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A PRECONDITIONER FOR A FETI-DP METHOD FOR MORTAR ELEMENT DISCRETIZATION OF A 4TH ORDER PROBLEM IN 2D*

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Abstract. In this paper a parallel preconditioner for a FETI-DP formulation for a mortar discretization of a fourth order problem is presented and analyzed. We show that the condition number of the preconditioned FETI-DP operator is proportional to $(1 + \log(H/h)^2$, where H and h are mesh sizes.

Key words. FETI-DP, mortar, nonmatching meshes, finite element, preconditioner

AMS subject classifications. 65N30, 65N55

1. Introduction.

Many technical problems are modeled by partial differential equations. A way of constructing an effective approximation of the differential problem is to introduce a global conforming mesh and then to set up an approximate discrete problem. However it is often required to use different approximation methods or independent local meshes in some subregions of the original domain. Then one can make adaptive changes of the local mesh in a substructure without modifying meshes in the other subdomains. This type of technique requires matching conditions on the interfaces between adjacent substructures to ensure some type of weak continuity of the solution. One possible way of enforcing such matching conditions is to impose some integral conditions on the jumps of the traces of finite element functions across subdomain interfaces. This approach is taken by a mortar method which is an effective method of constructing approximation on nonconforming triangulations, cf. [6].

In this paper we extend the results of Kim and Lee [22], where the case of a FETI-DP method for mortar discretization for a second order problem is analyzed, to a mortar discretization of a fourth order problem. We present and analyze a preconditioned FETI-DP (dual primal Finite Element Tearing and Interconnecting) method for solving the system of equations arising from the mortar element discretization of a model fourth order problem in 2D. In each subdomain a reduced Hsieh-Clough-Tocher (RHCT) conforming finite element space on an independent local mesh is defined, and then a discrete mortar problem of saddle point type is introduced.

The original problem is reduced to a smaller dual FETI-DP problem. We eliminate first the unknowns associated with the degrees of freedom at interior nodal points, and then the unknowns related to the degrees of freedom at the interface nodes. The resulting dual FETI-DP problem is solved iteratively using a fully parallel preconditioner. We prove that our method is almost optimal, i.e., the polylogarithmic bound with respect to the local number of degrees of freedom holds for the condition number of the preconditioned problem. There are many papers in which the mortar method was studied for coupling nonmatching meshes for discretizations of second order problems; see, e.g., [4, 5, 7, 36]. The mortar technique for discretizations of fourth order problems is considered in [2, 21, 29]. The domain decomposition methods and especially the FETI-DP methods form a class of fast and efficient iterative solvers for algebraic systems of equations arising from the finite element discretizations of PDEs of second and fourth order, cf. [20, 24, 26, 27]. There are many works about iterative solvers for mortar method for second order problem; see, e.g., [1, 8, 9, 15, 23] and

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the references therein. But there are only few papers investigating fast solvers for mortar discretizations of fourth order elliptic problems, cf. [28, 31, 37]. Recently a few FETI-DP type algorithms have been presented for mortar discretization of second order problems, cf. [12, 14, 16, 17, 18, 19, 22, 33].

The formulation of the FETI-DP system of equations for the same discrete problem was presented in a proceedings paper [32]. Some condition number estimates for another preconditioner were also given there without proofs under very restrictive assumptions, e.g., the ratio h_k/h_l remain constant for the mesh sizes on the edge $\overline{\Gamma}_{kl} = \partial\Omega_k \cap \partial\Omega_l$. To our knowledge there are no FETI type algorithms for solving systems of equations arising from a mortar discretization of a fourth order problem in the literature.

The remainder of the paper is organized as follows. In Section 2 the mortar discretization of a model problem is presented. The FETI-DP method is presented in Section 3. In Section 4 we present a Neumann-Dirichlet preconditioner, and finally in Sections 5 and 6 we prove our main theorem.

In this paper the following notation is used. $u \asymp v$, $x \succeq y$ and $w \preceq z$ mean that there exist positive constants c and C independent of the parameter of the fine triangulation of any substructure and the number of subdomains, such that

$$c\,u \leq \ v \ \leq \ C\,u, \quad x \geq c\,y \quad \text{and} \quad w \leq C\,z, \quad \text{respectively}.$$

2. Discrete problem. Let Ω be a bounded polygonal domain in 2D. We assume that we have a geometrically conforming decomposition of Ω into polygonal subdomains, i.e.,

$$\overline{\Omega} = \bigcup_{k=1}^{N} \overline{\Omega}_k$$

with $\overline{\Omega}_k \cap \overline{\Omega}_l = \emptyset,$ a common edge or a common vertex. The decomposition will be further referred to as the coarse triangulation and we assume its shape regularity in the sense of [10, Section 2, page 5]. The interface Γ is defined as the sum of all open edges of substructures which are not contained in $\partial\Omega$.

Our model problem is to find $u^* \in H_0^2(\Omega)$, such that

$$a(u^*, v) = f(v) \quad \forall v \in H_0^2(\Omega),$$

where $f \in L^2(\Omega)$, $H_0^2(\Omega) = \{v \in H^2(\Omega) \, : \, v = \partial_n v = 0 \quad \text{on} \quad \partial\Omega\}$, and $a(u,v) = \sum_{k=1}^N a_k(u,v)$ with

$$a_k(u,v) = \int_{\Omega_k} \rho_k \left[u_{x_1x_1} v_{x_1x_1} + 2\, u_{x_1x_2} v_{x_1x_2} + u_{x_2x_2} v_{x_2x_2} \right] \, dx.$$

Here ρ_k are positive constants, ∂_n represents the outward normal derivative to $\partial\Omega$, and $u_{x_ix_j} := \frac{\partial^2 u}{\partial x_i \partial x_j}$, for $i,j = 1,2$.

From the Lax-Milgram theorem, the continuity and ellipticity of the bilinear form $a(\cdot,\cdot)$ yield the existence and uniqueness of the solution; see, e.g., [11] or [13].

We introduce in each subdomain Ω_k a quasiuniform triangulation made of triangles $T_h(\Omega_k)$ with parameter $h_k = \max_{\tau \in T_h(\Omega_k)} \operatorname{diam} \tau$ (e.g., cf. [11]) and let $H_k = \operatorname{diam}(\Omega_k)$. On each Ω_k we introduce local reduced Hsieh-Clough-Tocher (RHCT) macro finite element spaces (see [13]) as follows: let the local RHCT space $W^h(\Omega_k) \subset H^2(\Omega_k)$ be formed by C^1 continuous functions, such that $v \in W^h(\Omega_k)$, where for each triangle $\tau \in T_h(\Omega_k)$,

(i) $v_{|\tau_i} \in P_3(\tau_i)$ for three subtriangles $\tau_i \subset \tau$, $\ i = 1,2,3$, formed by connecting the vertices of τ to its centroid; see Figure 2.1,

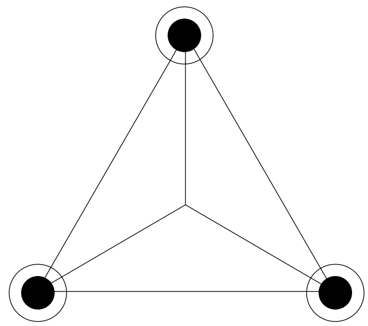

FIG. 2.1. *RHCT macro element.*

(ii) $\partial_n v$ are linear 1D polynomials on each edge $e \in \partial\tau$,

(iii) $v = \partial_n v = 0$ on $\partial\Omega_k \cap \partial\Omega$.

