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A ROBUST FEM-BEM MINRES SOLVER FOR DISTRIBUTED MULTIHARMONIC EDDY CURRENT OPTIMAL CONTROL PROBLEMS IN UNBOUNDED DOMAINS*

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Abstract. This work is devoted to distributed optimal control problems for multiharmonic eddy current problems in unbounded domains. We apply a multiharmonic approach to the optimality system and discretize in space by means of a symmetrically coupled finite and boundary element method, taking care of the different physical behavior in conducting and non-conducting subdomains, respectively. We construct and analyze a new preconditioned MinRes solver for the system of frequency domain equations. We show that this solver is robust with respect to the space discretization and time discretization parameters as well as the involved "bad" parameters like the conductivity and the regularization parameters. Furthermore, we analyze the asymptotic behavior of the error in terms of the discretization parameters for our special discretization scheme.

Key words. time-periodic optimization, eddy current problems, finite element discretization, boundary element discretization, symmetric coupling, MinRes solver.

AMS subject classifications. 49N20, 35Q61, 65M38, 65M60, 65F08

1. Introduction. The multiharmonic finite element method or harmonic-balanced finite element method has been used by many authors in different applications; see, e.g., [4, 15, 17, 37, 46]. Switching from the time domain to the frequency domain allows us to replace expensive time-integration procedures by the solution of a system of partial differential equations for the amplitudes belonging to the sine- and to the cosine-excitation. Following this strategy, Copeland et al. [11, 12], Bachinger et al. [5, 6], and Kolmbauer and Langer [32] applied harmonic and multiharmonic approaches to parabolic initial-boundary value problems and the eddy current problem.

Furthermore, the multiharmonic finite element method has been generalized to multiharmonic parabolic and multiharmonic eddy current optimal control problems [28, 31]. Indeed, in [31] a MinRes solver for the solution of multiharmonic eddy current optimal control problems is constructed that is robust with respect to the discretization parameter h and all involved parameters like frequency, conductivity, reluctivity, and the regularization parameter. This solver is based on a pure finite element discretization of a bounded domain. Furthermore, in [30] the results of [32] for the time-harmonic eddy current problem are extended to the case of unbounded domains using a symmetric coupling of the finite element method (FEM) and the boundary element method (BEM) [24]. Even in this case, parameter-robust block-diagonal preconditioners can be constructed.

The aim of this work is to generalize these ideas of combining the multiharmonic approach and the FEM-BEM coupling method to multiharmonic eddy current optimal control problems:

$$\min J(\mathbf{y}, \mathbf{u}), \text{ s.t. } \sigma \frac{\partial \mathbf{y}}{\partial t} + \mathbf{curl}\,(\nu\, \mathbf{curl}\, \mathbf{y}) = \mathbf{u},$$

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with appropriate periodicity and boundary (radiation) conditions for $\mathbf{y}$. The fast solution of the corresponding large linear system of finite element equations is crucial for the competitiveness of this method. Hence, appropriate (parameter-robust) preconditioning is an important issue. Deriving the optimality system of the optimal control problem naturally results in a saddle point system. Due to the special structure of the multiharmonic time-discretization and the finite element-boundary element space discretization, we finally obtain a three-fold saddle point structure. A new technique of parameter-robust preconditioning of saddle point problems was introduced by Zulehner in [47]. We explore this technique to construct a parameter-robust preconditioned MinRes solver for our huge linear system of algebraic equations resulting from the multiharmonic finite element-boundary element discretization.

The outline of this work is as follows: in Section 2, we summarize some results concerning the appropriate trace spaces [8, 9] and the framework of boundary integral operators [24] for eddy current computations. In Section 3, we introduce the model problem. Section 4 is devoted to the variational formulation of the model problem. Therein we compute the optimality system and derive a space-time variational formulation. In Section 5, we discretize the optimality system in time and space in terms of a multiharmonic finite element-boundary element coupling method. The construction of a parameter-robust preconditioner for the discretized problem is addressed in Section 6. Finally, the results presented in Section 7 prove that the discretization scheme is convergent and provides the expected order of convergence.

2. Preliminaries.

Throughout this work, c is a generic constant that is independent of any discretization (h, N) and model parameters (ω, σ, ν, and λ). Furthermore, we use the generic constant C that is independent of h and N, but may depend on the other parameters.

