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PERTURBATION ANALYSIS FOR COMPLEX SYMMETRIC, SKEW SYMMETRIC, EVEN AND ODD MATRIX POLYNOMIALS*

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Abstract. In this work we propose a general framework for the structured perturbation analysis of several classes of structured matrix polynomials in homogeneous form, including complex symmetric, skew-symmetric, even and odd matrix polynomials. We introduce structured backward errors for approximate eigenvalues and eigenvectors and we construct minimal structured perturbations such that an approximate eigenpair is an exact eigenpair of an appropriately perturbed matrix polynomial. This work extends previous work of Adhikari and Alam for the non-homogeneous case (we include infinite eigenvalues), and we show that the structured backward errors improve the known unstructured backward errors.

Key words. Polynomial eigenvalue problem, even matrix polynomial, odd matrix polynomial, complex symmetric matrix polynomial, complex skew-symmetric matrix polynomial, perturbation theory, backward error.

AMS subject classifications. 65F15, 15A18, 65F35, 15A12

1. Introduction.

In this paper we study the perturbation analysis for eigenvalues and eigenvectors of matrix polynomials of degree m

$$\mathbf{L}(c,s) := \sum_{j=0}^{m} c^{m-j} s^j A_j, \tag{1.1}$$

with coefficient matrices, $A_j \in \mathbb{C}^{n\times n}$. In contrast to previous work on this topic [2, 3, 4], we consider the homogeneous form of matrix polynomials, where the eigenvalues are represented as pairs $(c,s) \in \mathbb{C}^2 \setminus \{0\}$, which for $c \neq 0$ correspond to finite eigenvalues $\lambda = \frac{s}{c}$, while $(0,1)$ corresponds to the eigenvalue ∞.

The eigenvalue problem for matrix polynomials arises naturally in a large number of applications; see, e.g., [17, 18, 23, 24, 27, 29, 36, 37] and the references therein. In many applications, the coefficient matrices have further structure which reflects the properties of the underlying physical model; see [9, 11, 12, 19, 28, 30, 32, 37]. Since the polynomial eigenvalue problems typically arise from physical modelling, including numerical discretization methods such as finite element modelling [10, 31], and since the eigenvalue problem is usually solved with numerical methods that are subject to round-off as well as approximation errors, it is very important to study the perturbation analysis of these problems. This analysis is necessary to study the sensitivity of the eigenvalue/eigenvectors under the modelling, discretization, approximation, and roundoff errors, but also to judge whether the numerical methods that are used yield reliable results.

While the perturbation analysis for classical and generalized eigenvalue problems is well studied (see [20, 33, 38]), for polynomial eigenvalue problems the situation is much less satisfactory and most research is very recent; see [22, 23, 24, 35, 36]. Here we are particularly interested in the behavior of the eigenvalues and eigenvectors under perturbations which preserve the structure of the matrix polynomial. This has recently been an important research topic [1, 2, 3, 6, 11, 12].

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In this paper we will focus on complex matrix polynomials, where the coefficient matrices are complex symmetric or skew-symmetric, i.e., $\mathbf{L}(c,s) = \pm\mathbf{L}^T(c,s)$, or where the matrix polynomials are T-even or T-odd, i.e., $\mathbf{L}(c,s) = \pm\mathbf{L}^T(c,-s)$. Complex (skew)-symmetric problems arise in the finite element modelling of the acoustic field in car interiors and in the design of axisymmetric VCSEL devices; see, e.g., [8, 34]. Complex T-even or T-odd problems arise in the vibration analysis for high-speed trains; see, e.g., [25, 26]. Many applications only need finite eigenvalues and associated eigenvectors, but the eigenvectors associated with the eigenvalue infinity play an important role as well, since quite often the infinite spectrum has to be deflated before classical methods can be employed; see [13, 14].

While the perturbation analysis and the construction of backward errors for finite eigenvalues have been studied in detail, there are only few results associated with the eigenvalue infinity. We will present a systematic general perturbation framework that covers finite and infinite eigenvalues and extends the structured theory of [1, 2, 3, 6, 11, 12] as well as the unstructured theory for the homogeneous case studied in [5, 6, 7, 16, 24, 33]. In particular, to present the backward error analysis for a given approximation to an eigenvalue/eigenvector pair of a matrix polynomial $\mathbf{L}$, we will construct an appropriately structured minimal (in the Frobenius and the spectral norm) perturbation polynomial $\Delta\mathbf{L}$ such that the given eigenvalue/eigenvector pair is exact for $\mathbf{L} + \Delta\mathbf{L}$. It will turn out that the so constructed minimal perturbation is unique in the case of the Frobenius norm and that there are infinitely many such minimal perturbations in the case of the spectral norm. We will compare the so constructed perturbations with those constructed for matrix pencils and matrix polynomials in [2, 3, 4] and show that our results generalize these results and provide the following further information on the eigenvalues 0 and ∞ of $\mathbf{L} + \Delta\mathbf{L}$.

- For the case of complex symmetric or skew-symmetric matrix polynomials, we show that the nearest perturbed matrix polynomial can have all kinds of eigenvalues including 0 and ∞.
- When the degree is $m = 1$, we present the perturbation analysis for the case of T-even and T-odd matrix pencils and we show that the nearest perturbed pair can have 0 and ∞ as eigenvalues depending on the choice of (λ, μ) for which we want to compute the backward error. Furthermore, when $\lambda = 0$ or $\mu = 0$, then we show that the perturbed pair is the same for the spectral and the Frobenius norm.
- When the degree is $m > 1$ and even, then for the case of T-even matrix polynomials we show that the nearest perturbed polynomial can have both 0 and ∞ eigenvalues depending on the choice of (λ, μ) for which we want to compute the backward error. Again, when $\lambda = 0,\ \mu \neq 0$ or $\lambda \neq 0,\ \mu = 0$, then the perturbed polynomial is the same for the spectral and the Frobenius norm.
- When $m > 1$ is odd, then for the case of T-even matrix polynomials we show that the nearest perturbed matrix polynomial can have all possible finite eigenvalues including 0 but not the eigenvalue ∞.
- When $m > 1$ is even, then for the case of T-odd matrix polynomials we show that the nearest perturbed polynomial can have non-zero finite eigenvalues but not the eigenvalue ∞.
- When $m > 1$ is odd, then for the case of T-odd matrix polynomials we show that the perturbed polynomial can have only ∞ and non-zero finite eigenvalues.

The paper is organized as follows: In Section 2, we review some known techniques that were developed in [5, 6, 7] for matrix pencils and identify the types of structured homogeneous matrix polynomials that we will analyze as well as the eigenvalue symmetry that arises for these structured matrix polynomials. In Section 3 and in Section 4 we present the structured backward error analysis of an approximate eigenpair for complex symmetric, complex skew-symmetric, T-even, and T-odd matrix polynomials and compare these results with the corresponding unstructured backward errors. We also present a systematic general procedure for the construction of an appropriate structured minimal complex symmetric, complex skew-symmetric, T-even, and T-odd polynomial $\Delta\mathbf{L}$ such that the given approximate eigenvalue and eigenvector are exact for $\mathbf{L}+\Delta\mathbf{L}$. These results cover finite and infinite eigenvalues and generalize results of [1, 2, 3, 4, 11] in a systematic way.

2. Notation and preliminaries. We denote by $\mathbb{R}^{n\times n}, \mathbb{C}^{n\times n}$ the sets of real and complex $n \times n$ matrices, respectively. For an integer p, $1 \leq p \leq \infty$, and an elementwise nonnegative vector $w = [w_1, \ldots, w_n]^T \in \mathbb{R}^n$, we define a weighted p-(semi)norm of a real or complex vector $x = [x_1, \ldots, x_n]^T$ via

$$\|x\|_{w,p} := \|[w_1x_1, w_2x_2, \ldots, w_nx_n]^T\|_p.$$

If w is elementwise strictly positive, then this is a norm, and if w has zero components then it is a seminorm. We define the componentwise inverse of w via $w^{-1} := [w_1^{-1}, \ldots, w_m^{-1}]^T$, where we use the convention that $w_i^{-1} = 0$ if $w_i = 0$.

We will consider structured and unstructured backward errors both in the spectral norm and the Frobenius norm on $\mathbb{C}^{n\times n}$, which are given by

$$\|A\|_2 := \max_{\|x\|_2=1} \|Ax\|, \quad \|A\|_F := (\text{trace} A^*A)^{1/2},$$

respectively.

By $\sigma_{\max}(A)$ and $\sigma_{\min}(A)$ we denote the largest and smallest singular value of a matrix A, respectively. The identity matrix is denoted by I and $\overline{A}$, A^T, and A^H stand for the conjugate, transpose, and conjugate transpose of a matrix A, respectively.

The set of all matrix polynomials of degree $m \geq 0$ with coefficients in $\mathbb{C}^{n\times n}$ is denoted by $\mathbf{L}_m(\mathbb{C}^{n\times n})$. This is a vector space which we can equip with weighted (semi)norms (given a nonnegative weight vector $w := [w_0, w_1, \ldots, w_m]^T \in \mathbb{R}^{m+1} \setminus \{0\}$) defined as

$$|\!|\!|\mathbf{L}|\!|\!|_{w,F} := \|(A_0, \ldots, A_m)\|_{w,F} = (w_0^2\|A_0\|_F^2 + \ldots + w_m^2\|A_m\|_F^2)^{1/2},$$

for the Frobenius norm and

$$|\!|\!|\mathbf{L}|\!|\!|_{w,2} := \|(A_0, \ldots, A_m)\|_{w,2} = (w_0^2\|A_0\|_2^2 + \ldots + w_m^2\|A_m\|_2^2)^{1/2},$$

for the spectral norm. A matrix polynomial is called *regular* if $\det(\mathbf{L}(\lambda, \mu)) \neq 0$ for some $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{(0, 0)\}$, otherwise it is called *singular*. The *spectrum* of a homogeneous matrix polynomial $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ is defined as

$$\Lambda(\mathbf{L}) := \{(c, s) \in \mathbb{C}^2 \setminus \{(0, 0)\} : \ \text{rank}(\mathbf{L}(c, s)) < n\}.$$

In the following we normalize the set of points $(c, s) \in \mathbb{C}^2 \setminus \{(0, 0)\}$, such that c is real and $|c|^2 + |s|^2 = 1$. With this normalization, it follows that the spectrum $\Lambda(\mathbf{L})$ of a matrix polynomial $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ can be identified with a subset of the Riemann sphere; see, e.g., [6].

In the following we will compute backward errors for structured matrix polynomials. These were introduced, e.g., in [21, 35], but here we follow [5, 6, 7] and define the backward error of an approximate eigenpair as follows. Let $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ be an approximate eigenvalue of $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ with corresponding normalized approximate right eigenvector $x \neq 0$ with $x^Hx = 1$, i.e., $\mathbf{L}(\lambda, \mu)x = 0$. Then we consider the Frobenius and spectral norm backward errors associated with a given nonnegative weight vector $[w_0, w_1, \ldots, w_m]^T$

$$\eta_{w,F}(\lambda, \mu, x, \mathbf{L}) := \inf\{|\!|\!|\Delta\mathbf{L}|\!|\!|_{w,F}, \ \Delta\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n}), \ (\mathbf{L}(\lambda, \mu) + \Delta\mathbf{L}(\lambda, \mu))x = 0\},$$
$$\eta_{w,2}(\lambda, \mu, x, \mathbf{L}) := \inf\{|\!|\!|\Delta\mathbf{L}|\!|\!|_{w,2}, \ \Delta\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n}), (\mathbf{L}(\lambda, \mu) + \Delta\mathbf{L}(\lambda, \mu))x = 0\},$$

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respectively. When $w := [1, 1, \ldots, 1]^T$, then we just leave off the index w for convenience. The backward errors for structured matrix polynomials from a set $\mathbf{S} \subset \mathbf{L}_m(\mathbb{C}^{n\times n})$ are defined analogously as

$$\begin{aligned} \eta^{\mathbf{S}}_{w,F}(\lambda,\mu,x,\mathbf{L}) &:= \inf\{|\!|\!|\Delta\mathbf{L}|\!|\!|_{w,F},\ \Delta\mathbf{L}\in\mathbf{S},\ (\mathbf{L}(\lambda,\mu)+\Delta\mathbf{L}(\lambda,\mu))x=0\},\\ \eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathbf{L}) &:= \inf\{|\!|\!|\Delta\mathbf{L}|\!|\!|_{w,2},\ \Delta\mathbf{L}\in\mathbf{S},\ (\mathbf{L}(\lambda,\mu)+\Delta\mathbf{L}(\lambda,\mu))x=0\}, \end{aligned}$$

respectively.

In order to compute the backward errors, we will need the partial derivative $\nabla_i\|z\|_{w,2}$ of the map

$$\begin{aligned} \mathbb{C}^{m+1} &\to \mathbb{R},\\ z &\mapsto \|z\|_{w,2} = \|(z_0, z_1, \ldots, z_m)\|_{w,2}, \end{aligned} \tag{2.1}$$

which is the derivative of (2.1) with respect to the variable z_j obtained by fixing the variables $z_0, z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m$ as constants. The gradient of the map (2.1) is then defined as

$$\nabla(\|z\|_{w,2}) = [\nabla_0\|z\|_{w,2}, \nabla_1\|z\|_{w,2}, \ldots, \nabla_m\|z\|_{w,2}]^T \in \mathbb{C}^{m+1}.$$

For a given $(\lambda,\mu)\in\mathbb{C}^2\setminus\{(0,0)\}$ and $x\in\mathbb{C}^n$ with $x^Hx=1$, we set $k := -\mathbf{L}(\lambda,\mu)x$ and, with a given nonnegative weight vector $[w_0, w_1, \ldots, w_m]^T$, we introduce

$$H_{w,2} := H_{w,2}(\lambda,\mu) := \|(\lambda^m\mu^0, \lambda^{m-1}\mu, \ldots, \lambda^0\mu^m)\|_{w,2},$$

and we use the notation $\nabla_j H_{w,2}$ for the partial derivative (with respect to z_j) of the map (2.1) at $(\lambda^m\mu^0, \lambda^{m-1}\mu, \ldots, \lambda^0\mu^m)$. Then we have

$$\eta_{w,2}(\lambda,\mu,x,\mathbf{L}) = \frac{\|\mathbf{L}(\lambda,\mu)x\|}{H_{w^{-1},2}(\lambda,\mu)}. \tag{2.2}$$

Defining for each of the coefficients

$$z_{A_j} := \frac{\nabla_j H_{w^{-1},2}}{H_{w^{-1},2}} \tag{2.3}$$

and introducing the perturbations $\Delta A_j := \overline{z_{A_j}}kx^H$ for the coefficients, we form the matrix polynomial

$$\Delta\mathbf{L}(c,s) = \sum_{j=0}^{m} c^{m-j}s^j\Delta A_j,$$

with

$$|\!|\!|\Delta\mathbf{L}|\!|\!|_{w,2} = \frac{\|k\|}{H_{w^{-1},2}}.$$

For $z\in\mathbb{C}$ we set $\mathrm{sign}(z) := \overline{z}/|z|$, when $z\neq 0$ and $\mathrm{sign}(z) := 0$ when $z=0$. With these definitions we have the following preliminary results which generalize the corresponding results of [5, 6] to matrix polynomials.