The degrees of freedom of RHCT macro elements are given by

$$\{v(p_i), v_{x_1}(p_i), v_{x_2}(p_i)\}, \qquad i = 1, 2, 3, \tag{2.1}$$

for the three vertices p_i of any $\tau \in T_h(\Omega_k)$; see Figure 2.1. We further call all vertices the nodal points or nodes.

Next we introduce an auxiliary global space

$$W^h(\Omega) = \prod_{k=1}^{N} W^h(\Omega_k).$$

We define $\widetilde{W}^h(\Omega)$ as the subspace of $W^h(\Omega)$ formed by all functions which are continuous at the crosspoints (i.e., the common vertices of the subdomain) and have continuous gradients at the crosspoints.

Since the triangulations of two adjacent subdomains Ω_k and Ω_l are independent, their common edge, denoted by Γ_{kl}, inherits two independent one dimensional meshes $T_h^k(\Gamma_{kl})$ induced by $T_h(\Omega_k)$ and $T_h^l(\Gamma_{lk})$ induced by $T_h(\Omega_l)$; see Figure 2.2. Thus we have to distinguish between the two sides (or meshes) of the interface Γ_{kl}. According to the rule $\rho_k \geq \rho_l$, we name the side on the h_k mesh as the mortar (master) side and denote it by γ_{kl}, and name the other side associated with the h_l mesh the slave (nonmortar) side denoted by δ_{lk}. Let $\gamma_{kl,h}$ be the set of all nodal vertices of elements of the $T_h^k(\Gamma_{kl})$ on the open edge Γ_{kl}, and $\overline{\gamma}_{kl,h}$ be the set of nodes of the same triangulation on $\overline{\Gamma}_{kl}$, respectively. Similarly, the sets $\delta_{lk,h}$ and $\overline{\delta}_{lk,h}$ consist of nodes of the h_l triangulation of Γ_{lk} and $\overline{\Gamma}_{lk}$, respectively.

We also introduce two test function spaces associated with $T_h^l(\delta_{lk})$: $M_t^l(\delta_{lk})$ represents the space of C^1 smooth functions that are piecewise cubic on $T_h^l(\delta_{lk})$ and are piecewise linear on the two end elements of $T_h^l(\delta_{lk})$, and $M_n^l(\delta_{lk})$ represents the space formed by continuous functions that are piecewise linear on $T_h^l(\delta_{lk})$ and are piecewise constant on the two end elements of $T_h^l(\delta_{lk})$.

Our discrete space V^h is defined as the subspace of $\widetilde{W}^h(\Omega)$ which satisfy two mortar conditions on each interface $\Gamma_{lk} \subset \Gamma$ with δ_{lk} its slave side and γ_{kl} master side:

$$\int_{\delta_{lk}} (u_k - u_l)\phi \, ds = 0 \quad \forall \phi \in M_t^l(\delta_{lk}), \tag{2.2a}$$

$$\int_{\delta_{lk}} (\partial_n u_k - \partial_n u_l)\psi \, ds = 0 \quad \forall \psi \in M_n^l(\delta_{lk}). \tag{2.2b}$$

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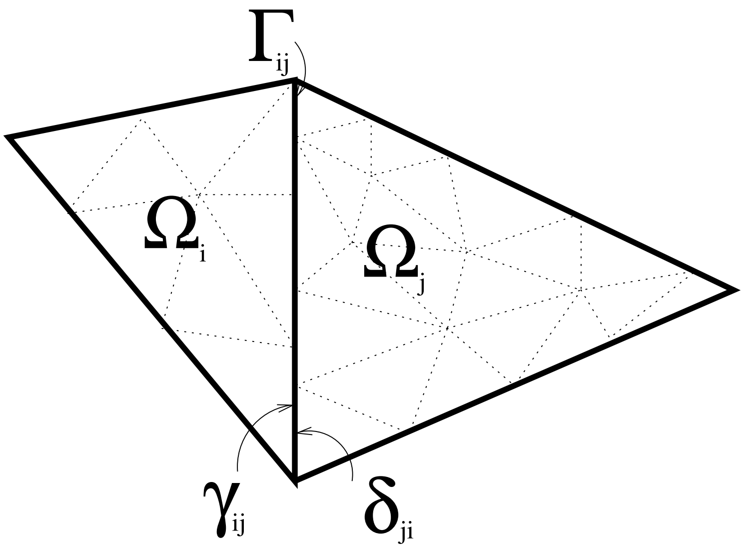

FIG. 2.2. *Nonconforming meshes.*

We can now formulate our discrete problem: find $u_h^* \in V^h$, such that

$$a(u_h^*, v) = f(v) \quad \forall v \in V^h. \tag{2.3}$$

We see that $a(u,u) = 0$ implies that u is linear over each subdomain. Then the continuity of u and ∇u at crosspoints yields that u is linear over Ω. Finally, the boundary conditions yields that $u = 0$. Hence the discrete problem has a unique solution.

We introduce the space $M = \prod_{\delta_{lk} \subset \Gamma} M_{lk}$, where $M_{lk} = M_t^l(\delta_{lk}) \times M_n^l(\delta_{lk})$, and define the bilinear form $b(u,\psi)$ on $\widetilde{W}^h(\Omega) \times M$ by: let $u = (u_1, \dots, u_N) \in \widetilde{W}^h(\Omega)$ and $\psi = (\psi_{lk})_{\delta_{lk}} = (\psi_{lk,1}, \psi_{lk,2})_{\delta_{lk}} \in M$,

$$\begin{aligned} b(u,\psi) &= \sum_{\delta_{lk} \subset \Gamma} \sum_{s=1,2} b_{lk,s}(u, \psi_{lk,s}), \\ b_{lk,1}(u, \psi_{lk,1}) &= \int_{\delta_{lk}} (u_k - u_l)\psi_{lk,1}\, ds, \\ b_{lk,2}(u, \psi_{lk,2}) &= \int_{\delta_{lk}} (\partial_n u_k - \partial_n u_l)\psi_{lk,2}\, ds. \end{aligned}$$

We can rewrite (2.3) as the following saddle point problem (cf. [30]): find a pair $(u_h^*, \lambda^*) \in \widetilde{W}^h(\Omega) \times M$, such that

$$\begin{array}{rcll} a(u_h^*, v) + b(v, \lambda^*) &=& f(v) & \forall v \in \widetilde{W}^h(\Omega), \\ b(u_h^*, \phi) &=& 0 & \forall \phi \in M. \end{array} \tag{2.4}$$

We see that u_h^* is a solution of (2.3), cf. [30]. In this paper we use the same notation to represent both a function and the vector containing the values of degrees of freedom of this function.

3. FETI-DP systems of equations. In this section, we formulate a FETI-DP operator for problem (2.4). We follow the approach taken in [22] and [26] for the formulation of the FETI-DP problem and the construction of the preconditioner.

3.1. Trace spaces. In this subsection we introduce the trace (in H^2 sense) of functions in $W^h(\Omega_k)$. For any $u \in W^h(\Omega_k)$, let $Tr_{|\partial\Omega_k}\, u$ be a linear trace operator mapping $W^h(\Omega_k)$ onto the triple $L^2(\partial\Omega_k) \times L^2(\partial\Omega_k) \times L^2(\partial\Omega_k)$, defined by $Tr_{|\partial\Omega_k}\, u = (u_{|\partial\Omega_k}, \nabla u_{|\partial\Omega_k})$, where $v_{|\partial\Omega_k}$ is the trace of $v \in H^1(\Omega_k)$ onto the boundary of Ω_k. Note that $\nabla u_{|\Gamma_{kl}}(s) = (\partial_t u(s), \partial_n u(s))$ over an edge $\Gamma_{kl} \subset \partial\Omega_k$, thus we get $Tr_{|\partial\Omega_k} u(s) = (u(s), \partial_t u(s), \partial_n u(s))$ for $s \in \Gamma_{kl}$. Because $\partial_t u(s)$ along this edge is uniquely defined by u, we can define the trace onto an edge Γ_{kl} by $Tr_{|\Gamma_{kl}}\, u = (u_{|\Gamma_{kl}}, \partial_n u_{|\Gamma_{kl}}) \subset L^2(\Gamma_{kl}) \times L^2(\Gamma_{kl})$.

