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A POSTERIORI ERROR ESTIMATION FOR THE LEGENDRE COLLOCATION METHOD APPLIED TO INTEGRAL-ALGEBRAIC VOLTERRA EQUATIONS*

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Abstract. In this work, we analyze the Legendre collocation method for a mixed system of Volterra integral equations of the first and second kind which is known as Integral Algebraic Equations (IAEs). In order to obtain the approximate solution, the kernels in the system of integral equations are approximated by using the discrete Legendre expansion. A posteriori error estimate is obtained which is based on the Lebesgue constants corresponding to the Lagrange interpolation polynomials and some well-known results of orthogonal polynomials theory. The spectral rate of convergence for the described method applied to linear and nonlinear IAEs is also established in the L^2-norm. Finally, the proposed method is illustrated by several test problems which confirm the theoretical prediction of the error estimation.

Key words. integral algebraic equations, system of Volterra integral equations, Legendre collocation method, error analysis, numerical teratments

AMS subject classifications. 65R20, 34A05

1. Introduction.

Consider the following system of integral equation

$$\left\{\begin{aligned} y(t) &= f_1(t) + \int_0^t K_{11}(t,s)y(s)ds + \int_0^t K_{12}(t,s)z(s)ds, \\ 0 &= f_2(t) + \int_0^t K_{21}(t,s)y(s)ds + \int_0^t K_{22}(t,s)z(s)ds, \quad t \in I = [0,T], \end{aligned}\right. \tag{1.1}$$

where $K_{ij}(.,.)$, $i,j = 1,2$, are $d_i \times d_j$ matrices, f_1 and f_2 are given d_1, d_2 dimensional vector functions, respectively and (y,z) is a solution to be determined. Here, we assume that the data functions f_i and K_{ij}, $i,j = 1,2$, are sufficiently smooth such that $f_2(0) = 0$ and $|detK_{22}(t,t)| \geq k_0 > 0$ for all $t \in I$. The existence and uniqueness results for the solution of the system (1.1) have been discussed in [1]. However, the solvability and regularity of the solution of (1.1) may be utilized if we differentiate the second equation and consider the resulting equation as an equation of the second kind for z. Then, we formally solve for z and replace the resulting expressions in the first equation of the system, and we obtain an equation of the second kind for y. We emphasize that this reduction to an integral equation of the second kind is not practical from a numerical point of view.

The system (1.1) is a particular case of the general form of the Integral Algebraic Equations (IAEs)

$$A(t)X(t) = G(t) + \int_0^t K(t,s,X(s))ds,$$

which has been introduced in [1], where $detA(t) = 0$ and Rank $A(t) \geq 1$ on I. An initial investigation of these equations indicates that they have properties very similar to Differential Algebraic Equations (DAEs). In analogy with the theory of DAEs (see, e.g., [13]), Kauthen [16] in 2000 has called the system (1.1) the semi-explicit IAEs of index 1.

An important remark concerning the analysis of index-1 IAEs with respect to IAEs of higher index (index larger than one) is their wide range of applications in the mathematical

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modelling of problems in engineering and sciences, e.g., the controlled heat equation which represents a boundary reaction in diffusion of chemicals [12], the two dimensional biharmonic equation in a semi-infinite strip [5, 11], dynamic processes in chemical reactors [15] and deformed Pohlmeyer equation [21]. Also a good source of information (including numerous additional references) on applications of IAEs system is the monograph by Brunner [1].

As far as we know, there are some papers which have considered the theory of IAEs system. The existence and uniqueness results of continuous solution to linear IAEs system have been investigated by Chistyakov [8]. Gear [9] defined the index notion of IAEs system by considering the effect of perturbation of the equations on the solutions. He has also introduced the "index reduction procedure" for IAEs system in [9] similar to that in [10] for DAEs in which if the reduction process is terminated, then the index is determined. This means that under suitable conditions, there is a solution for the resulting regular system of integral equations. Bulatov [2] in 1997, gave the existence and uniqueness conditions of the solution for IAEs systems with convolution kernels and defined the index notion in analogy to Gear's approach. Further details of his investigation may be found in [3, 4]. A few numerical based works, e.g., the spline collocation method and its convergence properties [16], are also available in the literature for the semi-explicit IAEs system (1.1). Brunner [1] defined the index-1 tractable for the IAEs system (1.1) which is analogous to that defined for DAEs by März [17] and he also investigated the existence of a unique solution for this type of systems. Recently, the authors in [12] have defined the index-2 tractable for a class of IAEs and presented the Jacobi collocation method including the matrix-vector multiplication representation of the equation and its convergence analysis.

On the other hand, the classical Legendre polynomials have been used extensively in mathematical analysis and practical applications, and play an important role in the analysis and implementation of the spectral methods. Here, the numerical solvability of the index-1 semi-explicit IAEs (1.1) using the Legendre collocation scheme is investigated. We will provide a posteriori error estimate in the L^2-norm which theoretically justifies the spectral rate of convergence. To do so, we use some well-known results of the approximation theory from [6, 14, 18] relevant to the Legendre polynomials, Gronwall inequality and the Lebesgue constant regarding the Lagrange interpolation polynomials .

This paper is organized as follows. In Section 2, we carry out the Legendre collocation approach for the IAEs system (1.1). A posteriori error estimation of the method in the L^2-norm as a main result of the paper is given in Section 3. In Section 4, we generalize our results to the semi-explicit IAEs of index-1 in the Volterra-Hammerstein form and finally, some numerical experiments are reported in Section 5 to verify the theoretical results obtained in the previous sections.

2. The Legendre collocation method. We turn our attention to the formulation of the polynomial spectral method for solving the IAEs system (1.1), using the collocation approach. We first use some variable transformations to change the equation into a new system of integral equations defined on the standard interval [-1,1], so that the Legendre orthogonal polynomial theory can be applied conveniently. In order to describe the key ideas without having to resort to complex notations involving Kronecker products of matrices and vectors, we will consider the IAEs system (1.1) with $d_1 = d_2 = 1$.

For the sake of applying the theory of orthogonal polynomials, we use the change of variables:

$$\eta = \frac{2}{T}s - 1, \quad -1 \leq \eta \leq \tau, \qquad\qquad \tau = \frac{2}{T}t - 1, \quad -1 \leq \tau \leq 1, \tag{2.1}$$

to rewrite the IAEs system (1.1) as:

$$\left\{\begin{aligned} \widehat{y}(\tau) &= \widehat{f}_1(\tau) + \int_{-1}^{\tau} \widehat{K}_{11}(\tau,\eta)\widehat{y}(\eta)d\eta + \int_{-1}^{\tau} \widehat{K}_{12}(\tau,\eta)\widehat{z}(\eta)d\eta, \\ 0 &= \widehat{f}_2(\tau) + \int_{-1}^{\tau} \widehat{K}_{21}(\tau,\eta)\widehat{y}(\eta)d\eta + \int_{-1}^{\tau} \widehat{K}_{22}(\tau,\eta)\widehat{z}(\eta)d\eta, \end{aligned}\right. \tag{2.2}$$

where $\widehat{f}_i(\tau) = f_i(\frac{T}{2}(\tau+1))$, $\widehat{K}_{ij}(\tau,\eta) = \frac{T}{2}K_{ij}(\frac{T}{2}(\tau+1), \frac{T}{2}(\eta+1))$, $i,j = 1,2$, $\widehat{y}(\tau) = y(\frac{T}{2}(\tau+1))$ and $\widehat{z}(\tau) = z(\frac{T}{2}(\tau+1))$.

We consider the discrete expansion of $\widehat{K}_{ij}(\tau,\eta)$ as follows:

$$\mathcal{P}_N(\widehat{K}_{ij}(\tau_n,\eta)) = \sum_{k=0}^{N} (\widehat{K}_{ij})_k P_k(\eta), \quad (i,j = 1,2) \tag{2.3}$$

where $\mathcal{P}_N$ is a projection to the finite dimensional space $\mathcal{B}_N = span\{P_n(x)\}_{n=0}^N$ and P_n is the Legendre polynomial such that

$$(\widehat{K}_{ij})_k = \frac{1}{\gamma_k}\sum_{l=0}^{N} w_l \widehat{K}_{ij}(\tau_n,\tau_l)P_k(\tau_l). \tag{2.4}$$

Moreover, the quadrature points τ_l are the Legendre Gauss quadrature points, i.e., the zeros of P_{N+1}, where the normalization constant γ_k and the weights w_l are given by

$$\begin{aligned} \gamma_k &= \frac{2}{2k+1}\,, \qquad k = 0,\dots,N, \\ w_l &= \frac{2}{(1-\tau_l^2)(P'_{N+1}(\tau_l))^2}\,, \qquad l = 0,\dots,N. \end{aligned}$$

In the Legendre collocation method, we seek a solution of the form

$$\mathcal{P}_N(\widehat{y}(\eta)) = \widehat{y}_N(\eta) = \sum_{k=0}^{N} \widehat{y}_k P_k(\eta), \qquad \mathcal{P}_N(\widehat{z}(\eta)) = \widehat{z}_N(\eta) = \sum_{k=0}^{N} \widehat{z}_k P_k(\eta). \tag{2.5}$$