PROPOSITION 2.1. *Consider the map $\|z\|_{w,2}$ given by (2.1). Then $\|z\|_{w,2}$ is differentiable on $\mathbb{C}^{m+1}$ and*

$$\nabla_i\|z\|_{w,2} = \frac{w_i^2 z_i}{\|z\|_{w,2}},\ i = 0, 1, 2, \ldots, m.$$

Proof. The assertion follows from the fact that $\nabla(|z_i|^2) = 2z_i$. ∎

The proof of the following two propositions is analogous.

PROPOSITION 2.2. *Let m be an integer and let $\tilde{m} = \frac{m}{2} + 1,\ \hat{m} = m$ if m is even and $\tilde{m} = \frac{m+1}{2},\ \hat{m} = m - 1$ if m is odd. Consider the mapping*

$$\begin{aligned} K_{w,2} : \mathbb{C}^{\tilde{m}} &\to \mathbb{R} \\ z &\mapsto \|[z_0, z_2, z_4, \ldots, z_{\hat{m}}]^T\|_{w,2}. \end{aligned}$$

Then $K_{w,2}$ is differentiable and

$$\nabla_i K_{w,2}(z) = \frac{w_i^2 z_i}{K_{w,2}(z)},\ i = 0, 2, 4, \ldots, \hat{m}.$$

PROPOSITION 2.3. *Let m be an integer and let $\tilde{m} = \frac{m}{2},\ \hat{m} = m$ if m is even and $\tilde{m} = \frac{m+1}{2},\ \hat{m} = m - 1$ if m is odd. Consider the mapping*

$$\begin{aligned} N_{w,2} : \mathbb{C}^{\tilde{m}} &\to \mathbb{R} \\ z &\mapsto \|[z_1, z_3, z_5, \ldots, z_{\hat{m}}]^T\|_{w,2}. \end{aligned}$$

Then $N_{w,2}$ is differentiable and

$$\nabla_i N_{w,2}(z) = \frac{w_i^2 z_i}{N_{w,2}(z)},\ i = 1, 3, 5, \ldots, \hat{m}.$$

PROPOSITION 2.4. *Consider the functions*

$$\begin{aligned} H_{w,2}(c^m s^0, c^{m-1}s, \ldots, c^0 s^m) &= \|[c^m s^0, c^{m-1}s, \ldots, c^0 s^m]^T\|_{w,2}, \\ K_{w,2}(c^m s^0, c^{m-2}s^2, \ldots, c^0 s^m) &= \|[c^m s^0, c^{m-2}s^2, \ldots, c^0 s^m]^T\|_{w,2} && \text{if } m \text{ is even}, \\ K_{w,2}(c^m s^0, c^{m-2}s^2, \ldots, cs^{m-1}) &= \|[c^m s^0, c^{m-2}s^2, \ldots, cs^{m-1}]^T\|_{w,2} && \text{if } m \text{ is odd}, \\ N_{w,2}(c^{m-1}s, c^{m-3}s^3, \ldots, cs^{m-1}) &= \|[c^{m-1}s, c^{m-3}s^3, \ldots, cs^{m-1}]^T\|_{w,2} && \text{if } m \text{ is even}, \\ N_{w,2}(c^{m-1}s, c^{m-3}s^3, \ldots, c^0 s^m) &= \|[c^{m-1}s, c^{m-3}s^3, \ldots, c^0 s^m]^T\|_{w,2} && \text{if } m \text{ is odd}. \end{aligned}$$

For even m, the following formulas hold:

$$\begin{aligned} &\sum_{j=0,j\text{ even}}^{m} c^{m-j}s^j \frac{\overline{\nabla_j H_{w,2}}}{H_{w,2}} = \frac{K_{w,2}^2}{H_{w,2}^2} && \sum_{j=0}^{m} w_j^{-2}|\nabla_j K_{w,2}|^2 = 1, \\ &\sum_{j=1,j\text{ odd}}^{m-1} c^{m-j}s^j \frac{\overline{\nabla_j H_{w,2}}}{H_{w,2}} = \frac{N_{w,2}^2}{H_{w,2}^2} && \sum_{j=1}^{m-1} w_j^{-2}|\nabla_j N_{w,2}|^2 = 1, \\ &\sum_{j=0,j\text{ even}}^{m} c^{m-j}s^j \frac{\overline{\nabla_j K_{w,2}}}{K_{w,2}} = 1, && \sum_{j=1,j\text{ odd}}^{m-1} c^{m-j}s^j \frac{\overline{\nabla_j N_{w,2}}}{N_{w,2}} = 1, \\ &\sum_{j=0,j\text{ even}}^{m} c^{m-j}s^j \frac{\overline{\nabla_j H_{w,2}}}{H_{w,2}} + \sum_{j=1,j\text{ odd}}^{m-1} c^{m-j}s^j \frac{\overline{\nabla_j H_{w,2}}}{H_{w,2}} = 1. \end{aligned}$$

For odd m, the following formulas hold:

$$\sum_{j=0,j\text{ even}}^{m-1} c^{m-j}s^j\frac{\overline{\nabla_j H_{w,2}}}{H_{w,2}}=\frac{K_{w,2}^2}{H_{w,2}^2},\qquad \sum_{j=1,j\text{ odd}}^{m} c^{m-j}s^j\frac{\overline{\nabla_j H_{w,2}}}{H_{w,2}}=\frac{N_{w,2}^2}{H_{w,2}^2},$$

$$\sum_{j=0,j\text{ even}}^{m-1} c^{m-j}s^j\frac{\overline{\nabla_j K_{w,2}}}{K_{w,2}}=1,\qquad \sum_{j=1,j\text{ odd}}^{m} c^{m-j}s^j\frac{\overline{\nabla_j N_{w,2}}}{N_{w,2}}=1,$$

$$\sum_{j=0,j\text{ even}}^{m-1} c^{m-j}s^j\frac{\overline{\nabla_j H_{w,2}}}{H_{w,2}}+\sum_{j=1,j\text{ odd}}^{m} c^{m-j}s^j\frac{\overline{\nabla_j H_{w,2}}}{H_{w,2}}=1.$$

For all m, the following formulas hold:

$$\sum_{j=0}^{m} c^{m-j}s^j\frac{\overline{\nabla_j H_{w,2}}}{H_{w,2}}=1,\qquad \sum_{j=0}^{m} w_j^{-2}|\nabla_j H_{w,2}|^2=1.$$

Proof. By Proposition 2.1, we have

$$\nabla_j H_{w,2}(c^m,c^{m-1}s,\ldots,s^m)=\frac{w_j^2 c^{m-j}s^j}{H_{w,2}(c^m,c^{m-1}s,\ldots,s^m)}.$$

Then, we obtain

$$\sum_{j=0,\ j\text{ even}}^{m} c^{m-j}s^j\frac{\overline{\nabla_j H_{w,2}}}{H_{w,2}}=\sum_{j=0,\ j\text{ even}}^{m} w_j^2 c^{m-j}s^j\frac{\overline{c^{m-j}s^j}}{H_{w,2}^2}=\frac{K_{w,2}^2}{H_{w,2}}.$$

The other parts follow analogously, using Propositions 2.1–2.3. ◻

After establishing these formulas for general matrix polynomials, we now turn to the structured classes. These classes were discussed in detail in [28] but not in homogeneous form. So let us first introduce the homogeneous versions.

DEFINITION 2.5. *Let $(c,s)\in\mathbb{C}^2\setminus\{(0,0)\}$. A matrix polynomial $\mathbf{L}\in\mathbf{L}_m(\mathbb{C}^{n\times n})$ is called*

1. Symmetric/skew-symmetric *if* $\mathbf{L}(c,s)=\pm\mathbf{L}^T(c,s)$,
2. T-even/T-odd *if* $\mathbf{L}(c,s)=\pm\mathbf{L}^T(c,-s)$.

The spectra of these classes of structured matrices have a symmetry structure that is summarized in the following proposition which follows directly from the results for the non-homogeneous case in [28].

PROPOSITION 2.6.

1. *Let $\mathbf{L}\in\mathbf{L}_m(\mathbb{C}^{n\times n})$ be a complex symmetric or complex skew-symmetric matrix polynomial of the form (1.1). If $x\in\mathbb{C}^n$ is a right eigenvector of $\mathbf{L}$ corresponding to an eigenvalue $(\lambda,\mu)\in\mathbb{C}^2\setminus\{(0,0)\}$, then $\overline{x}$ is a left eigenvector corresponding to the eigenvalue (λ,μ).*
2. *Let $\mathbf{L}\in\mathbf{L}_m(\mathbb{C}^{n\times n})$ be a complex T-even or T-odd matrix polynomial of the form (1.1). If $x\in\mathbb{C}^n$ and $y\in\mathbb{C}^n$ are right and left eigenvector associated to an eigenvalue $(\lambda,\mu)\in\mathbb{C}^2\setminus\{(0,0)\}$ of $\mathbf{L}$, then $\overline{y}$ and $\overline{x}$ are right and left eigenvectors associated to the eigenvalue $(\lambda,-\mu)$.*

Since T-odd and T-even matrix polynomials have coefficients that are alternating between symmetric and skew-symmetric matrices, it is clear that in the product $x^T(\mathbf{L}(\lambda,\mu))x$ all terms associated with skew-symmetric coefficients vanish; these are the coefficients with

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TABLE 2.1
Eigenvalues and eigenvectors of structured matrix polynomials.

S	Eigenvalues	Eigenpairs	$x^T A_j x$
symmetric	(λ, μ)	$((\lambda, \mu), x, \overline{x})$	
skew-symm.	(λ, μ)	$((\lambda, \mu), x, \overline{x})$	0
T-even	$((\lambda, \mu), (\lambda, -\mu))$	$((\lambda, \mu), x, \overline{y}), ((\lambda, -\mu), y, \overline{x})$	0 for all odd j
T-odd	$((\lambda, \mu), (-\lambda, \mu))$	$((\lambda, \mu), x, \overline{y}), ((-\lambda, \mu), y, \overline{x})$	0 for all even j

odd index for T-even matrix polynomials, and the ones with even index for T-odd matrix polynomials. We summarize the properties of these structured matrix polynomials in Table 2.1.

To derive the backward error formulas, we will frequently need the following completion results in which for a symmetric matrix X, $X^{\frac{1}{2}}$ denotes the positive square root.

THEOREM 2.7 ([15]). *Consider a block matrix $T := \begin{bmatrix} A & C \\ B & X \end{bmatrix}$. Then for any positive number*

$$\chi \geq \max\left\{ \left\| \begin{bmatrix} A \\ B \end{bmatrix} \right\|_2, \left\| \begin{bmatrix} A & C \end{bmatrix} \right\|_2 \right\},$$

the block X can be chosen such that

$$\left\| \begin{bmatrix} A & C \\ B & X \end{bmatrix} \right\|_2 \leq \chi,$$

where X is of the form $X = -KA^H L + \chi(I - KK^H)^{1/2} Z (I - L^H L)^{1/2}$, and where $K := ((\chi^2 I - A^H A)^{-1/2} B^H)^H$, $L := (\chi^2 I - A^H A)^{-1/2} C$ with Z an arbitrary matrix such that $\|Z\|_2 \leq 1$.

As a Corollary of Theorem 2.7 one has the following result for complex matrices.

COROLLARY 2.8. *Let $A = \pm A^T$, $C = \pm B^T \in \mathbb{C}^{n \times n}$ and $\chi := \sigma_{\max}\left(\begin{bmatrix} A \\ B \end{bmatrix}\right)$. Then there exists a symmetric/skew-symmetric matrix $X \in \mathbb{C}^{n \times n}$ such that*

$$\sigma_{max}\left(\begin{bmatrix} A & \pm B^T \\ B & X \end{bmatrix}\right) = \chi,$$

and X has the form

$$X := -K\overline{A}K^T + \chi(I - KK^H)^{1/2} Z (I - \overline{K}K^T)^{1/2},$$

$K := B(\chi^2 I - \overline{A}A)^{-1/2}$ and where $Z = \pm Z^T \in \mathbb{C}^{n \times n}$ is an arbitrary matrix such that $\|Z\|_2 \leq 1$.

In the results presented below, we always use $Z = 0$. In the following section we derive backward errors for the different classes of structured matrix polynomials.

3. Backward errors for complex symmetric and skew-symmetric matrix polynomials. In this section we derive backward error formulas for homogeneous complex symmetric and skew-symmetric matrix polynomials. Throughout this section, we will make use of the partial derivatives $\dfrac{\nabla_j H_{w^{-1},2}}{H_{w^{-1},2}}$ of $H_{w^{-1},2}$ and of z_{A_j} as defined in (2.3).

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THEOREM 3.1. *Let $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ be a regular, symmetric matrix polynomial of the form (1.1), let $(\lambda,\mu) \in \mathbb{C}^2 \setminus \{(0,0)\}$, let $x \in \mathbb{C}^n$ be such that $x^H x = 1$ and $k := -\mathbf{L}(\lambda,\mu)x$. Introduce the perturbation matrices*

$$\Delta A_j = -\overline{x}x^T A_j x x^H + \overline{z_{A_j}}\left[\overline{x}k^T + kx^H - 2(x^T k)\overline{x}x^H\right], \ j = 0,1,\ldots,m$$

and define

$$\Delta\mathbf{L}(c,s) = \sum_{j=0}^{m} c^{m-j}s^j \Delta A_j \in \mathbf{L}_m(\mathbb{C}^{n\times n}).$$

Then $\Delta\mathbf{L}$ is a symmetric matrix polynomial and $(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu))x = 0$.

