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STIELTJES INTERLACING OF ZEROS OF JACOBI POLYNOMIALS FROM DIFFERENT SEQUENCES*

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Abstract. A theorem of Stieltjes proves that, given any sequence $\{p_n\}_{n=0}^{\infty}$ of orthogonal polynomials, there is at least one zero of p_n between any two consecutive zeros of p_k if $k < n$, a property called Stieltjes interlacing. We show that Stieltjes interlacing extends, under certain conditions, to the zeros of Jacobi polynomials from different sequences. In particular, we prove that the zeros of $P_{n+1}^{\alpha,\beta}$ interlace with the zeros of $P_{n-1}^{\alpha+k,\beta}$ and with the zeros of $P_{n-1}^{\alpha,\beta+k}$ for $k \in \{1,2,3,4\}$ as well as with the zeros of $P_{n-1}^{\alpha+t,\beta+k}$ for $t,k \in \{1,2\}$; and, in each case, we identify a point that completes the interlacing process. More generally, we prove that the zeros of the kth derivative of $P_n^{\alpha,\beta}$, together with the zeros of an associated polynomial of degree k, interlace with the zeros of $P_{n+1}^{\alpha,\beta}$, $k,n \in N$, $k < n$.

Key words. Interlacing of zeros; Stieltjes' Theorem; Jacobi polynomials.

AMS subject classifications. 33C45, 42C05

1. Introduction.

It is well known that if $\{p_n\}_{n=0}^{\infty}$ is a sequence of orthogonal polynomials, the zeros of p_n are real and simple, and if $x_{1,n} < x_{2,n} < \cdots < x_{n,n}$ are the zeros of p_n while $x_{1,n-1} < x_{2,n-1} < \cdots < x_{n-1,n-1}$ are the zeros of p_{n-1}, then

$$x_{1,n} < x_{1,n-1} < x_{2,n} < x_{2,n-1} < \cdots < x_{n-1,n-1} < x_{n,n},$$

a property called the interlacing of zeros. Another classical result on interlacing of zeros of orthogonal polynomials is due to Stieltjes who proved that if $m < n$, then between any two successive zeros of p_m there is at least one zero of p_n, a property called Stieltjes interlacing [13, Theorem 3.3.3]. Clearly, if $m < n-1$, there are not enough zeros of p_m to interlace fully with the n zeros of p_n. Nevertheless, using the same argument as Stieltjes, one can show that for $m < n-1$, provided p_m and p_n have no common zeros, there exist m open intervals, with endpoints at successive zeros of p_n, each of which contains exactly one zero of p_m. Moreover, in [3], Beardon shows that for each $m < n-1$, if p_m and p_n are co-prime, there exists a real polynomial S_{n-m-1} of degree $n-m-1$ whose real simple zeros provide a set of points that completes the interlacing picture. An important feature of the polynomials S_{n-m-1} is that they are completely determined by the coefficients in the three term recurrence relation satisfied by the orthogonal sequence $\{p_n\}_{n=0}^{\infty}$. The polynomials S_{n-m-1} are the dual polynomials introduced by de Boor and Saff in [4] or, equivalently, the associated polynomials analyzed by Vinet and Zhedanov in [15].

The interlacing property of zeros of polynomials is important in numerical quadrature applications, and in [12], Segura proved that interlacing of zeros holds, under certain assumptions, within sequences of classical orthogonal polynomials even when the parameter(s) on which they depend lie outside the value(s) required to ensure orthogonality. He also considered the interlacing of zeros of polynomials p_{n-1} and p_{n+1} in any orthogonal sequence $\{p_n\}_{n=0}^{\infty}$ and showed that interlacing of zeros occurs to the left and to the right of a specified

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point [12, Theorem 1]. Segura identified this point in terms of the coefficients in the three term recurrence relation satisfied by $\{p_n\}_{n=0}^{\infty}$; equivalently, it is the zero of the linear de Boor-Saff polynomial [3, Theorem 3]. Stieltjes interlacing was studied for the zeros of polynomials from different sequences of one-parameter orthogonal families, namely, Gegenbauer polynomials C_n^{λ} and Laguerre polynomials L_n^{α} in [5] and [6], respectively, and associated polynomials analogous to the de Boor-Saff polynomials were identified in each case. Related work in which recurrence relations for ${}_2F_1$ functions are considered can be found in [9].

In a generalization that is complementary to that of Segura in [12], it was proved in [7] that the zeros of $P_n^{\alpha,\beta}$ interlace with the zeros of polynomials from some different Jacobi sequences, including those of $P_n^{\alpha-t,\beta+k}$ and $P_{n-1}^{\alpha+t,\beta+k}$ for $0 \le t, k \le 2$, thereby confirming and extending a conjecture made by Richard Askey in [2]. Numerical examples were given to illustrate that, in general, if t or k is greater than 2, interlacing of zeros need not necessarily occur.

In this paper, we investigate the extent to which Stieltjes interlacing holds between the zeros of two Jacobi polynomials if each polynomial belongs to a sequence generated by a different value of the parameter α and/or β. We also identify, in each case, a polynomial that plays the role of the de Boor-Saff polynomial [3, 4], in the sense that its zeros provide a (non-unique) set of points that complete the interlacing process.

