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COMPUTATION OF THE TORSIONAL MODES IN AN AXISYMMETRIC ELASTIC LAYER*

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Abstract. This paper is devoted to the numerical study of an eigenvalue problem modeling the torsional modes in an infinite and axisymmetric elastic layer. In the cylindrical coordinates (r, z), without θ, the problem is posed in a semi-infinite strip $\Omega = \mathbb{R}_+^* \times \,]0, L[$. For the numerical approximation, we formulate the problem in the bounded domain $\Omega_R = \,]0, R[\times \,]0, L[$. To this end, we use the localized finite element method, which links two representations of the solution: the analytic solution in the exterior domain $\Omega'_R = \,]R, +\infty[\times \,]0, L[$ and the numerical solution in the interior domain Ω_R.

Key words. Torsional modes, spectra, localized finite elements

AMS subject classifications. 35P15, 65N30, 47A70

1. Introduction.

For $L > 0$ and $\Omega = \mathbb{R}_+^* \times \,]0, L[$, we consider the following eigenvalue problem:

$$(\mathrm{P}_0) \qquad \begin{cases} \text{Find } u \in D'(\Omega), u \neq 0 \text{ and } \omega \in \mathbb{R}_+ \text{ such that} \\ B_0 u = \omega^2 \rho u \text{ for } (r, z) \in \Omega, \\ u(r, 0) = 0, \left(\mu \dfrac{\partial u}{\partial z}\right)(r, L) = 0, \ \forall r > 0, \end{cases}$$

where the differential operator B_0 is defined by

$$B_0 u = -\frac{1}{\rho r}\left[\frac{\partial}{\partial r}\left(\mu r \frac{\partial u}{\partial r} - \mu u\right) + \mu \frac{\partial u}{\partial r} + \frac{\partial}{\partial z}\left(\mu r \frac{\partial u}{\partial z}\right) - \mu \frac{u}{r}\right]. \tag{1.1}$$

We use the definitions $\mathbb{R}_+^* = \,]0, +\infty[$ and $\mathbb{R}_+ = [0, +\infty[$. This problem models the vibrations of torsional modes $u_\theta(r, \theta, z, t) = u(r, z)e^{i\omega t}$ in an infinite and axisymmetric elastic layer occupying the domain $\widetilde{\Omega} = \left\{(x, y, z) \in \mathbb{R}^3 : \ 0 < z < L\right\}$ or $\widetilde{\Omega} = \Omega \times [0, 2\pi[$ in the cylindrical coordinates (r, θ, z) [1], where ω is the frequency. We suppose the layer is stratified and perturbed with a local perturbation, which means that it is characterized by a density $\rho(r, z)$ and a shearing coefficient $\mu(r, z)$ which satisfy the assumptions

(A1) $\qquad \mu, \rho \in L^\infty(\Omega), 0 < \mu_- = \inf \mu, \text{ and } 0 < \rho_- = \inf \rho,$

(A2) $\qquad \exists R > 0 \text{ such that } (\mu(r, z), \rho(r, z)) = (\mu_\infty(z), \rho_\infty(z)) \text{ for } r > R.$

The boundary conditions mean the layer is fixed on the face $z = 0$ and is free on the face $z = L$.

In this article, we propose a numerical method to compute the eigenvalues and the eigenmodes of the problem (P_0). As the domain Ω is unbounded, the simplest method is to impose the condition $u = 0$ on the fictitious boundary $r = R_0$ then discretize the problem on Ω_{R_0}. This technique is not accurate, especially when the mode is badly confined. If we are constrained to choose R_0 rather large, the dimension of the related algebraic system increases rapidly. To overcome this difficulty, we propose an exact method which consists of setting an

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equivalent problem in a bounded domain via the transmission condition on a fictitious boundary $r = R$ (R being the size of the perturbation). The idea is to use the Dirichlet-Neumann operator to link the analytic solution for the exterior domain $\Omega'_R =]R, +\infty[\times]0, L[$ to the numerical solution for the interior domain $\Omega_R =]0, R[\times]0, L[$. The transmission condition is expressed in terms of series which will be truncated at an order N for the numerical approximation. This method is well known as the localized finite element method, and has been used by several authors. We refer to the works [10, 13, 16], respectively, for the hydrodynamic problem, the resolution of the Helmholtz equation, and the Schrödinger equation. We mention also the report [5] for the computation of the guided modes in elasticity and [3, 6, 11] for the computation of the cutoff-frequencies in electromagnetism. Note that the differential operator B_0 in our model is singular at the origin, which makes the analysis more difficult.

The paper is organized as follows. In Section 2, we give a variational formulation (P_1) of the spectral problem (P_0). In Section 3, we formulate an equivalent problem (P_R) set in a bounded domain Ω_R by using the Dirichlet-Neumann operator. In Section 4, we truncate the series and discretize (P_R) by the finite element method, then we perform a convergence analysis as the rank of truncation $N \to +\infty$ and the discretization parameter $h \to 0$. Finally, we show in Section 5 some numerical results which validate the method.

2. Variational formulation. In the paper [2], we introduce (P_0) as a spectral problem: $Bu = \omega^2 u$, where B is a self-adjoint operator characterized by a variational triplet (H, V, b). We recall the essential results given there. We introduced the real Hilbert space (L^2 with weights)

$$H(\Omega) = \left\{u \in L^2_{loc}(\Omega) : \sqrt{r}u \in L^2(\Omega)\right\}$$

with the inner product $(u, v)_{H(\Omega)} = \iint_\Omega \rho uv\, r\, dr\, dz$ and the norm $\|u\|_{H(\Omega)} = (u, v)^{1/2}_{H(\Omega)}$, and the weighted Sobolev space

$$V(\Omega) = \left\{u \in H(\Omega) : \frac{u}{\sqrt{r}} \in L^2(\Omega),\ \sqrt{r}|\nabla u| \in L^2(\Omega),\ u(r, 0) = 0\right\}$$

equipped with the norm

$$\|u\|^2_{V(\Omega)} = \iint\limits_\Omega \left(|\nabla u|^2 + \frac{|u|^2}{r^2} + |u|^2\right) r\, dr\, dz.$$