Proof. Since for all j we have $\Delta A_j = \Delta A_j^T$, it follows that $\Delta\mathbf{L}$ is symmetric and we have that

$$\begin{aligned}(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu))x &= \sum_{j=0}^{m}\lambda^{m-j}\mu^j(A_j + \Delta A_j)x \\ &= \sum_{j=0}^{m}\lambda^{m-j}\mu^j\left[A_j x - \overline{x}x^T A_j x + \overline{z_{A_j}}\left[\overline{x}k^T x + k - 2(x^T k)\overline{x}\right]\right] \\ &= -k(I - \overline{x}x^T) + \left[\overline{x}k^T x + k - 2(x^T k)\overline{x}\right]\sum_{j=0}^{m}\lambda^{m-j}\mu^j\overline{z_{A_j}}.\end{aligned}$$

By Proposition 2.4 we have that $\sum_{j=0}^{m}\lambda^{m-j}\mu^j\overline{z_{A_j}} = 1$. Then

$$(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu))x = -(I - \overline{x}x^T)k + \overline{x}k^T x + k - 2(x^T k)\overline{x} = 0,$$

since $k^T x = x^T k$. ∎

Theorem 3.1 with $c = 1$ and $w = [1,1,\ldots,1]^T$ implies Theorem 4.2.1 of [2] for the case of non-homogeneous matrix polynomials that have only finite eigenvalues, i.e., for which $\det(A_m) \neq 0$. Theorem 3.1 also implies Theorem 2.2 of [3] for matrix pencils. Using Theorem 3.1 we then obtain the following backward errors for complex symmetric matrix polynomials.

THEOREM 3.2. *Let $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ be a complex symmetric matrix polynomial of the form (1.1), let $(\lambda,\mu) \in \mathbb{C}^2 \setminus \{(0,0)\}$, let $x \in \mathbb{C}^n$ be such that $x^H x = 1$, and set $k := -\mathbf{L}(\lambda,\mu)x$.*

i) The structured backward error with respect to the Frobenius norm is given by

$$\eta^{\mathbf{S}}_{w,F}(\lambda,\mu,x,\mathbf{L}) = \frac{\sqrt{2\|k\|_2^2 - |x^T k|^2}}{H_{w^{-1},2}}.$$

There exists a unique complex symmetric polynomial $\Delta\mathbf{L}(c,s) := \sum_{j=0}^{m}c^{m-j}s^j\Delta A_j$ with coefficients

$$\Delta A_j = \overline{z_{A_j}}\left[\overline{x}k^T + kx^H - (x^T k)\overline{x}x^H\right], \ j = 0,1,\ldots,m$$

such that the structured backward error satisfies $\eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathbf{L}) = \|\Delta\mathbf{L}\|_{w,2}$ and $\overline{x}$, x are left and right eigenvectors corresponding to the eigenvalue (λ,μ) of $\mathbf{L} + \Delta\mathbf{L}$, respectively.

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ii) The structured backward error with respect to the spectral norm is given by

$$\eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathbf{L}) = \frac{\|k\|_2}{H_{w^{-1},2}}$$

and there exist a complex symmetric polynomial $\Delta\mathbf{L}(c,s) := \sum_{j=0}^{m} c^{m-j}s^j\Delta A_j$ *with coefficients*

$$\Delta A_j := \overline{z_{A_j}}\left[\overline{x}k^T + kx^H - (k^Tx)\overline{x}x^H - \frac{\overline{x^Tk}(I-\overline{x}x^T)kk^T(I-xx^H)}{\|k\|_2^2 - |x^Tk|^2}\right]$$

such that $\|\Delta\mathbf{L}\|_{w,2} = \eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathbf{L})$*, and* $(\mathbf{L}(\lambda,\mu)+\Delta\mathbf{L}(\lambda,\mu))x = 0$.

Proof. By Theorem 3.1 we have $(\mathbf{L}(\lambda,\mu)+\Delta\mathbf{L}(\lambda,\mu))x = 0$ and hence $k = \Delta\mathbf{L}(\lambda,\mu)x$. Now we construct a unitary matrix U which has x as its first column, $U = [x, U_1] \in \mathbb{C}^{n\times n}$ and let $\widetilde{\Delta A_j} := U^T\Delta A_j U = \begin{bmatrix} d_{j,j} & d_j^T \\ d_j & D_{j,j}\end{bmatrix}$, where $D_{j,j} = D_{j,j}^T \in \mathbb{C}^{(n-1)\times(n-1)}$. Then

$$\overline{U}\widetilde{\Delta\mathbf{L}(\lambda,\mu)}U^H = \overline{U}U^T(\Delta\mathbf{L}(\lambda,\mu))U^HU = \Delta\mathbf{L}(\lambda,\mu),$$

and hence

$$\overline{U}\widetilde{\Delta\mathbf{L}(\lambda,\mu)}U^Hx = \Delta\mathbf{L}(\lambda,\mu)x = k,$$

which implies that

$$\widetilde{\Delta\mathbf{L}(\lambda,\mu)}U^Hx = U^Tk = \begin{bmatrix} x^Tk \\ U_1^Tk\end{bmatrix}.$$

Therefore, we get that

$$\begin{bmatrix} \sum_{j=0}^m \lambda^{m-j}\mu^j d_{j,j} \\ \sum_{j=0}^m \lambda^{m-j}\mu^j d_j \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^m w_j d_{j,j}\dfrac{\lambda^{m-j}\mu^j}{w_j} \\ \sum_{j=0}^m w_j\lambda^{m-j}\mu^j\dfrac{d_j}{w_j}\end{bmatrix} = \begin{bmatrix} x^Tk \\ U_1^Tk\end{bmatrix}.$$

To minimize the norm of the perturbation, we solve this system for the parameters $d_{j,j}, d_j$ in a least squares sense, and obtain

$$\begin{bmatrix} w_0d_{0,0} \\ w_1d_{1,1} \\ w_2d_{2,2} \\ \vdots \\ w_md_{m,m}\end{bmatrix} = \begin{bmatrix} \overline{z_{A_0}} \\ \overline{z_{A_1}} \\ \overline{z_{A_2}} \\ \vdots \\ \overline{z_{A_m}}\end{bmatrix} x^Tk, \text{ and } \begin{bmatrix} w_0d_0 \\ w_1d_1 \\ \vdots \\ w_md_m\end{bmatrix} = \begin{bmatrix} \overline{z_{A_0}} \\ \overline{z_{A_1}} \\ \overline{z_{A_2}} \\ \vdots \\ \overline{z_{A_m}}\end{bmatrix} U_1^Tk,$$

Applying Proposition 2.1, we then get the following relations

$$d_{j,j} = \overline{z_{A_j}}x^Tk,\ d_j = \overline{z_{A_j}}U_1^Tk, \quad j = 0,1,\ldots,m.$$

From this we obtain

$$\begin{aligned} \Delta A_j &= \overline{U}\widetilde{\Delta A}U^H = \overline{x}d_{j,j}x^H + \overline{U_1}d_jx^H + \overline{x}d_j^TU_1^H + \overline{U_1}D_{j,j}U_1^H \\ &= \overline{z_{A_j}}[(\overline{x}x^Tkx^H) + \overline{U_1}U_1^Tkx^H + \overline{x}k^TU_1U_1^H)] + \overline{U_1}D_{j,j}U_1^H \\ &= \overline{z_{A_j}}[(\overline{x}x^Tkx^H) + (I-\overline{x}x^T)kx^H + \overline{x}k^T(I-xx^H))] + \overline{U_1}D_{j,j}U_1^H \\ &= \overline{z_{A_j}}\left[kx^H + \overline{x}k^T - (k^Tx)\overline{x}x^H\right] + \overline{U_1}D_{j,j}U_1^H. \end{aligned} \tag{3.1}$$

In the Frobenius norm, the minimal perturbation is obtained by taking $D_{j,j} = 0$, and hence we get

$$\begin{aligned}\|\Delta A_j\|_F^2 &= |d_{j,j}|^2 + 2\|d_j\|_2^2 = |z_{A_j}|^2(|x^T k|^2 + 2\|U_1^T k\|_2^2)\\ &= |\nabla_j H_{w^{-1},2}|^2 \frac{2\|k\|_2^2 - |x^T k|^2}{H_{w^{-1},2}^2},\end{aligned}$$

since $\|U^T k\|_2^2 = |x^T k|^2 + \|U_1^T k\|_2^2$. Using $\sum_{j=0}^{m} w_j^2 |\nabla_j H_{w^{-1},2}|^2 = 1$ from Proposition 2.4, we obtain that in the case of the Frobenius norm

$$\|\Delta \mathbf{L}\|_{w,F}^2 = \sum_{j=0}^{m} w_j^2 |\nabla_j H_{w^{-1},2}|^2 \frac{2\|k\|_2^2 - |x^T k|^2}{H_{w^{-1},2}^2} = \frac{2\|k\|_2^2 - |x^T k|^2}{H_{w^{-1},2}^2},$$

and hence,

$$\|\Delta \mathbf{L}\|_{w,F} = \sqrt{\frac{2\|k\|_2^2 - |x^T k|^2}{H_{w^{-1},2}^2}}.$$

As $k^T x$ is a scalar constant, it follows that all ΔA_j and thus also $\Delta \mathbf{L}$ are symmetric and

$$\begin{aligned}(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu))x &= \sum_{j=0}^{m} \lambda^{m-j}\mu^j (A_j + \Delta A_j)x = -k + (\sum_{j=0}^{m} \lambda^{m-j}\mu^j \Delta A_j)x\\ &= -k + \sum_{j=0}^{m} \lambda^{m-j}\mu^j \overline{z_{A_j}}[k x^H + \overline{x} k^T - \overline{x} k^T x x^H]x\\ &= -k + k + \overline{x} k^T x - \overline{x} k^T x = 0.\end{aligned}$$

Here we have used that by Proposition 2.4 we have that $\sum_{j=0}^{m} \lambda^{m-j}\mu^j \overline{z_{A_j}} = 1$. Similarly, it follows that $\overline{x}^H(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu)) = 0$.

For the spectral norm we can apply Corollary 2.8 to (3.1) and get

$$\begin{aligned}D_{j,j} =& -\frac{\overline{z_{A_j}}}{P^2}\left[\overline{x^T k}(U_1^T k)(U_1^T k)^T\right]\\ &+ \chi\left[I - \frac{(U_1^T k)(U_1^T k)^H}{P^2}\right]^{1/2} Z \left[I - \frac{\overline{U_1^T k}(U_1^T k)^T}{P^2}\right]^{1/2},\end{aligned}$$

where $Z = Z^T$ and $\|Z\|_2 \le 1$, $P^2 = \|k\|_2^2 - |x^T k|^2$, $\chi := \sqrt{\|d_{j,j}\|^2 + \|d_j\|_2^2}$. With the special choice $Z = 0$ we get $D_{j,j} = -\frac{\overline{z_{A_j}}}{P^2}\left[\overline{x^T k}(U_1^T k)(U_1^T k)^T\right]$ and

$$\overline{U_1} D_{j,j} U_1^H = -\frac{\overline{z_{A_j}}}{P^2} \overline{U_1} U_1^T k k^T U_1 U_1^H = -\frac{\overline{z_{A_j}}}{P^2}(I - \overline{x} x^T) k k^T (I - x x^H).$$

Hence,

$$\Delta A_j = \overline{z_{A_j}}\left[k x^H + \overline{x} k^T - \overline{x}(k^T x) x^H\right] - \frac{\overline{z_{A_j}}}{P^2}(I - \overline{x} x^T) k k^T (I - x x^H),$$

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$\Delta\mathbf{L}(c,s)$ is symmetric, and $(\mathbf{L}(\lambda,\mu)+\Delta\mathbf{L}(\lambda,\mu))x=0$. With

$$\chi := \sigma_{\max}\left(\begin{bmatrix} d_{j,j} \\ d_j \end{bmatrix}\right) = |z_{A_j}|\sqrt{|x^T k|^2 + \|U_1^T k\|^2} = \frac{|\nabla_j H_{w^{-1},2}|}{H_{w^{-1},2}}\|k\|_2,$$

and Corollary 2.8 we have $\chi = \|\Delta A_j\|_2$, and by Proposition 2.4, $\sum_{j=0}^m w_j^2 |\nabla_j H_{w^{-1},2}|^2 = 1$, it follows that

$$\eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathbf{L}) = \|\Delta\mathbf{L}\|_{w,2} = \frac{\|k\|_2}{H_{w^{-1},2}}. \qquad \square$$

Note that in the construction of a minimal spectral norm backward error we have infinitely many choices of an appropriate completion Z for which $\|Z\|_2 \le 1$, but here and in the following we always take $Z=0$ to simplify the formulas.

Remark 3.3. If $w_j = 0$ for $j = 0,\ldots,m$, then $z_{A_j} = \dfrac{\nabla_j H_{w^{-1},2}(\lambda,\mu)}{H_{w^{-1},2}(\lambda,\mu)} = 0$ and hence by Theorem 3.2 we have that $\Delta A_j = 0$, $j=0,\ldots,m$. This shows that $w_j = 0$ implies that A_j remains unperturbed.

We then have the following relations between structured and unstructured backward errors.

COROLLARY 3.4. *Let $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ be a regular, symmetric matrix polynomial of the form (1.1), let $(\lambda,\mu) \in \mathbb{C}^2 \setminus \{(0,0)\}$, let $x \in \mathbb{C}^n$ be such that $x^H x = 1$. Then,*

$$\eta^{\mathbf{S}}_{w,F}(\lambda,\mu,x,\mathbf{L}) \le \sqrt{2}\eta_{w,2}(\lambda,\mu,x,\mathbf{L})$$
$$\eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathbf{L}) = \eta_{w,2}(\lambda,\mu,x,\mathbf{L}).$$

Proof. By Theorem 3.2 with $k := -\mathbf{L}(\lambda,\mu)x$, we have that

$$\eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathbf{L}) = \frac{\|k\|_2}{H_{w^{-1},2}}, \quad \text{and } \eta^{\mathbf{S}}_{w,F}(\lambda,\mu,x,\mathbf{L}) = \frac{\sqrt{2\|k\|_2^2 - |x^T k|^2}}{H_{w^{-1},2}}$$

and from (2.2) we have that $\eta_{w,2}(\lambda,\mu,x,\mathbf{L}) = \dfrac{\|k\|_2}{H_{w^{-1},2}}$. Thus the assertion follows. $\square$

As a corollary we obtain the results of [2, 3, 4] for the case of non-homogeneous matrix polynomials that have no infinite eigenvalues, as well as the result for homogeneous matrix pencils $\mathbf{L}(c,s) = cA + sB \in \mathbf{L}_1(\mathbb{C}^{n\times n})$ and in the special case, i.e., for $c = 1$, we obtain results given in Theorems 3.1, and 3.2 of [3].