2. Results. We recall that, for $\alpha, \beta > -1$, the sequence of Jacobi polynomials $\{P_n^{\alpha,\beta}\}_{n=0}^{\infty}$ is orthogonal with respect to the weight function $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$ on $(-1,1)$ and satisfies the three term recurrence relation [13]

$$\begin{aligned} (2.1)\quad & \tfrac{2(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} P_{n+1}^{\alpha,\beta}(x) \\ &= \left(x - \tfrac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}\right) P_n^{\alpha,\beta}(x) - \tfrac{2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)} P_{n-1}^{\alpha,\beta}(x). \end{aligned}$$

Our first four results consider cases when Stieljes interlacing occurs between the zeros of Jacobi polynomials from different sequences whose degrees differ by two.

THEOREM 2.1.

(i) If $P_{n-1}^{\alpha+t,\beta}$ and $P_{n+1}^{\alpha,\beta}$ are co-prime, then

- *(a) the zeros of $P_{n-1}^{\alpha+t,\beta}$ and $\frac{\beta^2-\alpha^2+t(\beta-\alpha+2n(n+\beta+1))}{(2n+\alpha+\beta+t)(2n+\alpha+\beta+2)}$ interlace with the zeros of $P_{n+1}^{\alpha,\beta}$ for fixed $t \in \{0,1,2\}$;*
- *(b) the zeros of $P_{n-1}^{\alpha+3,\beta}$ and $\frac{n(n+\alpha+\beta+2)+(\alpha+2)(n-\alpha+\beta)}{(n+\alpha+2)(n+\alpha+\beta+2)}$ interlace with the zeros of $P_{n+1}^{\alpha,\beta}$;*
- *(c) the zeros of $P_{n-1}^{\alpha+4,\beta}$ and $\frac{2n(n+\alpha+\beta+3)+(\alpha+3)(\beta-\alpha)}{2n(n+\alpha+\beta+3)+(\alpha+3)(\alpha+\beta+2)}$ interlace with the zeros of $P_{n+1}^{\alpha,\beta}$.*

(ii) If $P_{n-1}^{\alpha+t,\beta}$ and $P_{n+1}^{\alpha,\beta}$ are not co-prime, they have one common zero located at the respective points identified in (i) (a) to (c) and the $n-1$ zeros of $P_{n-1}^{\alpha+t,\beta}$ interlace with the remaining n (non-common) zeros of $P_{n+1}^{\alpha,\beta}$.

REMARK 2.2. A theorem due to Gibson [8] proves that if $\{p_n\}_{n=0}^{\infty}$ is any orthogonal sequence, the polynomials p_{n+1} and p_m, $m = 1, 2, \ldots, n-1$ can have at most $\min\{m, n-m\}$ common zeros. Theorem 2.1 (ii) extends Gibson's result to Jacobi polynomials of degree $n-1$ and $n+1$ from different orthogonal sequences.

REMARK 2.3. The case $t = 0$ in Theorem 2.1 (i) was proved by Segura [12, Section 3.1]. For completeness and the convenience of the reader, we provide an alternative proof of this case.

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Since Jacobi polynomials satisfy the symmetry property [10, p. 82, Equation (4.1.1)]

$$P_n^{\alpha,\beta}(x) = (-1)^n P_n^{\beta,\alpha}(-x), \tag{2.2}$$

we immediately obtain the following Corollary of Theorem 2.1.

COROLLARY 2.4.

(i) If $P_{n-1}^{\alpha,\beta+t}$ and $P_{n+1}^{\alpha,\beta}$ are co-prime, then

- *(a) The zeros of $P_{n-1}^{\alpha,\beta+t}$ and $\frac{\beta^2-\alpha^2-t(\alpha-\beta+2n(n+\alpha+1))}{(2n+\alpha+\beta+t)(2n+\alpha+\beta+2)}$ interlace with the zeros of $P_{n+1}^{\alpha,\beta}$ for fixed $t \in \{1,2\}$;*
- *(b) The zeros of $P_{n-1}^{\alpha,\beta+3}$ and $-\frac{n(n+\alpha+\beta+2)+(\beta+2)(n-\beta+\alpha)}{(n+\beta+2)(n+\alpha+\beta+2)}$ interlace with the zeros of $P_{n+1}^{\alpha,\beta}$;*
- *(c) The zeros of $P_{n-1}^{\alpha,\beta+4}$ and $-\frac{2n(n+\alpha+\beta+3)+(\beta+3)(\alpha-\beta)}{2n(n+\alpha+\beta+3)+(\beta+3)(\alpha+\beta+2)}$ interlace with the zeros of $P_{n+1}^{\alpha,\beta}$.*

(ii) If $P_{n-1}^{\alpha,\beta+t}$ and $P_{n+1}^{\alpha,\beta}$ are not co-prime, they have one common zero located at the respective points identified in (i) (a) to (c) and the $n-1$ zeros of $P_{n-1}^{\alpha,\beta+t}$ interlace with the remaining n (non-common) zeros of $P_{n+1}^{\alpha,\beta}$.

Numerical experiments suggest that results analogous to those proved in Theorem 2.1 and its Corollary also hold as t varies continuously between 0 and 4.

