_h\phi)\,dx,$$

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the estimate follows immediately from (7.3). ∎

Next, we will derive error estimates in the norm of the Sobolev space $H^{2-\alpha}(\Omega)$. The following duality formula is useful:

$$\|\psi\|_{H^{2-\alpha}(\Omega)} = \max_{\phi \in H^{-2+\alpha}(\Omega)\setminus\{0\}} \frac{\phi(\psi)}{\|\phi\|_{H^{-2+\alpha}(\Omega)}} \qquad \forall\, \psi \in H_0^{2-\alpha}(\Omega). \tag{7.16}$$

Given any $\phi \in H^{-2+\alpha}(\Omega)$, let $\xi \in H_0^2(\Omega)$ satisfy

$$a(\xi, v) = \phi(v) \qquad \forall\, v \in H_0^2(\Omega). \tag{7.17}$$

Then elliptic regularity (cf. (5.14)) implies

$$\|\xi\|_{H^{2+\alpha}(\Omega)} \le C_\Omega \|\phi\|_{H^{-2+\alpha}(\Omega)}. \tag{7.18}$$

LEMMA 7.6. *Let α be the index of elliptic regularity for the biharmonic problem. We have*

$$\|\zeta - E_h(E_h^*\zeta)\|_{H^{2-\alpha}(\Omega)} \le Ch^{2\alpha}\|\zeta\|_{H^{2+\alpha}(\Omega)} \qquad \forall\, \zeta \in H^{2+\alpha}(\Omega) \cap H_0^2(\Omega). \tag{7.19}$$

Proof. Let $\phi \in H^{-2+\alpha}(\Omega)$ be arbitrary and $\xi \in H_0^2(\Omega)$ satisfy (7.17). It follows from (2.5), (7.1), Lemma 7.4, Lemma 7.5, and (7.18) that

$$\begin{aligned}
\phi(\zeta - E_hE_h^*\zeta) &= a(\xi, \zeta - E_hE_h^*\zeta) \\
&= a(\xi, \zeta) - a_h(E_h^*\xi, E_h^*\zeta) \\
&= a_h(\xi - E_h^*\xi, \zeta) + a_h(E_h^*\xi, \zeta - E_h^*\zeta) \\
&= a_h(\xi - E_h^*\xi, \zeta) + a_h(\xi, \zeta - E_h^*\zeta) - a_h(\xi - E_h^*\xi, \zeta - E_h^*\zeta) \\
&= a_h(\xi - E_h^*\xi, \mathcal{I}_h\zeta) + a_h(\xi - E_h^*\xi, \zeta - \mathcal{I}_h\zeta) \\
&\qquad + a_h(\mathcal{I}_h\xi, \zeta - E_h^*\zeta) + a_h(\xi - \mathcal{I}_h\xi, \zeta - E_h^*\zeta) \\
&\qquad\quad - a_h(\xi - E_h^*\xi, \zeta - E_h^*\zeta) \\
&\lesssim h^{2\alpha}\|\zeta\|_{H^{2+\alpha}(\Omega)}\|\phi\|_{H^{-2+\alpha}(\Omega)},
\end{aligned}$$

which together with (7.16) implies (7.19). ∎

THEOREM 7.7. *The following error estimate holds for the post-processed solution E_hu_h :*

$$\|u - E_hu_h\|_{H^{2-\alpha}(\Omega)} \le Ch^{2\alpha}\|f\|_{L_2(\Omega)},$$

where α is the index of elliptic regularity for the biharmonic problem.