In an analogous way we can derive the results for complex skew-symmetric matrix polynomials.

THEOREM 3.5. *Let $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ be a complex skew-symmetric matrix polynomial of the form (1.1), let $(\lambda,\mu) \in \mathbb{C}^2\setminus\{(0,0)\}$, let $x \in \mathbb{C}^n$ such that $x^H x = 1$ and $k := -\mathbf{L}(\lambda,\mu)x$. Introduce the perturbation matrices*

$$\Delta A_j := -\overline{z_{A_j}}\left[\overline{x}k^T - kx^H\right], \; j = 0,1,2,\ldots,m.$$

Then the matrix polynomial $\Delta\mathbf{L}(c,s) = \displaystyle\sum_{j=0}^{m} c^{m-j}s^j \Delta A_j$, is complex skew-symmetric and $(\mathbf{L}(\lambda,\mu)+\Delta\mathbf{L}(\lambda,\mu))x = 0$.

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Proof. By construction $\Delta\mathbf{L}$ is complex skew-symmetric and by Proposition 2.4, we have $\sum_{j=0}^{m}\lambda^{m-j}\mu^j\overline{z_{A_j}}=1$. Thus, we have

$$\begin{aligned}
&(\mathbf{L}(\lambda,\mu)+\Delta\mathbf{L}(\lambda,\mu))x\\
&\quad=-k+\Delta\mathbf{L}(\lambda,\mu)x=-k+\sum_{j=0}^{m}\lambda^{m-j}\mu^j\overline{z_{A_j}}\left[\overline{x}k^T-kx^H\right]x\\
&\quad=-k+\overline{x}k^Tx+k=0,
\end{aligned}$$

as $\overline{x}k^Tx=0$, since the polynomial has skew-symmetric coefficients. ☐

THEOREM 3.6. *Let $\mathbf{L}\in\mathbf{L}_m(\mathbb{C}^{n\times n})$ be a complex skew-symmetric matrix polynomial of the form (1.1), let $(\lambda,\mu)\in\mathbb{C}^2\setminus\{(0,0)\}$, let $x\in\mathbb{C}^n$ be such that $x^Hx=1$ and let $k:=-\mathbf{L}(\lambda,\mu)x$. The structured backward errors with respect to the Frobenius norm and spectral norm are given by*

$$\begin{aligned}
\eta^{\mathbf{S}}_{w,F}(\lambda,\mu,x,\mathbf{L})&=\frac{\sqrt{2\|k\|_2^2}}{H_{w^{-1},2}},\\
\eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathbf{L})&=\frac{\|k\|_2}{H_{w^{-1},2}},
\end{aligned}$$

respectively. Introducing the perturbation matrices

$$\Delta A_j=-\overline{z_{A_j}}\left[\overline{x}k^T-kx^H\right],j=0,1,\dots,m,$$

then $\Delta\mathbf{L}(c,s):=\sum_{j=0}^{m}c^{m-j}s^j\Delta A_j$ is skew-symmetric, $(\Delta\mathbf{L}(\lambda,\mu)+\mathbf{L}(\lambda,\mu))x=0$, $\eta^{\mathbf{S}}_{w,F}(\lambda,\mu,x,\mathbf{L})=|||\Delta\mathbf{L}|||_{w,F}$ and $\eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathbf{L})=|||\Delta\mathbf{L}|||_{w,2}$.

Proof. By Theorem 3.5 we have $(\mathbf{L}(\lambda,\mu)+\Delta\mathbf{L}(\lambda,\mu))x=0$ and hence we have that $k=\Delta\mathbf{L}(\lambda,\mu)x$. We choose a unitary matrix U of the form $U=[x,U_1]$, $U_1\in\mathbb{C}^{n\times n-1}$ and define $\widetilde{\Delta A_j}:=U^T\Delta A_jU=\begin{bmatrix}0 & d_j^T\\ -d_j & \Delta D_{j,j}\end{bmatrix}$, where

$$\Delta D_{j,j}=-\Delta D_{j,j}^T\in\mathbb{C}^{(n-1)\times(n-1)}.$$

Then

$$\overline{U}\widetilde{\Delta\mathbf{L}(\lambda,\mu)}U^H=\overline{U}U^T(\Delta\mathbf{L}(\lambda,\mu))U^HU=\Delta\mathbf{L}(\lambda,\mu),$$

and hence

$$\overline{U}\widetilde{\Delta\mathbf{L}(\lambda,\mu)}U^Hx=\Delta\mathbf{L}(\lambda,\mu)x=k,$$

which implies that

$$\widetilde{\Delta\mathbf{L}(\lambda,\mu)}U^Hx=U^Tk=\begin{bmatrix}x^Tk\\ U_1^Tk\end{bmatrix}.$$

Since $U^Hx=e_1$, it follows that $x^Tk=0$ and

$$U_1^Tk=-\sum_{j=0}^{m}\lambda^{m-j}\mu^jd_j=\sum_{j=0}^{m}w_j\lambda^{m-j}\mu^j\frac{d_j}{w_j}.$$

To minimize the perturbation we solve the system for the parameters $d_{j,j}, d_j$ in a least squares sense, and obtain $x^T k = 0$ and

$$\begin{bmatrix} w_0 d_0 \\ w_1 d_1 \\ \vdots \\ w_m d_m \end{bmatrix} = - \begin{bmatrix} \overline{z_{A_0}} \\ \overline{z_{A_1}} \\ \vdots \\ \overline{z_{A_m}} \end{bmatrix} U_1^T k,$$

where $H_{w,2} = \| \left[\lambda^m \mu^0, \lambda^{m-1}\mu, \ldots, \lambda^0 \mu^m\right]^T \|_{w,2}$. This yields $d_{j,j} = 0, d_j = -\overline{z_{A_j}} U_1^T k$ and then

$$\widetilde{\Delta A_j} = \begin{bmatrix} 0 & -\left(\overline{z_{A_j}} U_1^T k\right)^T \\ \overline{z_{A_j}} U_1^T k & \Delta D_{j,j} \end{bmatrix}.$$

The Frobenius norm can be minimized by taking $\Delta D_{j,j} = 0$ and then we have

$$\|\Delta A_j\|_F^2 = 2\|d_j\|_2^2 = 2|z_{A_j}|^2 \|U_1^T k\|^2 = |\nabla_j H_{w^{-1},2}|^2 \frac{2\|k\|_2^2}{H_{w^{-1},2}^2},$$

since $\|k\|_2^2 = \|U^T k\|_2^2 = |x^T k|^2 + \|U_1^T k\|_2^2 = \|U_1^T k\|_2^2$. Also by Proposition 2.4, we have that $\sum_{j=0}^m w_j^2 |\nabla_j H_{w^{-1},2}|^2 = 1$. Thus we obtain $\|\Delta \mathbf{L}\|_{w,F} = \dfrac{\sqrt{2\|k\|_2^2}}{H_{w^{-1},2}}$ and

$$\begin{aligned}
\Delta A_j &= \overline{U} \widetilde{\Delta A} U^H = \begin{bmatrix} \overline{x} & \overline{U_1} \end{bmatrix} \begin{bmatrix} 0 & d_j^T \\ -d_j & \Delta D_{j,j} \end{bmatrix} \begin{bmatrix} x^H \\ U_1^H \end{bmatrix} \\
&= -\overline{U_1} d_j x^H + \overline{x} d_j^T U_1^H + \overline{U_1} \Delta D_{j,j} U_1^H \\
&= \overline{U_1}\, \overline{z_{A_j}} U_1^T k x^H - \overline{x} (\overline{z_{A_j}} U_1^T k)^T U_1^H + \overline{U_1} \Delta D_{j,j} U_1^H \\
&= \overline{z_{A_j}} \left[\overline{U_1} U_1^T k x^H - \overline{x} k^T U_1 U_1^H)\right] + \overline{U_1} \Delta D_{j,j} U_1^H \\
&= \overline{z_{A_j}} [(I - \overline{x} x^T) k x^H - \overline{x} k^T (I - x x^H))]. \qquad (3.2)
\end{aligned}$$

Therefore

$$\Delta A_j = \overline{z_{A_j}} [k x^H - \overline{x} k^T]$$

is complex skew-symmetric and we have $(\mathbf{L}(\lambda, \mu) + \Delta \mathbf{L}(\lambda, \mu))x = 0$.

To minimize the spectral norm we make use of Corollary 2.8 and obtain

$$\begin{aligned}
\Delta D_{j,j} = &-\frac{\overline{z_{A_j}}}{P^2} [\overline{x^T k} (U_1^T k)(U_1^T k)^T] \\
&+ \left[I - \frac{(U_1^T k)(U_1^T k)^H}{P^2}\right] Z \left[I - \frac{\overline{U_1^T k} (U_1^T k)^T}{P^2}\right],
\end{aligned}$$

where $Z = -Z^T$ with $\|Z\|_2 \leq 1$, and $P^2 = \|k\|_2^2 - |x^T k|^2$. Choosing $Z = 0$, we get

$$\Delta D_{j,j} = -\frac{\overline{z_{A_j}}}{P^2} [\overline{x^T k} (U_1^T k)(U_1^T k)^T]$$

and using (3.2), we get

$$\overline{U_1} \Delta D_{j,j} U_1^H = -\frac{\overline{z_{A_j}}}{P^2} \overline{x^T k}\, \overline{U_1} U_1^T k k^T U_1 U_1^H = -\frac{\overline{z_{A_j}}}{P^2} \overline{x^T k} (I - \overline{x} x^T) k k^T (I - x x^H),$$

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and hence

$$\Delta A_j = \overline{z_{A_j}}\left[-kx^H + \overline{x}k^T - 2\overline{x}(k^Tx)x^H\right] - \frac{\overline{z_{A_j}}}{P^2}\overline{x^Tk}(I-\overline{x}x^T)kk^T(I-xx^H).$$

The skew-symmetry of A_j implies that $x^Tk = 0$ and thus $\Delta A_j = \overline{z_{A_j}}\left[kx^H - \overline{x}k^T\right]$ is complex skew-symmetric. Then $\Delta\mathbf{L}(c,s)$ is complex skew-symmetric as well and with $\chi_{\Delta A_j} = |\overline{z_{A_j}}|\|U_1^Tk\|_2$ we have that $(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu))x = 0$.

By Corollary 2.8 we obtain

$$\|\Delta A_j\|_2 = |z_{A_j}|\|U_1^Tk\|_2 = |z_{A_j}|\sqrt{\|k\|_2^2 - |x^Tk|^2} = |z_{A_j}|\|k\|_2$$

and hence $\eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathbf{L}) = \|\Delta\mathbf{L}\|_{w,2}$. $\square$

As a direct corollary of Theorem 3.6 we have the following relation between structured and unstructured backward errors of an approximate eigenpair.

COROLLARY 3.7. *Let $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ be a skew-symmetric matrix polynomial of the form (1.1), let $(\lambda,\mu) \in \mathbb{C}^2 \setminus \{(0,0)\}$, let $x \in \mathbb{C}^n$ satisfy $x^Hx = 1$, and set $k := -\mathbf{L}(\lambda,\mu)x$. Then the structured and unstructured backward errors are related via*

$$\eta^{\mathbf{S}}_{w,F}(\lambda,\mu,x,\mathbf{L}) = \sqrt{2}\eta_{w,2}(\lambda,\mu,x,\mathbf{L}),$$
$$\eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathbf{L}) = \eta_{w,2}(\lambda,\mu,x,\mathbf{L}).$$

As a further corollary we obtain Theorem 4.3.4 of [2]; see also [4] for non-homogeneous matrix polynomials with no infinite eigenvalues.

For matrix pencils $\mathbf{L}(c,s) = cA_0 + sA_1 \in \mathbf{L}_1(\mathbb{C}^{n\times n})$, Theorem 3.6 in the special case $c = 1$ also implies the results given in Theorem 3.3 and Theorem 4.2 of [3].

To illustrate our results, in the following we present some examples.

Example 3.8. Consider the complex symmetric pencil $\mathbf{L} \in \mathbf{L}_1(\mathbb{C}^{2\times 2})$ with coefficients $A_0 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $A_1 := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, and take $x = \begin{bmatrix} -\imath/\sqrt{2} \\ \imath/\sqrt{2} \end{bmatrix}$, $(\lambda,\mu) = (0,1)$.

For the Frobenius norm we obtain the coefficients of the perturbation pencil $\Delta\mathbf{L}$ as $\Delta A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\Delta A_1 := \begin{bmatrix} 0.25 & 0.25 \\ 0.25 & -0.75 \end{bmatrix}$. Then $(0,1)$ is an eigenvalue of $\mathbf{L} + \Delta\mathbf{L}$ and $\|\Delta\mathbf{L}\|_F = \eta^{\mathbf{S}}_F(\lambda,\mu,x,\mathbf{L}) = 0.8660$.

For the spectral norm we obtain $\Delta A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and $\Delta A_1 = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}$. Again $(0,1)$ is an eigenvalue of $\mathbf{L} + \Delta\mathbf{L}$ and $\|\Delta\mathbf{L}\|_2 = \eta^{\mathbf{S}}_2(\lambda,\mu,x,\mathbf{L}) = 0.7071$; see also Table 3.1.

Example 3.9. Consider the complex skew-symmetric pencil $\mathbf{L} \in \mathbf{L}_1(\mathbb{C}^{2\times 2})$ with coefficients $A_0 := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $A_1 := \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$ and take $x = \begin{bmatrix} -\imath/\sqrt{2} \\ \imath/\sqrt{2} \end{bmatrix}$, $(\lambda,\mu) = (0,1)$.

For the Frobenius norm and spectral norm, the coefficients or the perturbation pencil are $\Delta A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\Delta A_1 = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$, $(0,1)$ is an eigenvalue of $\mathbf{L} + \Delta\mathbf{L}$. The norm of the perturbation is $\|\Delta\mathbf{L}\|_F = \eta^{\mathbf{S}}_F(\lambda,\mu,x,\mathbf{L}) = 2.8284$, while for the spectral norm we obtain $\|\Delta\mathbf{L}\|_2 = \eta^{\mathbf{S}}_2(\lambda,\mu,x,\mathbf{L}) = 2$, see also Table 3.1.