CONJECTURE 2.5. *For $t \in (0,2)$, if $P_{n-1}^{\alpha+t,\beta}$ and $P_{n+1}^{\alpha,\beta}$ are co-prime, the zeros of $P_{n-1}^{\alpha+t,\beta}$ and $\frac{\beta^2-\alpha^2+t(\beta-\alpha+2n(n+\beta+1))}{(2n+\alpha+\beta+t)(2n+\alpha+\beta+2)}$ interlace with the zeros of $P_{n+1}^{\alpha,\beta}$.*

Our next two results prove that Stieltjes interlacing of the zeros of Jacobi polynomials from different sequences also holds when both the parameters α and β change within certain constraints.

THEOREM 2.6.

(i) For each fixed $j,k \in \{1,2\}$, if $P_{n-1}^{\alpha+j,\beta+k}$ and $P_{n+1}^{\alpha,\beta}$

- *(a) are co-prime, then the zeros of $P_{n-1}^{\alpha+j,\beta+k}$ and $\frac{\beta-\alpha-n(k-j)}{\alpha+\beta+2+n(4-j-k)}$ interlace with the zeros of $P_{n+1}^{\alpha,\beta}$;*
- *(b) are not co-prime, they have one common zero located at the point identified in (i) (a) and the $n-1$ zeros of $P_{n-1}^{\alpha+j,\beta+k}$ interlace with the n remaining (non-common) zeros of $P_{n+1}^{\alpha,\beta}$.*

(ii) If $P_{n-1}^{\alpha+3,\beta+1}$ and $P_{n+1}^{\alpha,\beta}$

- *(a) are co-prime, then the zeros of $P_{n-1}^{\alpha+3,\beta+1}$ and $\frac{n^2+n(\alpha+\beta+3)-(\alpha+2)(\alpha-\beta)}{n^2+n(\alpha+\beta+3)+(\alpha+2)(\alpha+\beta+2)}$ interlace with the zeros of $P_{n+1}^{\alpha,\beta}$;*
- *(b) are not co-prime, then they have one common zero located at the point identified in (ii) (a) and the $n-1$ zeros of $P_{n-1}^{\alpha+3,\beta+1}$ interlace with the n remaining (non-common) zeros of $P_{n+1}^{\alpha,\beta}$.*

(iii) If $P_{n-1}^{\alpha+1,\beta+3}$ and $P_{n+1}^{\alpha,\beta}$

- *(a) are co-prime, then the zeros of $P_{n-1}^{\alpha+1,\beta+3}$ and $\frac{-n^2-n(\alpha+\beta+3)-(\beta+2)(\alpha-\beta)}{n^2+n(\alpha+\beta+3)+(\beta+2)(\alpha+\beta+2)}$ interlace with the zeros of $P_{n+1}^{\alpha,\beta}$;*
- *(b) are not co-prime, then they have one common zero located at the point identified in (iii) (a) and the $n-1$ zeros of $P_{n-1}^{\alpha+1,\beta+3}$ interlace with the n remaining (non-common) zeros of $P_{n+1}^{\alpha,\beta}$.*

THEOREM 2.7.

(i) If the respective pairs of polynomials are co-prime, then

- *(a) the zeros of $P_{n-1}^{\alpha-1,\beta+1}$ and $\frac{\alpha+\beta}{2n+\alpha+\beta}$ interlace with the zeros of $P_{n+1}^{\alpha,\beta}$;*

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(b) the zeros of $P_{n-1}^{\alpha-1,\beta+2}$ *and* $\frac{-n+\beta+1}{n+\beta+1}$ *interlace with the zeros of* $P_{n+1}^{\alpha,\beta}$*;*
(c) the zeros of $P_{n-1}^{\alpha+1,\beta-1}$ *and* $\frac{-\alpha-\beta}{2n+\alpha+\beta}$ *interlace with the zeros of* $P_{n+1}^{\alpha,\beta}$*;*
(d) the zeros of $P_{n-1}^{\alpha+2,\beta-1}$ *and* $\frac{n-\alpha-1}{n+\alpha+1}$ *interlace with the zeros of* $P_{n+1}^{\alpha,\beta}$*.*

(ii) If the respective pairs of polynomials in (i) (a) to (d) are not co-prime, then they have one common zero located at the points identified in (i) (a) to (d) and the $n-1$ *zeros of the respective polynomial of degree* $n-1$ *in each case interlace with the* n *(non-common) zeros of* $P_{n+1}^{\alpha,\beta}$*.*

REMARK 2.8. Restrictions on the ranges of t and k are required in our theorems since, in general, Stieltjes interlacing is not retained between the zeros of Jacobi polynomials from different sequences, whose degrees differ by two.

Using Mathematica, we see that

When $n = 5$, $\alpha = 20.7$ and $\beta = 0.5$, the zeros of $P_6^{\alpha,\beta}$ and $P_4^{\alpha+5,\beta}$ or $P_4^{\alpha,\beta-1}$ do not interlace, illustrating that Stieltjes interlacing does not hold in general for $t > 4$, $k = 0$ or $t = 0$, $k < 0$.

When $t = k = -1$ and n, α and β are chosen as in the example above, the zeros of $P_4^{\alpha-1,\beta-1}$ and $P_6^{\alpha,\beta}$ do not interlace.

The zeros of $P_{n+1}^{\alpha,\beta}$ and those of $P_{n-1}^{\alpha+4,\beta+1}$ or $P_{n-1}^{\alpha+3,\beta+2}$ do not interlace when $n = 7$, $\alpha = -0.9$ and $\beta = 329.3$.