We also define the spaces

$$W_k(\Gamma_{kl}) := Tr_{|\Gamma_{kl}} W^h(\Omega_k) \quad \text{and} \quad W_k := \left(Tr_{|\Omega_k}\right) W^h(\Omega_k),$$

and

$$W = W_1 \times \ldots \times W_N.$$

Note that $W_k(\Gamma_{kl}) = W_k^{1,3}(\Gamma_{kl}) \times W_k^{0,1}(\Gamma_{kl})$, where $W_k^{1,3}(\Gamma_{kl})$ is the space of C^1 continuous piecewise cubic functions on the 1D triangulation $T_h^k(\Gamma_{kl})$, and $W_k^{0,1}(\Gamma_{kl})$ is a space of continuous piecewise linear functions on $T_h^k(\Gamma_{kl})$. Thus a function in W_k is defined by all degrees of freedom associated with all nodal points, i.e., vertices on $\partial\Omega_k$, cf. (2.1).

We also introduce

$$W_{0,k}^{1,3}(\Gamma_{kl}) = W_k^{1,3}(\Gamma_{kl}) \cap H_0^2(\Gamma_{kl}), \quad W_{0,k}^{0,1}(\Gamma_{kl}) = W_k^{0,1}(\Gamma_{kl}) \cap H_0^1(\Gamma_{kl}), \tag{3.1}$$

i.e., the subspaces of the trace spaces with zero boundary conditions in H^2 and H^1 sense, respectively.

3.2. Matrix form of mortar conditions. We introduce matrix forms of mortar conditions. On a slave $\delta_{lk} \subset \partial\Omega_l$, $u_{\delta_{lk}}$ and $\partial_n u_{\delta_{lk}}$ are split into two vectors representing tangential and normal traces

$$\begin{aligned} u_{\delta_{lk}} &= u_{\delta_{lk}}^{(r)} + u_{\delta_{lk}}^{(c)}, \\ \partial_n u_{\delta_{lk}} &= \partial_n u_{\delta_{lk}}^{(r)} + \partial_n u_{\delta_{lk}}^{(c)}, \end{aligned} \tag{3.2}$$

where $u_{\delta_{lk}}^{(r)}$ represents the respective degrees of freedom of the tangential trace function at interior nodal points (interior to δ_{lk}) and $u_{\delta_{lk}}^{(c)}$ four degrees of freedom at the ends of this slave. We have also an analogous splitting of two vectors representing tangential and normal traces onto the master γ_{kl}.

We can now rewrite (2.2a) and (2.2b) in a matrix form using nodal basis functions as

$$\begin{aligned} B_{t,\gamma_{kl}} u_{\gamma_{kl}} - B_{t,\delta_{lk}} u_{\delta_{lk}} &= 0, \\ B_{n,\gamma_{kl}} \partial_n u_{\gamma_{kl}} - B_{n,\delta_{lk}} \partial_n u_{\delta_{lk}} &= 0. \end{aligned} \tag{3.3}$$

Using the splitting (3.2) we get

$$\begin{aligned} B_{t,\gamma_{kl}}^{(r)} u_{\gamma_{kl}}^{(r)} + B_{t,\gamma_{kl}}^{(c)} u_{\gamma_{kl}}^{(c)} - B_{t,\delta_{lk}}^{(r)} u_{\delta_{lk}}^{(r)} - B_{t,\delta_{lk}}^{(c)} u_{\delta_{lk}}^{(c)} &= 0, \\ B_{n,\gamma_{kl}}^{(r)} \partial_n u_{\gamma_{kl}}^{(r)} + B_{n,\gamma_{kl}}^{(c)} \partial_n u_{\gamma_{kl}}^{(c)} - B_{n,\delta_{lk}}^{(r)} \partial_n u_{\delta_{lk}}^{(r)} - B_{n,\delta_{lk}}^{(c)} \partial_n u_{\delta_{lk}}^{(c)} &= 0, \end{aligned} \tag{3.4}$$

where the matrices $B_{t,\delta_{lk}}, B_{n,\delta_{lk}}$ are mass matrices resulting by substituting standard nodal basis functions of $W_l^{1,3}(\delta_{lk}), W_l^{0,1}(\delta_{lk})$ and $M_t^l(\delta_{lk}), M_n^l(\delta_{lk})$ into (2.2a) and (2.2b), respectively, i.e.,

$$\begin{aligned} B_{t,\delta_{lk}} &= \{(\phi_{x,s}, \psi_{y,r})\}_{\substack{x,y\in\delta_{lk,h} \\ s,r=0,1}} \quad && \phi_{x,s} \in W_l^{1,3}(\delta_{lk}), \psi_{y,r} \in M_t^l(\delta_{lk}), \\ B_{n,\delta_{lk}} &= \{(\phi_x, \psi_y)\}_{x,y\in\delta_{lk,h}} && \phi_x \in W_l^{0,1}(\delta_{lk}), \psi_y \in M_n^l(\delta_{lk}), \end{aligned}$$

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where $\phi_{x,s}$ and $\psi_{x,s}$ are nodal basis functions of $W_l^{1,3}(\delta_{lk})$ and $M_t^l(\delta_{lk})$, respectively, associated with a node x of $\overline{\delta}_{lk,h}$. They represent a *value* degree of freedom if $s=0$, or a *derivative* degree of freedom if $s=1$, i.e., e.g.,

$$\frac{d^r}{dt^r}\phi_{x,s}(y)=\begin{cases}1, & \text{if} \quad r=s \text{ and } y=x,\\ 0, & \text{otherwise,}\end{cases} \qquad \text{for } r=0,1, \quad y\in\overline{\delta}_{lk,h},$$

$\phi_x\in W_l^{0,1}(\delta_{lk})$ and $\psi_x\in M_n^l(\delta_{lk})$ are nodal basis function of these respective spaces, which equal one at the node x and zero at all remaining nodal points of $\overline{\delta}_{lk,h}$. The matrices $B_{t,\gamma_{kl}},B_{n,\gamma_{kl}}$ are defined analogously by replacing the basis functions of $W_l^{1,3}(\delta_{lk})$ by the nodal basis of $W_k^{1,3}(\gamma_{kl})$ and the basis of $W_l^{0,1}(\delta_{lk})$ by the basis of $W_k^{0,1}(\gamma_{kl})$, respectively. The matrices with superscripts (c) and (r) are submatrices of respective mass matrices corresponding to the splitting (3.2).

Note that $B_{t,\delta_{lk}}^{(r)},B_{n,\delta_{lk}}^{(r)}$ are positive definite square matrices, cf. [29]. But in general $B_{t,\gamma_{kl}}^{(r)},B_{n,\gamma_{kl}}^{(r)}$ and all other matrices in (3.3) and (3.4) are rectangular. Then we can define block-diagonal matrices

$$B_{\delta_{lk}}^{(r)}=\begin{bmatrix}B_{t,\delta_{lk}}^{(r)} & 0\\ 0 & B_{n,\delta_{lk}}^{(r)}\end{bmatrix},\quad B_{\gamma_{k,l}}^{(r)}=\begin{bmatrix}B_{t,\gamma_{k,l}}^{(r)} & 0\\ 0 & B_{n,\gamma_{k,l}}^{(r)}\end{bmatrix}, \tag{3.5}$$

and analogously $B_{\delta_{lk}}^{(c)}$ and $B_{\gamma_{k,l}}^{(c)}$ replacing (r) by (c), and $B_{\delta_{lk}}$ and $B_{\gamma_{k,l}}$ removing the superscript (r).