Proof. Let $\phi \in H^{-2+\alpha}(\Omega)$ be arbitrary and $\xi \in H_0^2(\Omega)$ satisfy (7.17). It follows from (1.1), (2.5), (3.1), (4.11), (5.14), (5.15), (7.1), Lemma 7.4, and (7.18) that

$$\begin{aligned}
\phi(E_h(u_h - E_h^*u)) &= a(\xi, E_h(u_h - E_h^*u)) \\
&= a_h(E_h^*\xi - \mathcal{I}_h\xi, u_h - E_h^*u) + a_h(\mathcal{I}_h\xi, u_h - E_h^*u) \\
&= a_h(E_h^*\xi - \mathcal{I}_h\xi, u_h - E_h^*u) + a_h(\mathcal{I}_h\xi, u_h) - a(E_h\mathcal{I}_h\xi, u) \\
&= a_h(E_h^*\xi - \mathcal{I}_h\xi, u_h - E_h^*u) + (f, \mathcal{I}_h\xi - E_h\mathcal{I}_h\xi) \\
&\lesssim h^{2\alpha}\|\xi\|_{H^{2+\alpha}(\Omega)}\|f\|_{L_2(\Omega)} + h^{2+\alpha}\|\xi\|_{H^{2+\alpha}(\Omega)}\|f\|_{L_2(\Omega)} \\
&\lesssim h^{2\alpha}\|\phi\|_{H^{-2+\alpha}(\Omega)}\|f\|_{L_2(\Omega)},
\end{aligned}$$

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which together with (7.16) implies

$$\|E_h(u_h - E_h^* u)\|_{H^{2-\alpha}(\Omega)} \leq Ch^{2\alpha}\|f\|_{L_2(\Omega)},$$

and hence, in view of (5.14) and Lemma 7.6,

$$\begin{aligned}\|u - E_h u_h\|_{H^{2-\alpha}(\Omega)} &\leq \|u - E_h E_h^* u\|_{H^{2-\alpha}(\Omega)} + \|E_h(u_h - E_h^* u)\|_{H^{2-\alpha}(\Omega)} \\ &\leq Ch^{2\alpha}\|f\|_{L_2(\Omega)}. \qquad \square\end{aligned}$$

The following corollary is immediate.

COROLLARY 7.8. *Let α be the index of elliptic regularity for the biharmonic problem. The following error estimate holds for the post-processed solution $E_h u$:*

$$\|u - E_h u_h\|_{L_2(\Omega)} \leq Ch^{2\alpha}\|f\|_{L_2(\Omega)}.$$

REMARK 7.9. Since $\alpha = 1$ when Ω is convex, we have

$$\|u - E_h u\|_{L_2(\Omega)} \leq \|u - E_h u\|_{H^1(\Omega)} \leq Ch^2\|f\|_{L_2(\Omega)} \tag{7.20}$$

for a convex domain Ω.

REMARK 7.10. We see from Theorem 7.1, Theorem 7.7, and Corollary 7.8 that the post-processed solution $E_h u_h$ satisfies all the correct error estimates. Therefore the WOPSIP method is also relevant for computing C^1 solutions of (1.1).

We can now establish an L_2 error estimate for the solution u_h of (3.1).

COROLLARY 7.11. *Let α be the index of elliptic regularity for the biharmonic problem. We have*

$$\|u - u_h\|_{L_2(\Omega)} \leq Ch^{2\alpha}\|f\|_{L_2(\Omega)}.$$

Proof. Using Corollary 4.2, (5.1), Theorem 5.2, and Theorem 7.7, we find

$$\begin{aligned}\|u - u_h\|_{L_2(\Omega)} &\leq \|u - E_h u_h\|_{L_2(\Omega)} + \|u_h - E_h u_h\|_{L_2(\Omega)} \\ &\lesssim h^{2\alpha}\|f\|_{L_2(\Omega)} + h^2\|u_h\|_h \\ &\lesssim h^{2\alpha}\|f\|_{L_2(\Omega)} + h^2\big[\|u - u_h\|_h + |u|_{H^2(\Omega)}\big] \\ &\lesssim h^{2\alpha}\|f\|_{L_2(\Omega)}. \qquad \square\end{aligned}$$

Finally, we have a convergence theorem for the modified WOPSIP method (Remark 7.2) when the right-hand side f is in $H^{-2+\alpha}(\Omega)$.