We now state a general result for Stieltjes interlacing between the zeros of $P_{n+1}^{\alpha,\beta}$ and the $n-k$ zeros of the kth derivative of $P_n^{\alpha,\beta}$.

THEOREM 2.9. *Let* $P_n^{\alpha,\beta}$*,* $\alpha, \beta > -1$*,* $n \in \mathbb{N}$*, denote the Jacobi polynomial of degree* n*.*

(i) For each $k \in \{1, 2, \ldots, n-1\}$*, there exist polynomials* G_k *and* H_k *of degree* k *such that*

$$(1-x^2)^k Q_{n,k} P_{n-k}^{\alpha+k,\beta+k}(x) = (n+1)H_{k-1}(x)P_{n+1}^{\alpha,\beta}(x) + G_k(x)P_n^{\alpha,\beta}(x), \tag{2.3}$$

where $Q_{n,k} = \frac{(n+\alpha+\beta+2)_{k-1}(2n+\alpha+\beta+2)}{2^{2k}}$ *and* $(\ )_k$ *denotes the Pochhammer symbol [10, p. 8, Equation (1.3.6)].*

(ii) Let $k \in \{1, 2, \ldots, n-1\}$*,* k *fixed. If* $P_{n+1}^{\alpha,\beta}$ *and* $P_{n-k}^{\alpha+k,\beta+k}$ *are co-prime, then the zeros of the* k*th derivative of* $P_n^{\alpha,\beta}$*, together with the* k *real zeros of* G_k*, interlace with the zeros of* $P_{n+1}^{\alpha,\beta}$*.*

(iii) Let $k \in \{1, 2, \ldots, n-1\}$*,* k *fixed. If* $P_{n+1}^{\alpha,\beta}$ *and* $P_{n-k}^{\alpha+k,\beta+k}$ *have* r *common zeros, then the* $(n-2r)$ *non-common zeros of the product* $G_k P_{n-k}^{\alpha+k,\beta+k}$*, together with the* r *common zeros of* $P_{n+1}^{\alpha,\beta}$ *and* $P_{n-k}^{\alpha+k,\beta+k}$*, interlace with the* $(n+1-r)$ *non-common zeros of* $P_{n+1}^{\alpha,\beta}$*.*

3. Proofs. Jacobi polynomials are linked with the ${}_2F_1$ Gauss hypergeometric polynomials via the following identity [1, p. 99]

$$P_n^{\alpha,\beta}(x) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right). \tag{3.1}$$

In our proofs, we make use of this connection between Jacobi and ${}_2F_1$ hypergeometric polynomials, as well as the following contiguous function relations satisfied by ${}_2F_1$ polynomials.

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LEMMA 3.1. *Let* $F_n = {}_2F_1(-n, b; c; z)$ *and denote* ${}_2F_1(-n-1, b+1; c; z)$ *by* $F_{n+1}(b+)$, ${}_2F_1(-n+1, b+1; c-3; z)$ *by* $F_{n-1}(b+, c-3)$ *and so on. Then,*

$$\left(\frac{b(c+n)}{(b+n)(b+n+1)} - z\right) F_n = \frac{b(c+n)}{(b+n)(b+n+1)} F_{n+1}(b+) + \frac{n(b-c)z}{c(b+n)} F_{n-1}(c+) \tag{3.2}$$

$$\left(\frac{c}{b+n+1} - z\right) F_n = \frac{c}{(b+n+1)} F_{n+1}(b+) + \frac{(b-c)nz^2}{c(c+1)} F_{n-1}(b+, c+2) \tag{3.3}$$

$$\left(\frac{c+n}{b+1} - z\right) \frac{1+b-c}{b+n-1} F_n = \frac{(1+b-c-nz)(c+n)}{(b+1)(b+n-1)} F_{n+1}(b+) + \frac{nz(1-z)^2}{c} F_{n-1}(b+2, c+) \tag{3.4}$$

$$\left(\frac{c+n}{b+n+1} - z\right) F_n = \frac{c+n}{b+n+1} F_{n+1}(b+) - \frac{z(z-1)n}{c} F_{n-1}(b+, c+) \tag{3.5}$$

$$\begin{aligned}\left(\frac{c(c+1)}{(b+1)(c+n+1)} - z\right) F_n &= \frac{c+c^2-bnz+cnz}{(b+1)(c+n+1)} F_{n+1}(b+) \\ &\quad + \frac{n(b-c)(b+n+1)z^3}{c(c+1)(c+2)} F_{n-1}(b+2, c+3)\end{aligned} \tag{3.6}$$

$$\frac{(b-c+1)}{(b+1)(b+n+1)} F_n = \frac{b-c+1-z(b+n+1)}{(b+1)(b+n+1)} F_{n+1}(b+) - \frac{z^2-z}{c} F_n(b+2, c+) \tag{3.7}$$

$$\begin{aligned}(c - z(b+1-n)) F_n &= \left(c + 2nz - nz^2 \frac{1+b+n}{b+1-c}\right) F_{n+1}(b+) \\ &\quad + \frac{n(b+1)(b+2)((z-1)z)^2(b+1+n)}{(b+1-c)c(c+1)} F_{n-1}(b+3, c+2)\end{aligned} \tag{3.8}$$