For a mortar γ_{kl} let w_k^{kl} denote the vector representing all degrees of freedom related to nodes in $\gamma_{kl,h}$ and w_l^{lk} in $\delta_{lk,h}$. Then (3.3) can be rewritten as

$$B_{\gamma_{kl}}w_k^{kl}-B_{\delta_{lk}}w_l^{lk}=0.$$

Let us define $R_{kl}^k:W_k\to W_k(\Gamma_{kl})$ as the restriction operator, and $E_{kl}:M\to M_{kl}$ the extension by zero operator, and

$$B_i=\sum_{\gamma_{ij}\subset\partial\Omega_i}E_{ij}B_{\gamma_{ij}}R_{ij}^i-\sum_{\delta_{ik}\subset\partial\Omega_i}E_{ik}B_{\delta_{ik}}R_{ik}^i. \tag{3.6}$$

The matrices $B_i^{(r)}$ and $B_i^{(c)}$ are defined analogously by adding respective superscripts.

3.3. Elimination of variables. Here we eliminate all variables related to degrees of freedom of the u_h^* component in the solution of (2.4) and obtain a FETI-DP system of equations.

We split any function $u\in W^h(\Omega)$ into two parts

$$u=\mathcal{H}u+\mathcal{P}u,$$

where $\mathcal{P}u=(\mathcal{P}_1u,\ldots,\mathcal{P}_Nu)$ with $\mathcal{P}_ku\in H_0^2(\Omega_k)\cap W^h(\Omega_k)$, such that

$$a_k(\mathcal{P}_ku,v)=(f,v)\quad\forall v\in H_0^2(\Omega_k)\cap W^h(\Omega_k),$$

and $\mathcal{H}u=u-\sum_{k=1}^N\mathcal{P}_ku$, which is a discrete biharmonic part of u, equivalently defined by $\mathcal{H}u=(\mathcal{H}_1u,\ldots,\mathcal{H}_Nu)$, where $\mathcal{H}_ku\in W^h(\Omega_k)$ satisfies

$$\begin{cases}a_k(\mathcal{H}_ku,v)=0 & \forall v\in H_0^2(\Omega_k)\cap W^h(\Omega_k),\\ Tr_{|\partial\Omega_k}\mathcal{H}_ku=Tr_{|\partial\Omega_k}u & \text{on}\quad \partial\Omega_k.\end{cases} \tag{3.7}$$

We also have the so called the minimal property of discrete biharmonic functions, i.e.,

$$|\mathcal{H}_k u|_{H^2(\Omega_k)} = \min\{|v|_{H^2(\Omega_k)} : v \in W^h(\Omega_k),\ \ Tr_{|\partial\Omega_k} v = Tr_{|\partial\Omega_k} u\}.$$

Thus we split the solution u^* of (2.3) into the discrete biharmonic part and the local solutions:

$$u_h^* = u_{bh}^* + \mathcal{P} u_h^*.$$

where $u_{bh}^* = \mathcal{H} u_h^*$.

Note that a discrete biharmonic function $w = \mathcal{H} w$ is uniquely defined by the values of all degrees of freedom at nodes in $\overline{\Gamma}$. Thus it remains to find the values degrees of freedom of u_{bh}^* related to the nodes on $\overline{\Gamma}$.

If we represent a local matrix of the local bilinear form $a_k(u, v)$ in the standard basis of RHCT as $K^{(k)}$ and reorder the unknowns into interior and boundary unknowns, i.e.,

$$K^{(k)} = \left[\begin{array}{cc} K_{ii}^{(k)} & K_{ib}^{(k)} \\ K_{bi}^{(k)} & K_{bb}^{(k)} \end{array} \right],$$

then we can define a Schur complement matrix $S^{(k)}$ by

$$S^{(k)} = K_{bb}^{(k)} - K_{bi}^{(k)} (K_{ii}^{(k)})^{-1} K_{ib}^{(k)}.$$

For any vector $w \in W_k(\partial\Omega_k)$ we can write $w = \left[\begin{array}{c} w_r \\ w_c \end{array} \right]$, where w_c represents the values of degrees of freedom associated with the crosspoints and w_r the remaining degrees of freedom related to nodes interior to edges on $\partial\Omega_k$. We order the matrix $S^{(k)}$ in the following way:

$$S^{(k)} = \left[\begin{array}{cc} S_{rr}^{(k)} & S_{rc}^{(k)} \\ S_{cr}^{(k)} & S_{cc}^{(k)} \end{array} \right].$$

Next we introduce

$$\widetilde{W} \subset W$$

formed by functions with continuous degrees of freedom at crosspoints. Equivalently we can say that $\widetilde{W}$ is the space formed by all local traces of functions from $\widetilde{W}^h(\Omega)$. We can split a vector $w \in \widetilde{W}$ into $w = \left[\begin{array}{c} w_r \\ w_c \end{array} \right]$ where $w_r = (w_{1,r}, \ldots, w_{N,r})$ and w_c represents the values of degrees of freedom at crosspoints (global vertices of subdomains). Here $w_{i,r}$ represents the values of degrees of freedom related to nodes in $\partial\Omega_i$ which are not vertices of Ω_i.

Let L_c^i represents a matrix made of zeros and ones, such that $L_c^i w_c$ restricts the values of degrees of freedom of w_c to the respective degrees of freedom at the vertices of $\partial\Omega_i$, i.e., for any $w \in \widetilde{W}$ we can write $w = (w_1, \ldots, w_N)$ with $w_k \in W_k(\partial\Omega_k)$, such that $w_k = \left[\begin{array}{c} w_{k,r} \\ L_c^k w_c \end{array} \right]$.

We equip the space $\widetilde{W}$ with the norm

$$\|w\|_S^2 = \sum_{k=1}^N \|w_k\|_{S^{(k)}}^2.$$

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Taking $B_c = \sum_{i=1}^N B_i^{(c)} L_i^c$ and $B_r = (B_1^{(r)}, \ldots, B_N^{(r)})$, the second equation in (2.4) has the following matrix form (cf. (3.6)):

$$B_r w_r + B_c w_c = 0.$$

Thus we can rewrite the system (2.4) as

$$\begin{aligned} S_{rr} w_r + S_{rc} w_c + B_r^T \lambda &= g_r, \\ S_{cr} w_r + S_{cc} w_c + B_c^T \lambda &= g_c, \\ B_r w_r + B_c w_c &= 0, \end{aligned}$$

where $w \in \widetilde{W}$ is the vector representing the degrees of freedom of u_{bh}^* corresponding to all nodal points on Γ. The vectors g_r, g_c are the respective Schur complement right hand side vectors, $S_{rr} = \mathrm{diag}_{k=1,\ldots N}(S_{rr}^{(k)})$, $S_{rc}^T = S_{cr} = ((L_1^c)^T S_{cr}^{(1)}, \ldots, (L_N^c)^T S_{cr}^{(N)})$, and $S_{cc} = \sum_{k=1}^N (L_k^c)^T S_{cc}^{(k)} L_k^c$.