We can write problem (P_0) in the following variational form:

$$(\mathrm{P}_1) \qquad \begin{cases} \text{Find } u \in V(\Omega), u \neq 0, \text{ and } \omega > 0 \text{ such that} \\ b(u, v) = \omega^2 (u, v)_{H(\Omega)},\ \forall v \in V(\Omega), \end{cases}$$

where the bilinear form is defined by

$$b(u, v) = \iint\limits_\Omega \mu \left(r\nabla u \cdot \nabla v + \frac{uv}{r} - u\frac{\partial v}{\partial r} - v\frac{\partial u}{\partial r}\right) dr\, dz,\ \forall u, v \in V(\Omega).$$

This form is obviously continuous and symmetric. Using Poincaré's inequality

$$\forall u \in V(\Omega),\ \iint\limits_\Omega \left|\frac{\partial u}{\partial z}\right|^2 r\, dr\, dz \geq \frac{L^2}{2} \iint\limits_\Omega |u|^2 r\, dr\, dz, \tag{2.1}$$

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we can establish that $b(\cdot,\cdot)$ is V-coercive. Hence, from the representation theorem [8], there exists an unbounded self-adjoint operator B such that $b(u,v) = (Bu,v)_{H(\Omega)}$ for all $u \in D(B)$ and $v \in V$. The domain of B is given by

$$D(B) = \left\{ u \in V(\Omega) : B_0 u \in L^2(\Omega),\ \mu \frac{\partial u}{\partial z}(r,L) = 0 \right\} \text{ and } Bu = B_0 u.$$

The spectral formulation of the problem (P_0) is then:

$$\text{(P)} \qquad \left\{ \begin{array}{l} \text{Find } u \in D(B), u \neq 0, \text{ and } \omega > 0 \text{ such that} \\ B(u) = \omega^2 u. \end{array} \right.$$

REMARK 2.1. We can see from $u \in D(B)$ that $\operatorname{div}(\mu \nabla u) \in H(\Omega)$, hence the trace $(\mu \frac{\partial u}{\partial z})(r,L)$ exists in the generalized sense (in the space $H_{loc}^{-1/2}(\mathbb{R}_+^*)$).

The spectrum of B is described in the following proposition.

PROPOSITION 2.2. *The spectrum of B is $\sigma = \sigma_{\text{ess}} \cup \sigma_{\text{dis}}$, where*

(i) The essential spectrum of B is

$$\sigma_{\text{ess}} = [\gamma, +\infty[\ ,$$

where

$$\gamma = \inf_{g \in W(0,L), g\neq 0} \frac{\int_0^L \mu_\infty(z) \left|g'(z)\right|^2 dz}{\int_0^L \rho_\infty(z) \left|g(z)\right|^2 dz}$$

with $W(0,L) = \left\{ g \in H^1(0,L),\ g(0) = 0 \right\}$.

(ii) The discrete spectrum satisfies

$$\sigma_{\text{dis}} \subset [C, \gamma[\ , \quad \textit{with the lower bound } C = \left(\frac{\mu_-}{\rho_+}\right) \cdot \frac{L^2}{2} \textit{ and } \rho_+ = \sup \rho.$$

Proof. The assertion (i) is proven in [2]. The inclusion (ii) follows from (2.1); indeed, we have for $u \in V(\Omega)$

$$b(u,u) \geq \mu_- \iint\limits_{\Omega} \left(r \left| \frac{\partial u}{\partial z} \right|^2 - 2u \frac{\partial u}{\partial r} \right) dr\, dz.$$

Since $D(\Omega)$ is dense in $V(\Omega)$, it follows that

$$\iint\limits_{\Omega} 2u \frac{\partial u}{\partial r}\, dr\, dz = \int_0^L (|u(\infty,z)|^2 - |u(0,z)|^2) dz = 0,$$

hence

$$b(u,u) \geq \mu_- \iint\limits_{\Omega} \left| \frac{\partial u}{\partial z} \right|^2 r\, dr\, dz \geq \left(\frac{\mu_-}{\rho_+}\right) \frac{L^2}{2} \iint\limits_{\Omega} |u|^2 r \rho\, dr\, dz. \qquad \square$$

3. Formulation in a bounded domain. Before exhibiting the method, we introduce some notation. We let

$$\Omega_R =]0,R[\times]0,L[\quad \text{and} \quad \Omega'_R =]R,+\infty[\times]0,L[\,.$$

For $D = \Omega_R$ or Ω'_R, we denote by $H(D)$ (resp. $V(D)$) the space of the functions which are the restrictions to D of the elements of $H(\Omega)$ (resp. $V(\Omega)$) equipped with the induced norm. For simplicity, we also make the following assumption:

(A3) The velocity $c_\infty(z) := \left(\dfrac{\mu_\infty(z)}{\rho_\infty(z)}\right)^{1/2}$, $0<z<L,$ is a constant c_∞.

We set

$$u(r,z) = \begin{cases} u_1(r,z) & \text{for } (r,z)\in\Omega_R, \\ u_2(r,z) & \text{for } (r,z)\in\Omega'_R. \end{cases}$$

If $u \in D(B)$ is solution of (P), then the pair (u_1,u_2) satisfies the transmission problem

$$\begin{cases} B_0u_1 = \omega^2u_1 & \text{for } (r,z)\in\ \Omega_R, \\ B_0u_2 = \omega^2u_2 & \text{for } (r,z)\in\ \Omega'_R, \\ u_1(R,z) = u_2(R,z) & \text{for } 0<z<L, \\ \mu t(u_1)(R,z) = \mu_\infty t(u_2)(R,z) & \text{for } 0<z<L, \end{cases} \tag{3.1}$$

where $t(u) = r\dfrac{\partial u}{\partial r} - u.$

3.1. Exterior problem. We now exhibit the analytical form of the solution in the exterior domain Ω'_R. If $u \in V(\Omega)$ then the trace $u(R,z)$ belongs to the space

$$H_0^{\frac12}(\,]0,L[\,) = \left\{\varphi\in H^{\frac12}(\,]0,L[\,),\ \frac{\varphi}{\sqrt{z}}\in L^2(\,]0,L[\,)\right\}.$$

For ω^2 and $\varphi(z)\in H_0^{\frac12}(\,]0,L[\,)$ given, we consider the following boundary value problem

$$\begin{cases} B_0u_2 = \omega^2u_2 & \text{in } \Omega'_R, \\ u_2(R,z) = \varphi(z) & \text{for } z\in]0,L[\,. \end{cases} \tag{Q(ω)}$$

We also introduce the Sturm-Liouville problem

$$\begin{cases} \text{Find } g\in H^1(\,]0,L[\,), g\neq 0,\ \text{and } \beta>0 \text{ such that} \\ -\frac{d}{dz}\left(\mu_\infty(z)\frac{dg}{dz}\right) = \beta\mu_\infty(z)g,\ \ \forall z\in]0,L[\,, \\ g(0) = \left(\mu_\infty\frac{dg}{dz}\right)(L) = 0. \end{cases} \tag{3.2}$$

Since $\mu_\infty(z)\geq\mu_->0$, the problem (3.2) is regular in the sense that it admits a sequence of eigenvalues ($\beta_n>0$, $\beta_n\to+\infty$) and an orthogonal system of eigenfunctions $(g_n(z))$ which is complete in $L^2(0,L)$.