$$\begin{aligned}\left(z - \frac{c(c+1)}{(1+c+n)(1+b)-cn}\right) F_n &= -\frac{c+c^2-bnz+2cnz+nz^2+bnz^2+n^2z^2}{1+c-cn+n+b+bc+bn} F_{n+1}(b+) \\ &\quad + \frac{(b+1)(b+2)(1+b+n)(1+c+n)n(z-1)z^3}{c(c+1)(c+2)(1+c-cn+n+b+bc+bn)} F_{n-1}(b+3, c+3)\end{aligned} \tag{3.9}$$

$$\begin{aligned}\left(1 - \frac{(b+1)(2+c+2n)-cn}{c(c+2)} z\right) F_n &= \left(1 - \frac{2(b-c)n}{c(c+2)} z - \frac{n(b-c)(1+b+n)}{c(c+1)(c+2)} z^2\right) F_{n+1}(b+) \\ &\quad + \frac{a}{c^2(c+1)^2(c+2)^2(c+3)} F_{n-1}(b+3, c+4)\end{aligned} \tag{3.10}$$

where $a = (b+1)(b+2)(b-c)(c+n+1)(c+n+2)(1+b+n)z^4 n$.

Proof. For each $j = 1, 2, \ldots, n$, the coefficient of z^j on the left-hand side of (3.2) is

$$\begin{aligned}&\frac{2b(c+n)(-n)_j(b)_j}{(b+n)(b+n+1)(c)_j(j)!} - \frac{2(-n)_{j-1}(b)_{j-1}}{(c)_{j-1}(j-1)!} \\ &= \frac{2(-n)_{j-1}(b)_{j-1}}{(b+n)(b+n+1)(c)_j j!} \left(b(c+n)(-n+j-1)(b+j-1) - j(c+j-1)(b+n)(b+n+1)\right)\end{aligned}$$

while the coefficient of z^j on the right-hand side of (3.2) is given by

$$\begin{aligned}&\frac{2b(c+n)(-n-1)_j(b+1)_j}{(b+n)(b+n+1)(c)_j j!} + \frac{2n(b-c)}{c(b+n)} \frac{(-n+1)_{j-1}(b)_{j-1}}{(c+1)_{j-1}(j-1)!} \\ &= \frac{2(-n)_{j-1}(b)_{j-1}}{(b+n)(b+n+1)(c)_j j!} \left((c+n)(-n-1)(b+j-1)(b+j) - j(b-c)(-n+j+1)(b+n+1)\right).\end{aligned}$$

A straightforward calculation shows that these coefficients are equal and the result follows. The other identities can be proved in the same way by comparing coefficients. ☐

REMARK 3.2. The identities in Lemma 3.1 follow from the contiguous relations for ${}_2F_1$ hypergeometric polynomials [11, p. 71]. A useful algorithm in this regard is available as a computer package [14].

The following Lemma simplifies the proofs of Theorem 2.1 and Theorem 2.6.

LEMMA 3.3. *Let* $\{p_n\}_{n=0}^{\infty}$ *be a sequence of polynomials orthogonal on the (finite or infinite) interval* (c, d). *Let* g_{n-1} *be any polynomial of degree* $n-1$ *that for each* $n \in \mathbb{N}$ *satisfies*

$$g_{n-1}(x) = a_n(x)p_{n+1}(x) - (x - A_n)b_n(x)p_n(x) \tag{3.11}$$

for some constant A_n and some functions $a_n(x)$ and $b_n(x)$, with $b_n(x) \neq 0$ for $x \in (c,d)$. Then, for each $n \in \mathbb{N}$,

(i) the zeros of g_{n-1} are all real and simple and, together with the point A_n, they interlace with the zeros of p_{n+1} if g_{n-1} and p_{n+1} are co-prime;

(ii) if g_{n-1} and p_{n+1} are not co-prime, they have one common zero located at $x = A_n$ and the $n-1$ zeros of g_{n-1} interlace with the n (non-common) zeros of p_{n+1}.

Proof. Let $w_1 < w_2 < \cdots < w_{n+1}$ denote the zeros of p_{n+1}.

(i) Since p_n and p_{n+1} are always co-prime, and by assumption $b_n(x) \neq 0$ for $x \in (c,d)$ and p_{n+1} and g_{n-1} are co-prime, we deduce from (3.11) that $A_n \neq w_k$ for any $k \in \{1, 2, \ldots, n+1\}$. Evaluating (3.11) at w_k and w_{k+1}, we obtain

$$\frac{g_{n-1}(w_k)g_{n-1}(w_{k+1})}{p_n(w_k)p_n(w_{k+1})} = (w_k - A_n)(w_{k+1} - A_n)b_n(w_k)b_n(w_{k+1}), \tag{3.12}$$

for each $k \in \{1, 2, \ldots, n\}$. Since w_k and $w_{k+1} \in (c,d)$ while b_n does not change sign in (c,d), we know that $b_n(w_k)b_n(w_{k+1}) > 0$. Hence, the right-hand side of (3.12) is positive if and only if $A_n \notin (w_k, w_{k+1})$. Since $p_n(w_k)p_n(w_{k+1}) < 0$ for each $k \in \{1, 2, \ldots, n\}$ because the zeros of p_n and p_{n+1} are interlacing, we deduce that, provided $A_n \notin (w_k, w_{k+1})$, g_{n-1} has a different sign at consecutive zeros of p_{n+1} and therefore has an odd number of zeros (counting multiplicity) in each interval (w_k, w_{k+1}), $k \in \{1, 2, \ldots, n\}$, apart from one interval that may contain the point A_n. It follows from the Intermediate Value Theorem that for each $n \in \mathbb{N}$ the $n-1$ real simple zeros of g_{n-1}, together with the point A_n, interlace with the $n+1$ zeros of p_{n+1}.