Since S_{rr} is block diagonal and positive definite, we can eliminate w_r and obtain the new system

$$\begin{aligned} F_{cc} w_c + F_{c\lambda} \lambda &= d_c, \\ F_{\lambda c} w_c + F_{\lambda\lambda} \lambda &= d_\lambda, \end{aligned}$$

where $F_{\lambda\lambda} = B_r S_{rr}^{-1} B_r^T$, $F_{c\lambda} = F_{\lambda c}^T = S_{cr} S_{rr}^{-1} B_r^T - B_c^T$, $F_{cc} = S_{cr} S_{rr}^{-1} S_{rc} - S_{cc}$, $d_c = -g_c + S_{cr} S_{rr}^{-1} g_r$, and $d_\lambda = B_r S_{rr}^{-1} g_r$. Finally, we eliminate w_c and get

$$F_{DP} \lambda = d_\lambda - F_{\lambda c} F_{cc}^{-1} d_c, \tag{3.8}$$

where $F_{DP} = F_{\lambda\lambda} - F_{\lambda c} F_{cc}^{-1} F_{c\lambda}$ is the FETI-DP operator.

4. Preconditioner. Before defining the preconditioner we introduce some auxiliary spaces and operators. For a slave δ_{lk} we define $W_{0,l}(\delta_{lk}) = W_{0,l}^{1,3}(\Gamma_{kl}) \times W_{0,l}^{0,1}(\Gamma_{kl})$, cf. (3.1), let

$$W_\Delta = \Pi_{\delta_{lk} \subset \Gamma} W_{0,l}(\delta_{lk})$$

and let $\widetilde{W}_\Delta \subset \widetilde{W}$ be the space of functions extended from functions in W_Δ by zero onto the trace spaces corresponding to mortars. Note that the dimensions of both W_Δ and $\widetilde{W}_\Delta$ are the same as the dimension of M. We equip W_Δ with the norm

$$\|w\|_{S_\Delta} := \langle S_\Delta w, w \rangle = \|\tilde{w}\|_S,$$

where $S_\Delta = \mathrm{diag}_k(S_\Delta^{(k)})$, and $S_\Delta^{(k)}$ is the matrix built locally from $S^{(k)}$ by proper restrictions and extensions and $\tilde{w} \in \widetilde{W}$ is the extension of w by zero onto the trace spaces associated with mortars. We could equivalently define $\widetilde{W}_\Delta$ as the subspace of $\widetilde{W}$ of all functions, which equal zero on both master nodes and vertices of subdomains. Note that S_Δ is block diagonal with nonsingular blocks due to the fact that functions in $\widetilde{W}_\Delta$ equal zero on the vertices.

We also define, cf. (3.5),

$$B_\Delta := \mathrm{diag}_{\delta_{lk} \subset \Gamma}(B_{\delta_{lk}}^{(r)}).$$

Note that for any $\lambda = (\lambda_1, \lambda_2) \in M_{lk}$ and $w = (w_1, w_2) \in W_{0,l}(\delta_{lk})$ we have

$$\langle w, (B_{\delta_{lk}}^{(r)})^T \lambda \rangle = \langle B_{\delta_{lk}}^{(r)} w, \lambda \rangle = \sum_{j=1,2} \int_{\Gamma_{lk}} w_j \lambda_j \, ds. \tag{4.1}$$

Hence, B_Δ, B_Δ^T are block diagonal matrices with invertible blocks, cf. [29], Finally, we introduce the inverse of the preconditioner as:

$$\mathcal{M}_{DN} := B_\Delta S_\Delta^{-1} B_\Delta^T,$$

which is nonsingular, and thus we choose

$$\mathcal{M}_{DN}^{-1} = B_\Delta^{-T} S_\Delta B_\Delta^{-1}$$

as the preconditioner for problem (3.8). Note that $\mathcal{M}_{DN}^{-1}$ is a fully parallel preconditioner; application of $\mathcal{M}_{DN}^{-1}$ to a vector involves solving N local independent Dirichlet type problems.

5. Technical tools. In this section we present some technical results needed for the proof of the main theorem. For the analysis we need an equivalent definition of the preconditioner defined in the terms of a dual norm in W_Δ.

LEMMA 5.1. *For any $\lambda \in M$ we have*

$$\langle \mathcal{M}_{DN}\lambda, \lambda\rangle^{1/2} = \sup_{w \in W_\Delta \setminus \{0\}} \sum_{\delta_{lk} \subset \Gamma} \sum_{i=1,2.} \frac{\int_{\delta_{lk}} w_i \lambda_i \, ds}{\|w\|_{S_\Delta}} = \sup_{w \in W_\Delta \setminus \{0\}} \frac{b(\tilde{w}, \lambda)}{\|\tilde{w}\|_{S}.},$$

where $(\lambda_1, \lambda_2) \in M_{lk}$ and $(w_1, w_2) \in W_{0,l}(\delta_{lk})$ are the respective restrictions of $\lambda \in M$ and $w \in W_\Delta$ to the slave δ_{lk}, cf. (4.1), and $\tilde{w} \in \widetilde{W}$ is an extension of w by zeros.

Proof. The second equality follows from the definition of $b(\cdot,\cdot)$. The proof of the first equality follows from the definitions of B_Δ and S_Δ, (4.1), and a standard algebraic argument; see, e.g., [35]. $\square$

The formula in the next lemma is analogous to the one in [22, Lemma 4.2], and it can be proved similarly; see, e.g., the proof of [34, Lemma 37].

LEMMA 5.2. *For any $\lambda \in M$, we have*

$$\langle F_{DP}\lambda, \lambda\rangle^{1/2} = \sup_{w \in \widetilde{W} \setminus \{0\}} \frac{b(w, \lambda)}{\|w\|_S}.$$

The next three lemmas are well known. The first lemma is a discrete analog of the extension theorem for Sobolev spaces.

LEMMA 5.3. *Let $v \in W_k(\partial\Omega_k)$. Then there exists $Ext(v) \in W^h(\Omega_k)$, such that*

$$Tr_{|\partial\Omega_k} Ext(v) = v \quad \text{and} \quad |Ext(v)|_{H^2(\Omega_k)} \preceq |\nabla v|_{H^{1/2}(\partial\Omega_k)},$$

where $Tr_{|\partial\Omega_k} v = (v_{|\partial\Omega_k}, \nabla v_{|\partial\Omega_k})$ for $v \in H^2(\Omega_k)$.

Proof. See [25, Theorem 4.4]. $\square$

LEMMA 5.4. *For any $w = (w_{\delta_{ji}}) \in W_\Delta$, we have*

$$\|w\|_{S_\Delta}^2 \preceq \sum_{j=1}^N \rho_j \sum_{\delta_{jk} \subset \partial\Omega_j} \left(\|\partial_t w_1\|^2_{H_{00}^{1/2}(\Gamma_{kj})} + \|w_2\|^2_{H_{00}^{1/2}(\Gamma_{kj})} \right),$$

where $w_{\delta_{ji}} = (w_1, w_2) \in W_{0,k}^{1,3}(\Gamma_{kl}) \times W_{0,k}^{0,1}(\Gamma_{kl})$ is the restriction of w onto a slave $\delta_{ji} \subset \Gamma$.