REMARK 3.1. We notice that under the hypothesis (A3), the lower bound of the essential spectrum is $\gamma = \beta_1c_\infty^2$. Moreover, if $c_-^2 := \inf_\Omega(\frac{\mu}{\rho}) < c_\infty^2$ we prove by the *Min-Max principle* that the discrete spectrum is not empty; see [2].

PROPOSITION 3.2. *For any real* $\omega^2\in[\beta_1c_-^2,\beta_1c_\infty^2[\,$,

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1. (Q(ω)) *has a unique solution* $u_2(r,z) = R(\omega)\varphi(z)$*. Moreover, the operator* $R(\omega)$ *is linear and continuous from* $H_0^{\frac{1}{2}}(\,]0,L[\,)$ *into* $V(\Omega'_R)$*.*
2. $u_2(r,z)$ *has the following representation for* $r > R$*:*

$$u_2(r,z) = \sum_{n\geq 1} c_n \frac{K_1(\lambda_n(\omega)r)}{K_1(\lambda_n(\omega)R)} g_n(z), \tag{3.3}$$

where $\lambda_n(\omega) = \left(\beta_n - \frac{\omega^2}{c_\infty^2}\right)^{1/2}$, $c_n = \dfrac{1}{L}\displaystyle\int_0^L \mu_\infty(z)\varphi(z)g_n(z)\,dz$*, and* K_1 *is the modified Bessel function of the first order. The series converges in* $V(\Omega'_R)$*.*

Proof. The first part results from the variational formulation and coercivity of the bilinear form associated with the problem (Q(ω)). More precisely, there exists $v_\varphi \in V(\Omega'_R)$ such that $v_\varphi|_{r=R} = \varphi$. Setting $\tilde{u} = u_2 - v_\varphi$, $f = (B_0 - \omega^2)v_\varphi$, and $X = \{v \in V(\Omega'_R), v(R,z) = 0\}$, then (Q($\omega$)) is equivalent to

$$\begin{cases} \text{Find } \tilde{u} \in X \text{ such that} \\ b_\infty(\omega,\tilde{u},v) = \langle f, v\rangle, \quad \forall v \in X, \end{cases} \tag{3.4}$$

where

$$b_\infty(\omega,\tilde{u},v) = \iint\limits_{\Omega'_R} \mu_\infty\left(r\nabla\tilde{u}\cdot\nabla v + \frac{\tilde{u}v}{r}\right) dr\,dz - \omega^2(\tilde{u},v) \text{ and } (\tilde{u},v) = \iint\limits_{\Omega'_R} \rho_\infty \tilde{u} v\, r\,dr\,dz.$$

The brackets $\langle\cdot,\cdot\rangle$ designate the duality between X and X'.

If $\omega^2 < \beta_1 c_\infty^2$, then $b_\infty(\omega,\tilde{u},v)$ is X-coercive and bounded and $L(v) = \langle f, v\rangle$ is linear and continuous. By the Lax-Milgram theorem, there exists a unique solution $\tilde{u}$ such that

$$\|\tilde{u}\|_X \leq C_1\|f\|_{X'} \leq C_2\|v_\varphi\|_X \leq C_3\|\varphi\|_{H^{\frac{1}{2}}(0,L)},$$

which means that

$$\|u_2\|_{V(\Omega'_R)} \leq \|\tilde{u}\|_X + \|v_\phi\|_{V(\Omega'_R)} \leq C\|\varphi\|_{H^{\frac{1}{2}}(0,L)}.$$

For the second part, we use the method of separation of variables. To this end, we introduce the following space:

$$W_R = \left\{u \in L^2(\,]R,+\infty[\,) : \sqrt{r}u \in H^1(\,]R,+\infty[\,)\right\}$$

equipped with the norm $\|u\|_{W_R} = \|\sqrt{r}u\|_{H^1(\,]R,+\infty[\,)}$.

The solution u_2 admits the Fourier expansion $u_2(r,z) = \sum_{n\geq 1} u_n(r)g_n(z)$, which converges in $V(\Omega'_R)$, with the Fourier coefficients $u_n \in W_R$ and with $g_n(z)$ the sequence of eigenfunctions of the Sturm-Liouville problem (3.2); for details, see [9]. Inserting this form in the equation of (Q(ω)), we see that, for all $n \geq 1$, u_n is a solution of the modified Bessel equation

$$u_n''(r) + \frac{1}{r}u_n'(r) + \left(-\frac{1}{r^2} + \lambda_n^2(\omega)\right)u_n(r) = 0 \text{ for } r > R \text{ with } \lambda_n^2(\omega) = \beta_n - \frac{\omega^2}{c_\infty^2}.$$

As $u_n \in W_R$, we have $\sqrt{r}u_n \in L^2(\,]R,+\infty[\,)$ and $u_n(r) = d_n K_1(\lambda_n(\omega)r), \forall n \geq 1$ (according to the Bessel asymptotic formulas). The constant d_n is determined by the boundary condition. Finally, we get

$$u_2(r,z) = \sum_{n\geq 1} c_n \frac{K_1(\lambda_n(\omega)r)}{K_1(\lambda_n(\omega)R)} g_n(z), \quad r > R,$$

where c_n is the Fourier coefficient of φ. The previous series converges in $V(\Omega_R)$ if the numerical series $\sum n^2 \|u_n\|^2_{W_R}$ converge. We can see that $\|u_n\|^2_{W_R} = \lambda_n^2 \|\sqrt{r} u_n\|^2_{L^2(]R,+\infty[)}$ and $\lambda_n = O(n)$, hence $\sum n^2 \|u_n\|^2_{W_R} \le C_1 \sum n c_n^2 \le C_2 \|\varphi\|_{\frac{1}{2}}$. ∎

Note that the hypothesis (A3) is essential to the separation of variables in the equation $B_0 u = \omega^2 u$ in Ω'_R.