Inserting the discrete expansion (2.3) and (2.5) into (2.2), we obtain

$$\left\{\begin{aligned} \widehat{y}_N(\tau) &= \widehat{f}_1(\tau) + \sum_{k=0}^{N}\sum_{l=k}^{N}(c_{kl}+c'_{kl})V_{kl}, \\ 0 &= \widehat{f}_2(\tau) + \sum_{k=0}^{N}\sum_{l=k}^{N}(q_{kl}+q'_{kl})V_{kl}, \end{aligned}\right. \tag{2.6}$$

where $V_{kl}(\tau) = \int_{-1}^{\tau} P_k(\eta)P_l(\eta)d\eta$, and

$$c_{kl} = \begin{cases} \widehat{y}_k(\widehat{K}_{11})_k & k = l, \\ \widehat{y}_k(\widehat{K}_{11})_l + \widehat{y}_l(\widehat{K}_{11})_k & k \neq l, \end{cases} \quad c'_{kl} = \begin{cases} \widehat{z}_k(\widehat{K}_{12})_k & k = l, \\ \widehat{z}_k(\widehat{K}_{12})_l + \widehat{z}_l(\widehat{K}_{12})_k & k \neq l, \end{cases}$$

$$q_{kl} = \begin{cases} \widehat{y}_k(\widehat{K}_{21})_k & k = l, \\ \widehat{y}_k(\widehat{K}_{21})_l + \widehat{y}_l(\widehat{K}_{21})_k & k \neq l, \end{cases} \quad q'_{kl} = \begin{cases} \widehat{z}_k(\widehat{K}_{22})_k & k = l, \\ \widehat{z}_k(\widehat{K}_{22})_l + \widehat{z}_l(\widehat{K}_{22})_k & k \neq l. \end{cases}$$

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The unknown coefficients $\widehat{y}_k$ and $\widehat{z}_k$, $k = 0, \cdots, N$, are defined by the solution of the following system of equations which is obtained by substituting the collocation points τ_n in the system (2.6) and employing the discrete representation (2.5):

$$\left\{ \begin{aligned} \sum_{k=0}^{N} \widehat{y}_k P_k(\tau_n) &= \widehat{f}_1(\tau_n) + \sum_{k=0}^{N} \sum_{l=k}^{N} (c_{kl} + c'_{kl}) V_{kl}(\tau_n), \\ 0 &= \widehat{f}_2(\tau_n) + \sum_{k=0}^{N} \sum_{l=k}^{N} (q_{kl} + q'_{kl}) V_{kl}(\tau_n), \end{aligned} \right. \tag{2.7}$$

for $n = 0, 1, \cdots, N$.

Now, the coefficients $\widehat{y}_k$ and $\widehat{z}_k$ are obtained by solving the linear system (2.7) and finally the approximate solutions $\widehat{y}_N(\eta)$ and $\widehat{z}_N(\eta)$ will be computed by substituting these coefficients into (2.5).

3. Error estimation. In this section, we present a posteriori error estimate for the proposed scheme in the L^2-norm. At first, we recall some preliminaries and useful Lemmas from [6] and [18].

Following [6], the inverse inequality concerning differentiability of the algebraic polynomials on the interval $(-1, 1)$ can be expressed in terms of L^p-norms. Let $\phi \in P_N$, where P_N denotes the space of all polynomials of degree less than or equal to N. Then for any integer $r \geq 1$ and $2 \leq p \leq \infty$, there exists a positive constant C independent of N such that

$$\|\phi^{(r)}\|_{L^p(-1,1)} \leq CN^{2r} \|\phi\|_{L^p(-1,1)}. \tag{3.1}$$

We also give some error bounds for the Legendre system in terms of the Sobolev norms. The Sobolev norm and semi-norm of order $m \geq 0$, considered in this section are given by

$$\|u(x)\|^2_{H^m(-1,1)} = \sum_{k=0}^{m} \|\frac{\partial^k u}{\partial x^k}\|^2_{L^2(-1,1)}, \tag{3.2}$$

$$|u|_{H^{m,N}(-1,1)} = \left(\sum_{j=min(m,N+1)}^{m} \|u^{(j)}\|^2_{L^2(-1,1)} \right)^{\frac{1}{2}}. \tag{3.3}$$

The truncation error $u - \mathcal{P}_N u$, where $\mathcal{P}_N u = \sum_{k=0}^{N} \widehat{u}_k P_k$, can be estimated as follows:

$$\|u - \mathcal{P}_N u\|_{L^2(-1,1)} \leq CN^{-m} |u|_{H^{m,N}(-1,1)}, \qquad \forall u \in H^m(-1, 1). \tag{3.4}$$

In those cases for which the truncation error of the derivatives is relevant, the following estimate extends (3.4) to higher order Sobolev norms:

$$\|u - \mathcal{P}_N u\|_{H^l(-1,1)} \leq CN^{2l-\frac{1}{2}-m} |u|_{H^{m,N}(-1,1)}, \tag{3.5}$$

for $u \in H^m(-1, 1)$, $m \geq 1$ and $1 \leq l \leq m$.

The following main theorem reveals the convergence results of the presented scheme in L^2-norm:

THEOREM 3.1. *Consider the system of integral algebraic equation (2.2) where $\widehat{y}, \widehat{z} \in H^m(-1, 1)$ and the data functions $\widehat{f}_i$ and $\widehat{K}_{ij}$, $i, j = 1, 2$, are sufficiently smooth and*

$|\widehat{K}_{22}(\tau,\tau)| \geq k_0 > 0$ *for all* $\tau \in (-1,1)$. *Let* $(\widehat{y}_N, \widehat{z}_N)$ *be the Legendre collocation approximation of* $(\widehat{y}, \widehat{z})$ *which is defined by* (2.5). *Then the following estimates hold:*

$$\begin{aligned}\|\widehat{y}_N - \widehat{y}\|_{L^2(-1,1)} \leq{}& CN^{-m}\Big((N^{-1} + \Omega_{11}N^{\frac{1}{2}-m})|\widehat{y}|_{H^{m,N}(-1,1)} + |\widehat{f}_1|_{H^{m,N}(-1,1)}\Big)\\ &+CN^{-m}\Big((N^{-1} + \Omega_{12}N^{\frac{1}{2}-m})|\widehat{z}|_{H^{m,N}(-1,1)}\Big)\\ &+ CN^{\frac{1}{2}-m}\Big(\Omega_{11}\|\widehat{y}\|_{L^2(-1,1)} + \Omega_{12}\|\widehat{z}\|_{L^2(-1,1)}\Big)\\ &+CN^{-1}\Big(\|\widehat{y}\|_{L^2(-1,1)} + \|\widehat{z}\|_{L^2(-1,1)}\Big),\end{aligned} \tag{3.6}$$

$$\begin{aligned}\|\widehat{z}_N - \widehat{z}\|_{L^2(-1,1)} \leq{}& CN^{-m}\Big((N^{\frac{1}{2}} + \Omega_{21}N^{\frac{5}{2}-m})|\widehat{y}|_{H^{m,N}(-1,1)}\Big)\\ &+ CN^{-m}\Big((N^{\frac{1}{2}} + \Omega_{22}N^{\frac{5}{2}-m})|\widehat{z}|_{H^{m,N}(-1,1)}\Big)\\ &+ CN^{\frac{5}{2}-m}\Big(\Omega_{21}\|\widehat{y}\|_{L^2(-1,1)} + \Omega_{22}\|\widehat{z}\|_{L^2(-1,1)}\Big),\end{aligned} \tag{3.7}$$

provided that N *is sufficiently large and*

$$\Omega_{kj} = \max_{0\leq n\leq N} |\widehat{K}_{kj}(\tau_n,\eta)|_{H^{m,N}(-1,1)}, \quad k,j = 1,2.$$

Proof. Rewriting the first equation of (2.7) as:

$$\begin{aligned}\widehat{y}_N(\tau_n) = \widehat{f}_1(\tau_n) &+ \int_{-1}^{\tau_n} \widehat{K}_{11}(\tau_n,\eta)y_N(\eta)d\eta + S_1(\tau_n)\\ &+ \int_{-1}^{\tau_n} \widehat{K}_{12}(\tau_n,\eta)z_N(\eta)d\eta + S_2(\tau_n),\end{aligned} \tag{3.8}$$

where

$$S_1(\tau_n) = \sum_{k=0}^{N}\sum_{l=k}^{N} c_{kl}V_{kl}(\tau_n) - \int_{-1}^{\tau_n} \widehat{K}_{11}(\tau_n,\eta)y_N(\eta)d\eta,$$

$$S_2(\tau_n) = \sum_{k=0}^{N}\sum_{l=k}^{N} c'_{kl}V_{kl}(\tau_n) - \int_{-1}^{\tau_n} \widehat{K}_{12}(\tau_n,\eta)z_N(\eta)d\eta.$$

Then it follows from (3.8) that

$$\begin{aligned}&\widehat{y}_N(\tau_n) = \widehat{f}_1(\tau_n) + \int_{-1}^{\tau_n} \widehat{K}_{11}(\tau_n,\eta)e(\eta)d\eta + \int_{-1}^{\tau_n} \widehat{K}_{11}(\tau_n,\eta)\widehat{y}(\eta)d\eta\\ &+ \int_{-1}^{\tau_n} \widehat{K}_{12}(\tau_n,\eta)\varepsilon(\eta)d\eta + \int_{-1}^{\tau_n} \widehat{K}_{12}(\tau_n,\eta)\widehat{z}(\eta)d\eta + S_1(\tau_n) + S_2(\tau_n),\end{aligned} \tag{3.9}$$

such that $e(s) = \widehat{y}_N(s) - \widehat{y}(s)$ and $\varepsilon(s) = \widehat{z}_N(s) - \widehat{z}(s)$.