Proof. Let $\tilde{w} \in \widetilde{W}_\Delta$ be an extension of w to $\widetilde{W}$ by zero. Then by the definition $\|w\|_{S_\Delta}^2 = \|\tilde{w}\|_S^2$. Let $u \in \widetilde{W}^h(\Omega)$ be a discrete biharmonic function, such that $Tr_{|\partial\Omega_k} u_k = \tilde{w}$

on $\partial\Omega_k$ for $k = 1,\ldots,N$. Here $\tilde{w}$ is the unique function corresponding to the vector $\tilde{w}$ (denoted by the same symbol). Thus by the definition of Schur complement we have $\|\tilde{w}_k\|_{S^{(k)}} = \rho_k^{1/2}|u_k|_{H^2(\Omega_k)}$. Next by Lemma 5.3 and the minimal property of the discrete biharmonic functions we get

$$\|w\|^2_{S_\Delta} = \|\tilde{w}\|^2_S = \sum_{j=1}^N \rho_j |u|^2_{H^2(\Omega_j)} \preceq \sum_{j=1}^N \rho_j |\nabla u|^2_{H^{1/2}(\partial\Omega_j)}$$

$$\preceq \sum_{j=1}^N \rho_j \sum_{\delta_{jk}\subset\partial\Omega_j} (\|\partial_t w_1\|^2_{H^{1/2}_{00}(\Gamma_{kj})} + \|w_2\|^2_{H^{1/2}_{00}(\Gamma_{kj})}). \qquad \Box$$

The following lemma gives an estimate of the $H^{1/2}_{00}$ norm over an edge by $H^{1/2}$ and L^∞ norms; see [25, Lemma 4.1].

LEMMA 5.5. *If $u \in H^{1/2}_{00}(\Gamma_{kl})$ satisfies $\|\partial_t u\|_{L^\infty(\Gamma_{kl})} \preceq h_k^{-1}\|u\|_{L^\infty(\Gamma_{kl})}$, then*

$$\|u\|^2_{H^{1/2}_{00}(\Gamma_{kl})} \preceq |u|^2_{H^{1/2}(\Gamma_{kl})} + (1+\log(|\Gamma_{kl}|/h_k))\|u\|^2_{L^\infty(\Gamma_{kl})},$$

where Γ_{kl} is an edge of Ω_k.

The next result is a Sobolev like inequality; see, e.g., [38, Lemma 4.2.2].

LEMMA 5.6. *For any function $u \in W^h(\Omega_k)$, we have*

$$|u|^2_{W^{1,\infty}(\Omega_k)} \preceq (1+\log(H_k/h_k))\left(H_k^{-2}|u|^2_{H^1(\Omega_k)} + |u|^2_{H^2(\Omega_k)}\right).$$

Next we introduce two auxiliary operators and show their stability properties in the trace norms. We first define an operator associated with a slave $\delta_{lk} \subset \Gamma_{lk}$, which is a common edge of Ω_k and $\Omega_l a$, and show its stability property which is crucial for the analysis of our preconditioner.

DEFINITION 5.7. *Let $\pi^t_{lk} : L^2(\delta_{lk}) \to W^{1,3}_{0,l}(\Gamma_{kl})$, cf. (3.1), be defined by*

$$\int_{\delta_{lk}} \pi^t_{lk} u\,\phi\,ds = \int_{\delta_{lk}} u\,\phi\,ds \quad \forall\phi \in M^l_t(\delta_{lk}). \tag{5.1}$$

The following lemma states the stability of π^t_{lk}; see the proof of [29, Lemma 6].

LEMMA 5.8. *The following estimate holds for the operator π^t_{lk}:*

$$\|\partial_t \pi^t_{lk} u\|_{H^{1/2}_{00}(\delta_{lk})} \preceq \|\partial_t u\|_{H^{1/2}_{00}(\delta_{lk})} \quad \forall u \in H^{3/2}_{00}(\delta_{lk}).$$

We also introduce another operator $\pi^n_{lk} : L^2(\delta_{lk}) \to H^1_0(\delta_{lk})$, cf. [3].

DEFINITION 5.9. *Let $\pi^n_{lk} : L^2(\delta_{lk}) \to W^{0,1}_{0,l}(\Gamma_{kl})$, be a linear operator defined by, cf. (3.1),*

$$\int_{\delta_{lk}} \pi^n_{lk} u\,\psi\,ds = \int_{\delta_{lk}} u\,\psi\,ds \qquad \forall\psi \in M^l_n(\delta_{lk}). \tag{5.2}$$

The stability of π^n_{lk} is stated in the following lemma, cf. [3, Lemma 1].

LEMMA 5.10. *For the operator π^n_{lk}, we have*

$$\|\pi^n_{lk} u\|_{H^{1/2}_{00}(\delta_{lk})} \preceq \|u\|_{H^{1/2}_{00}(\delta_{lk})} \quad \forall u \in H^{1/2}_{00}(\delta_{lk}).$$

We also need the following technical lemmas. For any $u \in W^h(\Omega_k)$, we define a cubic interpolant $I_{H,3}u \in L^2(\Gamma_{kl})$, by $I_{H,3}u(x) = u_k(x)$ and $\partial_t I_{H,3}u(x) = \partial_t u(x)$, for x being an end of Γ_{kl}. Note that $\partial_t(u_{|\Gamma_{kl}} - I_{H,3}u_k) \in H_{00}^{1/2}(\Gamma_{kl})$.

LEMMA 5.11. *For any $u \in W^h(\Omega_k)$, we have*

$$\|\partial_t(u_{|\Gamma_{kl}} - I_{H,3}u)\|_{H_{00}^{1/2}(\Gamma_{kl})} \preceq \left(1 + \log\left(\frac{H_k}{h_k}\right)\right) |u|_{H^2(\Omega_k)}.$$

Proof. Let $w = u - I_{H,3}u$. Note that if we replace u by $u+p_1$, for any linear polynomial p_1, then w is unchanged since $I_{H,3}p_1 = p_1$. Lemma 5.5 yields that

$$\|\partial_t w\|^2_{H_{00}^{1/2}(\Gamma_{kl})} \preceq |\partial_t w|^2_{H^{1/2}(\Gamma_{kl})} + (1 + \log(H_k/h_k))\|\partial_t w\|^2_{L^\infty(\Gamma_{kl})}.$$

Note that by a scaling argument we have

$$|\partial_t I_{H,3}u|_{H^{1/2}(\Gamma_{kl})} + |I_{H,3}u|_{W^{1,\infty}(\Gamma_{kl})} \preceq H_k^{-2}\|u\|_{L^\infty(\Omega_k)} + |u|_{W^{1,\infty}(\Gamma_{kl})}.$$

Hence using a triangle inequality we get

$$\begin{aligned}\|\partial_t w\|^2_{H_{00}^{1/2}(\Gamma_{kl})} \preceq\ & |\partial_t u|^2_{H^{1/2}(\Gamma_{kl})} \\ & + (1 + \log(H_k/h_k))\left(H_k^{-4}\|u\|^2_{L^\infty(\Omega_k)} + |u|^2_{W^{1,\infty}(\Omega_k)}\right).\end{aligned}$$

We estimate the first term by the trace theorem, the second one by the embedding $H^2 \hookrightarrow L^\infty$ and a scaling argument, and the last term by Lemma 5.6 and again a scaling argument, which all together gives

$$\|\partial_t w\|^2_{H_{00}^{1/2}(\Gamma_{kl})} \preceq (1 + \log(H_k/h_k))^2 \left(\sum_{s=0}^{2} H_k^{-4+2s}|u|^2_{H^s(\Omega_k)}\right).$$

Finally, a scaling argument and a quotient space argument yield that

$$\|\partial_t w\|^2_{H_{00}^{1/2}(\Gamma_{kl})} \preceq (1 + \log(H_k/h_k))^2 |u|^2_{H^2(\Omega_k)}. \qquad \square$$

The next lemma can be shown following the lines of the proof of the previous lemma, cf. also [26, Lemma 5.1].