4. Backward errors for complex T-odd and T-even matrix polynomials. In this section we derive backward error formulas for homogeneous T-odd and T-even matrix polynomials. Throughout this section we assume that the coefficient matrix A_0 is in the even position, i.e., it is symmetric for a T-even and skew-symmetric for a T-odd matrix polynomial. The other case can be treated analogously via a multiplication with the imaginary unit $\imath$.

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TABLE 3.1
Structured and unstructured backward errors for Examples 3.8 and 3.9.

Example	$\mathbf{S}$	$\eta_2^{\mathbf{S}}(\lambda,\mu,x,\mathbf{L})$	$\eta_F^{\mathbf{S}}(\lambda,\mu,x,\mathbf{L})$	$\eta_2(\lambda,\mu,x,\mathbf{L})$
1	symmetric	0.7071	0.8660	0.7071
2	skew-symmetric	2	2.8284	2

For a given nonnegative vector w, an eigenvalue (λ,μ) and the partial derivatives as introduced in Propositions 2.1–2.4, we use the following abbreviations.

$$z_{A_j} := \frac{\nabla_j H_{w^{-1},2}(\lambda,\mu)}{H_{w^{-1},2}(\lambda,\mu)},\; n_{A_j} := \frac{\nabla_j N_{w^{-1},2}(\lambda,\mu)}{N_{w^{-1},2}(\lambda,\mu)},\; k_{A_j} := \frac{\nabla_j K_{w^{-1},2}(\lambda,\mu)}{K_{w^{-1},2}(\lambda,\mu)}.$$

We then have the following backward errors.

THEOREM 4.1. *Let $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ be a complex T-even or T-odd matrix polynomial of the form (1.1), let $(\lambda,\mu) \in \mathbb{C}^2 \setminus \{(0,0)\}$, let $x \in \mathbb{C}^n$ be such that $x^H x = 1$ and set $k := -\mathbf{L}(\lambda,\mu)x$. For $j = 0,1,2,\dots,m$, and different cases, we introduce the following perturbation matrices.*

- *In the case that m is even and $\lambda \neq 0$, or when $m > 1$ is odd then let for T-even matrix polynomials*

$$\Delta A_j := \begin{cases} \overline{k_{A_j}}(x^T k)(\overline{x}x^H) + \overline{z_{A_j}}\left[(I-\overline{x}x^T)kx^H + \overline{x}k^T(I-xx^H)\right] & \textit{for even } j,\\ -\overline{z_{A_j}}\left[-(I-\overline{x}x^T)kx^H + \overline{x}k^T(I-xx^H)\right] & \textit{for odd } j,\end{cases}$$

  so that the perturbation preserves the structure,
- *in the case that $m > 1$ is even and both $\lambda \neq 0, \mu \neq 0$, or in the case that m is odd and $\mu \neq 0$, let for T-odd matrix polynomials*

$$\Delta A_j := \begin{cases} \overline{n_{A_j}}(x^T k)(\overline{x}x^H) + \overline{z_{A_j}}\left[(I-\overline{x}x^T)kx^H + \overline{x}k^T(I-xx^H)\right] & \textit{for odd } j,\\ -\overline{z_{A_j}}\left[-(I-\overline{x}x^T)kx^H + \overline{x}k^T(I-xx^H)\right] & \textit{for even } j,\end{cases}$$

  so that the perturbation again preserves the structure,
- *in the case that $\lambda \neq 0, \mu \neq 0$ consider perturbation matrices for symmetric or skew-symmetric coefficients*

$$\Delta A_j := \begin{cases} -\overline{x}x^T A_j x x^H + \overline{z_{A_j}}\left[(I-\overline{x}x^T)kx^H + \overline{x}k^T(I-xx^H)\right] & \textit{symm.},\\ -\overline{z_{A_j}}\left[-(I-\overline{x}x^T)kx^H + \overline{x}k^T(I-xx^H)\right] & \textit{skew-symm.}\end{cases}$$

Then there exists a matrix polynomial $\Delta\mathbf{L}(c,s) = \sum_{j=0}^{m} c^{m-j}s^j \Delta A_j \in \mathbb{C}^{n\times n}$ that is structure preserving T-odd or T-even and satisfies $(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu))x = 0$.

Proof. Let $\Delta\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ be of the form $\Delta\mathbf{L}(c,s) = \sum_{j=0}^{m} c^{m-j}s^j\Delta A_j$. Then by the construction it is easy to see that $\Delta\mathbf{L}$ is either T-even or T-odd and it remains to show that $(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu))x = 0$. We begin with a T-odd polynomial $\mathbf{L}$. In both cases that

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m is even or odd, we have

$$\begin{aligned}
(\mathbf{L}(\lambda,\mu)+\Delta\mathbf{L}(\lambda,\mu))x &= \sum_{j=0}^{m}\lambda^{m-j}\mu^j(A_j+\Delta A_j)x \\
&= \left(\sum_{j=0, j \text{ even}}^{m}\lambda^{m-j}\mu^j A_j\right)x - \left[-k+\overline{x}x^Tk\right]\sum_{j=0,\, j \text{ even}}^{m}\lambda^{m-j}\mu^j\overline{z_{A_j}} \\
&\quad + \sum_{j=1,\, j \text{ odd}}^{m-1}\lambda^{m-j}\mu^j A_j x + [(x^Tk)\overline{x} + \sum_{j=1,\, j \text{ odd}}^{m-1}\lambda^{m-j}\mu^j\overline{z_{A_j}}(I-\overline{x}x^T)k] \\
&= -k + \left[\sum_{j=0, j \text{ even}}^{m}\lambda^{m-j}\mu^j\overline{z_{A_j}} + \sum_{j=1,\, j \text{ odd}}^{m-1}\lambda^{m-j}\mu^j\overline{z_{A_j}}\right](I-\overline{x}x^T)k + x^Tk\overline{x} \\
&= -k + k - \overline{x}x^Tk + x^Tk\overline{x} = 0,
\end{aligned}$$

since by Proposition 2.4 we have that

$$\sum_{j=0,\, j \text{ even}}^{m}\lambda^{m-j}\mu^j\overline{z_{A_j}} + \sum_{j=1,\, j \text{ odd}}^{m-1}\lambda^{m-j}\mu^j\overline{z_{A_j}} = 1.$$

The proof for T-even polynomials is analogous. ☐

In the special case of linear matrix polynomials, i.e., for $m = 1$, we have the following expressions. For even pencils we have

$$\begin{aligned}
\Delta A_0 &:= -|\mathrm{sign}(\mu)|^2\overline{x}x^TA_0xx^H + \overline{z_{A_0}}\left[(I-\overline{x}x^T)kx^H + \overline{x}k^T(I-xx^H)\right], \\
\Delta A_1 &:= -\overline{z_{A_1}}\left[-(I-\overline{x}x^T)kx^H + \overline{x}k^T(I-xx^H)\right],
\end{aligned}$$

and for odd pencils we have

$$\begin{aligned}
\Delta A_1 &:= -|\mathrm{sign}(\lambda)|^2\overline{x}x^TA_1xx^H + \overline{z_{A_1}}\left[(I-\overline{x}x^T)kx^H + \overline{x}k^T(I-xx^H)\right], \\
\Delta A_0 &:= -\overline{z_{A_0}}\left[-(I-\overline{x}x^T)kx^H + \overline{x}k^T(I-xx^H)\right],
\end{aligned}$$

where $|\mathrm{sign}(z)| = 1$, if $z \neq 0$ and $|\mathrm{sign}(z)| = 0$, for $z = 0$.

As a corollary we obtain the results for the case of non-homogeneous matrix polynomials with no infinite eigenvalues of Theorem 4.2.1 in [2], see also [3, 4]. This case follows by setting $c = 1, \mathbf{L}(s) = \mathbf{L}(1,s), \Lambda = [1,\mu,\dots,\mu^m]^T$ and $w = [1,1,\dots,1]^T$.

The minimal backward errors for complex T-even polynomials and $m > 1$ are as follows.

THEOREM 4.2. *Let $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ be a T-even matrix polynomial of the form (1.1), let $(\lambda,\mu) \in \mathbb{C}^2 \setminus \{(0,0)\}$, let $x \in \mathbb{C}^n$ be such that $x^Hx = 1$ and set $k := -\mathbf{L}(\lambda,\mu)x$.*

i) The structured backward error with respect to the Frobenius norm is given by

$$\eta^{\mathbf{S}}_{w,F}(\lambda,\mu,x,\mathbf{L}) = \begin{cases} \sqrt{\dfrac{|x^Tk|^2}{K^2_{w^{-1},2}} + 2\dfrac{\|k\|_2^2 - |x^Tk|^2}{H^2_{w^{-1},2}}} & \text{if } m \text{ is even or if } \mu \neq 0 \text{ and } m \text{ is odd} \\ \sqrt{\dfrac{2\|k\|_2^2 - |x^Tk|^2}{H^2_{w^{-1},2}}} & \text{if } \lambda = 0 \text{ and, } m \text{ is even,} \\ \sqrt{\dfrac{2\|k\|_2^2 - |x^Tk|^2}{H^2_{w^{-1},2}}} & \text{if } \mu = 0 \text{ and, } m \text{ is odd.} \end{cases}$$

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ii) The structured backward error with respect to the spectral norm is given by

$$\eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathbf{L}) = \begin{cases} \sqrt{\dfrac{|x^Tk|^2}{K^2_{w^{-1},2}} + \dfrac{\|k\|_2^2 - |x^Tk|^2}{H^2_{w^{-1},2}}} & \text{if } m \text{ is even or if } \mu \neq 0 \text{ and } m \text{ is odd} \\ \dfrac{\|k\|_2}{H_{w^{-1},2}} & \text{if } \lambda = 0 \text{ and, } m \text{ is even,} \\ \dfrac{\|k\|_2}{H_{w^{-1},2}} & \text{if } \mu = 0 \text{ and, } m \text{ is odd.} \end{cases}$$

When m is even, or when m is odd and $\lambda \neq 0$, introduce the perturbation matrices

$$\Delta A_j := \begin{cases} \overline{k_{A_j}}(x^Tk)(\overline{x}x^H) + \overline{z_{A_j}}\left[(I-\overline{x}x^T)kx^H + \overline{x}k^T(I-xx^H)\right] & \text{for even } j, \\ -\overline{z_{A_j}}\left[-(I-\overline{x}x^T)kx^H + \overline{x}k^T(I-xx^H)\right] & \text{for odd } j. \end{cases}$$

Then $\Delta\mathbf{L}(c,s) = \sum_{j=0}^m c^{m-j}s^j\Delta A_j$ is the unique T-even matrix polynomial satisfying $(\mathbf{L}(\lambda,\mu)+\Delta\mathbf{L}(\lambda,\mu))x = 0$, and $\|\Delta\mathbf{L}\|_{w,F} = \eta^{\mathbf{S}}_{w,F}(\lambda,\mu,x,\mathbf{L})$. Similarly, for the spectral norm, when m is even or when m is odd and $\lambda \neq 0$, introduce the perturbation matrices

$$\Delta A_j := \begin{cases} \Delta A_j - \dfrac{\overline{k_{A_j}}\,\overline{x^Tk}(I-xx^H)kk^T(I-\overline{x}x^T)}{\|k\|^2 - |x^Tk|^2} & \text{for even } j, \\ \Delta A_j & \text{for odd } j. \end{cases}$$

Then the matrix polynomial $\Delta\mathbf{L}(c,s) = \sum_{j=0}^m c^{m-j}s^j\Delta A_j$ is T-even, has spectral norm $\|\Delta\mathbf{L}\|_{w,2} = \eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathbf{L})$, and satisfies $(\mathbf{L}(\lambda,\mu)+\Delta\mathbf{L}(\lambda,\mu))x = 0$.

Proof. Theorem 4.1 implies that $(\mathbf{L}(\lambda,\mu)+\Delta\mathbf{L}(\lambda,\mu))x = 0$ and hence $k = \Delta\mathbf{L}(\lambda,\mu)x$. Now choose a unitary matrix $U = [x, U_1]$, $U_1 \in \mathbb{C}^{n\times n-1}$ and let

$$\widetilde{\Delta A_j} := U^T\Delta A_jU = \begin{bmatrix} d_{j,j} & d_j^T \\ d_j & \Delta D_{j,j} \end{bmatrix}, \quad \Delta D_{j,j} = \Delta D_{j,j}^T \in \mathbb{C}^{(n-1)\times(n-1)}$$

when j is even and

$$\Delta A_j = \overline{U}\begin{bmatrix} 0 & b_j^T \\ -b_j & \Delta B_{j,j} \end{bmatrix}U^H, \quad \Delta B_{j,j}^T = -\Delta B_{j,j}$$

when j is odd. Then, since $U\widetilde{\Delta\mathbf{L}(\lambda,\mu)}U^T = UU^T(\Delta\mathbf{L}(\lambda,\mu))U^TU = \Delta\mathbf{L}(\lambda,\mu)$, it follows that $U\widetilde{\Delta\mathbf{L}(\lambda,\mu)}U^Tx = \Delta\mathbf{L}(\lambda,\mu)x = k$, and hence $\widetilde{\Delta\mathbf{L}(\lambda,\mu)}U^Tx = U^Tk = \begin{bmatrix} x^Tk \\ U_1^Tk \end{bmatrix}$. Using

$$\begin{bmatrix} \displaystyle\sum_{j=0}^m w_jd_{j,j}\frac{\lambda^{m-j}\mu^j}{w_j} \\ \displaystyle\sum_{j=0,\ j \text{ even}}^m w_j\lambda^{m-j}\mu^j\frac{d_j}{w_j} - \sum_{j=1,j \text{ odd}}^m w_j\lambda^{m-j}\mu^j\frac{b_j}{w_j} \end{bmatrix} = \begin{bmatrix} x^Tk \\ U_1^Tk \end{bmatrix},$$

to minimize the perturbation, we solve this system for the parameters $d_{j,j}, d_j$ in a least square sense, and we obtain

$$\begin{bmatrix} w_0a_{0,0} \\ w_2a_{2,2} \\ \vdots \\ w_ma_{m,m} \end{bmatrix} = \begin{bmatrix} \overline{z_{A_m}} \\ \overline{z_{A_2}} \\ \vdots \\ \overline{z_{A_m}} \end{bmatrix} x^Tk.$$