Now, using (2.3), we first multiply both sides of (3.9) by $\frac{1}{\gamma_k}w_nP_k(\tau_n)$ and sum up from $n = 0$ to N and then multiply the altered equation by $P_k(\tau)$ and sum up from $k = 0$ to N, and we get

$$\begin{aligned}\widehat{y}_N(\tau) ={}& \mathcal{P}_N\left(\widehat{f}_1(\tau)\right) + \mathcal{P}_N\left(\int_{-1}^{\tau} \widehat{K}_{11}(\tau,\eta)\widehat{y}(\eta)d\eta\right)\\ &+ \mathcal{P}_N\left(\int_{-1}^{\tau} \widehat{K}_{11}(\tau,\eta)e(\eta)d\eta\right) + \mathcal{P}_N\left(\int_{-1}^{\tau} \widehat{K}_{12}(\tau,\eta)\varepsilon(\eta)d\eta\right)\\ &+ \mathcal{P}_N\left(\int_{-1}^{\tau} \widehat{K}_{12}(\tau,\eta)\widehat{z}(\eta)d\eta\right) + \mathcal{P}_N\left(S_1(\tau)\right) + \mathcal{P}_N\left(S_2(\tau)\right).\end{aligned} \tag{3.10}$$

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Subtracting the first equation of (2.2) from (3.10), we obtain

$$\begin{aligned} e(\tau) &= \int_{-1}^{\tau} \widehat{K}_{11}(\tau,\eta)e(\eta)d\eta + \int_{-1}^{\tau} \widehat{K}_{12}(\tau,\eta)\varepsilon(\eta)d\eta \\ &+ \left(\mathcal{P}_N(\widehat{f}_1(\tau)) - \widehat{f}_1(\tau)\right) + \mathcal{P}_N\left(S_1(\tau)\right) + \mathcal{P}_N\left(S_2(\tau)\right) + F_1 + F_2 + F_3 + F_4, \end{aligned} \tag{3.11}$$

where

$$F_1 = \mathcal{P}_N\left(\int_{-1}^{\tau} \widehat{K}_{11}(\tau,\eta)e(\eta)d\eta\right) - \int_{-1}^{\tau} \widehat{K}_{11}(\tau,\eta)e(\eta)d\eta, \tag{3.12}$$

$$F_2 = \mathcal{P}_N\left(\int_{-1}^{\tau} \widehat{K}_{12}(\tau,\eta)\varepsilon(\eta)d\eta\right) - \int_{-1}^{\tau} \widehat{K}_{12}(\tau,\eta)\varepsilon(\eta)d\eta, \tag{3.13}$$

$$F_3 = \mathcal{P}_N\left(\int_{-1}^{\tau} \widehat{K}_{11}(\tau,\eta)\widehat{y}(\eta)d\eta\right) - \int_{-1}^{\tau} \widehat{K}_{11}(\tau,\eta)\widehat{y}(\eta)d\eta,$$

$$F_4 = \mathcal{P}_N\left(\int_{-1}^{\tau} \widehat{K}_{12}(\tau,\eta)\widehat{z}(\eta)d\eta\right) - \int_{-1}^{\tau} \widehat{K}_{12}(\tau,\eta)\widehat{z}(\eta)d\eta.$$

The second equation of (2.7) can be rewritten as follows:

$$\begin{aligned} 0 &= \widehat{f}_2(\tau_n) + \int_{-1}^{\tau_n} \widehat{K}_{21}(\tau_n,\eta)y_N(\eta)d\eta + \int_{-1}^{\tau_n} \widehat{K}_{22}(\tau_n,\eta)z_N(\eta)d\eta \\ &\quad + S_3(\tau_n) + S_4(\tau_n), \end{aligned} \tag{3.14}$$

where

$$S_3(\tau_n) = \sum_{k=0}^{N}\sum_{l=k}^{N} q_{kl}V_{kl}(\tau_n) - \int_{-1}^{\tau_n} \widehat{K}_{21}(\tau_n,\eta)y_N(\eta)d\eta,$$

$$S_4(\tau_n) = \sum_{k=0}^{N}\sum_{l=k}^{N} q'_{kl}V_{kl}(\tau_n) - \int_{-1}^{\tau_n} \widehat{K}_{22}(\tau_n,\eta)z_N(\eta)d\eta.$$

From the second equation of (2.2), we can write

$$\mathcal{P}_N\left(\int_{-1}^{\tau} \widehat{K}_{21}(\tau,\eta)\widehat{y}(\eta)d\eta + \int_{-1}^{\tau} \widehat{K}_{22}(\tau,\eta)\widehat{z}(\eta)d\eta\right) = -\mathcal{P}_N(\widehat{f}_2(\tau)). \tag{3.15}$$

Consequently, using a similar procedure as outlined in the first part and (3.15), equation (3.14) gives

$$\begin{aligned} 0 &= \int_{-1}^{\tau} \widehat{K}_{21}(\tau,\eta)e(\eta)d\eta + \int_{-1}^{\tau} \widehat{K}_{22}(\tau,\eta)\varepsilon(\eta)d\eta \\ &+ \mathcal{P}_N\left(S_3(\tau)\right) + \mathcal{P}_N\left(S_4(\tau)\right) + F_5 + F_6, \end{aligned} \tag{3.16}$$

where F_5 and F_6 are defined by

$$F_5 = \mathcal{P}_N\left(\int_{-1}^{\tau} \widehat{K}_{21}(\tau,\eta)e(\eta)d\eta\right) - \int_{-1}^{\tau} \widehat{K}_{21}(\tau,\eta)e(\eta)d\eta,$$

$$F_6 = \mathcal{P}_N\left(\int_{-1}^{\tau} \widehat{K}_{22}(\tau,\eta)\varepsilon(\eta)d\eta\right) - \int_{-1}^{\tau} \widehat{K}_{22}(\tau,\eta)\varepsilon(\eta)d\eta.$$

Differentiating (3.16) with respect to τ, yields:

$$\begin{aligned} &- \widehat{K}_{21}(\tau,\tau)e(\tau) - \widehat{K}_{22}(\tau,\tau)\varepsilon(\tau) = \int_{-1}^{\tau} \frac{\partial \widehat{K}_{21}(\tau,\eta)}{\partial \tau} e(\eta)d\eta \\ &+ \int_{-1}^{\tau} \frac{\partial \widehat{K}_{22}(\tau,\eta)}{\partial \tau}\varepsilon(\eta)d\eta + \mathcal{P}'_N\left(S_3(\tau)\right) + \mathcal{P}'_N\left(S_4(\tau)\right) + F'_5 + F'_6. \end{aligned} \tag{3.17}$$

Equations (3.11) and (3.17) can be written as the equivalent compact matrix representation:

$$\mathbf{A}(\tau)\mathbf{E}(\tau) = \int_{-1}^{\tau} \mathbf{K}(\tau,\eta)\mathbf{E}(\eta)d\eta + \mathbf{B}, \tag{3.18}$$

with

$$\mathbf{A}(\tau) = \begin{bmatrix} 1 & 0 \\ -\widehat{K}_{21}(\tau,\tau) & -\widehat{K}_{22}(\tau,\tau) \end{bmatrix}, \qquad \mathbf{K}(\tau,\eta) = \begin{bmatrix} \widehat{K}_{11}(\tau,\eta) & \widehat{K}_{12}(\tau,\eta) \\ \dfrac{\partial \widehat{K}_{21}(\tau,\eta)}{\partial \tau} & \dfrac{\partial \widehat{K}_{22}(\tau,\eta)}{\partial \tau} \end{bmatrix},$$

$$\mathbf{E}(\tau) = \begin{bmatrix} e(\tau) \\ \varepsilon(\tau) \end{bmatrix},$$

and

$$\mathbf{B} = \begin{bmatrix} \left(\mathcal{P}_N(\widehat{f}_1(\tau)) - \widehat{f}_1(\tau)\right) + \mathcal{P}_N(S_1(\tau)) + \mathcal{P}_N(S_2(\tau)) + F_1 + F_2 + F_3 + F_4 \\ \mathcal{P}'_N(S_3(\tau)) + \mathcal{P}'_N(S_4(\tau)) + F'_5 + F'_6 \end{bmatrix}.$$

Noting that, since $|\widehat{K}_{22}(\tau,\tau)| \geq k_0 > 0$, the inverse of the matrix $\mathbf{A}(\tau)$ exists and is bounded.