2.1. Differential operators and traces.

Throughout this work, we use boldface letters to denote vectors and vector-valued functions. In this section, Ω is a generic bounded Lipschitz polyhedral domain of $\mathbb{R}^3$. We denote by Γ its boundary and by $\mathbf{n}$ the unit outward normal to Ω. Let $(\cdot,\cdot)_{L_2(\Omega)}$ be the inner product in $L_2(\Omega)$ and $\|\cdot\|_{L_2(\Omega)}$ the corresponding norm. Furthermore, we denote the product space by $\mathbf{L_2}(\Omega) := L_2(\Omega)^3$. The underlying Hilbert space is the space

$$\mathbf{H}(\mathbf{curl}\,,\Omega) := \{\mathbf{v} \in \mathbf{L_2}(\Omega) : \mathbf{curl}\,\mathbf{v} \in \mathbf{L_2}(\Omega)\}\,,$$

endowed with the graph norm

$$\|\mathbf{v}\|^2_{\mathbf{H}(\mathbf{curl}\,,\Omega)} := \|\mathbf{v}\|^2_{\mathbf{L_2}(\Omega)} + \|\mathbf{curl}\,\mathbf{v}\|^2_{\mathbf{L_2}(\Omega)}.$$

For the traces of a function $\mathbf{u} \in \mathbf{H}(\mathbf{curl}\,,\Omega)$, we fix the following notations: Let γ_D and γ_N denote the Dirichlet trace $\gamma_D\mathbf{u} := \mathbf{n} \times (\mathbf{u} \times \mathbf{n})$ and the Neumann trace $\gamma_N\mathbf{u} := \mathbf{curl}\,\mathbf{u} \times \mathbf{n}$ on the interface Γ, respectively. For the definition of the appropriate trace spaces, we use the definitions of the surface differential operators $\mathbf{grad}_\Gamma, \mathbf{curl}\,_\Gamma, \mathrm{curl}_\Gamma, \mathrm{div}_\Gamma$; see, e.g., [8, 9]. The appropriate trace spaces for polyhedral domains have been introduced by Buffa and Ciarlet in [8, 9]. The spaces for the Dirichlet trace γ_D and the Neumann trace γ_N are given by

$$\mathbf{H}_\perp^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) := \{\boldsymbol{\lambda} \in \mathbf{H}_\perp^{-\frac{1}{2}}(\Gamma), \mathrm{curl}_\Gamma \boldsymbol{\lambda} \in H^{-\frac{1}{2}}(\Gamma)\}$$

and

$$\mathbf{H}_\parallel^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) := \{\boldsymbol{\lambda} \in \mathbf{H}_\parallel^{-\frac{1}{2}}(\Gamma), \mathrm{div}_\Gamma \boldsymbol{\lambda} \in H^{-\frac{1}{2}}(\Gamma)\},$$

respectively. These spaces are equipped with the corresponding graph norms. Furthermore, $\mathbf{H}_\perp^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma)$ is the dual of $\mathbf{H}_\parallel^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ and vice versa. The corresponding duality

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product is the extension of the $\mathbf{L_2}(\Gamma)$ duality product, and in the following it will be denoted by a subscript τ

$$\langle \cdot, \cdot \rangle_\tau := \langle \cdot, \cdot \rangle_{\mathbf{H}_{\parallel}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)\times \mathbf{H}_{\perp}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Gamma)} .$$

We also need the space

$$\mathbf{H}_{\parallel}^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0,\Gamma) := \left\{ \boldsymbol{\lambda} \in \mathbf{H}_{\parallel}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma) : \mathrm{div}_\Gamma \boldsymbol{\lambda} = 0 \right\}$$

that turns out to be the correct space for the Neumann trace in our setting.