3.2. The Dirichlet-Neumann operator. We first introduce some tools. For $s \in \mathbb{R}$, we have the (equivalent) definition

$$H_0^s(]0,L[) = \left\{ v(z) = \sum_{p=1}^{+\infty} v_p g_p(z) : \ \|v\|_s^2 = \sum_{p=1}^{+\infty} |v_p|^2 p^{2s} < +\infty \right\}.$$

The dual product between H_0^s and $H_0^{-s} = (H_0^s)'$ is $\langle v, u\rangle_s = L \sum_{p=1}^{+\infty} v_p \bar{u}_p$.

Recall that $t(u) = r\dfrac{\partial u}{\partial r} - u$, for $u \in D(B)$. The Dirichlet-Neumann operator is defined as follows:

$$T_\omega : H_0^{\frac{1}{2}}(]0,L[) \to H_0^{-\frac{1}{2}}(]0,L[) \quad \text{such that} \quad T_\omega(\varphi) = t(R(\omega)\varphi)\,|_{r=R},$$

where $R(\omega)\varphi$ is the solution of the problem (Q(ω)) associated with the data $\varphi(z)$.

PROPOSITION 3.3. *We have:*

1. *T_ω is linear and continuous and the bilinear form $\langle -T_\omega(u_0), v_0\rangle$ is symmetric and positive.*
2. *T_ω admits the expansion*

$$T_\omega(u_0)(z) = \sum_{n\ge 1} (u_0)_n \left(\frac{\lambda_n(\omega) R K_1'(\lambda_n(\omega)R)}{K_1(\lambda_n(\omega)R)} - 1 \right) g_n(z) \text{ for } r > R, \tag{3.5}$$

where the series converges in the space $H_0^{-\frac{1}{2}}(]0,L[)$.

Proof. The first part follows from the identity

$$\langle -T_\omega(u_0), v_0\rangle = \iint_{\Omega'_R} \mu_\infty \nabla u \cdot \nabla v \, r \, dr\, dz - \omega^2 (u,v) + \sum_{p\ge 1} (u_0)_p (v_0)_p,$$

where u is the solution of the problem (Q(ω)) associated with the data u_0. The second part results from the application of the differential operator t to the series (3.3). ∎

REMARK 3.4. If the medium is homogeneous, we have:

$$g_n(z) = \sin\left((n+0.5)\frac{\pi z}{L}\right), \quad \lambda_n^2(\omega) = -\frac{\omega^2}{c_\infty^2} + (n+0.5)^2 \left(\frac{\pi}{L}\right)^2, \quad n \ge 1. \tag{3.6}$$

3.3. Problem (P$_\mathrm{R}$). The transmission conditions (3.1) allow us to formulate the problem

$$(\mathrm{P_R}) \qquad \begin{cases} \text{Find } u_1 \in V(\Omega_R), u \ne 0, \text{ and } \omega^2 \in I = [\beta_1 c_-^2, \beta_1 c_\infty^2[\text{ such that} \\ B_0 u_1 = \omega^2 u_1 \text{ in } \Omega_R, \\ \mu t(u_1)|_{r=R} = \mu_\infty T_\omega(u_1|_{r=R}). \end{cases}$$

The problems (P$_\mathrm{R}$) and (P) are equivalent in the the following sense:

PROPOSITION 3.5. *We have:*

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1. *If the pair* (ω^2, u) *is a solution of the problem* (P) *then* $(\omega^2, u|_{\Omega_R})$ *is a solution of the problem* ($\mathrm{P_R}$).

2. *Conversely, if the pair* (ω^2, u_1) *is a solution of the problem* ($\mathrm{P_R}$) *then* u_1 *can be extended uniquely to a solution* (ω^2, u) *of the problem* (P).

REMARK 3.6. The eigenvalue problem ($\mathrm{P_R}$) is nonlinear since $T(\omega)$ is a nonlinear function.

3.4. Study of the nonlinearity. For $\alpha \in I = [\beta_1 c_-^2, \beta_1 c_\infty^2[$ fixed, we consider the linear problem:

$$(\mathrm{P_R}(\alpha)) \qquad \begin{cases} \text{Find } u_1 \in V(\Omega_R), u_1 \neq 0, \text{ and } \omega^2(\alpha) \in I \text{ such that} \\ Bu_1 = \omega^2(\alpha) u_1 \text{ in } \Omega_R, \\ \mu t(u_1)|_{r=R} = \mu_\infty T_\alpha(u_1|_{r=R}). \end{cases}$$

Suppose that $\alpha \to \omega^2(\alpha)$ is a curve having a fixed point $\alpha_0 \in I$ ($\omega^2(\alpha_0) = \alpha_0$); then (u_1, α_0) is a solution of ($\mathrm{P_R}$). We shall examine the question of existence of such curves. To this end, we use the variational form of ($\mathrm{P_R}(\alpha)$):

$$(\widetilde{\mathrm{P}}_{\mathrm{R}}(\alpha)) \qquad \begin{cases} \text{Find } u \in V(\Omega_R), u \neq 0, \text{ and } \omega^2 \in I \text{ such that} \\ C(\alpha, u, v) := A(u,v) + D(\alpha, u, v) \ = \omega^2 (u,v)_{H(\Omega_R)}, \ \ \forall\, v \in V(\Omega_R), \end{cases}$$

where

$$A(u,v) = \iint\limits_{\Omega_R} \mu \left(r\nabla u \cdot \nabla v + \frac{uv}{r} - u\frac{\partial v}{\partial r} - v\frac{\partial u}{\partial r} \right) dr\, dz$$

and

$$D(\alpha, u, v) = \sum_{n\geq 1} \left(\frac{\lambda_n(\omega) R K_1'(\lambda_n(\omega)R)}{K_1(\lambda_n(\omega)R)} - 1 \right) (u_0)_n (v_0)_n.$$

We prove in [9] that $C(\alpha, u, v)$ is coercive and characterizes a family of operators $C(\alpha)$.