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Then $d_{j,j} = \overline{k_{A_j}} x^T k$, $d_j = \overline{z_{A_j}} U_1^T k$ for even j and $b_j = \overline{z_{A_j}} U_1^T k$ for odd j and we obtain

$$\Delta A_j := \begin{cases} \overline{U} \begin{bmatrix} \overline{k_{A_j}} x^T k & \left(\overline{z_{A_j}} U_1^T k\right)^T \\ \overline{z_{A_j}} U_1^T k & \Delta D_{j,j} \end{bmatrix} U^H & \text{for even } j, \\ \overline{U} \begin{bmatrix} 0 & -\left(\overline{z_{A_j}} U_1^T k\right)^T \\ \overline{z_{A_j}} U_1^T k & \Delta B_{j,j} \end{bmatrix} U^H & \text{for odd } j. \end{cases}$$

This implies that

$$\Delta A_j = -\overline{z_{A_j}} \left[-(I - \overline{x} x^T) k x^H + \overline{x} k^T (I - x x^H) \right] + \overline{U_1} \Delta D_{j,j} U_1^H, \tag{4.1}$$

when j is odd. For even j, we get

$$\begin{aligned} \Delta A_j &= \begin{bmatrix} \overline{x} & \overline{U_1} \end{bmatrix} \begin{bmatrix} \overline{k_{A_j}} x^T k & \left(\overline{z_{A_j}} U_1^T k\right)^T \\ \overline{z_{A_j}} U_1^T k & \Delta D_{j,j} \end{bmatrix} \begin{bmatrix} x^H \\ U_1^H \end{bmatrix} \\ &= \overline{k_{A_j}} (x^T k)(\overline{x} x^H) + \overline{z_{A_j}} \left[\overline{U_1} (U_1^T) k x^H + \overline{x} k^T U_1 U_1^H \right] + \overline{U_1} \Delta D_{j,j} U_1^H, \end{aligned}$$

and thus

$$\Delta A_j = \overline{k_{A_j}} (x^T k)(\overline{x} x^H) + \overline{z_{A_j}} \left[(I - \overline{x} x^T) k x^H + \overline{x} k^T (I - x x^H) \right] + \overline{U_1} \Delta D_{j,j} U_1^H. \tag{4.2}$$

The Frobenius norm can be minimized by taking $\Delta A_{j,j} = 0$, so we obtain

$$\Delta A_j := \begin{cases} \overline{k_{A_j}} (x^T k)(\overline{x} x^H) + \overline{z_{A_j}} \left[(I - \overline{x} x^T) k x^H + \overline{x} k^T (I - x x^H) \right] & \text{for even } j, \\ -\overline{z_{A_j}} \left[-(I - \overline{x} x^T) k x^H + \overline{x} k^T (I - x x^H) \right] & \text{for odd } j. \end{cases}$$

Since the Frobenius norm is unitarily invariant, it follows that for even j we have

$$\begin{aligned} \|\Delta A_j\|_F &= \sqrt{|a_{j,j}|^2 + 2\|a_j\|_2^2} = \sqrt{|k_{A_j}|^2 |x^T k|^2 + 2|z_{A_j}|^2 \|U_1^T k\|_2^2} \\ &= \sqrt{\frac{|\nabla_j K_{w^{-1},2}|^2 |x^T k|^2}{K_{w^{-1},2}^2} + 2\frac{|\nabla_j H_{w^{-1},2}|^2 \|U_1^T k\|_2^2}{H_{w^{-1},2}^2}}. \end{aligned}$$

Similarly for odd j, we have $\|\Delta A_j\|_F = \sqrt{2}|z_{A_j}|\|U_1^T k\|_2$. Furthermore, by Proposition 2.4, we have $\sum_{j=\text{even}}^{m} w_j^2 |\nabla_j K_{w^{-1},2}|^2 = 1$ and $\sum_{j=0}^{m} w_j^2 |\nabla_j H_{w^{-1},2}|^2 = 1$ when m is even. Then it follows that

$$\begin{aligned} \|\Delta \mathbf{L}\|_{w,F} &= \sqrt{\sum_{j=0}^{m} w_j^2 \|\Delta A_j\|_F^2} = \sqrt{\frac{|x^T k|^2}{K_{w^{-1},2}^2} + \frac{2\|U_1^T k\|_2^2}{H_{w^{-1},2}^2}} \\ &= \sqrt{\frac{|x^T k|^2}{K_{w^{-1},2}^2} + \frac{2(\|k\|_2^2 - |x^T k|^2)}{H_{w^{-1},2}^2}}. \end{aligned}$$

For the spectral norm, we have from (4.1) and (4.2) that

$$\Delta A_j := \begin{cases} \overline{k_{A_j}} (x^T k)(\overline{x} x^H) + \overline{z_{A_j}} \left[(I - \overline{x} x^T) k x^H + \overline{x} k^T (I - x x^H) \right] + S_j & \text{for even } j, \\ -\overline{z_{A_j}} \left[-(I - \overline{x} x^T) k x^H + \overline{x} k^T (I - x x^H) \right] & \text{for odd } j, \end{cases}$$

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where $S_j := \overline{U_1}\Delta D_{j,j}U_1^H = \frac{\overline{z_{A_j}}}{P^2}\overline{x^T k}(I - \overline{x}x^T)kk^T(I - xx^H)$, and $P^2 = \|k\|_2^2 - |x^T k|^2$. Now let

$$\chi_{\Delta A_j} := \begin{cases} \sqrt{|k_{A_j}|^2|x^T k|^2 + |z_{A_j}|^2(\|k\|^2 - |x^T k|^2)} & \text{for even } j, \\ \sqrt{|z_{A_j}|^2(\|k\|_2^2 - |x^T k|^2)} & \text{for odd } j. \end{cases}$$

Hence, by Corollary 2.8 it follows that $\|A\|_2 = \chi_{\Delta A_j}$. Then

$$\|\Delta \mathbf{L}\|_{w,2} = \sqrt{\sum_{j=0}^{m} w_j^2 \|\Delta A\|_2^2} = \sqrt{\frac{|x^T k|^2}{K_{w^{-1},2}^2} + \frac{\|k\|^2 - |x^T k|^2}{H_{w^{-1},2}^2}},$$

and

$$\eta_{w,2}^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L}) = \sqrt{\frac{|x^T k|^2}{K_{w^{-1},2}^2} + \frac{\|k\|^2 - |x^T k|^2}{H_{w^{-1},2}^2}}. \qquad \square$$

We obtain the following relations between the structured and unstructured backward errors.

COROLLARY 4.3. *Let $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ be a T-even matrix polynomial of the form (1.1), let $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{(0,0)\}$, let $x \in \mathbb{C}^n$ be such that $x^H x = 1$, and set $k := -\mathbf{L}(\lambda, \mu)x$.*

1. *If $w := [1, 1, \ldots, 1]^T$, $|\lambda| = |\mu| = 1$ and if m is odd, then $H_{w^{-1},2}^2 = 2K_{w^{-1},2}^2$ and for the Frobenius norm we get*

$$\eta_{w,F}^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L}) = \sqrt{2}\eta_{w,2}(\lambda, \mu, x, \mathbf{L}).$$

Similarly, for the spectral norm we have

$$\eta_{w,2}^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L}) = \frac{\sqrt{\|k\|_2^2 + |x^T k|^2}}{H_{w^{-1},2}}.$$

2. *If m is even or if m is odd and $\lambda \neq 0$, then for the Frobenius and the spectral norm we have*

$$\eta_{w,F}^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L}) \leq \sqrt{2}\eta_{w,2}(\lambda, \mu, x, \mathbf{L}),$$
$$\eta_{w,2}^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L}) = \eta_{w,2}(\lambda, \mu, x, \mathbf{L}),$$

respectively.

Proof. Consider the case that $|\lambda| = |\mu| = 1$, $w = [1, 1, \ldots, 1]^T$ and that m is odd. Then $H_{w^{-1},2}^2 = 2K_{w^{-1},2}^2$. Substituting these in Theorem 4.2 and then applying (2.2), we get for the Frobenius norm that

$$\eta_{w,F}^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L}) = \sqrt{2}\eta_{w,2}(\lambda, \mu, x, \mathbf{L})$$

and for the spectral norm that

$$\eta_{w,2}^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L}) = \sqrt{\frac{\|k\|_2^2 + |x^T k|^2}{H_{w^{-1},2}^2}}.$$

If m is even and $\lambda = 0$, then we have $K_{w^{-1},2} = w_m^{-1}|\mu|^m$ and $H_{w^{-1},2} = w_m^{-1}|\mu|^m$ and hence

$$\eta_{w,F}^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L}) \leq \sqrt{2}\eta_{w,2}(\lambda, \mu, x, \mathbf{L}),$$
$$\eta_{w,2}^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L}) = \eta_{w,2}(\lambda, \mu, x, \mathbf{L}).$$

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Similarly, for $\mu = 0$ we have $K_{w^{-1},2} = w_0^{-1}|\lambda|^m$ and $H_{w^{-1},2} = w_0^{-1}|\lambda|^m$, and hence

$$\eta^{\mathbf{S}}_{w,F}(\lambda,\mu,x,\mathbf{L}) \le \sqrt{2}\eta_{w,2}(\lambda,\mu,x,\mathbf{L}),$$
$$\eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathbf{L}) = \eta_{w,2}(\lambda,\mu,x,\mathbf{L}).$$

The assertion for the case that $\lambda \neq 0$ and m is odd follows analogously. □

As a corollary we obtain the results for non-homogeneous matrix polynomials with no infinite eigenvalues of [2, 4], using the notation $\Lambda_e := [1, \mu^2, \ldots, \mu^m]^T$ if m is even and $\Lambda_e := [1, \mu^2, \ldots, \mu^{m-1}]^T$ if m is odd.

COROLLARY 4.4. *Let $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ be a T-even matrix polynomial of the form $\mathbf{L}(s) = \sum_{j=0}^m s^j A_j \in \mathbb{C}^{n\times n}$ that has only finite eigenvalues. Let $\mu \in \mathbb{C}$, let $x \in \mathbb{C}^n$ be such that $x^H x = 1$ and set $k := -\mathbf{L}(\mu)x$.*

i) The structured backward error with respect to the Frobenius norm is given by

$$\eta^{\mathbf{S}}_F(\mu,x,\mathbf{L}) = \begin{cases} \sqrt{\dfrac{|x^Tk|_2^2}{\|\Lambda_e\|_2^2} + 2\dfrac{\|k\|_2^2 - |x^Tk|^2}{\|\Lambda\|_2^2}} & \text{if } \mu \in \mathbb{C}\setminus\{0\}, \\ \sqrt{2\|k\|_2^2 - |x^Tk|^2} & \text{if } \mu = 0. \end{cases}$$

ii) The structured backward error with respect to the spectral norm is given by

$$\eta^{\mathbf{S}}_2(\mu,x,\mathbf{L}) = \begin{cases} \sqrt{\dfrac{|x^Tk|^2}{\|\Lambda_e\|_2^2} + \dfrac{\|k\|_2^2 - |x^Tk|^2}{\|\Lambda\|_2^2}} & \text{if } \mu \in \mathbb{C}\setminus\{0\}, \\ \eta_2(\mu,x,\mathbf{L}) & \text{if } \mu = 0. \end{cases}$$

In particular, if $|\mu| = 1$ and m is odd, then we have $\|\Lambda\|_2^2 = 2\|\Lambda_e\|_2^2$. Moreover, for the Frobenius norm we have $\eta^{\mathbf{S}}_F(\mu,x,\mathbf{L}) = \sqrt{2}\eta_2(\mu,x,\mathbf{L})$ and for the spectral norm we obtain $\eta^{\mathbf{S}}_2(\mu,x,\mathbf{L}) = \dfrac{\sqrt{\|k\|_2^2 + |x^Tk|^2}}{\|\Lambda\|_2}$.

If we introduce the perturbation matrices

$$\Delta A_j := \begin{cases} \dfrac{\overline{\mu^j}(x^Tk)(\overline{x}x^H)}{\|\Lambda_e\|_2^2} + \dfrac{\overline{\mu^j}}{\|\Lambda\|_2^2}\left[(I-\overline{x}x^T)kx^H + \overline{x}k^T(I-xx^H)\right] & \text{for even } j, \\ -\dfrac{\overline{\mu^j}}{\|\Lambda\|_2^2}\left[-(I-\overline{x}x^T)kx^H + \overline{x}k^T(I-xx^H)\right] & \text{for odd } j, \end{cases}$$

then $\Delta\mathbf{L}(s) = \sum_{j=0}^m s^j \Delta A_j$ is the uniquely defined T-even matrix polynomial that satisfies $(\mathbf{L}(\mu) + \Delta\mathbf{L}(\mu))x = 0$ and $\|\Delta\mathbf{L}\|_F = \eta^{\mathbf{S}}_F(\mu,x,\mathbf{L})$ for the Frobenius norm.

For the spectral norm, we introduce

$$\Delta A_j := \begin{cases} \Delta A_j - \dfrac{\overline{\mu^j x^Tk}(I-xx^H)kk^T(I-\overline{x}x^T)}{\|\Lambda_e\|_2^2(\|k\|^2 - |x^Tk|^2)} & \text{for even } j, \\ \Delta A_j & \text{for odd } j. \end{cases}$$

Then $\Delta\mathbf{L}(s) = \sum_{j=0}^m s^j \Delta A_j$ is a T-even matrix polynomial such that $(\mathbf{L}(\mu)+\Delta\mathbf{L}(\mu))x = 0$ and $\|\Delta\mathbf{L}\|_2 = \eta^{\mathbf{S}}_2(\mu,x,\mathbf{L})$.

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Proof. The proof follows from Theorem 4.2 using $w = [1, 1, \ldots, 1]^T$, $c = 1$ and that $H_{w^{-1},2} := \|\Lambda\|_2$, $K_{w^{-1},2} := \|\Lambda_e\|_2$. □

Remark 4.5. Corollary 4.3 implies that for $|\mu| = 1$, and for the spectral norm we have that

$$\eta_2^{\mathbf{S}}(\mu, x, \mathbf{L}) = \frac{\sqrt{\|k\|_2^2 + |x^T k|^2}}{\|\Lambda\|_2},$$

while in [2, Theorem 4.3.6] and in [4, Theorem 3.7] it is shown that $\eta_2^{\mathbf{S}}(\mu, x, \mathbf{L}) = \eta_2(\mu, x, \mathbf{L})$ when $w = [1, 1, \ldots, 1]^T$ and m is odd.

For complex T-even pencils we obtain the following result.