LEMMA 5.12. *For any $u \in W^h(\Omega_k)$, we have*

$$\|\partial_n u_{|\Gamma_{kl}} - I_H \partial_n u_{|\Gamma_{kl}}\|_{H_{00}^{1/2}(\Gamma_{kl})} \preceq \left(1 + \log\left(\frac{H_k}{h_k}\right)\right) |u|_{H^2(\Omega_k)},$$

where $I_H \partial_n u_{|\Gamma_{kl}}$ is a linear interpolant of $\partial_n u_{|\Gamma_{kl}} \in W_k^{0,1}(\Gamma_{kl})$ defined by the values of the function at the ends of Γ_{kl}.

The following lemma is crucial for our analysis.

LEMMA 5.13. *For any $w \in \widetilde{W}$ and any slave $\delta_{lk} \subset \Gamma$, we have*

$$\begin{aligned}\rho_l \left(\|\partial_t w^{(k)}_{1|\gamma_{kl}} - \partial_t w^{(l)}_{1|\delta_{lk}}\|^2_{H_{00}^{1/2}(\Gamma_{kj})} + \|w^{(k)}_{2|\gamma_{kl}} - w^{(l)}_{2|\delta_{lk}}\|^2_{H_{00}^{1/2}(\Gamma_{kj})}\right) & \\ \preceq \left(1 + \log\left(\frac{H}{\underline{h}}\right)\right)^2 \sum_{j=k,l} \|w_j\|^2_{S^{(j)}},&\end{aligned}$$

where $(w^{(k)}_{1|\gamma_{kl}}, w^{(k)}_{2|\gamma_{kl}}) \in W_k(\Gamma_{kl})$ *is the restriction of* $w_k \in W_k$ *to the mortar* γ_{kl}*, and* $(w^{(l)}_{1|\delta_{lk}}, w^{(l)}_{2|\delta_{lk}}) \in W_l(\Gamma_{lk})$ *is the restriction of* $w_l \in W_l$ *to the slave* δ_{lk}*. Here* $H = \max_k h_k$ *and* $\underline{h} = \min_k h_k$*.*

Proof. Let $u \in \widetilde{W}^h(\Omega)$ be a discrete biharmonic function, such that $Tr_{|\partial\Omega_i} u_i = w_i$ on $\partial\Omega_i$, for any subdomain Ω_i. Then, in particular, we have

$$Tr_{|\gamma_{kl}} u_k = (u_{k|\gamma_{kl}}, \partial_n u_{k|\gamma_{kl}}) = (w^{(k)}_{1|\gamma_{kl}}, w^{(k)}_{2|\gamma_{kl}}),$$
$$Tr_{|\delta_{lk}} u_l = (u_{l|\delta_{lk}}, \partial_n u_{l|\delta_{lk}}) = (w^{(l)}_{1|\delta_{lk}}, w^{(l)}_{2|\delta_{lk}})$$

and $\|w_j\|^2_{S^{(j)}} = \rho_j |u_j|^2_{H^2(\Omega_j)}$, cf. the definition of Schur complement and (3.7). Note that $\partial_t u_{k|\gamma_{kl}} - \partial_t u_{l|\delta_{lk}} \in H^{1/2}_{00}(\Gamma_{kl})$ and $\partial_n u_{k|\gamma_{kl}} - \partial_n u_{l|\delta_{lk}} \in H^{1/2}_{00}(\Gamma_{kl})$.

By the continuity of all degrees of freedom of u at the ends of this edge we also have $I_{H,3} u_k = I_{H,3} u_l$ and $I_H \partial_n u_k = I_H \partial_n u_l$ on Γ_{kl}. Here $I_{H,3}$ is defined as in Lemma 5.11, and I_H is from Lemma 5.12.

We first estimate the first term: $\rho_l \|\partial_t w^{(k)}_{1|\gamma_{kl}} - \partial_t w^{(l)}_{1|\delta_{lk}}\|^2_{H^{1/2}_{00}(\Gamma_{kj})}$. We have

$$\begin{aligned}\rho_l \|\partial_t w^{(k)}_{1|\gamma_{kl}} - \partial_t w^{(l)}_{1|\delta_{lk}}\|_{H^{1/2}_{00}(\Gamma_{kj})} &= \rho_l \|\partial_t u_k - \partial_t u_l\|_{H^{1/2}_{00}(\Gamma_{kl})} \\ &\le \sum_{j=k,l} \rho_l \|\partial_t (u_j - I_{H,3} u_j)\|_{H^{1/2}_{00}(\Gamma_{kl})}.\end{aligned}$$

Then Lemma 5.11 and the assumption $\rho_l \le \rho_k$, for the master γ_{kl} and the slave δ_{lk}, yield that

$$\rho_l \|\partial_t u_k - \partial_t u_l\|^2_{H^{1/2}_{00}(\Gamma_{kl})} \preceq \left(1 + \log\left(\frac{H}{\underline{h}}\right)\right)^2 \sum_{j=k,l} \rho_j |u_j|^2_{H^2(\Omega_j)}. \tag{5.3}$$

Next we estimate the term corresponding to the normal derivative and get

$$\begin{aligned}\rho_l \|w^{(k)}_{2|\gamma_{kl}} - w^{(l)}_{2|\delta_{lk}}\|_{H^{1/2}_{00}(\Gamma_{kj})} &= \rho_l \|\partial_n u_k - \partial_n u_l\|_{H^{1/2}_{00}(\Gamma_{kl})} \\ &\le \sum_{j=k,l} \rho_l \|\partial_n u_j - I_H \partial_n u_j\|_{H^{1/2}_{00}(\Gamma_{kl})}.\end{aligned}$$

Applying twice Lemma 5.12 to the two terms on the right-hand side of this inequality and using the assumption $\rho_l \le \rho_k$, we have

$$\rho_l \|\partial_n u_k - \partial_n u_l\|^2_{H^{1/2}_{00}(\Gamma_{kl})} \preceq \left(1 + \log\left(\frac{H}{\underline{h}}\right)\right)^2 \sum_{j=k,l} \rho_j |u_j|^2_{H^2(\Omega_j)}.$$

This and (5.3) completes the proof. $\square$

We define a projection $P_\Delta : \widetilde{W} \to W_\Delta$ by

$$(P_\Delta w)_{\delta_{ji}} := (\pi^t_{ji}(w^{(i)}_{1|\gamma_{ij}} - w^{(j)}_{1|\delta_{ji}}), \pi^n_{ji}(w^{(i)}_{2|\gamma_{ij}} - w^{(j)}_{2|\delta_{ji}})) \quad \text{on} \quad \delta_{ji}, \tag{5.4}$$

where $(w^{(i)}_{1|\gamma_{ij}}, w^{(i)}_{2|\gamma_{ij}}) \in W_i(\gamma_{ij})$ and $(w^{(j)}_{1|\delta_{ji}}, w^{(j)}_{2|\delta_{ji}}) \in W_j(\delta_{ji})$ are the restriction of $w \in \widetilde{W}$ to the mortar γ_{ij} and the slave δ_{ji} of an interface Γ_{ij}.

LEMMA 5.14. *For all* $w \in \widetilde{W}$*, we have*

$$\|P_\Delta w\|_{S_\Delta} \preceq (1 + \log(H/\underline{h}))\|w\|_S,$$

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where $H = \max_k h_k$ *and* $\underline{h} = \min_k h_k$.