From now on, to simplify the notation, we denote $\|\cdot\|_{L^2(-1,1)}$ by $\|\cdot\|$. Using the Gronwall inequality (see, e.g., [19, Lemma 3.4]) on (3.18), we have

$$\|\mathbf{E}\| \leq C\|\mathbf{B}\|. \tag{3.19}$$

It follows from (3.4) that

$$\|\mathcal{P}_N(\widehat{f}_1(\tau)) - \widehat{f}_1(\tau)\| \leq CN^{-m}|\widehat{f}_1|_{H^{m,N}(-1,1)}. \tag{3.20}$$

Using (3.4) and (3.3) for $m = 1$, we obtain

$$\begin{aligned} \|F_1\| &\leq CN^{-1}\|\widehat{K}_{11}(\tau,\tau)e(\tau) + \int_{-1}^{\tau} \frac{\partial(\widehat{K}_{11}(\tau,\eta))}{\partial\tau}e(\eta)d\eta\| \\ &\leq CN^{-1}\left\{ \|\widehat{K}_{11}(\tau,\tau)\|\,\|e(\tau)\| + \|\int_{-1}^{\tau} \frac{\partial(\widehat{K}_{11}(\tau,\eta))}{\partial\tau}e(\eta)d\eta\| \right\} \\ &\leq CN^{-1}\left\{ \|\widehat{K}_{11}(\tau,\tau)\|\, CN^{-m}|\widehat{y}|_{H^{m,N}(-1,1)} + C\|e(\tau)\| \right\} \\ &\leq CN^{-1}\left\{ (\|\widehat{K}_{11}(\tau,\tau)\| + C)\, CN^{-m}|\widehat{y}|_{H^{m,N}(-1,1)} \right\} \\ &\leq CN^{-1-m}|\widehat{y}|_{H^{m,N}(-1,1)}, \end{aligned}$$

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and consequently

$$\|F_2\| \le CN^{-1-m}|\widehat{z}|_{H^{m,N}(-1,1)},$$

$$\|F_3\| \le CN^{-1}\|\widehat{y}\|,$$

$$\|F_4\| \le CN^{-1}\|\widehat{z}\|.$$

Also,

$$\begin{aligned}\mathcal{P}_N(S_1(\tau)) &= \sum_{k=0}^{N}\Big(\frac{1}{\gamma_k}\sum_{n=0}^{N} w_n S_1(\tau_n)P_k(\tau_n)\Big)P_k(\tau)\\ &= \sum_{n=0}^{N} S_1(\tau_n)\Big(w_n\sum_{k=0}^{N}\frac{1}{\gamma_k}P_k(\tau)P_k(\tau_n)\Big)\\ &= \sum_{n=0}^{N} S_1(\tau_n)L_n(\tau),\end{aligned}$$

where $L_n(\tau)$ is the Lagrange interpolation polynomial based on the Gauss quadrature nodes (see, e.g., [14, page 90]). Therefore, we have

$$\|\mathcal{P}_N(S_1(\tau))\| \le \max_{0\le n\le N}|S_1(\tau_n)| \max_{\tau\in(-1,1)}\sum_{n=0}^{N}|L_n(\tau)|. \tag{3.21}$$

Moreover, using the Cauchy-Schwarz inequality [6], $|S_1(\tau_n)|$ can be written as:

$$\begin{aligned}|S_1(\tau_n)| &= \Big|\int_{-1}^{\tau_n}\Big(\mathcal{P}_N(\widehat{K}_{11}(\tau_n,\eta)) - \widehat{K}_{11}(\tau_n,\eta)\Big)\widehat{y}_N(\eta)d\eta\Big|\\ &\le \|\mathcal{P}_N(\widehat{K}_{11}(\tau_n,\eta)) - \widehat{K}_{11}(\tau_n,\eta)\|\ \|\widehat{y}_N\|.\end{aligned}$$

It then follows from (3.4) that

$$|S_1(\tau_n)| \le CN^{-m}|\widehat{K}_{11}(\tau_n,\eta)|_{H^{m,N}(-1,1)}(\|e\| + \|\widehat{y}\|). \tag{3.22}$$

The expression $\max_{\tau\in(-1,1)}\sum_{n=0}^{N}|L_n(\tau)|$ in (3.21) can be estimated by considering the following result on the Lebesgue constant for Lagrange interpolation from [18]:

$$\max_{\eta\in(-1,1)}\sum_{i=0}^{N}|L_i(\eta)| = 1 + C_0N^{\frac{1}{2}} + C_1 + O(N^{\frac{-1}{2}}), \tag{3.23}$$

where $\{L_j(x)\}_{j=0}^{N}$ are Lagrange interpolation polynomials with the Legendre Gauss, Gauss-Radau, or Gauss-Lobatto points $\{x_j\}_{j=0}^{N}$.

Considering the above result and (3.22), the following relation for (3.21) holds

$$\|\mathcal{P}_N(S_1(\tau))\| \le CN^{\frac{1}{2}-m}\Omega_{11}(CN^{-m}|\widehat{y}|_{H^{m,N}(-1,1)} + \|\widehat{y}\|), \tag{3.24}$$

and similarly

$$\|\mathcal{P}_N(S_2(\tau))\| \le CN^{\frac{1}{2}-m}\Omega_{12}(CN^{-m}|\widehat{z}|_{H^{m,N}(-1,1)} + \|\widehat{z}\|). \tag{3.25}$$

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Set $l = 1$ in (3.5) and using (3.2), we have

$$||F_5'|| \le ||F_5||_{H^1(-1,1)} \le CN^{\frac{3}{2}-m}\Big|\int_{-1}^{\tau}\widehat{K}_{21}(\tau,\eta)e(\eta)d\eta\Big|_{H^{m,N}(-1,1)}.$$

Applying (3.3) for $m = 1$, the above relation can be written as

$$\begin{aligned}||F_5'|| &\le N^{\frac{1}{2}}||\widehat{K}_{21}(\tau,\tau)e(\tau) + \int_{-1}^{\tau}\frac{\partial(\widehat{K}_{21}(\tau,\eta))}{\partial\tau}e(\eta)d\eta||\\ &\le N^{\frac{1}{2}}\Big\{\left(||\widehat{K}_{21}(\tau,\tau)|| + C\right)CN^{-m}|\widehat{y}|_{H^{m,N}(-1,1)}\Big\} \le CN^{\frac{1}{2}-m}|\widehat{y}|_{H^{m,N}(-1,1)}.\end{aligned}$$

Consequently, we have

$$||F_6'|| \le CN^{\frac{1}{2}-m}|\widehat{z}|_{H^{m,N}(-1,1)}.$$

It then follows from (3.1), (3.24) and (3.25) that

$$||\mathcal{P}_N'(S_3(\tau))|| \le CN^2||\mathcal{P}_N(S_3(\tau))|| \le CN^{\frac{5}{2}-m}\Omega_{21}(CN^{-m}|\widehat{y}|_{H^{m,N}(-1,1)} + ||\widehat{y}||),$$

and

$$||\mathcal{P}_N'(S_4(\tau))|| \le CN^{\frac{5}{2}-m}\Omega_{22}(CN^{-m}|\widehat{z}|_{H^{m,N}(-1,1)} + ||\widehat{z}||).$$

Finally, combining the above estimates and (3.19), the desired error estimates (3.6) and (3.7) are obtained. □

4. Generalization to the semi-explicit IAEs of index-1 in the Volterra-Hammerstein form. As a matter of interest, it is remarked that this approach may be applicable for the nonlinear IAEs of index-1. For this purpose, here we will consider the nonlinear semi-explicit IAEs of index-1 and try to obtain an error estimation for the proposed method similar to Theorem 3.1. We emphasize that our error analysis does not cover the general nonlinear case which contains some complications and restrictions for establishing a convergent result that will be the subject of our future work.

Consider the nonlinear semi-explicit IAEs of index-1 in the Volterra-Hammerstein form

$$\text{(4.1)}\qquad \begin{cases} y(t) = f_1(t) + \displaystyle\int_0^t \mathbf{K}_1(t,s)\mathbf{G}_1(s,y(s),z(s))ds, \\ 0 = f_2(t) + \displaystyle\int_0^t \mathbf{K}_2(t,s)\mathbf{G}_2(s,y(s),z(s))ds, \qquad t \in I = [0,T], \end{cases}$$

where $\mathbf{K}_i(t,s) = [K_{i1}(t,s), K_{i2}(t,s)]$ are sufficiently smooth and $\mathbf{G}_i(s,y(s),z(s)) = [G_{i1}(s,y(s)), G_{i2}(s,z(s))]^T$ are nonlinear in $y(s)$ and $z(s)$, $i = 1,2$.