PROPOSITION 3.7 ([9]). $C(\alpha)$ *is a positive self-adjoint operator with a compact resolvent. The eigenvalues form an increasing sequence having the properties:*

1. $\omega_m^2(\alpha) \leq \omega_{m+1}^2(\alpha)$, $\omega_1^2(\alpha) \geq c_-^2 \beta_1$,
2. $\omega_m^2(\alpha) = \min_{V_m \in F_m} \max_{u \in V_m} \frac{C(\alpha,u,u)}{\|u\|^2}$, *where* F_m *denotes the family of the subspaces* $V_m \subset V(\Omega_R)$ *with dimension* m.
3. *the functions* $\alpha \to \omega_m^2(\alpha)$, $m \in \mathbb{N}^*$, *are strictly decreasing and Lipschitz continuous on the interval* I.

Proof. These properties are a consequence of the following coercivity results:

1. $C(\alpha, u, u) \geq c_-^2 \beta_1 (u,u)_{H(\Omega_R)}$,
2. for all $\epsilon > 0$, there exist positive constants $C_1(\epsilon)$ and $C_2(\epsilon)$ such that

$$C(\alpha, u, u) + C_1(\epsilon)(u,u)_{H(\Omega_R)} \geq C_2(\epsilon)\|u\|^2_{V(\Omega_R)}.$$

Then we use the Min-Max principle [15]. ☐

As a consequence of Proposition 3.7, we have

COROLLARY 3.8. *For* $\alpha \in I$*, the following two properties are equivalent:*

1. $\alpha = \omega^2$ *is an eigenvalue of* B.
2. $\exists m \in \mathbb{N}$ *such that* $\omega_m^2(\alpha) = \alpha$.

We conclude with the following regularity result.

THEOREM 3.9 (Regularity). *Suppose that* $\mu \in C^{0,1}(\Omega_R)$ *and let* $u \in V(\Omega_R)$ *be an eigenfunction of* ($\mathrm{P_R}(\alpha)$). *Then*

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1. $\sqrt{r}u \in H^2(\Omega_R)$ *and* $\|\sqrt{r}u\|_2 \leq C\|u\|_{H(\Omega_R)}$,
2. $\forall r \in]0, R/2[\ ,\ |u(r)| \leq Cr\|u\|_{H(\Omega_R)}$.

Proof. The proof is rather technical. We reproduce here the main steps of [9].

1) Let $u \in V(\Omega_R)$ be a solution of ($\mathrm{P_R}(\alpha)$). Then $v = \sqrt{r}u \in H^1(\Omega_R)$ satisfies the problem

$$\begin{cases} -\Delta v + \frac{3}{4}\frac{v}{r^2} = f(v) & \text{in } \Omega_R, \\ v(r,0) = \frac{\partial v}{\partial z}(r,L) = 0, & 0 \leq r \leq R, \\ v(0,z) = 0, \ \ R\frac{\partial v}{\partial r}(R,z) = \frac{3v(R,z)}{2R} + T_\alpha(v|_{r=R}), & 0 \leq z \leq L. \end{cases} \tag{3.7}$$

where $f(v) = \left[\rho\omega^2 + \frac{\partial \mu}{\partial r}\left(\frac{\partial v}{\partial r} - \frac{3v}{2r}\right) + \frac{\partial \mu}{\partial z}\frac{\partial v}{\partial r}\right]\mu^{-1}$. We can see that $f(v) \in L^2(\Omega_R)$ and $\|f(v)\|_0 \leq C\|u\|_{V(\Omega_R)}$.

2) We can decompose $v = v_1 + v_2$ such that the pair (v_1, v_2) solves the systems

$$\begin{cases} -\Delta v_1 + \frac{3}{4}\frac{v_1}{r^2} = f(v) & \text{in } \Omega_R, \\ v_1(r,0) = \frac{\partial v_1}{\partial z}(r,L) = 0, & 0 \leq r \leq R, \\ v_1(0,z) = \frac{\partial v_1}{\partial r}(R,z) = 0, & 0 \leq z \leq L. \end{cases} \tag{3.8}$$

and

$$\begin{cases} -\Delta v_2 + \frac{3}{4}\frac{v_2}{r^2} = 0 & \text{in } \Omega_R, \\ v_2(r,0) = \frac{\partial v_2}{\partial z}(r,L) = 0, & 0 \leq r \leq R, \\ v_2(0,z) = 0, \ \ R\frac{\partial v_2}{\partial r}(R,z) = \frac{3v(R,z)}{2R} + T_\alpha(v|_{r=R}), & 0 \leq z \leq L. \quad (T) \end{cases} \tag{3.9}$$

3) Using separation of variables we can express v_1 and $f(r,z) = f(v(r,z))$ as the series

$$v_1(r,z) = \sum_{n\geq 0} v_{1n}\sin(\sqrt{\beta_n}z), \quad f(r,z) = \sum_{n\geq 0} f_n(r)\sin(\sqrt{\beta_n}z) \quad \left(\beta_n = (2n+1)^2\frac{\pi^2}{4L^2}\right),$$

where v_{1n} is the solution of the boundary value problem

$$\begin{cases} -v_{1n}'' + (\beta_n + \frac{3}{4r^2})v_{1n} = f_n(r), & r \in]0,R[\ , \\ v_{1n}(0) = v_{1n}'(R) = 0. \end{cases} \tag{3.10}$$

The solution of (3.10) is given by

$$v_{1n}(r) = \int_0^R G(r,r')f_n(r')dr'$$

where $G(r,r')$ is the Green function of (3.10), which involves the modified Bessel functions $I_1(\sqrt{\beta_n}r)$ and $K_1(\sqrt{\beta_n}r)$. Using asymptotic formulas, we can prove the inequalities

$$|v_{1n}(r)| \leq Cr\|f_n\|_{L^2(]0,R[)} \ \text{ and } \ \|v_{1n}\|_{H^2(]0,R[)} \leq C\|f_n\|_{L^2(]0,R[)},$$

and as a consequence we obtain

$$|v_1(r)| \leq Cr\|f\|_{L^2(\Omega_R)} \ \text{ and } \ \|v_1\|_{H^2(\Omega_R)} \leq C\|f\|_{L^2(\Omega_R)}.$$

4) In the same manner, we obtain the expression

$$v_2(r,z) = \sqrt{\frac{r}{R}}\sum_{n\geq 0}\psi_{2n}\frac{I_1(\sqrt{\beta_n}r)}{I_1(\sqrt{\beta_n}R)}\sin(\sqrt{\beta_n}z)$$