THEOREM 7.12. *The following error estimates hold for the modified WOPSIP method*:

$$\|u - E_h^* u\|_h \leq Ch^{\alpha}\|f\|_{H^{-2+\alpha}(\Omega)}, \tag{7.21}$$

$$\|u - E_h E_h^* u\|_{H^{2-\alpha}(\Omega)} \leq Ch^{2\alpha}\|f\|_{H^{-2+\alpha}(\Omega)}, \tag{7.22}$$

$$\|u - E_h^* u\|_{L_2(\Omega)} \leq Ch^{2\alpha}\|f\|_{H^{-2+\alpha}(\Omega)}, \tag{7.23}$$

where α is the index of elliptic regularity for the biharmonic problem.

Proof. The estimates (7.21) and (7.22) follow directly from (5.14), Lemma 7.4, and Lemma 7.6. Together with Corollary 4.2, these estimates imply (7.23):

$$\begin{aligned}\|u - E_h^* u\|_{L_2(\Omega)} &\leq \|u - E_h E_h^* u\|_{L_2(\Omega)} + \|E_h E_h^* u - E_h^* u\|_{L_2(\Omega)} \\ &\lesssim \|u - E_h E_h^* u\|_{H^{2-\alpha}(\Omega)} + h^2\|E_h^* u\|_h \\ &\lesssim h^{2\alpha}\|u\|_{H^{2+\alpha}(\Omega)} + h^2\big(\|E_h^* u - u\|_h + |u|_{H^2(\Omega)}\big) \\ &\lesssim h^{2\alpha}\|f\|_{H^{-2+\alpha}(\Omega)}. \qquad \square\end{aligned}$$

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8. Numerical results. In this section we report the results of several numerical experiments. For the first set of numerical experiments, we take Ω to be the unit square $(0,1)\times(0,1)$ and the exact solution of (1.1) to be

$$u(x,y) = 100x^2(1-x)^2y^2(1-y)^2.$$

We compute the solution u_h of (3.1) on several uniform grids with mesh sizes $h = 1/2^i$ for $i = 1,2,\cdots 5$. The errors in the energy norm and the L_2 norm together with their orders of convergence are presented in Table 8.1. These numerical results clearly match the theoretical results in Theorem 5.2 and Corollary 7.11.

TABLE 8.1
Errors and orders of convergence for the WOPSIP method.

h	$\|u-u_h\|_h$	Order	$\|u-u_h\|_{L_2(\Omega)}$	Order
1/2	17.197247559437201	–	1.716371062750962	–
1/4	7.101544418782296	1.2759	0.277965054420832	2.6263
1/8	3.201633853279419	1.1493	0.058037854904340	2.2598
1/16	1.537650668307142	1.0580	0.013567912004911	2.0967
1/32	0.758213520169036	1.0200	0.003314694892577	2.0332

For comparison, we compute the solutions of the Morley nonconforming method on the same grids and tabulate the errors and their orders of convergence in Table 8.2. It is evident that the magnitudes of the errors of these two methods are similar.

TABLE 8.2
Errors and order of convergence for the Morley nonconforming method.

h	$\|u-u_h\|_h$	order	$\|u-u_h\|_{L_2(\Omega)}$	order
1/2	6.977229062890963	–	0.323439329204391	–
1/4	4.774226099723017	0.5474	0.141179892106251	1.1960
1/8	2.628590625177154	0.8610	0.041796714441572	1.7561
1/16	1.354472363700791	0.9566	0.011013849766270	1.9241
1/32	0.682957077321058	0.9879	0.002795365814910	1.9782

We also compute the post-processed solution $E_h u_h$ and present the L_2 error and the order of convergence in Table 8.3. These numerical results match the theoretical estimate (7.20). Table 8.4 contains the condition number of the preconditioned system $B_h^{-1}A_h$ and its order

TABLE 8.3
Error and order of convergence for the post-processed solution.

h	$\|u-E_h u_h\|_{L_2(\Omega)}$	Order
1/2	1.695407290104347	–
1/4	0.276528870979329	2.6161
1/8	0.058428415656511	2.2427
1/16	0.013603479656276	2.1027
1/32	0.003317111091565	2.0360

of growth (in terms of h^{-1}). The order of growth is clearly 4, as predicted by Lemma 3.5.