(ii) If p_{n+1} and g_{n-1} have common zeros, it follows from (3.11) that there can only be one common zero at $x = A_n$ since p_n and p_{n+1} are co-prime. For $x \neq A_n$ we can rewrite (3.11) as

$$G_{n-2}(x) = a_n(x)P_n(x) - b_n(x)p_n(x), \tag{3.13}$$

where $(x - A_n)G_{n-2}(x) = g_{n-1}(x)$ and $(x - A_n)P_n(x) = p_{n+1}(x)$. Note that the zeros of P_n are exactly the n (non-common) zeros of p_{n+1}, say $v_1 < \cdots < v_n$, and at most one interval of the form (v_k, v_{k+1}), $k \in \{1, \ldots, n-1\}$, can contain the point A_n. Evaluating (3.13) at v_k and v_{k+1}, for each $k \in \{1, \ldots, n-1\}$ such that $A_n \notin (v_k, v_{k+1})$, we obtain

$$G_{n-2}(v_k)G_{n-2}(v_{k+1}) = b_n(v_k)b_n(v_{k+1})p_n(v_k)p_n(v_{k+1}) < 0,$$

and it follows that G_{n-2} has an odd number of zeros in each interval (v_k, v_{k+1}), $k \in \{1, 2, \ldots, n\}$, that does not contain A_n. Since there are at least $n-2$ of these intervals and $\deg(G_{n-2}) = n-2$, there are at most $n-2$ such intervals and we deduce that $A_n = w_j$ where $j \in \{2, \ldots, n\}$ and the zeros of G_{n-2}, together with the point A_n, interlace with the n zeros of P_n. The stated result is then an immediate consequence of the definitions of G_{n-2} and P_n. □

Proof of Theorem 2.1.

(i) (a) If $t = 0$, the result follows from (2.1) and Lemma 3.3 (i). For $t = 1$, we use (3.2) with $b = \alpha + \beta + n + 1$ and $c = \alpha + 1$, together with (3.1), and then apply Lemma 3.3 (i). For $t = 2$, the stated result follows from (3.3) and (3.1) together with Lemma 3.3 (i).

(b) Replacing b by $n + \alpha + \beta + 1$, c by $\alpha + 1$ and z by $\frac{1-x}{2}$ in (3.6) and using

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(3.1), we obtain

$$\left(x - \frac{n^2+(2\alpha+\beta+4)-(\alpha+2)(\alpha-\beta)}{(n+\alpha+2)(n+\alpha+\beta+2)}\right) P_n^{\alpha,\beta}(x)$$
$$= \frac{(n+1)A(x)}{(n+\alpha+1)(n+\alpha+2)(n+\alpha+\beta+2)} P_{n+1}^{\alpha,\beta}(x) + \frac{(1-x)^3(2n+\alpha+\beta+2)(n+\beta)}{4(n+\alpha+1)(n+\alpha+2)} P_{n-1}^{\alpha+3,\beta}(x),$$

where $A(x) = n(n+\beta)(x-1) + 2(\alpha+1)(\alpha+2)$. Lemma 3.3 (i) then yields the result.

(c) From (3.10) and (3.1) we have

$$\left(x - \frac{2n^2-(\alpha+3)(\alpha-\beta)+2n(\alpha+\beta+3)}{C_n}\right) P_n^{\alpha,\beta}(x)$$
$$= \frac{-(n+1)B(x)}{2(n+\alpha+1)(\alpha+2)C_n} P_{n+1}^{\alpha,\beta}(x) + \frac{(1-x)^4 D_n}{8(n+\alpha+1)(\alpha+2)C_n} P_{n-1}^{\alpha+4,\beta}(x),$$

where

$$C_n = 2n(n+\alpha+\beta+3) + (\alpha+3)(\alpha+\beta+2),$$
$$D_n = (2n+\alpha+\beta+2)(n+\beta)(n+\alpha+\beta+2)(n+\alpha+\beta+3),$$

and $B(x)$ is a polynomial of degree 2 in x which depends on n, α, and β.

The result follows from Lemma 3.3 (i).

(ii) This follows immediately from Lemma 3.3 (ii) and the proofs of Theorem 2.1 (i) (a) to (c). ☐

Proof of Theorem 2.6.

(i) (a) The case when $j = k = 1$ will be proved in Theorem 2.9. For $j = k = 2$, (3.8) and (3.1) yield

$$\left(x - \frac{\beta-\alpha}{\alpha+\beta+2}\right) P_n^{\alpha,\beta}(x)$$
$$= \frac{2(n+1)C(x)}{(n+\alpha+1)(n+\beta+1)(\alpha+\beta+2)} P_{n+1}^{\alpha,\beta}(x) + E_n(1-x^2)^2 P_{n-1}^{\alpha+2,\beta+2}(x),$$

where

$$E_n = \frac{(n+\alpha+\beta+2)(n+\alpha+\beta+3)(2n+\alpha+\beta+2)}{8(n+\alpha+1)(n+\beta+1)(\alpha+\beta+2)}$$

and $C(x)$ is a polynomial of degree 2 in x which depends on n, α and β. The result follows from Lemma 3.3 (i).