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with $\psi_{jn} = (\psi_j, \sin(\sqrt{\beta_n} z))_{L^2(0,L)}$ and $\psi_j(z) = v_j(R, z)$ for $j = 1, 2$. Using the boundary condition (T), which relates v_1 to v_2, we establish that $\psi_2 \in H^{3/2}$ and $\|\psi_2\|_{3/2} \leq C\|\psi_1\|_{3/2}$. Then by a direct calculation we prove that $\Delta v_2 \in L^2$ and $\|\Delta v_2\|_0 \leq C\|\psi_2\|_{3/2}$. Finally, an asymptotic study when $r \to 0$ shows $|u_2(r)| \leq Cr\|\psi_2\|_{3/2}$, which concludes the proof. $\square$

4. Discretization.

4.1. Semi-discretized problem. For the numerical approximation of the problem ($\mathrm{P_R}(\alpha)$), we first truncate series (3.5) in the expression of T_α. This leads us to set the following *semi-discretized* problem:

$$(\mathrm{P_R^N}(\alpha)) \quad \begin{cases} \text{Find } u \in V(\Omega_R), u \neq 0, \text{ and } \omega^2 \in I \text{ such that} \\ C^N(\alpha, u, v) := A(u, v) + D^N(\alpha, u, v) = \omega^2 (u, v)_{H(\Omega_R)}, \ \ \forall v \in V(\Omega_R), \end{cases}$$

where

$$D^N(\alpha, u, v) = \sum_{n=1}^{n=N} \left(\frac{\lambda_n(\omega) R K_1'(\lambda_n(\omega) R)}{K_1(\lambda_n(\omega) R)} - 1 \right) (u_0)_n (v_0)_n. \tag{4.1}$$

This problem possesses a sequence of eigenvalues $\mu_m^N(\alpha) = \omega_m^N(\alpha)^2$ and eigenfunctions $u_m^N(\alpha)$, $m = 1, 2, \ldots$, having all the properties of the exact problem. Moreover, the sequence $\mu_m^N(\alpha)$ converges to $\omega_m(\alpha)^2$ as $N \to +\infty$. More precisely, we have the following result.

THEOREM 4.1 ([9]). *Suppose* $\mu \in C^{0,1}(\Omega_R)$ *and* $(u_m(\alpha), \omega_m^2(\alpha))$ *is a solution of the problem* ($\mathrm{P_R}(\alpha)$). *Then we have*

$$0 \leq \omega_m^2(\alpha) - \omega_m^N(\alpha)^2 \leq \frac{C}{N^2}, \tag{4.2}$$

and

$$\left\| u_m^N(\alpha) - u_m(\alpha) \right\| \leq \frac{C}{N^2}. \tag{4.3}$$

Proof. The proof is similar to that of [3, 4]. $\square$

4.2. Discretization by finite elements. The goal here is to approximate ($\mathrm{P_R^N}(\alpha)$) by finite elements. For this we consider a subspace $V_h \subset V(\Omega_R)$ of dimension $M = M(h)$, where h is a discretization parameter, and we consider the following discretized problem:

$$(\mathrm{P_R^{N,h}}(\alpha)) \qquad \begin{cases} \text{Find } u \in V_h, u \neq 0, \text{ and } \omega^2 \in I \text{ such that} \\ C^N(\alpha, u, v_h) = \omega^2 (u, v_h)_{H(\Omega_R)}, \ \ \forall v_h \in V_h. \end{cases}$$

We denote the eigenelements of ($\mathrm{P_R^{N,h}}(\alpha)$) by $(\mu_{m,h}^N, u_{m,h}^N)$, $m = 1, M$.

In practice, we define V_h as follows. Let $T_h = \{K_i\}_{i=1}^M$ be a regular triangulation of the rectangle Ω_R with vertices $\{a_i\}_{i=1}^M$, and define $\Gamma_0 = \{(0, z),\ 0 < z < H\}$ and $\Gamma_1 = \{(r, 0),\ 0 < r < R\}$. Then we define the spaces

$$\mathcal{M} = \left\{ \varphi \in C^0(\overline{\Omega_R}) : \ \varphi \equiv 0 \text{on } \Gamma_0 \cup \Gamma_1 \right\} \text{ and}$$
$$V_h = \{ \varphi \in \mathcal{M} \cap V(\Omega_R) : \varphi|_{K_i} \in P_1(K_i) \text{ for } 1 \leq i \leq M \}.$$

We introduce the interpolation operator

$$\Pi_h : \mathcal{M} \longrightarrow V_h, \ \text{such that } (\Pi_h \varphi)(a_i) = \varphi(a_i).$$

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As in the classical theory [7, 14], we can show the following interpolation property: for $u \in V(\Omega_R)$,

$$\lim_{h \longrightarrow 0} \inf_{v_h \in V_h(\Omega_R)} \|u - v_h\|_{V(\Omega_R)} = 0. \tag{4.4}$$

Let $\mathcal{O}$ be a regular open of $\mathbb{R}^2_+ = \{(r,z) : \; r > 0\}$. For $l \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, we recall the following weighted Sobolev spaces:

$$W^{l,2}_\alpha(\mathcal{O}) = \{u \in D'(\mathcal{O}) : r^\alpha D^\beta u \in L^2(\mathcal{O}) \text{ for } 0 \le |\beta| \le l\}$$

and

$$X^{l,2}_\alpha(\mathcal{O}) = \{u \in D'(\mathcal{O}) : r^{\alpha - l + |\beta|} D^\beta u \in L^2(\mathcal{O}) \text{ for } 0 \le |\beta| \le l\}$$

equipped with the natural norms $\|\cdot\|_{l,\alpha}$. These spaces are studied in [12].

We now recall a useful interpolation result.