For the second set of numerical experiments we take Ω to be the L-shaped domain $(-1,1)^2 \setminus \big([0,1)\times(-1,0]\big)$, and consider the model problem (1.1) on Ω with the following singular solution [21, p.107]:

$$u(r,\theta) = (r^2\cos^2\theta - 1)^2(r^2\sin^2\theta - 1)^2 r^{(1+z)} g(\theta),$$

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TABLE 8.4
Condition numbers of the preconditioned system and orders of growth.

h	Condition number of $B_h^{-1}A_h$	Order of growth
1/2	2.127683603246884e+001	–
1/4	4.501699323404198e+001	1.0812
1/8	2.910422778240135e+002	2.6927
1/16	3.526267672393217e+003	3.5988
1/32	5.927291290250981e+004	4.0712

where $z = 0.544483736782464$ is a noncharacteristic root of $\sin^2(z\omega) = z^2\sin^2(\omega)$ with $\omega = \frac{3\pi}{2}$,

$$\begin{aligned} g(\theta) = &\left[\frac{1}{z-1}\sin\left((z-1)\omega\right) - \frac{1}{z+1}\sin\left((z+1)\omega\right)\right] \\ &\times \left[\cos((z-1)\theta) - \cos((z+1)\theta)\right] \\ &- \left[\frac{1}{z-1}\sin((z-1)\theta) - \frac{1}{z+1}\sin((z+1)\theta)\right] \\ &\times \left[\cos\left((z-1)\omega\right) - \cos\left((z+1)\omega\right)\right], \end{aligned}$$

and (r, θ) are the polar coordinates. We compute the discrete solution u_k on a sequence of adaptive meshes $\mathcal{T}_k$ generated by bisecting the marked triangles and edges of $\mathcal{T}_{k-1}$, where the triangles and edges are marked according to the bulk criteria of Dörfler [18]. The error estimator captures the singularities of the solution throughout the mesh refinement process; cf. Figure 8.1.

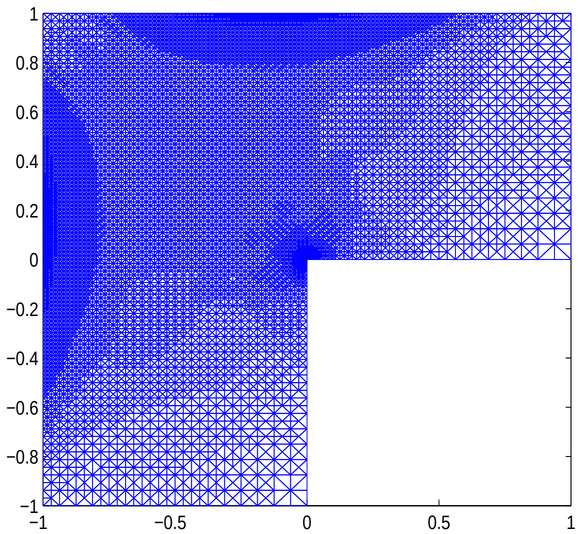

FIGURE 8.1. *Adaptive mesh after 30 refinement steps.*

The energy error and the error estimator are plotted against the number of dofs in the log-log plot in Figure 8.2, which demonstrates that the error estimator is reliable (Theorem 6.1) and that the performance of the adaptive algorithm is optimal.