COROLLARY 4.6. *Let $\mathbf{L}(c, s) = cA_0 + sA_1 \in \mathbf{L}_1(\mathbb{C}^{n\times n})$ be a T-even matrix pencil. Let $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{(0, 0)\}$, let $x \in \mathbb{C}^n$ be such that $x^H x = 1$, and set $k := -\mathbf{L}(\lambda, \mu)x$, $w := [1, 1]^T$.*

i) The structured backward error with respect to the Frobenius norm is given by

$$\eta_F^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L}) = \begin{cases} \sqrt{|x^T A_0 x|^2 + 2\dfrac{\|k\|_2^2 - |\lambda|^2 |x^T A_0 x|^2}{\|[\lambda, \mu]^T\|_2^2}} & \\ = \sqrt{\dfrac{\left(\frac{|\mu|^2}{|\lambda|^2} - 1\right)|x^T k|^2 + 2\|k\|_2^2}{\|[\lambda, \mu]^T\|_2^2}} & \text{if } \lambda \neq 0, \\ \sqrt{2}\eta_{w,2}(\lambda, \mu, x, \mathbf{L}) & \text{if } \mu = 0, \\ \sqrt{2}\eta_{w,2}(\lambda, \mu, x, \mathbf{L}) & \text{if } \lambda = 0, \\ \sqrt{2}\eta_{w,2}(\lambda, \mu, x, \mathbf{L}) & \text{if } |\lambda| = 1, |\mu| = 1. \end{cases}$$

ii) The structured backward error with respect to the spectral norm is given by

$$\eta_2^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L}) = \begin{cases} \sqrt{|x^T A_0 x|^2 + \dfrac{\|k\|_2^2 - |\lambda|^2 |x^T A_0 x|^2}{\|[\lambda, \mu]^T\|_2^2}} & \\ = \sqrt{\dfrac{|\mu|^2 |x^T A_0 x|^2 + \|k\|_2^2}{\|[\lambda, \mu]^T\|_2^2}} & \text{if } \lambda \neq 0, \\ \eta_2(\lambda, \mu, x, \mathbf{L}) & \text{if } \mu = 0, \\ \eta_2(\lambda, \mu, x, \mathbf{L}) & \text{if } \lambda = 0, \\ \sqrt{\dfrac{|x^T A_0 x|^2 + \|k\|_2^2}{2}} & \text{if } |\lambda| = |\mu| = 1. \end{cases}$$

Defining the perturbation matrices

$$\begin{aligned} \Delta A_0 &:= -|\mathrm{sign}(\lambda)|^2 \overline{x} x^T A_0 x x^H + \overline{z_{A_0}} \left[(I - \overline{x}x^T)kx^H + \overline{x}k^T(I - xx^H)\right], \\ \Delta A_1 &:= -\overline{z_{A_1}} \left[-(I - \overline{x}x^T)kx^H + \overline{x}k^T(I - xx^H)\right], \end{aligned}$$

we have for the Frobenius norm that $\Delta\mathbf{L}(c, s) = c\Delta A_0 + s\Delta A_1$ is the unique T-even matrix polynomial that satisfies $(\mathbf{L}(\lambda, \mu) + \Delta\mathbf{L}(\lambda, \mu))x = 0$ and $\|\Delta\mathbf{L}\|_{w,F} = \eta_{w,F}^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L})$.

For the spectral norm we introduce the perturbation matrices

$$\begin{aligned} \Delta A_0 &:= \Delta A_0 - \frac{\mathrm{sign}(\lambda^2)\overline{x^T A_0 x}(I - \overline{x}x^T)kk^T(I - xx^H)}{(\|k\|^2 - |x^T A_0 x|^2)}, \\ \Delta A_1 &:= -\overline{z_{A_1}} \left[-(I - \overline{x}x^T)kx^H + \overline{x}k^T(I - xx^H)\right], \end{aligned}$$

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then $\Delta\mathbf{L}(c,s) = c\Delta A_0 + s\Delta A_1$ *is* T*-even and satisfies* $(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu))x = 0$ *and* $\|\Delta\mathbf{L}\|_{w,2} = \eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathbf{L})$.

Proof. The proof follows as in Theorem 4.1, using $m = 1$ and $w := [1,1]^T$. ∎

It follows that for $\lambda = 0$ in the T-even case we have $\Delta A_0 = 0$ and

$$\Delta A_1 := -\overline{z_{A_1}}\left[-(I - \overline{x}x^T)kx^H + \overline{x}k^T(I - xx^H)\right].$$

These perturbations are the same for the spectral and the Frobenius norm. Furthermore, Corollary 4.6 shows that

$$\eta^{\mathbf{S}}_F(\lambda,\mu,x,\mathbf{L}) \leq \begin{cases} \sqrt{2}\eta_2(\lambda,\mu,x,\mathbf{L}) & \text{if } |\mu| < |\lambda|, \\ \|[\lambda,\mu]^T\|_2\, \eta_2(\lambda,\mu,x,\mathbf{L}) & \text{if } |\mu| > |\lambda|. \end{cases}$$

For a non-homogeneous pencil $\mathbf{L}(s) = A_0 + sA_1 \in \mathbf{L}_1(\mathbb{C}^{n\times n})$ we then have

$$\eta^{\mathbf{S}}_F(\mu,x,\mathbf{L}) \leq \begin{cases} \sqrt{2}\eta_2(\mu,x,\mathbf{L}) & \text{if } |\mu| < 1, \\ \|[1,\mu]^T\|_2\eta_2(\lambda,\mu,x,\mathbf{L}) & \text{if } |\mu| > 1, \end{cases}$$

which has been shown in Theorem 3.4 of [3] for the case that $\mu \neq 0$.

Example 4.7. Consider a T-even matrix pencil which has coefficients $A_0 := \begin{bmatrix} 2 & 1 \\ 1 & \imath \end{bmatrix}$, $A_1 := \begin{bmatrix} 0 & -\imath \\ \imath & 0 \end{bmatrix}$, let $x = \begin{bmatrix} -\imath/\sqrt{2} \\ \imath/\sqrt{2} \end{bmatrix}$ and $(\lambda,\mu) = (1,0)$. Then we obtain the following perturbation matrices.

For the Frobenius norm we have

$$\Delta A_0 = \begin{bmatrix} -1+0.25\imath & 0+0.25\imath \\ 0+0.25\imath & 1-0.75\imath \end{bmatrix}, \qquad \Delta A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$A_0 + \Delta A_0 = \begin{bmatrix} 1+0.25\imath & 1+0.25\imath \\ 1+0.25\imath & 1+0.25\imath \end{bmatrix}, \qquad A_1 + \Delta A_1 = \begin{bmatrix} 0 & -\imath \\ +\imath & 0 \end{bmatrix},$$

and $\|\Delta\mathbf{L}\|_F = \eta^{\mathbf{S}}_F(\lambda,\mu,x,\mathbf{L})$.

For the spectral norm we obtain

$$\Delta A_0 = \begin{bmatrix} -1.2+0.10\imath & -0.20+0.10\imath \\ -0.20+0.10\imath & 0.80-0.90\imath \end{bmatrix}, \qquad \Delta A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$A_0 + \Delta A_0 = \begin{bmatrix} 0.80+0.10\imath & 0.80+0.10\imath \\ 0.80+0.10\imath & 0.80+0.10\imath \end{bmatrix}, \qquad A_1 + \Delta A_1 = \begin{bmatrix} 0 & -\imath \\ \imath & 0 \end{bmatrix},$$

and $\eta^{\mathbf{S}}_2(\lambda,\mu,x,\mathbf{L}) = \|\Delta\mathbf{L}\|_2 = 1.2247$; see also Table 4.1.

In a similar way we can derive the results for T-odd matrix polynomials.

THEOREM 4.8. *Let* $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ *be a* T*-odd matrix polynomial of the form* (1.1)*, let* $(\lambda,\mu) \in \mathbb{C}^2 \setminus \{(0,0)\}$*, let* $x \in \mathbb{C}^n$ *be such that* $x^H x = 1$ *and set* $k := -\mathbf{L}(\lambda,\mu)x$.

i) The structured backward error with respect to the Frobenius norm is given by

$$\eta^{\mathbf{S}}_{w,F}(\lambda,\mu,x,\mathbf{L}) = \begin{cases} \sqrt{\dfrac{|x^Tk|^2}{N^2_{w^{-1},2}} + 2\dfrac{\|k\|_2^2 - |x^Tk|^2}{H^2_{w^{-1},2}}} & \begin{array}{l} \text{if } \mu \neq 0 \text{ and } m \text{ odd, or} \\ \text{if } \mu,\lambda \neq 0, \text{ and } m \text{ even,} \end{array} \\ \sqrt{\dfrac{2\|k\|_2^2 - |x^Tk|^2}{H^2_{w^{-1},2}}} & \text{if } \lambda = 0 \text{ and } m \text{ odd.} \end{cases}$$

TABLE 4.1
Computed structured and unstructured backward errors for Example 4.7.

(λ,μ)	$\mathbf{S}$	$\eta_2^{\mathbf{S}}(\lambda,\mu,x,\mathbf{L})$	$\eta_F^{\mathbf{S}}(\lambda,\mu,x,\mathbf{L})$	$\eta_2(\lambda,\mu,x,\mathbf{L})$
$(1,0)$	T-even	1.2247	1.6583	1.2247
$(0,1)$	T-even	1	1.414	1
$(2,1)$	T-even	1.0247	1.3601	1
$(4,3)$	T-even	0.9644	1.2689	0.9165
$(2i,i)$	T-even	1.0247	1.3601	1
$(2+3i,1+i)$	T-even	1.1255	1.5111	1.1106
$(1,2)$	T-even	0.9487	1.2450	0.8365
$(1,1)$	T-even	0.9354	1.2247	0.8660

ii) The structured backward error with respect to the spectral norm is given by

$$\eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathbf{L}) = \begin{cases} \sqrt{\dfrac{|x^Tk|^2}{N^2_{w^{-1},2}} + \dfrac{\|k\|_2^2 - |x^Tk|^2}{H^2_{w^{-1},2}}} & \text{if } \mu \neq 0 \text{ and } m \text{ odd, or} \\ & \text{if } \lambda\,\mu \neq 0, \text{ and } m \text{ even,} \\ \dfrac{\|k\|_2}{H_{w^{-1},2}} & \text{if } \lambda = 0 \text{ and } m \text{ odd.} \end{cases}$$

For $\mu \neq 0$ and odd m or for $\lambda \neq 0$, $\mu \neq 0$ and m even, introduce the perturbation matrices

$$\Delta A_j := \begin{cases} \overline{n_{A_j}}(x^Tk)(\overline{x}x^H) + \overline{z_{A_j}}\left[(I-\overline{x}x^T)kx^H + \overline{x}k^T(I-xx^H)\right] & \text{for odd } j, \\ -\overline{z_{A_j}}\left[-(I-\overline{x}x^T)kx^H + \overline{x}k^T(I-xx^H)\right] & \text{for even } j. \end{cases}$$

Then, for the Frobenius norm, $\Delta\mathbf{L}(c,s) = \sum_{j=0}^m c^{m-j}s^j\Delta A_j$ is the unique T-odd matrix polynomial such that $(\mathbf{L}(\lambda,\mu)+\Delta\mathbf{L}(\lambda,\mu))x = 0$ and $\|\Delta\mathbf{L}\|_{w,F} = \eta^{\mathbf{S}}_{w,F}(\lambda,\mu,x,\mathbf{L})$.

For $\mu \neq 0$ and odd m or for $\lambda \neq 0$, $\mu \neq 0$ and even m and the spectral norm consider the perturbation matrices

$$\Delta E_j := \begin{cases} \Delta A_j - \dfrac{\overline{n_{A_j}}\,\overline{x^Tk}(I-xx^H)kk^T(I-\overline{x}x^T)}{(\|k\|^2 - |x^Tk|^2)} & \text{for odd } j, \\ \Delta A_j & \text{for even } j. \end{cases}$$

Then $\Delta\mathbf{L}(c,s) = \sum_{j=0}^m c^{m-j}s^j\Delta E_j$ is T-odd, satisfies $(\mathbf{L}(\lambda,\mu)+\Delta\mathbf{L}(\lambda,\mu))x = 0$ and $\|\Delta\mathbf{L}\|_{w,2} = \eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathbf{L})$.

Proof. The proof is analogous to that for T-even matrix polynomials. □

We then obtain the following relations between structured and unstructured backward errors of an approximate eigenpair.

COROLLARY 4.9. *Let $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ be a T-even matrix polynomial of the form (1.1), let $(\lambda,\mu) \in \mathbb{C}^2 \setminus \{(0,0)\}$, let $x \in \mathbb{C}^n$ be such that $x^Hx = 1$, and set $k := -\mathbf{L}(\lambda,\mu)x$.*

1. *If $\lambda = 0$ and m is odd, then for the Frobenius norm we have*

$$\eta^{\mathbf{S}}_{w,F}(\lambda,\mu,x,\mathbf{L}) \leq \sqrt{2}\eta_{w,2}(\lambda,\mu,x,\mathbf{L}).$$

2. *If $\lambda = 0$ and m is odd, then for the spectral norm we have*

$$\eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathbf{L}) = \eta_{w,2}(\lambda,\mu,x,\mathbf{L}).$$

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3. *Let $w := [1,1,\dots,1]^T$ and $|\lambda| = |\mu| = 1$ for odd m. Then we have for the Frobenius-norm*

$$\eta^{\mathbf{S}}_{w,F}(\lambda,\mu,x,\mathbf{L}) = \sqrt{2}\eta_{w,2}(\lambda,\mu,x,\mathbf{L}),$$

and for the spectral-norm

$$\eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathbf{L}) = \frac{\sqrt{\|k\|_2^2 + |x^Tk|^2}}{H_{w^{-1};2}}.$$

Proof. The proof follows from the fact that if $w := [1,1,\dots,1]^T$ and $|\lambda| = |\mu| = 1$ and m is odd, then we have $H^2_{w^{-1},2} = 2N^2_{w^{-1},2}$ and then applying (2.2) the results follow. ∎

As a corollary we also obtain the results for the case of non-homogeneous matrix polynomials with no infinite eigenvalues of [2, 4]. By introducing the notation $\Lambda_o := [\mu,\mu^3,\dots,\mu^{m-1}]^T$ when m is even and $\Lambda_o := [\mu,\mu^3,\dots,\mu^m]^T$ when m is odd and by choosing the weight vector $w := [1,1,\dots,1]^T$, we have the following result similar to Theorem 4.3.8 of [2].