Proof. Take any $w \in \widetilde{W}$ and consider its components associated with an interface $\Gamma_{ij} \subset \Gamma$. We have four trace functions $w^{(i)}_{|\gamma_{ij}} = (w^{(i)}_{1|\gamma_{ij}}, w^{(i)}_{2|\gamma_{ij}})$ and $w^{(j)}_{|\delta_{ji}} = (w^{(j)}_{1|\delta_{ji}}, w^{(j)}_{2|\delta_{ji}})$. By the continuity of degrees of freedom at crosspoints, we have $w^{(i)}_{1|\gamma_{ij}} - w^{(j)}_{1|\delta_{ji}} \in H^2_0(\Gamma_{ij})$ and $w^{(i)}_{2|\gamma_{ij}} - w^{(j)}_{2|\delta_{ji}} \in H^1_0(\Gamma_{ij})$.

Let $\tilde{w} \in \widetilde{W}$ be the extension of $P_\Delta w$ by zero, and let the pair $(\tilde{w}_1, \tilde{w}_2)$ denotes the restriction of $\tilde{w}$ to a slave δ_{ji}. By Lemma 5.4 we get

$$\|P_\Delta w\|^2_{S_\Delta} = \|\tilde{w}\|^2_S \preceq \sum_{j=1}^{N} \sum_{\delta_{ji} \subset \partial\Omega_j} \rho_j (\|\partial_t \tilde{w}_{1|\delta_{ji}}\|^2_{H^{1/2}_{00}(\Gamma_{ij})} + \|\tilde{w}_{2|\delta_{ji}}\|^2_{H^{1/2}_{00}(\Gamma_{ij})}).$$

By the definition of $\tilde{w}$ and Lemmas 5.8 and 5.10 we get

$$\begin{aligned}
\|\tilde{w}\|^2_{S_\Delta} &\preceq \sum_{j=1}^{N} \sum_{\delta_{ji} \subset \partial\Omega_j} \rho_j (\|\partial_t \tilde{w}_{1|\delta_{ji}}\|^2_{H^{1/2}_{00}(\Gamma_{ij})} + \|\tilde{w}_{2|\delta_{ji}}\|^2_{H^{1/2}_{00}(\Gamma_{ij})}) \\
&= \sum_{\delta_{ji} \subset \Gamma} \rho_j \left(\|\partial_t \pi^t_{ij} (w^{(i)}_{1|\gamma_{ij}} - w^{(j)}_{1|\delta_{ji}})\|^2_{H^{1/2}_{00}(\Gamma_{ij})} + \right. \\
&\qquad\qquad \left. + \|\pi^n_{ij} (w^{(i)}_{2|\gamma_{ij}} - w^{(j)}_{2|\delta_{ji}})\|^2_{H^{1/2}_{00}(\Gamma_{ij})} \right) \\
&\preceq \sum_{\delta_{ji}} \rho_j \left(\|\partial_t w^{(i)}_{1|\gamma_{ij}} - \partial_t w^{(j)}_{1|\delta_{ji}}\|^2_{H^{1/2}_{00}(\Gamma_{ij})} + \|w^{(i)}_{2|\gamma_{ij}} - w^{(j)}_{2|\delta_{ji}}\|^2_{H^{1/2}_{00}(\Gamma_{ij})} \right).
\end{aligned}$$

Finally, using Lemma 5.13 and summing over all slaves we conclude that

$$\|P_\Delta w\|_{S_\Delta} = \|\tilde{w}\|_S \preceq \left(1 + \log\left(\frac{H}{\underline{h}}\right)\right) \|w\|_S. \qquad \square$$

6. Condition number bounds. In this section we give the condition number estimate of the preconditioned operator in the following main theorem of this paper.

THEOREM 6.1. *For any* $\lambda \in M$, *we have*

$$\langle \mathcal{M}_{DN} \lambda, \lambda \rangle \leq \langle F_{DP} \lambda, \lambda \rangle \preceq \left(1 + \log\left(\frac{H}{\underline{h}}\right)\right)^2 \langle \mathcal{M}_{DN} \lambda, \lambda \rangle,$$

where $H = \max_k h_k$ *and* $\underline{h} = \min_k h_k$.

Proof. Lower bound. For any nonzero $w \in W_\Delta$, define $\tilde{w} \in \widetilde{W}_\Delta$ as the extension of w by zero. Then we have $\|w\|_{S_\Delta} = \|\tilde{w}\|_S$. Thus by Lemmas 5.1 and 5.2, we have

$$\begin{aligned}
\langle \mathcal{M}_{DN} \lambda, \lambda \rangle^{1/2} &= \sup_{w \in W_\Delta \setminus \{0\}} \frac{b(\tilde{w}, \lambda)}{\|w\|_{S_\Delta}} = \sup_{\tilde{w} \in \widetilde{W}_\Delta \setminus \{0\}} \frac{b(\tilde{w}, \lambda)}{\|\tilde{w}\|_S} \\
&\leq \sup_{w \in \widetilde{W} \setminus \{0\}} \frac{b(w, \lambda)}{\|w\|_S} = \langle F_{DP} \lambda, \lambda \rangle^{1/2}.
\end{aligned}$$

Upper bound. For any $w \in \widetilde{W}$, we have four trace functions associated with the interface $\Gamma_{ij} \subset \Gamma$: $w^{(i)}_{|\gamma_{ij}} = (w^{(i)}_{1|\gamma_{ij}}, w^{(i)}_{2|\gamma_{ij}})$ and $w^{(j)}_{|\delta_{ji}} = (w^{(j)}_{1|\delta_{ji}}, w^{(j)}_{2|\delta_{ji}})$. Then by (5.4), and

Definitions 5.7 and 5.9 we have

$$\begin{aligned} b(w,\lambda) &= \sum_{\Gamma_{ij}\subset\Gamma}\sum_{k=1,2}\int_{\Gamma_{ij}} (w^{(i)}_{k|\gamma_{ij}} - w^{(j)}_{k|\delta_{ji}})\lambda_{k|\delta_{ji}}\, ds \\ &= \sum_{\Gamma_{ij}\subset\Gamma}\int_{\Gamma_{ij}} \pi^t_{ji}(w^{(i)}_{1|\gamma_{ij}} - w^{(j)}_{1|\delta_{ji}})\lambda_1\, ds + \int_{\Gamma_{ij}} \pi^n_{ji}(w^{(i)}_{2|\gamma_{ij}} - w^{(j)}_{2|\delta_{ji}})\lambda_2\, ds \\ &= \sum_{\Gamma_{ij}\subset\Gamma}\sum_{k=1,2}\int_{\Gamma_{ij}} (P_\Delta w)_{k|\delta_{ji}}\lambda_{k|\delta_{ji}}\, ds. \end{aligned}$$

Hence by Lemmas 5.2 and 5.1 we conclude that

$$\begin{aligned} \langle F_{DP}\lambda,\lambda\rangle^{1/2} &= \sup_{w\in\widetilde{W}\setminus\{0\}} \frac{b(w,\lambda)}{\|w\|_S} \\ &= \sup_{w\in\widetilde{W}\setminus\{0\}} \sum_{\Gamma_{ij}\subset\Gamma}\sum_{k=1,2} \frac{\int_{\Gamma_{ij}} (P_\Delta w)_{k|\delta_{ji}}\lambda_{k|\delta_{ji}}\, ds}{\|w\|_S} \\ &\le \langle \mathcal{M}_{DN}\lambda,\lambda\rangle^{1/2} \sup_{w\in\widetilde{W}\setminus\{0\}} \frac{\|P_\Delta w\|_{S_\Delta}}{\|w\|_S} \\ &\preceq (1+\log(H/\underline{h}))\langle \mathcal{M}_{DN}\lambda,\lambda\rangle^{1/2}. \end{aligned}$$

The last estimate follows from Lemma 5.14. $\square$

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