Using the change of variables (2.1), the system can be written in the general form:

$$\text{(4.2)}\qquad \widehat{\mathbf{A}}\widehat{X}(\tau) = \widehat{f}(\tau) + \int_{-1}^{\tau}\widehat{\mathbf{K}}(\tau,\eta)\widehat{\mathbf{G}}(\eta,\widehat{X}(\eta))d\eta, \qquad \tau \in [-1,1],$$

where $\widehat{X}(\tau) = [\widehat{y},\widehat{z}]^T$, $\widehat{f}(\tau) = \Big[\widehat{f}_1,\widehat{f}_2\Big]^T$, $\widehat{\mathbf{K}}(\tau,\eta) = \mathrm{diag}\Big[\widehat{\mathbf{K}}_1,\widehat{\mathbf{K}}_2\Big]$, $\widehat{\mathbf{G}}(\eta,\widehat{X}(\eta)) = \Big[\widehat{\mathbf{G}}_1,\widehat{\mathbf{G}}_2\Big]^T$ and $\widehat{\mathbf{A}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Using the discrete expansion of $\widehat{X}(\tau)$, we obtain:

$$\mathcal{P}_N(\widehat{X}(\eta)) = \widehat{X}_N(\eta) = \sum_{k=0}^{N} \widehat{x}_k P_k(\eta). \tag{4.3}$$

Assume that, the nonlinear analytic function $\widehat{\mathbf{G}}(\eta, \widehat{X}(\eta))$ can be expanded as:

$$\widehat{\mathbf{G}}(\eta, \widehat{X}(\eta))) \simeq \sum_{i=0}^{N} \gamma_i(\eta)\widehat{X}^i(\eta). \tag{4.4}$$

Substituting $\widehat{X}_N(\eta)$ in (4.4), yields

$$\widehat{\mathbf{G}}_N(\eta, \widehat{X}_N(\eta)) \simeq \sum_{i=0}^{N} \gamma_i(\eta)\widehat{X}_N^i(\eta). \tag{4.5}$$

Using the orthogonal Legendre series expansion of $\widehat{\mathbf{K}}(\tau, \eta)$, (4.3), (4.5), and inserting the Gauss Legendre quadrature points τ_n in the system (4.2), we conclude

$$\widehat{\mathbf{A}} \sum_{k=0}^{N} \widehat{x}_k P_k(\tau_n) = \widehat{f}(\tau_n) + \sum_{i=0}^{N} \int_{-1}^{\tau_n} \gamma_i(\eta) \Big(\sum_{l=0}^{N} \widehat{\mathbf{K}}_l P_l(\eta) \Big) \Big(\sum_{k=0}^{N} \widehat{x}_k P_k(\eta) \Big)^i d\eta, \qquad m = 0, 1, \cdots, N, \tag{4.6}$$

where $\{\gamma_i(\eta)\}_{i=0}^N$ are continuous functions and $\{\widehat{\mathbf{K}}_l\}_{l=0}^N$ can be obtaine from (2.4).

The above procedure leads to a nonlinear system of equations for $\widehat{x}_k$ whose solution yields the unknown coefficients.

We can now follow the strategy given in the previous section with some restrictions and new conditions for establishing a convergent result similar to Theorem 3.1:

THEOREM 4.1. *Assume that the hypotheses of Theorem 3.1 hold and let the function $\widehat{\mathbf{G}}(\eta, \widehat{X}(\eta)))$ in the nonlinear IAEs (4.2) satisfy (4.4). If $\widehat{X}_N = [\widehat{y}_N, \widehat{z}_N]$ be the Legendre collocation approximation of $\widehat{X} = [\widehat{y}, \widehat{z}]$ which is defined by (2.5), then the following estimates hold*

$$\begin{aligned} \|\widehat{y}_N - \widehat{y}\|_{L^2(-1,1)} \le{}& CN^{\frac{5}{2}-m}\Omega_{11} \sum_{i=0}^{N} \Big(CN^{\frac{3}{4}-m} |\widehat{y}|_{H^{m,N}(-1,1)} + \|\widehat{y}\|_{L^\infty(-1,1)} \Big)^i \\ &+ N^{\frac{5}{2}-m}\Omega_{12} \sum_{i=0}^{N} \Big(CN^{\frac{3}{4}-m} |\widehat{z}|_{H^{m,N}(-1,1)} + \|\widehat{z}\|_{L^\infty(-1,1)} \Big)^i \\ &+ CN^{\frac{1}{2}-m} \Big(|\widehat{y}|_{H^{m,N}(-1,1)} + |\widehat{z}|_{H^{m,N}(-1,1)} \Big) \\ &+ \widehat{W}_{11}(\widehat{y}) + \widehat{W}_{12}(\widehat{z}), \end{aligned} \tag{4.7}$$

$$\begin{aligned} \|\widehat{z}_N - \widehat{z}\|_{L^2(-1,1)} \le{}& CN^{\frac{5}{2}-m}\Omega_{21} \sum_{i=0}^{N} \Big(CN^{\frac{3}{4}-m} |\widehat{y}|_{H^{m,N}(-1,1)} + \|\widehat{y}\|_{L^\infty(-1,1)} \Big)^i \\ &+ CN^{\frac{5}{2}-m}\Omega_{22} \sum_{i=0}^{N} \Big(CN^{\frac{3}{4}-m} |\widehat{z}|_{H^{m,N}(-1,1)} + \|\widehat{z}\|_{L^\infty(-1,1)} \Big)^i \\ &+ CN^{\frac{1}{2}-m} \Big(|\widehat{y}|_{H^{m,N}(-1,1)} + |\widehat{z}|_{H^{m,N}(-1,1)} \Big) \\ &+ \widehat{W}_{21}(\widehat{y}) + \widehat{W}_{22}(\widehat{z}), \end{aligned} \tag{4.8}$$

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where

$$\widehat{W}_{kj}(\widehat{u}) = C\Phi_{kj} N^{\frac{13}{4}-m} |\widehat{u}|_{H^{m,N}(-1,1)} \sum_{i=2}^{N} \Big\{ \Big(C N^{\frac{3}{4}-m} |\widehat{u}|_{H^{m,N}(-1,1)} + \|\widehat{u}\|_{L^\infty(-1,1)} \Big)^{i-1} + \cdots + \|\widehat{u}\|^{i-1}_{L^\infty(-1,1)} \Big\},$$

and

$$\Phi_{kj} = \max_{0\le n\le N} \|\widehat{K}_{kj}(\tau_n,\eta)\|_{L^2(-1,1)}, \qquad k,j = 1,2,$$

provided that N is sufficiently large.

Proof. Rewriting the first equation of (4.6) as:

$$\begin{aligned}\hat{y}_N(\tau_n) = \widehat{f}_1(\tau_n) + \sum_{i=0}^{N} \int_{-1}^{\tau_n} (\gamma_{11})_i \widehat{K}_{11}(\tau_n,\eta)\hat{y}^i_N(\eta)d\eta + D_1(\tau_n)\\ + \sum_{i=0}^{N} \int_{-1}^{\tau_n} (\gamma_{12})_i \widehat{K}_{12}(\tau_n,\eta)\hat{z}^i_N(\eta)d\eta + D_2(\tau_n),\end{aligned} \tag{4.9}$$

where

$$D_1(\tau_n) = \sum_{i=0}^{N} \int_{-1}^{\tau_n} (\gamma_{11})_i \Big(\mathcal{P}_N(\widehat{K}_{11}(\tau_n,\eta)) - \widehat{K}_{11}(\tau_n,\eta) \Big) \hat{y}^i_N(\eta)d\eta,$$

$$D_2(\tau_n) = \sum_{i=0}^{N} \int_{-1}^{\tau_n} (\gamma_{12})_i \Big(\mathcal{P}_N(\widehat{K}_{12}(\tau_n,\eta)) - \widehat{K}_{12}(\tau_n,\eta) \Big) \hat{z}^i_N(\eta)d\eta.$$

The first equation of (4.2), together with (4.4), gives rise to

$$\begin{aligned}\mathcal{P}_N \left(\int_{-1}^{\tau} (\gamma_{11})_1 \widehat{K}_{11}(\tau,\eta)\widehat{y}(\eta)d\eta + \int_{-1}^{\tau} (\gamma_{12})_1 \widehat{K}_{12}(\tau,\eta)\widehat{z}(\eta)d\eta \right) =\\ + \mathcal{P}_N \Big(\widehat{y}(\tau) - \widehat{f}_1(\tau) - \int_{-1}^{\tau} ((\gamma_{11})_0 \widehat{K}_{11}(\tau,\eta) + (\gamma_{12})_0 \widehat{K}_{12}(\tau,\eta))d\eta \Big)\\ - \mathcal{P}_N \Big(\sum_{i=2}^{N} \int_{-1}^{\tau} ((\gamma_{11})_i \widehat{K}_{11}(\tau,\eta)\widehat{y}^i(\eta) + (\gamma_{12})_i \widehat{K}_{12}(\tau,\eta)\widehat{z}^i(\eta))d\eta \Big).\end{aligned} \tag{4.10}$$