For $\mathbf{u} \in \mathbf{H}(\mathbf{curl}\,\mathbf{curl}\,, \mathbb{R}^3\backslash\Omega) := \{\mathbf{u} \in \mathbf{H}(\mathbf{curl}\,, \mathbb{R}^3\backslash\Omega) : \mathbf{curl}\,\mathbf{curl}\,\mathbf{u} \in \mathbf{L_2}(\mathbb{R}^3\backslash\Omega)\}$, the integration by parts formula for the exterior domain $\mathbb{R}^3\backslash\Omega$ holds

$$\langle \gamma_N \mathbf{u}, \gamma_D \mathbf{v} \rangle_\tau = -(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v})_{\mathbf{L_2}(\mathbb{R}^3\backslash\Omega)} + (\mathbf{curl}\,\mathbf{curl}\,\mathbf{u}, \mathbf{v})_{\mathbf{L_2}(\mathbb{R}^3\backslash\Omega)}. \tag{2.1}$$

The Dirichlet and Neumann trace can be extended to continuous mappings:

LEMMA 2.1 ([8, 9, 24]). *The trace operators*

$$\gamma_D : \mathbf{H}(\mathbf{curl}\,, \Omega) \to \mathbf{H}_{\perp}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Gamma) \quad \textit{and} \quad \gamma_N : \mathbf{H}(\mathbf{curl}\,\mathbf{curl}\,, \Omega) \to \mathbf{H}_{\parallel}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)$$

are linear, continuous and surjective.

For more details, we refer the reader to [8, 9] for the precise definition of the trace spaces $\mathbf{H}_{\parallel}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)$ and $\mathbf{H}_{\perp}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Gamma)$ and the corresponding analytical framework.

2.2. Boundary integral operators and the Calderon projection. In order to deal with expressions on the interface Γ between the bounded and unbounded domains, we use the framework of symmetric FEM-BEM coupling for eddy current problems; see [24]. The boundary integral equations for the exterior problem emerge from a representation formula. In the case of Maxwell's equations, this is the Stratton-Chu formula for the exterior domain. Taking into account that $\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} = \mathbf{0}$ and $\mathrm{div}\,\mathbf{u} = 0$ in the exterior domain, the solution is given by

$$\begin{aligned}\mathbf{u}(\mathbf{x}) = &\int_\Gamma (\mathbf{n} \times \mathbf{curl}\ \mathbf{u})(\mathbf{y}) E(\mathbf{x},\mathbf{y})\, \mathrm{dS}_\mathbf{y} - \mathbf{curl}_\mathbf{x} \int_\Gamma (\mathbf{n} \times \mathbf{u})(\mathbf{y}) E(\mathbf{x},\mathbf{y})\, \mathrm{dS}_\mathbf{y} \\ &+ \nabla_\mathbf{x} \int_\Gamma (\mathbf{n} \cdot \mathbf{u})(\mathbf{y}) E(\mathbf{x},\mathbf{y})\, \mathrm{dS}_\mathbf{y},\end{aligned}$$

where $E(\cdot,\cdot)$ is the fundamental solution of the Laplacian in three dimensions given by

$$E(\mathbf{x},\mathbf{y}) := \frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{y}|}, \quad \mathbf{x},\mathbf{y} \in \mathbb{R}^3, \mathbf{x} \neq \mathbf{y}.$$

Introducing the notations

$$\begin{aligned}\boldsymbol{\psi}_A(\mathbf{u})(\mathbf{x}) &:= \int_\Gamma \mathbf{u}(\mathbf{y}) E(\mathbf{x},\mathbf{y})\, \mathrm{dS}_\mathbf{y}, \\ \psi_V(\mathbf{n}\cdot\mathbf{u})(\mathbf{x}) &:= \int_\Gamma (\mathbf{n}\cdot\mathbf{u})(\mathbf{y}) E(\mathbf{x},\mathbf{y})\, \mathrm{dS}_\mathbf{y}, \\ \boldsymbol{\psi}_M(\mathbf{n}\times\mathbf{u})(\mathbf{x}) &:= \mathbf{curl}_\mathbf{x} \int_\Gamma (\mathbf{n}\times\mathbf{u})(\mathbf{y}) E(\mathbf{x},\mathbf{y})\, \mathrm{dS}_\mathbf{y},\end{aligned}$$

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we can rewrite the representation formula as

$$\mathbf{u} = \boldsymbol{\psi}_M[\gamma_D \mathbf{u}] - \boldsymbol{\psi}_A[\gamma_N \mathbf{u}] - \nabla\psi_V[\gamma_{\mathbf{n}}\mathbf{u}]. \tag{2.2}$$