For $j = 1, k = 2$, the mixed recurrence relation

$$\left(x + \frac{n+\alpha-\beta}{n+\alpha+\beta+2}\right) P_n^{\alpha,\beta}(x)$$
$$= \frac{(n(x+1)+2\beta+2)(n+1)}{(n+\alpha+\beta+2)(n+\beta+1)} P_{n+1}^{\alpha,\beta}(x) - \frac{(x+1)^2(x-1)(2n+\alpha+\beta+2)}{4(n+\beta+1)} P_{n-1}^{\alpha+1,\beta+2}(x)$$

is obtained from (3.1) together with (3.4). Lemma 3.3 (i) then yields the stated result.

For $j = 2, k = 1$, the result follows from the symmetry property (2.2).

(b) From (3.1) and (3.9), we obtain the mixed recurrence relation

$$\left(x - \frac{n^2-(\alpha+2)(\alpha-\beta)+n(\alpha+\beta+3)}{n^2+n(\alpha+\beta+3)+(\alpha+2)(\alpha+\beta+2)}\right) P_n^{\alpha,\beta}(x)$$
$$= \frac{4(\alpha+1)(\alpha+2)+(3\alpha-\beta+4)n-2nx(n+2\alpha+3)+nx^2(2n+\alpha+\beta+2)}{2(n+\alpha+1)(n^2+(\alpha+2)(\alpha+\beta+2)+n(\alpha+\beta+3))}(n+1)P_{n+1}^{\alpha,\beta}(x)$$
$$+ \frac{n(1-x)^3(1+x)(n+\alpha+\beta+2)(n+\alpha+\beta+3)(2n+\alpha+\beta+2)}{8n(n^2+(\alpha+2)(\alpha+\beta+2)+n(\alpha+\beta+3))(n+\alpha+1)} P_{n-1}^{\alpha+3,\beta+1}(x),$$

and Lemma 3.3 (i) then yields the stated result.

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(c) This follows directly from the symmetry property (2.2).

(ii) This follows from Lemma 3.3 (ii) and the proofs of Theorem 2.6 (i) (a) to (c). □

We omit the proof of Theorem 2.7 which follows exactly the same reasoning as the proofs of Theorems 2.1 and 2.6.

Proof of Theorem 2.9.

(i) We use the mixed recurrence relations

$$\left(1-x^2\right)P_{n-1}^{\alpha+1,\beta+1}(x)=2\left(x+\tfrac{\alpha-\beta}{2n+\alpha+\beta+2}\right)P_n^{\alpha,\beta}(x)-\tfrac{4(n+1)}{2n+\alpha+\beta+2}P_{n+1}^{\alpha,\beta}(x) \tag{3.14}$$

and

$$\left(1-x^2\right)P_n^{\alpha+1,\beta+1}(x)= \tfrac{2}{n+\alpha+\beta+2}\left(\tfrac{2(n+\beta+1)(n+\alpha+1)}{2n+\alpha+\beta+2}P_n^{\alpha,\beta}(x)-(n+1)\left(x-\tfrac{\alpha-\beta}{2n+\alpha+\beta+2}\right)P_{n+1}^{\alpha,\beta}(x)\right), \tag{3.15}$$

which can be obtained from (3.1), (3.5), and (3.7). We prove our result by induction on k.

For $k=1$, equation (2.3) is the same as equation (3.14) with $H_0(x)=-1$, $G_1(x)=\frac{1}{2}\left((2n+\alpha+\beta+2)x+\alpha-\beta\right)$ and $Q_{n,1}=\frac{1}{4}(2n+\alpha+\beta+2)$. Therefore, (2.3) holds for $k=1$.

Next, we assume that the result holds for $m=1,2,\dots,k$, i.e we assume that

$$(1-x^2)^m Q_{n,m}P_{n-m}^{\alpha+m,\beta+m}(x)=(n+1)H_{m-1}(x)P_{n+1}^{\alpha,\beta}(x)+G_m(x)P_n^{\alpha,\beta}(x), \tag{3.16}$$

with G_m and H_m polynomials of degree m and $Q_{n,m}=\frac{(n+\alpha+\beta+2)_{m-1}(2n+\alpha+\beta+2)}{2^{2m}}$ for $m=1,2,\dots,k$.

For $m=k+1$, the left-hand side of (2.3) is equal to

$$(1-x^2)^{k+1}Q_{n,k+1}P_{n-k-1}^{\alpha+k+1,\beta+k+1}(x),$$

and, applying (3.14) and (3.15), a straightforward calculation shows that this equals

$$G_{k+1}(x)P_n^{\alpha,\beta}(x)+(n+1)H_k(x)P_{n+1}^{\alpha,\beta}(x)$$

with

$$H_k(x)=\tfrac{-n}{2}\left(x-\tfrac{\alpha-\beta}{2n+\alpha+\beta+2}\right)H_{k-1}(x)-\tfrac{n+\alpha+\beta+2}{2n+\alpha+\beta+2}G_k(x)$$

and

$$G_{k+1}(x)=\tfrac{n(n+\alpha+1)(n+\beta+1)}{2n+\alpha+\beta+2}H_{k-1}(x)+\tfrac{n+\alpha+\beta+2}{2}\left(x+\tfrac{\alpha-\beta}{2n+\alpha+\beta+2}\right)G_k(x),$$

which is the right-hand side of (2.3) for $m=k+1$. It follows that (3.16) holds for $m=k+1$, and the result follows by induction on k.