Considering (3.10), after some manipulations on (4.9) and inserting (4.10) into the resulted equation, we obtain

$$\begin{aligned}0 = \int_{-1}^{\tau} (\gamma_{11})_1 \widehat{K}_{11}(\tau,\eta)e(\eta)d\eta + \int_{-1}^{\tau} (\gamma_{12})_1 \widehat{K}_{12}(\tau,\eta)\varepsilon(\eta)d\eta + W_1 + W_2\\ + \mathcal{P}_N(D_1(\tau)) + \mathcal{P}_N(D_2(\tau)) + F_1 + F_2,\end{aligned} \tag{4.11}$$

where

$$W_1 = \sum_{i=2}^{N} \mathcal{P}_N \Big(\int_{-1}^{\tau} (\gamma_{11})_i \widehat{K}_{11}(\tau_n,\eta)(\hat{y}^i_N(\eta) - \hat{y}^i(\eta))d\eta \Big),$$

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$$W_2 = \sum_{i=2}^{N} \mathcal{P}_N\Big(\int_{-1}^{\tau} (\gamma_{12})_i \widehat{K}_{12}(\tau_n,\eta)(\hat{z}_N^i(\eta) - \hat{z}^i(\eta))d\eta\Big).$$

Differentiating (4.11) with respect to τ, yields:

$$-(\gamma_{11})_1\widehat{K}_{11}(\tau,\tau)e(\tau) - (\gamma_{12})_1\widehat{K}_{12}(\tau,\tau)\varepsilon(\tau) = \int_{-1}^{\tau}\frac{\partial\widehat{K}_{11}(\tau,\eta)(\gamma_{11})_1}{\partial\tau}e(\eta)d\eta + W_1' + W_2'$$
$$+\int_{-1}^{\tau}\frac{\partial\widehat{K}_{12}(\tau,\eta)(\gamma_{12})_1}{\partial\tau}\varepsilon(\eta)d\eta + \mathcal{P}'_N\left(D_1(\tau)\right) + \mathcal{P}'_N\left(D_2(\tau)\right) + F_1' + F_2'. \tag{4.12}$$

Consequently, the second equation of (4.6) can be written as:

$$-(\gamma_{21})_1\widehat{K}_{21}(\tau,\tau)e(\tau) - (\gamma_{22})_1\widehat{K}_{22}(\tau,\tau)\varepsilon(\tau) = \int_{-1}^{\tau}\frac{\partial\widehat{K}_{21}(\tau,\eta)(\gamma_{21})_1}{\partial\tau}e(\eta)d\eta + W_3' + W_4'$$
$$+\int_{-1}^{\tau}\frac{\partial\widehat{K}_{22}(\tau,\eta)(\gamma_{22})_1}{\partial\tau}\varepsilon(\eta)d\eta + \mathcal{P}'_N\left(D_3(\tau)\right) + \mathcal{P}'_N\left(D_4(\tau)\right) + F_5' + F_6', \tag{4.13}$$

where

$$W_3' = \sum_{i=2}^{N} \mathcal{P}'_N\Big(\int_{-1}^{\tau} (\gamma_{21})_i \widehat{K}_{21}(\tau_n,\eta)(\hat{y}_N^i(\eta) - \hat{y}^i(\eta))d\eta\Big),$$

$$W_4' = \sum_{i=2}^{N} \mathcal{P}'_N\Big(\int_{-1}^{\tau} (\gamma_{22})_i \widehat{K}_{22}(\tau_n,\eta)(\hat{z}_N^i(\eta) - \hat{z}^i(\eta))d\eta\Big),$$

$$D_3(\tau) = \sum_{i=0}^{N}\int_{-1}^{\tau}(\gamma_{21})_i\Big(\mathcal{P}_N(\widehat{K}_{21}(\tau,\eta)) - \widehat{K}_{21}(\tau,\eta)\Big)\hat{y}_N^i(\eta)d\eta,$$

$$D_4(\tau) = \sum_{i=0}^{N}\int_{-1}^{\tau}(\gamma_{22})_i\Big(\mathcal{P}_N(\widehat{K}_{22})(\tau,\eta)) - \widehat{K}_{22}(\tau,\eta)\Big)\hat{z}_N^i(\eta)d\eta.$$

Now, considering (4.12) and (4.13) in the matrix notation (3.18) and using the Gronwall inequality for the obtained equation, we conclude:

$$||\mathbf{E}|| \le C||\tilde{\mathbf{B}}||, \tag{4.14}$$

where

$$\tilde{\mathbf{B}} = \begin{bmatrix} \mathcal{P}'_N(D_1(\tau)) + \mathcal{P}'_N(D_2(\tau)) + W_1' + W_2' + F_1' + F_2' \\ W_3' + W_4' + \mathcal{P}'_N(D_3(\tau)) + \mathcal{P}'_N(D_4(\tau)) + F_5' + F_6' \end{bmatrix}.$$

In a similar manner to (3.21) and (3.22), we have

$$||\mathcal{P}_N(D_1(\tau))|| \le \max_{0\le n\le N}|D_1(\tau_n)| \max_{\tau\in(-1,1)}\sum_{n=0}^{N}|L_n(\tau)|, \tag{4.15}$$

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such that $|D_1(\tau_n)|$ can be written as:

$$\begin{aligned}
|D_1(\tau_n)| &= \sum_{i=0}^{N} \left| \int_{-1}^{\tau_n} (\gamma_{11})_i \Big(\mathcal{P}_N(\widehat{K}_{11}(\tau_n,\eta)) - \widehat{K}_{11}(\tau_n,\eta) \Big) \hat{y}_N^i(\eta) d\eta \right| \\
&\leq \| \Big(\mathcal{P}_N(\widehat{K}_{11}(\tau_n,\eta)) - \widehat{K}_{11}(\tau_n,\eta) \Big) \| \sum_{i=0}^{N} \|(\gamma_{11})_i \hat{y}_N^i\| \\
&\leq CN^{-m} |\widehat{K}_{11}(\tau_n,\eta)|_{H^{m,N}(-1,1)} \sum_{i=0}^{N} \|\hat{y}_N^i\|_{L^\infty(-1,1)} \\
&\leq CN^{-m} |\widehat{K}_{11}(\tau_n,\eta)|_{H^{m,N}(-1,1)} \sum_{i=0}^{N} \|\hat{y}_N\|^i_{L^\infty(-1,1)} \\
&\leq CN^{-m} |\widehat{K}_{11}(\tau_n,\eta)|_{H^{m,N}(-1,1)} \sum_{i=0}^{N} \Big(\|e\|_{L^\infty(-1,1)} + \|\widehat{y}\|_{L^\infty(-1,1)} \Big)^i.
\end{aligned}$$

We are now applying the well-known Sobolev inequality from [6]

$$\|u\|_{L^\infty(a,b)} \leq \left(\frac{1}{b-a} + 2 \right)^{\frac{1}{2}} \|u\|^{\frac{1}{2}}_{L^2(a,b)} \|u\|^{\frac{1}{2}}_{H^1(a,b)},$$

where $(a,b) \subset \mathbb{R}$ is a bounded interval of the real line and $u \in H^1(a,b)$.

It then follows from (3.4) and (3.5) that

$$|D_1(\tau_n)| \leq CN^{-m} |\widehat{K}_{11}(\tau_n,\eta)|_{H^{m,N}(-1,1)} \sum_{i=0}^{N} \Big(CN^{\frac{3}{4}-m} |\widehat{y}|_{H^{m,N}(-1,1)} + \|\widehat{y}\|_{L^\infty(-1,1)} \Big)^i.$$

On the other hand, using (3.1) and (3.23), similar to (3.24) and (3.25), we obtain

$$\begin{aligned}
\|\mathcal{P}'_N(D_1(\tau))\| &\leq CN^2 \|\mathcal{P}_N(D_1(\tau))\| \\
&\leq CN^{\frac{5}{2}-m} \Omega_{11} \sum_{i=0}^{N} \Big(CN^{\frac{3}{4}-m} |\widehat{y}|_{H^{m,N}(-1,1)} + \|\widehat{y}\|_{L^\infty(-1,1)} \Big)^i,
\end{aligned} \tag{4.16}$$

and

$$\|\mathcal{P}'{}_N(D_2(\tau))\| \leq CN^{\frac{5}{2}-m} \Omega_{12} \sum_{i=0}^{N} \Big(CN^{\frac{3}{4}-m} |\widehat{z}|_{H^{m,N}(-1,1)} + \|\widehat{z}\|_{L^\infty(-1,1)} \Big)^i. \tag{4.17}$$

Similarly,

$$\|W_1\| \leq \max_{0 \leq n \leq N} \sum_{i=2}^{N} \left| \Big(\int_{-1}^{\tau_n} (\gamma_{11})_i \widehat{K}_{11}(\tau_n,\eta)(\hat{y}_N^i(\eta) - \hat{y}^i(\eta)) d\eta \Big) \right| \max_{\tau \in (-1,1)} \sum_{n=0}^{N} |L_n(\tau)|,$$