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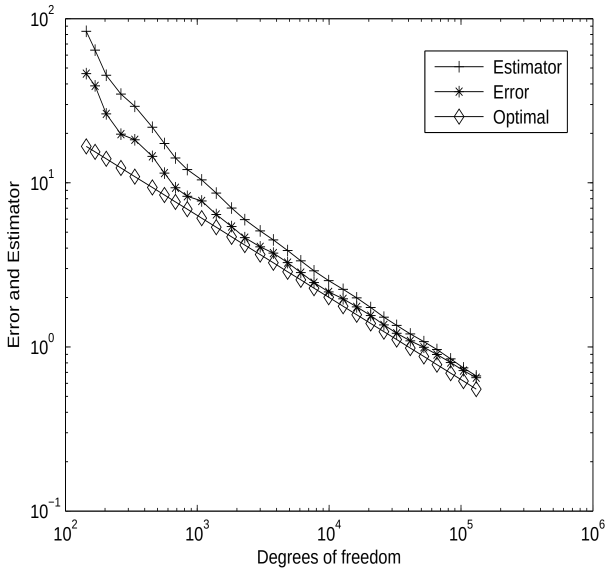

FIGURE 8.2. *Error and estimator for the problem on the L-shaped domain.*

The efficiency index given by $\eta_h/\|u-u_h\|_h$ is computed as a function of the number of dofs and then plotted in Figure 8.3, which shows that the error estimator is efficient (Theorem 6.2).

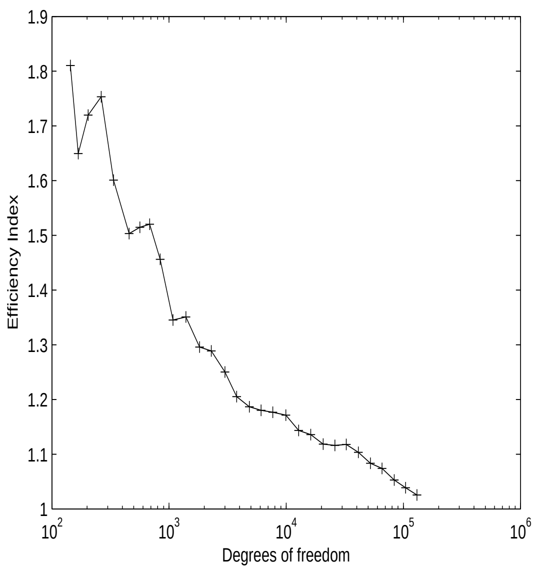

FIGURE 8.3. *Efficiency of the error estimator for the problem on the L-shaped domain.*

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9. Concluding remarks. Even though the number of dofs of the WOPSIP method is three times the number of dofs of the classical Morley nonconforming method, the intrinsic parallelism of the WOPSIP method can potentially be exploited to result in a much faster algorithm. As a first step, we have shown in this paper that the performance of the WOPSIP method is similar to the performance of the Morley method in terms of the magnitudes of the discretization errors, and that reliable and efficient error estimators are available for adaptive solvers.

The WOPSIP method developed in this paper can be applied to general fourth order problems. It can also be applied to a fourth order singular perturbation problem of the form $\epsilon^2\Delta^2 u - \Delta u = f$, provided the second order term is correctly discretized as in [30]. The WOPSIP approach can also be used to construct an intrinsically parallel version of the Nilssen-Tai-Winther finite element method [24] that is designed to handle fourth order singular perturbation problems. Research in this direction will be reported elsewhere.