THEOREM 4.2 ([12]). *If the triangulation T_h is regular, then there exists a constant $C > 0$ such that for every $u \in W^{2,2}_{1/2}(\Omega_R) \cap X^{1,2}_{1/2}(\Omega_R)$, we have*

$$\|u - \Pi_h u\|_{1,\frac{1}{2}} \le Ch \|u\|_{2,\frac{1}{2}} \tag{4.5}$$

and such that for all $u \in W^{2,2}_{1/2}(\Omega_R) \cap X^{1,2}_{1/2}(\Omega_R)$, we have

$$\left\| r^{-\frac{1}{2}}(u - \Pi_h u) \right\|_0 \le Ch \|u\|_{2,\frac{1}{2}}\,. \tag{4.6}$$

THEOREM 4.3. *Suppose that $\mu \in C^{0,1}(\Omega_R)$ and the triangulation is regular. Suppose u is a solution of* ($\mathrm{P}^{\mathrm{N}}_{\mathrm{R}}(\alpha)$). *Then there exists a constant $C > 0$ such that*

$$\|u - \Pi_h u\|_{V(\Omega_R)} \le C \|u\|_{1,\Omega_R}\,. \tag{4.7}$$

Proof. As μ is smooth, it follows from Theorem 3.9 and Hardy's inequality that $u \in W^{2,2}_{1/2}(\Omega_R) \cap X^{1,2}_{1/2}(\Omega_R)$. To conclude we use Theorem 4.2 by observing the following imbedding:

$$W^{2,2}_{1/2}(\Omega_R) \cap X^{1,2}_{1/2}(\Omega_R) \;\subset\; X^{2,2}_{1/2}(\Omega_R) \;\subset\; H^1(\Omega_R), \tag{4.8}$$

which is continuous; moreover, the norm $\|u\|_{V(\Omega_R)}$ is equivalent to $\left(\|u\|^2_{1,\frac{1}{2}} + \left\|r^{-1/2}u\right\|^2_0\right)^{1/2}$.
∎

We introduce the projection $\widetilde{\Pi}_h$ defined by the variational equation

$$C^N(\alpha, \widetilde{\Pi}_h u - u, u_h) + \beta_0(\widetilde{\Pi}_h u - u, u_h)_{H(\Omega_R)} = 0, \;\; \forall v_h \in V_h(\Omega_R). \tag{4.9}$$

The coercivity leads to the following interpolation result.

THEOREM 4.4. *Suppose that $\mu \in C^{0,1}(\Omega_R)$ and let u be an eigenfunction of* ($\mathrm{P}^{\mathrm{N}}_{\mathrm{R}}(\alpha)$). *Then there exists a constant $C > 0$ such that*

$$\left\|u - \widetilde{\Pi}_h u\right\|_{V(\Omega_R)} \le Ch \|u\|_{1,\Omega_R}\,. \tag{4.10}$$

THEOREM 4.5 (Convergence). *We have*

$$\lim_{h\longrightarrow 0}\left|\mu_m^N(\alpha)-\mu_{m,h}^N(\alpha)\right|=0; \tag{4.11}$$

furthermore, if the eigenvalue $\mu_m^N(\alpha)$ is simple, then

$$0\le \mu_m^N(\alpha)-\mu_{m,h}^N(\alpha)\le Ch^2 \text{ and } \left\|u_m^N(\alpha)-u_{m,h}^N(\alpha)\right\|_V\le\ Ch. \tag{4.12}$$

The previous theorem is analogous to Theorem 6.5.1 in [14].

THEOREM 4.6 (Global Error). *Suppose that $\mu\in C^{0,1}(\Omega_R)$. For each solution $(\mu_m(\alpha),u_m(\alpha))$ of* ($\mathrm{P_R}(\alpha)$) *we have, for all $\alpha\in I$,*

1. $0\le\mu_m(\alpha)-\mu_{m,h}^N(\alpha)\le C\left(h^2+\dfrac{1}{N^2}\right)$,
2. $\left\|u_m(\alpha)-u_{m,h}^N(\alpha)\right\|_{V(\Omega_R)}\le C\left(h+\dfrac{1}{N^2}\right)$.

4.3. Implementation of the method. Let $h_1=R/M_r$ and $h_2=L/M_z$ tend to zero where $M_r,M_z\in\mathbb{N}^*$, and let $M=M_z\times M_r$. We search for a solution to the problem ($\mathrm{P_R^{N,h}}(\alpha)$) in the form $u_h(\alpha)=\Sigma_j Y_j\varphi_j$, where $\{\varphi_j\}$ is the basis of V_h, which leads to the linear system

$$\begin{cases}\text{Find } Y\in\mathbb{R}^M, Y\neq 0, \text{ and } \lambda\in I \text{ such that}\\ \qquad (A+D^N(\alpha))Y=\lambda BY,\end{cases} \tag{4.13}$$

with the entries $A=(a_{ij})$, $D^N(\alpha)=(d_{ij})$, and $B=(b_{ij})$ given by

$$\begin{aligned}
a_{i,j}&=A(\varphi_i,\varphi_j)=\iint\limits_{K_{i,j}}\mu\left(r\nabla\varphi_i\cdot\nabla\varphi_j+\frac{\varphi_i\varphi_j}{r}-\varphi_i\frac{\partial\varphi_j}{\partial r}-\varphi_j\frac{\partial\varphi_i}{\partial r}\right)dr\,dz,\\
d_{i,j}&=D^N(\alpha,\varphi_i,\varphi_j)=\sum_{n=1}^{n=N}\left(\frac{\lambda_n(\alpha)RK_1'(\lambda_n(\alpha)R)}{K_1(\lambda_n(\alpha)R)}-1\right)(\varphi_{i0})_n(\varphi_{j0})_n,\\
b_{i,j}&=\iint\limits_{K_{i,j}}r\rho\varphi_i\varphi_j\,dr\,dz,\\
K_{i,j}&=\mathrm{supp}(\varphi_i)\cap\mathrm{supp}(\varphi_j).
\end{aligned}$$

$(\varphi_{i0})_n$ are the Fourier coefficients of order n of $\varphi_{i0}(z)=\varphi_i(z,R)$ associated with the system $\{g_n(z)\}$ (eigenfunctions of (3.2)) given by:

$$(\varphi_{i0})_n=\frac{2}{L}\int_0^L\mu_\infty(z)\varphi_i(z,R)g_n(z)dz.$$

REMARK 4.7. If $(\mu(z),\rho(z))$ are not constant we approximate $g_n(z)$ by discretizing the Sturm-Liouville problem (3.2) by the finite element method in the interval $]0,L[$.

For each α^2 in $\left[c_-^2\beta_1,c_\infty^2\beta_1\right[$, we solve the generalized eigenvalue problem (4.13). For that we perform the Cholesky factorization $B=L^TL$ and make the change of the coordinates $Z=L^TY$, which transforms the system into

$$L^{-T}\left(A+D^N(\alpha)\right)L^{-1}Z=\lambda Z. \tag{4.14}$$

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The latter system has a sequence of eigenvalues $\lambda_m^N(\alpha)$, $1 \leq m \leq M$. For m fixed, we put $g(\alpha) = \lambda_m^N(\alpha)$. The function g is decreasing (see Proposition 3.7), so g possesses a fixed point if and only if

$$g(c_-^2\beta_1) < c_\infty^2\beta_1. \tag{4.15}$$

If (4.15) holds, we approximate this point by the secant iteration

$$\alpha_0 = c_-^2\beta_1, \quad \alpha_{s+1} = \frac{c_\infty^2\beta_1 g(\alpha_s) - \alpha_s g(c_\infty^2\beta_1)}{g(\alpha_s) + c_\infty^2\beta_1 - g(c_\infty^2\beta_1) - \alpha_s} \text{ for } s = 0, 1, \ldots.$$

We stop the process when $|\alpha_{s+1} - \alpha_s| < \epsilon$, where ϵ is the desired accuracy.