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such that

$$\begin{aligned}
&\sum_{i=2}^{N}\Big|\Big(\int_{-1}^{\tau_n}(\gamma_{11})_i\widehat{K}_{11}(\tau_n,\eta)(\hat{y}_N^i(\eta)-\hat{y}^i(\eta))d\eta\Big)\Big| \le \|\widehat{K}_{11}(\tau_n,\eta))\|\sum_{i=2}^{N}\|(\gamma_{11})_i(\hat{y}_N^i-\hat{y}^i)\| \\
&\quad\le C\|\widehat{K}_{11}(\tau_n,\eta))\|\|\hat{y}_N-\hat{y}\|_{L^\infty(-1,1)}\sum_{i=2}^{N}\|\hat{y}_N^{i-1}+\hat{y}_N^{i-2}\hat{y}+\cdots+\hat{y}^{i-1}\|_{L^\infty(-1,1)} \\
&\quad\le C\|\widehat{K}_{11}(\tau_n,\eta))\|N^{\frac{3}{4}-m}|\widehat{y}|_{H^{m,N}(-1,1)}\sum_{i=2}^{N}\Big(\|\hat{y}_N\|_{L^\infty(-1,1)}^{i-1}+\cdots+\|\widehat{y}\|_{L^\infty(-1,1)}^{i-1}\Big) \\
&\quad\le C\|\widehat{K}_{11}(\tau_n,\eta))\|N^{\frac{3}{4}-m}|\widehat{y}|_{H^{m,N}(-1,1)}\sum_{i=2}^{N}\Big(\big(CN^{\frac{3}{4}-m}|\widehat{y}|_{H^{m,N}(-1,1)} \\
&\qquad\qquad\qquad\qquad+\|\widehat{y}\|_{L^\infty(-1,1)}\big)^{i-1}+\cdots+\|\widehat{y}\|_{L^\infty(-1,1)}^{i-1}\Big),
\end{aligned}$$

and

$$\begin{aligned}
\|W_1\| \le C\Phi_{11}N^{\frac{5}{4}-m}|\widehat{y}|_{H^{m,N}(-1,1)}\sum_{i=2}^{N}\Big(\big(CN^{\frac{3}{4}-m}|\widehat{y}|_{H^{m,N}(-1,1)} \\
+\|\widehat{y}\|_{L^\infty(-1,1)}\big)^{i-1}+\cdots+\|\widehat{y}\|_{L^\infty(-1,1)}^{i-1}\Big).
\end{aligned}$$

Using (3.1), we have

$$\begin{aligned}
\|W_1'\| \le C\Phi_{11}N^{\frac{13}{4}-m}|\widehat{y}|_{H^{m,N}(-1,1)}\sum_{i=2}^{N}\Big(\big(CN^{\frac{3}{4}-m}|\widehat{y}|_{H^{m,N}(-1,1)} \\
+\|\widehat{y}\|_{L^\infty(-1,1)}\big)^{i-1}+\cdots+\|\widehat{y}\|_{L^\infty(-1,1)}^{i-1}\Big).
\end{aligned}$$

and

$$\begin{aligned}
\|W_2'\| \le C\Phi_{12}N^{\frac{13}{4}-m}|\widehat{z}|_{H^{m,N}(-1,1)}\sum_{i=2}^{N}\Big(\big(CN^{\frac{3}{4}-m}|\widehat{z}|_{H^{m,N}(-1,1)} \\
+\|\widehat{z}\|_{L^\infty(-1,1)}\big)^{i-1}+\cdots+\|\widehat{z}\|_{L^\infty(-1,1)}^{i-1}\Big).
\end{aligned}$$

Due to (3.1), (4.16) and (4.17), we obtain

$$\begin{aligned}
\|\mathcal{P}_N'(D_3(\tau))\| &\le CN^2\|\mathcal{P}_N(D_3(\tau))\| \\
&\le CN^{\frac{5}{2}-m}\Omega_{21}\sum_{i=0}^{N}\Big(CN^{\frac{3}{4}-m}|\widehat{y}|_{H^{m,N}(-1,1)}+\|\widehat{y}\|_{L^\infty(-1,1)}\Big)^i,
\end{aligned}$$

$$\|\mathcal{P}_N'(D_4(\tau))\| \le CN^{\frac{5}{2}-m}\Omega_{22}\sum_{i=0}^{N}\Big(CN^{\frac{3}{4}-m}|\widehat{z}|_{H^{m,N}(-1,1)}+\|\widehat{z}\|_{L^\infty(-1,1)}\Big)^i,$$

and similarly

$$\begin{aligned}
\|W_3'\| \le C\Phi_{21}N^{\frac{13}{4}-m}|\widehat{y}|_{H^{m,N}(-1,1)}\sum_{i=2}^{N}\Big(\big(CN^{\frac{3}{4}-m}|\widehat{y}|_{H^{m,N}(-1,1)} \\
+\|\widehat{y}\|_{L^\infty(-1,1)}\big)^{i-1}+\cdots+\|\widehat{y}\|_{L^\infty(-1,1)}^{i-1}\Big).
\end{aligned}$$

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and

$$\|W_4'\| \le C\Phi_{22} N^{\frac{13}{4}-m} |\widehat{z}|_{H^{m,N}(-1,1)} \sum_{i=2}^{N} \Big(\big(CN^{\frac{3}{4}-m} |\widehat{z}|_{H^{m,N}(-1,1)} + \|\widehat{z}\|_{L^\infty(-1,1)}\big)^{i-1} + \cdots + \|\widehat{z}\|^{i-1}_{L^\infty(-1,1)} \Big).$$

Finally, according to (3.12) and (3.13) in the proof of Theorem 3.1 and applying (3.2) and (3.3) for $m = 1$, we have

$$\|F_1'\| \le \|F_1\|_{H^1(-1,1)} \le CN^{\frac{1}{2}-m} |\widehat{y}|_{H^{m,N}(-1,1)},$$
$$\|F_2'\| \le \|F_2\|_{H^1(-1,1)} \le CN^{\frac{1}{2}-m} |\widehat{z}|_{H^{m,N}(-1,1)}.$$

Also, using (4.14) and the upper bound which is obtained for $\|F_5'\|$ and $\|F_6'\|$, together with the above estimations, the desired estimates (4.7) and (4.8) are obtained. □

5. Numerical experiments. In the following, we implement the Legendre collocation method on some test problems and show the reliability and efficiency of the presented method and error estimation using program code written in Mathematica®. We will also show that the proposed scheme can provide reasonable results for nonlinear as well as linear IAEs systems.

EXAMPLE 5.1. Consider the following linear system of IAEs with index 1:

$$\mathbf{A}(t)X(t) = g(t) + \int_0^t \mathbf{K}(t,s)X(s)ds, \qquad t \in [0,1],$$

where

$$\mathbf{A}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{K}(t,s) = \begin{bmatrix} t^3+s+1 & \cos(3s)+1 \\ t+s+2 & \sin(3s)+2 \end{bmatrix},$$

$$X(t) = [y(t) \;\; z(t)]^T, \quad g(t) = [f_1(t) \;\; f_2(t)]^T$$

and

$$f_1(t) = 1 - (1+t+t^3)\sin t - \frac{1}{3}(3+\cos 3t)(\sin\frac{3t}{2})^2,$$
$$f_2(t) = 1 - \cos t - 2(1+t)\sin t + \frac{1}{12}(-8-6t+8\cos 3t+\sin 6t).$$

The exact solution of this system is

$$y(t) = \cos t, \qquad z(t) = \sin 3t.$$

Let $(\hat{y}_N, \hat{z}_N)$ and $(\hat{y}, \hat{z})$ be the approximate and the exact solution of the system, respectively, which is given by (2.5). The $L^2(-1,1)$ norm of the errors are reported in Table 5.1 and Figure 5.1.

EXAMPLE 5.2.

$$\mathbf{A}(t)X(t) = g(t) + \int_0^t \mathbf{K}(t,s)X(s)ds, \qquad t \in [0,1],$$

where

$$\mathbf{A}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{K}(t,s) = \begin{bmatrix} t^2+s^2+2 & s+t+1 \\ e^{2t+s} & s+t^2+2 \end{bmatrix},$$

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TABLE 5.1
$L^2(-1,1)$ errors for Example 5.1.