REFERENCES

[1] R. ADAMS AND J. FOURNIER, *Sobolev Spaces*, second ed., Academic Press, New York, 2003.
[2] M. AINSWORTH AND J. T. ODEN, *A Posteriori Error Estimation in Finite Element Analysis*, Wiley-Interscience, New York, 2000.
[3] J. ARGYRIS, I. FRIED, AND D. SCHARPF, *The TUBA family of plate elements for the matrix displacement method*, Aeronaut J., 72 (1968), pp. 701–709.
[4] J. BERGH AND J. LÖFSTRÖM, *Interpolation Spaces*, Springer, Berlin, 1976.
[5] H. BLUM AND R. RANNACHER, *On the boundary value problem of the biharmonic operator on domains with angular corners*, Math. Methods Appl. Sci., 2 (1980), pp. 556–581.
[6] J. BRAMBLE AND S. HILBERT, *Estimation of linear functionals on Sobolev spaces with applications to Fourier transforms and spline interpolation*, SIAM J. Numer. Anal., 7 (1970), pp. 113–124.
[7] S. BRENNER, *Two-level additive Schwarz preconditioners for nonconforming finite element methods*, Math. Comp., 65 (1996), pp. 897–921.
[8] ———, *Convergence of nonconforming multigrid methods without full elliptic regularity*, Math. Comp., 68 (1999), pp. 25–53.
[9] S. BRENNER, T. GUDI, L. OWENS, AND L.-Y. SUNG, *An intrinsically parallel finite element method*, J. Sci. Comput., 40 (2009), pp. 27–50.
[10] S. BRENNER, T. GUDI, AND L.-Y. SUNG, *A posteriori error control for a weakly over-penalized symmetric interior penalty method*, J. Sci. Comput., 40 (2009), pp. 37–50.
[11] ———, *An a posteriori error estimator for a quadratic C^0 interior penalty method for the biharmonic problem*, IMA J. Numer. Anal., 30 (2010), pp. 777–798.
[12] S. BRENNER, L. OWENS, AND L.-Y. SUNG, *A weakly over-penalized symmetric interior penalty method*, Electron. Trans. Numer. Anal., 30 (2008), pp. 107–127, http://etna.mcs.kent.edu/vol.30.2008/pp107-127.dir.
[13] S. BRENNER AND L. SCOTT, *The Mathematical Theory of Finite Element Methods*, third ed., Springer, New York, 2008.
[14] S. BRENNER, K. WANG, AND J. ZHAO, *Poincaré-Friedrichs inequalities for piecewise H^2 functions*, Numer. Funct. Anal. Optim., 25 (2004), pp. 463–478.
[15] P. CIARLET, *Sur l'élément de Clough et Tocher*, RAIRO Anal. Numér., 8 (1974), pp. 19–27.
[16] ———, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.
[17] M. DAUGE, *Elliptic Boundary Value Problems on Corner Domains*, Lecture Notes in Mathematics 1341, Springer, Berlin-Heidelberg, 1988.
[18] W. DÖRFLER, *A convergent adaptive algorithm for Poisson's equation*, SIAM J. Numer. Anal., 33 (1996), pp. 1106–1124.
[19] T. DUPONT AND R. SCOTT, *Polynomial approximation of functions in Sobolev spaces*, Math. Comp., 34 (1980), pp. 441–463.
[20] P. GRISVARD, *Elliptic Problems in Non Smooth Domains*, Pitman, Boston, 1985.
[21] ———, *Singularities in Boundary Value Problems*, Masson, Paris, 1992.
[22] T. GUDI, *A new error analysis for discontinuous finite element methods for linear elliptic problems*, Math. Comp., 79 (2010), pp. 2169–2189.
[23] L. MORLEY, *The triangular equilibrium problem in the solution of plate bending problems*, Aeronaut. Q., 19 (1968), pp. 149–169.

[24] T. NILSSEN, X.-C. TAI, AND R. WINTHER, *A robust nonconforming H^2-element*, Math. Comp., 70 (2001), pp. 489–505.
[25] L. OWENS, *Multigrid Methods for Weakly Over-Penalized Interior Penalty Methods*, Ph.D. thesis,Department of Mathematics, University of South Carolina, 2007.
[26] Z. SHI, *Error estimates of Morley element*, Chinese J. Numer. Math. Appl., 12 (1990), pp. 9–15.
[27] L. TARTAR, *An Introduction to Sobolev Spaces and Interpolation Spaces*, Springer, Berlin, 2007.
[28] H. TRIEBEL, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, Amsterdam, 1978.
[29] R. VERFÜRTH, *A Review of A Posteriori Error Estmation and Adaptive Mesh-Refinement Techniques*, Wiley-Teubner, Chichester, 1995.
[30] M. WANG, J. XU, AND Y. HU, *Modified Morley element method for a fourth order elliptic singular perturbation problem*, J. Comput. Math., 24 (2006), pp. 113–120.