COROLLARY 4.10. *Let $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ be a T-odd matrix polynomial of the form $\mathbf{L}(s) = \sum_{j=0}^m s^jA_j \in \mathbb{C}^{n\times n}$ with $\det(A_m) \neq 0$, let $\mu \in \mathbb{C}\setminus\{0\}$ and let $x \in \mathbb{C}^n$ be such that $x^Hx = 1$ and set $k := -\mathbf{L}(\mu)x$.*

i) The structured backward error with respect to the Frobenius norm is given by

$$\eta^{\mathbf{S}}_F(\mu,x,\mathbf{L}) = \left\{ \sqrt{\frac{|x^Tk|^2}{\|\Lambda_o\|_2^2} + 2\frac{\|k\|_2^2 - |x^Tk|^2}{\|\Lambda\|_2^2}}. \right.$$

ii) The structured backward error with respect to the spectral norm is given by

$$\eta^{\mathbf{S}}_2(\mu,x,\mathbf{L}) = \left\{ \sqrt{\frac{|x^Tk|^2}{\|\Lambda_o\|_2^2} + \frac{\|k\|_2^2 - |x^Tk|^2}{\|\Lambda\|_2^2}}. \right.$$

In particular, for odd m and $|\mu| = 1$ we have for the Frobenius norm $\|\Lambda\|_2^2 = 2\|\Lambda_o\|_2^2$ and $\eta^{\mathbf{S}}_F(\mu,x,\mathbf{L}) = \sqrt{2}\eta_2(\mu,x,\mathbf{L})$ and for the spectral norm $\eta^{\mathbf{S}}_2(\mu,x,\mathbf{L}) = \dfrac{\sqrt{\|k\|_2^2 + |x^Tk|^2}}{\|\Lambda\|_2}$.

Defining the perturbation matrices

$$\Delta A_j := \begin{cases} \dfrac{\overline{\mu^j}(x^Tk)(\overline{x}x^H)}{\|\Lambda_o\|_2^2} + \dfrac{\overline{\mu^j}}{\|\Lambda\|_2^2}\left[(I-\overline{x}x^T)kx^H + \overline{x}k^T(I-xx^H)\right] & \text{for odd } j, \\ -\dfrac{\overline{\mu^j}}{\|\Lambda\|_2^2}\left[-(I-\overline{x}x^T)kx^H + \overline{x}k^T(I-xx^H)\right] & \text{for even } j, \end{cases}$$

then $\Delta\mathbf{L}(s) = \sum_{j=0}^m s^j\Delta A_j$ is the uniquely defined T-odd matrix polynomial such that $(\mathbf{L}(\mu) + \Delta\mathbf{L}(\mu))x = 0$ and $\|\Delta\mathbf{L}\|_F = \eta^{\mathbf{S}}_F(\mu,x,\mathbf{L})$ in the Frobenius norm.

For the spectral-norm, we introduce the perturbation matrices

$$\Delta E_j := \begin{cases} \Delta A_j - \dfrac{\overline{\mu^j x^Tk}(I-xx^H)kk^T(I-\overline{x}x^T)}{\|\Lambda_o\|_2^2(\|k\|^2 - |x^Tk|^2)} & \text{for odd } j, \\ \Delta A_j & \text{for even } j. \end{cases}$$

Then $\Delta\mathbf{L}(s) = \sum_{j=0}^m s^j\Delta E_j$ is a T-odd matrix polynomial such that $(\mathbf{L}(\mu)+\Delta\mathbf{L}(\mu))x = 0$ and $\|\Delta\mathbf{L}\|_2 = \eta^{\mathbf{S}}_2(\mu,x,\mathbf{L})$.

Proof. The proof follows from Theorem 4.8 using the fact that $H_{w^{-1},2} := \|\Lambda\|_2$, $K_{w^{-1},2} := \|\Lambda_o\|_2$ when $w = [1, 1, \ldots, 1]^T$ and $c = 1$. □

Remark 4.11. The case that $\mu = 0$ is not covered by the formulas in Corollary 4.10 for the case $m > 1$. But it has been shown in Theorem 4.3.8 of [2] that for $\mu = 0$, it hold that $\eta_F^{\mathbf{S}}(\mu, x, \mathbf{L}) = \sqrt{2}\eta_2(\mu, x, \mathbf{L})$ and $\eta_2^{\mathbf{S}}(\mu, x, \mathbf{L}) = \eta_2(\mu, x, \mathbf{L})$, respectively, for the Frobenius norm and the spectral norm. For $|\mu| = 1$ and the spectral norm we have

$$\eta_2^{\mathbf{S}}(\mu, x, \mathbf{L}) = \frac{\sqrt{\|k\|_2^2 + |x^T k|^2}}{\|\Lambda\|_2},$$

while again it has been shown in Theorem 4.3.8 of [2] that $\eta_2^{\mathbf{S}}(\mu, x, \mathbf{L}) = \eta_2(\mu, x, \mathbf{L})$.

For the pencil case we have the following Corollary.

COROLLARY 4.12. *Let $\mathbf{L}(c, s) = cA_0 + sA_1 \in \mathbf{L}_1(\mathbb{C}^{n\times n})$ be a T-odd matrix pencil, let $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{(0, 0)\}$, let $x \in \mathbb{C}^n$ be such that $x^H x = 1$ and set $k := -\mathbf{L}(\lambda, \mu)x$.*

i) The structured backward error with respect to the Frobenius norm is given by

$$\eta_F^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L}) = \begin{cases} \sqrt{|x^T A_1 x|^2 + 2\dfrac{\|k\|_2^2 - |\mu|^2 |x^T A_1 x|^2}{\|[\lambda, \mu]^T\|_2^2}} \\ \quad = \sqrt{\dfrac{\left(\frac{|\lambda|^2}{|\mu|^2} - 1\right)|x^T k|^2 + 2\|k\|_2^2}{\|[\lambda, \mu]^T\|_2^2}} & \text{if } \mu \neq 0, \\ \sqrt{2}\eta_2(\lambda, \mu, x, \mathbf{L}) & \text{if } \lambda = 0, \\ \sqrt{2}\eta_2(\lambda, \mu, x, \mathbf{L}) & \text{if } \mu = 0, \\ \sqrt{2}\eta_2(\lambda, \mu, x, \mathbf{L}) & \text{if } |\lambda| = 1, |\mu| = 1. \end{cases}$$

ii) The structured backward error with respect to the spectral norm is given by

$$\eta_2^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L}) = \begin{cases} \sqrt{|x^T A_1 x|^2 + \dfrac{\|k\|_2^2 - |\mu|^2 |x^T A_1 x|^2}{\|[\lambda, \mu]^T\|_2^2}} \\ \quad = \sqrt{\dfrac{|\lambda|^2 |x^T A_1 x|^2 + \|k\|_2^2}{\|[\lambda, \mu]^T\|_2^2}} & \text{if } \mu \neq 0, \\ \eta_2(\lambda, \mu, x, \mathbf{L}) & \text{if } \lambda = 0, \mu \neq 0, \\ \eta_2(\lambda, \mu, x, \mathbf{L}) & \text{if } \lambda \neq 0, \mu = 0, \\ \sqrt{\dfrac{|x^T A_1 x|^2 + \|k\|_2^2}{2}} & \text{if } |\lambda| = 1, |\mu| = 1. \end{cases}$$

iii) Introduce the perturbation matrices

$$\begin{aligned} \Delta A_0 &:= -\overline{z_{A_0}}\left[-(I - \overline{x}x^T)kx^H + \overline{x}k^T(I - xx^H)\right], \\ \Delta A_1 &:= -|\mathrm{sign}(\mu)|^2 \overline{x}x^T A_1 x x^H + \overline{z_{A_1}}\left[(I - \overline{x}x^T)kx^H + \overline{x}k^T(I - xx^H)\right]. \end{aligned}$$

Then for the Frobenius norm we obtain the unique T-odd pencil $\Delta\mathbf{L}(c, s) = c\Delta A_0 + s\Delta A_1$ such that $(\mathbf{L}(\lambda, \mu) + \Delta\mathbf{L}(\lambda, \mu))x = 0$ and $\|\Delta\mathbf{L}\|_F = \eta_F^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L})$.

For the spectral norm, defining

$$\Delta E_1 := \Delta A_1 - \frac{\mathrm{sign}(\mu^2)\overline{x^T A_1 x}(I - \overline{x}x^T)kk^T(I - xx^H)}{(\|k\|^2 - |x^T A_1 x|^2)} \quad \text{and } \Delta E_0 := \Delta A_0,$$

then we obtain a T-odd pencil $\Delta\mathbf{L}(c, s) = c\Delta E_0 + s\Delta E_1$ with $(\mathbf{L}(\lambda, \mu) + \Delta\mathbf{L}(\lambda, \mu))x = 0$ and $\|\Delta\mathbf{L}\|_2 = \eta_2^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L})$.

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TABLE 4.2
Computed structured and unstructured backward errors for Example 4.13.

(λ,μ)	$\mathbf{S}$	$\eta_2^{\mathbf{S}}(\lambda,\mu,x,\mathbf{L})$	$\eta_F^{\mathbf{S}}(\lambda,\mu,x,\mathbf{L})$	$\eta_2(\lambda,\mu,x,\mathbf{L})$
$(0,1)$	T-odd	1	1.2247	1
$(1,0)$	T-odd	2.2361	3.1623	2.2361
$(2,1)$	T-odd	2.2361	3.0822	2.1448
$(4,3)$	T-odd	2.0881	2.8671	2.0100
$(2i,i)$	T-odd	2.2361	3.0822	2.1448
$(2+3i,1+i)$	T-odd	2.3310	3.2197	2.2361
$(1,2)$	T-odd	1.5166	2.0248	1.4832
$(1,1)$	T-odd	1.9365	2.6458	1.8708

Proof. The proof is analogous to that of Theorem 4.2 using $m=1$ and $w:=[1,1]^T$. $\square$

By the above results it is clear that if $\mu=0$, then for the T-odd case we have $\Delta A_1 = 0$ and $\Delta A_0 = -\overline{z_{A_0}}\left[-(I-\overline{x}x^T)kx^H + \overline{x}k^T(I-xx^H)\right]$. These perturbations are the same for the spectral and Frobenius norm.

Furthermore, Corollary 4.12 shows that

$$\eta_F^{\mathbf{S}}(\lambda,\mu,x,\mathbf{L}) \leq \begin{cases} \sqrt{2}\eta_2(\lambda,\mu,x,\mathbf{L}) & \text{when } |\mu|>|\lambda|, \\ \|[\lambda,\mu]^T\|_2\eta_2(\lambda,\mu,x,\mathbf{L}) & \text{when } |\mu|<|\lambda|. \end{cases}$$

Now consider a pencil $\mathbf{L}(z)=A_0+zA_1 \in \mathbf{L}_1(\mathbb{C}^{n\times n})$. Then for given $\mu\in\mathbb{C}$ and $x\in\mathbb{C}^n$ such that $x^Hx=1$, we have

$$\eta_F^{\mathbf{S}}(\mu,x,\mathbf{L}) \leq \begin{cases} \sqrt{2}\eta_2(\mu,x,\mathbf{L}) & \text{when } |\mu|>1, \\ \|[1,\mu^{-1}]^T\|_2\eta_2(\lambda,\mu,x,\mathbf{L}) & \text{when } |\mu|<1, \end{cases}$$

which has been shown in [3].

As another corollary we obtain the results for T-odd matrix pencils$\mathbf{L}(z) := A_0 + zA_1$ presented in [2].

Let us illustrate these perturbation results with a few examples.

Example 4.13. Consider a T-odd matrix pencil with coefficients $A_0 := \begin{bmatrix} 0 & -2+\imath \\ 2-\imath & 0 \end{bmatrix}$ and $A_1 := \begin{bmatrix} 1+\imath & 0 \\ 0 & 0 \end{bmatrix}$. Let $x = \begin{bmatrix} -\imath/\sqrt{2} \\ \imath/\sqrt{2} \end{bmatrix}$ and $(\lambda,\mu)=(0,1)$.

i) For the Frobenius norm we obtain the minimal perturbation coefficients

$$\Delta A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \Delta A_1 = \begin{bmatrix} -0.75-0.75\imath & 0.25+0.25\imath \\ 0.25+0.25\imath & 0.25+0.25\imath \end{bmatrix},$$
$$A_0+\Delta A_0 = \begin{bmatrix} 0 & -2+\imath \\ 2-\imath & 0 \end{bmatrix}, \quad A_1+\Delta A_1 = \begin{bmatrix} 0.25+0.25\imath & 0.25+0.25\imath \\ 0.25+0.25\imath & 0.25+0.25\imath \end{bmatrix},$$

and $\|\Delta\mathbf{L}\|_F = \eta_F^{\mathbf{S}}(\lambda,\mu,x,\mathbf{L})$.

ii) For the spectral norm we obtain

$$\Delta A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \Delta A_1 = \begin{bmatrix} -0.5-0.5\imath & 0.5+0.5\imath \\ 0.5+0.5\imath & 0.5+0.5\imath \end{bmatrix},$$
$$A_0+\Delta A_0 = \begin{bmatrix} 0 & -2+\imath \\ 2-\imath & 0 \end{bmatrix}, \quad A_1+\Delta A_1 = \begin{bmatrix} 0.5+0.5\imath & 0.5+0.5\imath \\ 0.5+0.5\imath & 0.5+0.5\imath \end{bmatrix},$$

and $\|\Delta\mathbf{L}\|_F = \eta_F^{\mathbf{S}}(\lambda,\mu,x,\mathbf{L}) = 1$; see also Table 4.2

5. Conclusion. The structured backward errors for an approximate eigenpair and the construction of minimal structured matrix polynomials have been introduced in [1, 2, 3, 4] such that an approximate eigenpair of $\mathbf{L}$ becomes exact for $\mathbf{L}+\Delta\mathbf{L}$ in the Frobenius and the spectral norm. However, this theory has been based on the condition that the polynomial eigenvalue problem has no eigenvalue at ∞. Also for T-odd matrix pencil case there is no information on the backward error for the 0 eigenvalue. In this paper we have extended these results in the homogeneous setup of matrix polynomials which is a more convenient way to do the general perturbation analysis for matrix polynomials in that it equally treats all eigenvalues of a regular matrix polynomial. We have presented a systematic general procedure for the construction of appropriately structured minimal norm polynomials $\Delta\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ such that approximate eigenvector and eigenvalue become exact ones of the polynomial $\mathbf{L}+\Delta\mathbf{L}$. The resulting minimal perturbation is unique in the case of the Frobenius norm and there are infinitely many solutions for the case of the spectral norm. Furthermore, we derived the known results for matrix pencils and polynomials of [2, 3, 4] as corollaries and we have illustrated the results with several examples.

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