N	$\|\|\hat{y}_N - \hat{y}\|\|_{L^2(-1,1)}$	$\|\|\hat{z}_N - \hat{z}\|\|_{L^2(-1,1)}$
4	2.30×10^{-4}	9.60×10^{-4}
6	2.25×10^{-6}	1.53×10^{-5}
8	1.41×10^{-8}	1.33×10^{-7}
10	6.04×10^{-11}	7.48×10^{-10}
12	1.87×10^{-13}	2.92×10^{-12}

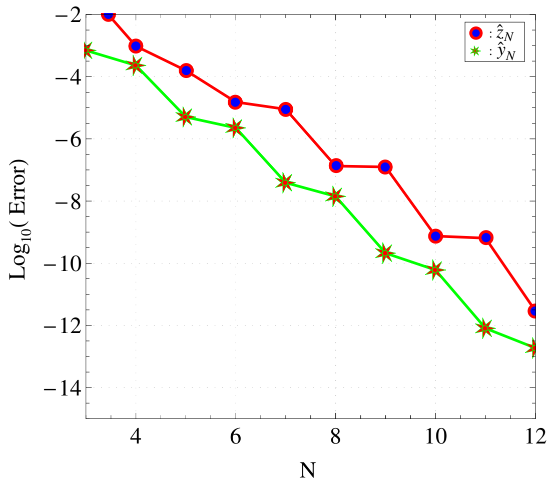

FIG. 5.1. *$L^2(-1,1)$ errors versus N for Example 5.1.*

$$X(t) = [y(t) \;\; z(t)]^T, \qquad\qquad g(t) = [f_1(t) \;\; f_2(t)]^T,$$

and

$$f_1(t) = e^{2t}\frac{1}{4}(5 + 2t^2 - e^{2t}(5 - 2t + 4t^2)) - (1+t)\arctan t - \frac{1}{2}\ln(1+t^2),$$
$$f_2(t) = -\frac{1}{3}e^{2t}(-1 + e^{3t}) - (2+t^2)\arctan t - \frac{1}{2}\ln(1+t^2),$$

with the exact solution: $y(t) = e^{2t}$, $z(t) = \dfrac{1}{1+t^2}$.

Table 5.2 and Figure 5.2 show the errors for several values of N and the associated rate of convergence, which confirms the expected convergence of the method as described in Theorem 3.1.

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TABLE 5.2
$L^2(-1,1)$ *errors for Example 5.2.*

N	$\|\hat{y}_N - \hat{y}\|_{L^2(-1,1)}$	$\|\hat{z}_N - \hat{z}\|_{L^2(-1,1)}$
4	1.38×10^{-3}	4.80×10^{-3}
6	7.89×10^{-6}	2.33×10^{-5}
8	2.71×10^{-8}	5.90×10^{-6}
10	7.10×10^{-11}	3.10×10^{-7}
12	8.81×10^{-13}	5.62×10^{-9}

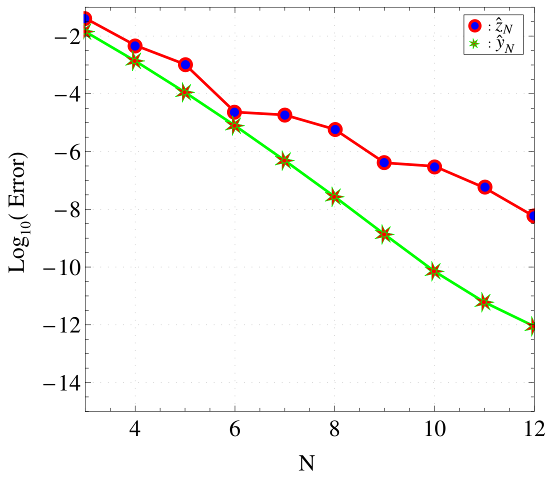

FIG. 5.2. $L^2(-1,1)$ *errors versus* N *for Example 5.2.*

The last test problem considers the following nonlinear IAEs system of index-1 from [16].
EXAMPLE 5.3.

$$\begin{cases} y(t) = f_1(t) + \displaystyle\int_0^t e^{t-s} y^2(s) z(s) ds, \\ 0 = f_2(t) + \displaystyle\int_0^t (1+t-s) y(s) z(s) ds, \qquad t \in [0,1], \end{cases}$$

where

$$f_1(t) = e^{-t} - \frac{1}{10} e^{-2t} (3e^{3t} - 3\cos t + \sin t),$$
$$f_2(t) = \frac{1}{2}(-1 - t + e^{-t}\cos t),$$

with the exact solution: $y(t) = e^{-t}, \ z(t) = \cos t$.

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We work with the same conditions as in [16]. Let (u, v) be the approximation of the exact solution (y, z). The errors for the numerical results obtained by using the spline collocation method with $m = 3, 4$ and $N = 4, 6, 8, 10$ for Radau II collocation parameters are presented in Table 5.3; the L^2- norm of the errors for the proposed method with several values of N are also reported in the same Table. Figure 5.3 displays the exponential rate of convergence which confirms the prediction of Theorem 4.1.

TABLE 5.3
Comparison of the spline collocation and Legendre spectral methods for Example 5.3.

	Spline collocation method [16]			
	$\|\|y-u\|\|_\infty$		$\|\|z-v\|\|_\infty$	
N	$m=3$	$m=4$	$m=3$	$m=4$
4	4.99×10^{-6}	4.45×10^{-9}	1.69×10^{-4}	7.60×10^{-6}
6	5.95×10^{-7}	2.35×10^{-10}	5.29×10^{-5}	1.49×10^{-6}
8	1.33×10^{-7}	2.96×10^{-11}	2.28×10^{-5}	4.71×10^{-7}
10	4.24×10^{-8}	6.01×10^{-12}	1.11×10^{-5}	1.92×10^{-7}
	Present method			
	$\|\|\hat{y}_N-\hat{y}\|\|_{L^2(-1,1)}$		$\|\|\hat{z}_N-\hat{z}\|\|_{L^2(-1,1)}$	
4	8.74×10^{-6}		5.72×10^{-5}	
6	1.29×10^{-8}		2.84×10^{-8}	
8	1.12×10^{-11}		1.01×10^{-10}	
10	6.40×10^{-15}		3.00×10^{-14}	

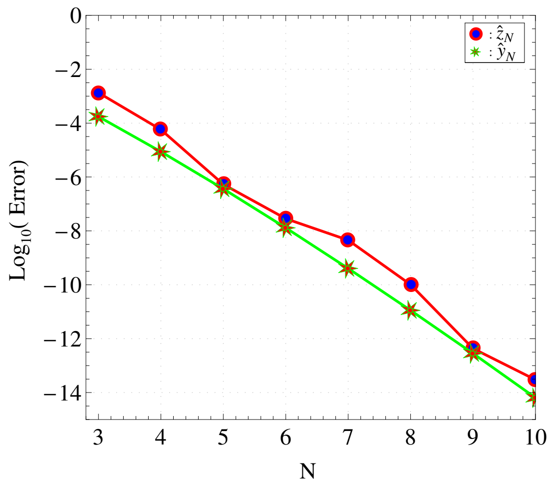

FIG. 5.3. $L^2(-1,1)$ *errors versus* N *for Example 5.3.*

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Figure 5.4, presents a comparison between the error behaviours of the spline collocation scheme with $m = 4, N = 10$ and the proposed method with $N = 10$. From Table 5.3 and Figure 5.4, we can see that the Legendre spectral method coincide to a very high degree of accuracy over the spline collocation method. This is also consistent with the prediction of Theorem 4.1.

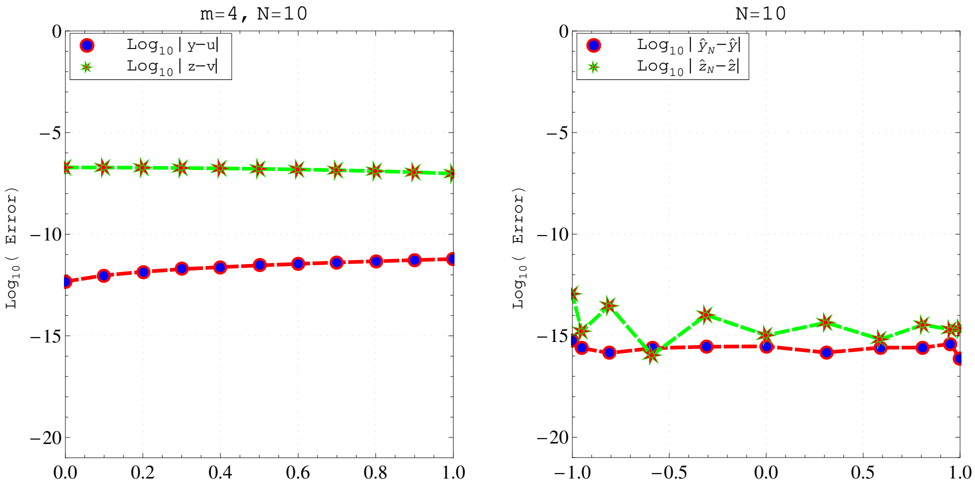

FIG. 5.4. *The error behaviors for the spline collocation method with $N = 10, m = 4$ (Left) and the Legendre spectral method for $N = 10$ (Right) in Example 5.3.*

6. Conclusions. This paper studies the Legendre collocation method for the semi-explicit IAEs system of index-1. The scheme consists of finding an explicit expression for the integral terms of the equations associated with the Legendre collocation method. A posteriori error estimation of the method in the L^2 norm was obtained. It should be noticed that, the IAEs systems are coupled systems consisting of the first and second kind Volterra equations, so that in our considered numerical tests, we can not use the Legendre-Gauss-Radau or Gauss-Lobatto points as the collocation points. An optimal error estimate may be obtained by choosing a suitable collocation points. We may need to follow some other ways, in modifying the proposed method in Section 2, to obtain the optimal rate of convergence which is the subject of our future work.

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