(ii) We note that $D^k[P_n^{\alpha,\beta}]=\frac{1}{2^k}(n+\alpha+\beta+1)_k P_{n-k}^{\alpha+k,\beta+k}$, where D^k denotes the k-th derivative [13, p. 63]. From (2.3), provided $P_{n+1}^{\alpha,\beta}(x)\neq 0$, we have

$$\frac{(1-x^2)^k Q_{n,k}P_{n-k}^{\alpha+k,\beta+k}(x)}{P_{n+1}^{\alpha,\beta}(x)}=(n+1)H_{k-1}(x)+\frac{G_k(x)P_n^{\alpha,\beta}(x)}{P_{n+1}^{\alpha,\beta}(x)}. \tag{3.17}$$

Now, if $w_1 < w_2 < \cdots < w_{n+1}$ are the zeros of $P_{n+1}^{\alpha,\beta}$, we have

$$\frac{P_n^{\alpha,\beta}(x)}{P_{n+1}^{\alpha,\beta}(x)} = \sum_{j=1}^{n+1} \frac{A_j}{x - w_j}$$

where $A_j > 0$ for each $j \in \{1, \dots, n+1\}$ [13, Theorem 3.3.5]. Therefore (3.17) can be written as

(3.18)
$$\frac{(1-x^2)^k Q_{n,k} P_{n-k}^{\alpha+k,\beta+k}(x)}{P_{n+1}^{\alpha,\beta}(x)} = (n+1)H_{k-1}(x) + \sum_{j=1}^{n+1} \frac{G_k(x)A_j}{x - w_j}, \quad x \neq w_j.$$

Since $P_{n+1}^{\alpha,\beta}$ and $P_n^{\alpha,\beta}$ are always co-prime while $P_{n+1}^{\alpha,\beta}$ and $P_{n-k}^{\alpha+k,\beta+k}$ are co-prime by assumption, it follows from (2.3) that $G_k(w_j) \neq 0$ for any $j \in \{1, 2, \dots, n+1\}$. Suppose that G_k does not change sign in $I_j = (w_j, w_{j+1})$ where $j \in \{1, 2, \dots, n\}$. Since $A_j > 0$ and the polynomial H_{k-1} is bounded on I_j while the right hand side of (3.18) takes arbitrarily large positive and negative values, it follows that $P_{n-k}^{\alpha+k,\beta+k}$ must have an odd number of zeros in each interval in which G_k does not change sign. Since G_k is of degree k, there are at least $n-k$ intervals (w_j, w_{j+1}), $j \in \{1, \dots, n\}$ in which G_k does not change sign, and so each of these intervals must contain exactly one of the $n-k$ real, simple zeros of $P_{n-k}^{\alpha+k,\beta+k}$. We deduce that the k zeros of G_k are real and simple and, together with the zeros of $P_{n-k}^{\alpha+k,\beta+k}$, interlace with the $n+1$ zeros of $P_{n+1}^{\alpha,\beta}$.

(iii) Assume that $P_{n+1}^{\alpha,\beta}$ and $P_{n-k}^{\alpha+k,\beta+k}$ have r common zeros. From (2.3), it follows that if $P_{n-k}^{\alpha+k,\beta+k}$ and $P_{n+1}^{\alpha,\beta}$ have any common zeros, these must also be zeros of G_k since $P_n^{\alpha,\beta}$ and $P_{n+1}^{\alpha,\beta}$ are co-prime. It follows that $r \leq \min\{k, n-k\}$ and there are at least $(n-2r)$ open intervals $I_j = (w_j, w_{j+1})$ with endpoints at successive zeros w_j and w_{j+1} of $P_{n+1}^{\alpha,\beta}$ where neither w_j or w_{j+1} is a zero of $P_{n-k}^{\alpha+k,\beta+k}$ or $G_k(x)$. If G_k does not change sign in an interval $I_j = (w_j, w_{j+1})$, it follows from (3.18), since $A_j > 0$ and H_{k-1} is bounded while the right hand side takes arbitrarily large positive and negative values for $x \in I_j$, that $P_{n-k}^{\alpha+k,\beta+k}$ must have an odd number of zeros in that interval. Since this applies to at least $(n-2r)$ intervals I_j and $P_{n-k}^{\alpha+k,\beta+k}$ has exactly $(n-k-r)$ simple zeros that are not zeros of $P_{n+1}^{\alpha,\beta}$ while G_k has at most $(k-r)$ zeros that are not zeros of $P_{n+1}^{\alpha,\beta}$, it follows that there must be exactly $(n-2r)$ intervals $I_j = (w_j, w_{j+1})$ with endpoints at successive zeros w_j and w_{j+1} of $P_{n+1}^{\alpha,\beta}$ where neither w_j or w_{j+1} is a zero of $P_{n-k}^{\alpha+k,\beta+k}$. This implies that the common zeros of $P_{n+1}^{\alpha,\beta}$ and $P_{n-k}^{\alpha+k,\beta+k}$ cannot be two consecutive zeros of $P_{n+1}^{\alpha,\beta}$, and the stated result now follows using the same argument as in (ii). □

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