5. Numerical results.

We present two simple numerical experiment to verify and illustrate the result in this paper.

5.1. An example with piecewise constant profile.

In the first example, the domain is $\Omega_R =]0, R[\times]0, L[$ where $R = L = 1$. Define the piecewise constant coefficients:

$$\begin{aligned} \rho_1 &= 1.0 \times 10^3 \text{ kg/m}^3, & \rho_2 &= 1.0 \times 10^3 \text{ kg/m}^3 \\ \mu_1 &= 0.5 \times 10^{11} \text{ N/m}^3, & \mu_2 &= 1.0 \times 10^{11} \text{ N/m}^3 \end{aligned}$$

In this case there exists a hierarchy of eigenmodes $u_p(r, z) = u_p(r)\sin(\lambda_p z)$, $\lambda_p = \dfrac{2p+1}{2L}\pi$, indexed with an integer p, such that

$$u_p(r) = A \begin{cases} \dfrac{J_1(\alpha_p r)}{J_1(\alpha_p R)} & \text{if } r < R, \\[2ex] \dfrac{K_1(\beta_p r)}{K_1(\beta_p R)} & \text{if } r > R, \end{cases}$$

where

$$\alpha_p^2 = \frac{\omega^2}{c_1^2} - \lambda_p^2, \quad \beta_p^2 = \lambda_p^2 - \frac{\omega^2}{c_2^2} \text{ with } c_1^2 = \frac{\mu_1}{\rho_1}, \quad c_2^2 = \frac{\mu_2}{\rho_2}.$$

and $\{J_\nu(z), K_\nu(z)\}$ are Bessel and modified Bessel functions of order ν. The eigenvalues ω^2 are the roots of the characteristic equation, in the interval $I_p = \left[c_1^2\lambda_p^2, c_2^2\lambda_p^2\right[$,

$$G_p(\omega^2) := \alpha_p R\frac{J_0(\alpha_p R)}{J_1(\alpha_p R)} + \frac{\mu_2}{\mu_1}\beta_p R\frac{K_0(\beta_p R)}{K_1(\beta_p R)} + 2\left(\frac{\mu_2}{\mu_1} - 1\right) = 0.$$

$G_p(\omega^2)$ possesses p roots in the interval I_p.

We have computed numerically the first frequencies and compared with exact ones. Results are shown in the Table 5.1.

TABLE 5.1
Convergence of the method for the first eigenvalues.

p	1	2		3		
ω	40877	60899	71439	81840	90600	102303
ω_h	40916	60794	71992	81865	89956	100915
$\frac{\lvert\omega-\omega_h\rvert}{\omega}$	0.0010	0.0017	0.0077	0.0003	0.0071	0.0135

The approximation ω_h^N is computed with the data $N = 1$, $M_r = 24$, $M_z = 30$. We have used the command *spec* of the software *Scilab 5* based on the routine DGEEV of LAPACK. We observe that the result is insensible of higher orders $N \geq 2$.

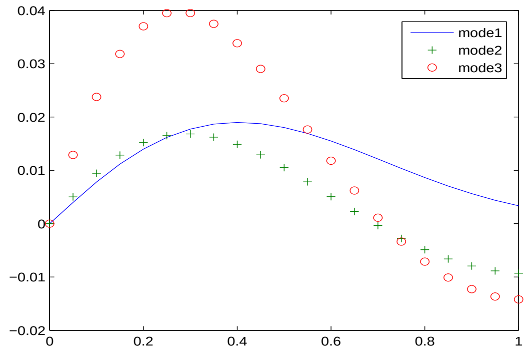

FIGURE 5.1. *First modes for $z = L$.*

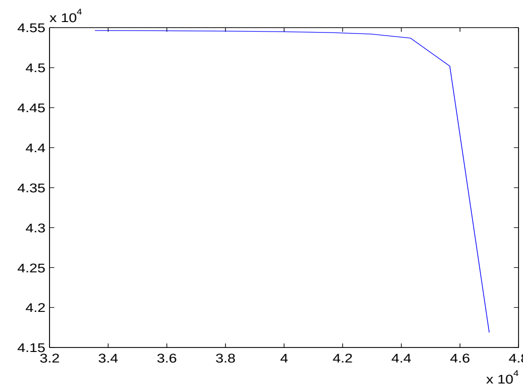

FIGURE 5.2. *The first dispersion curve $\alpha \to \omega_{1,h}^N(\alpha)$.*

5.2. An example with linear profile. As a second example, we consider a problem with coefficient $\mu(r,z)$ which is affine in $\Omega_1 =]0,1[\times]0,1[$:

$$\mu(r,z) = \begin{cases} a(r+z)+\mu_{\min} & \text{for } 0 < r < 1, \\ \mu_\infty & \text{for } r > 1, \end{cases}$$

with $a = 0.2 \times 10^{11}$, $\mu_{min} = 0.5 \times 10^{11}$ and $\mu_\infty = 1.0 \times 10^{11}$. With $N = 10$ and $M_r = M_z = 23$, we have computed the first frequencies

$$\omega_{1,h} = 45230, \quad \omega_{2,h} = 57054, \quad \omega_{3,h} = 72183.$$

In Table 5.2, we show the evolution of $\omega_{1,h}^N$ with N, h fixed. We notice that the contribution of the ranks $N = 1, 2$ is essential.

TABLE 5.2
Evolution of ω_i^N with N

N	1	2	3	4	5
ω_1	45227.233	45230.235	45230.237	45230.238	45230.242
ω_2	58321.369	58321.424	58321.443	58321.459	58321.461
ω_3	72296.395	72302.084	72302.100	72302.105	72302.107

The corresponding eigenvectors are plotted, for $z = L$, in Figure 5.1. Figure 5.2 shows that the dispersion curve $\alpha \to \omega_1(\alpha)$ is decreasing.

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