Taking the Dirichlet and the Neumann trace in the representation formula (2.2) and deriving a variational framework, allows us to state a Calderon mapping in a weak setting:

$$\begin{aligned} \langle \boldsymbol{\mu}, \gamma_D \mathbf{u}\rangle_\tau &= \langle \boldsymbol{\mu}, \mathbf{C}(\gamma_D \mathbf{u})\rangle_\tau - \langle \boldsymbol{\mu}, \mathbf{A}(\gamma_N \mathbf{u})\rangle_\tau\,, \qquad \forall \boldsymbol{\mu} \in \mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0, \Gamma),\\ \langle \gamma_N \mathbf{u}, \boldsymbol{\theta}\rangle_\tau &= \langle \mathbf{N}(\gamma_D \mathbf{u}), \boldsymbol{\theta}\rangle_\tau - \langle \mathbf{B}(\gamma_N \mathbf{u}), \boldsymbol{\theta}\rangle_\tau\,, \qquad \forall \boldsymbol{\theta} \in \mathbf{H}_{\perp}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma), \end{aligned} \tag{2.3}$$

where the well-known boundary integral operators are given by

$$\begin{aligned} \mathbf{A}\boldsymbol{\lambda} &:= \gamma_D \boldsymbol{\psi}_{\mathbf{A}}(\boldsymbol{\lambda}), \qquad & \mathbf{B}\boldsymbol{\lambda} &:= \gamma_N \boldsymbol{\psi}_{\mathbf{A}}(\boldsymbol{\lambda}),\\ \mathbf{C}\boldsymbol{\mu} &:= \gamma_D \boldsymbol{\psi}_{\mathbf{M}}(\boldsymbol{\mu}), \qquad & \mathbf{N}\boldsymbol{\mu} &:= \gamma_N \boldsymbol{\psi}_{\mathbf{M}}(\boldsymbol{\mu}). \end{aligned}$$

In the following we collect several useful results; see [24]. The mappings

$$\begin{aligned} \mathbf{A} &: \mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{H}_{\perp}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma),\\ \mathbf{B} &: \mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma),\\ \mathbf{C} &: \mathbf{H}_{\perp}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) \to \mathbf{H}_{\perp}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma),\\ \mathbf{N} &: \mathbf{H}_{\perp}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) \to \mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) \end{aligned}$$

are linear and bounded. The bilinear form on $\mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0, \Gamma)$ induced by the operator $\mathbf{A}$ is symmetric and positive definite, i.e.,

$$\langle \boldsymbol{\lambda}, \mathbf{A}\boldsymbol{\lambda}\rangle_\tau \geq c_1^{\mathbf{A}} \|\boldsymbol{\lambda}\|^2_{\mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)}, \qquad \forall \boldsymbol{\lambda} \in \mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0, \Gamma).$$

The bilinear form on $\mathbf{H}_{\perp}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma)$ induced by the operator $\mathbf{N}$ is symmetric and negative semi-definite, i.e.,

$$-\langle \mathbf{N}\boldsymbol{\mu}, \boldsymbol{\mu}\rangle_\tau \geq c_1^{\mathbf{N}} \|\mathrm{curl}_\Gamma \boldsymbol{\mu}\|^2_{H^{-\frac{1}{2}}(\Gamma)}, \qquad \forall \boldsymbol{\mu} \in \mathbf{H}_{\perp}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma).$$

We have the symmetry property

$$\langle \mathbf{B}(\boldsymbol{\mu}), \boldsymbol{\lambda}\rangle_\tau = \langle \boldsymbol{\mu}, (\mathbf{C} - \mathbf{Id})(\boldsymbol{\lambda})\rangle_\tau, \qquad \forall \boldsymbol{\mu} \in \mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0, \Gamma), \boldsymbol{\lambda} \in \mathbf{H}_{\perp}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma).$$

3. The model problem. In this work we consider an optimal control problem with distributed control of the form: find the state $\mathbf{y}$ and the control $\mathbf{u}$ that minimizes the cost functional

$$J(\mathbf{y}, \mathbf{u}) = \frac{1}{2} \int_{\Omega_1 \times (0,T)} |\mathbf{y} - \mathbf{y_d}|^2 \mathrm{d}\mathbf{x}\, \mathrm{d}t + \frac{\lambda}{2} \int_{\Omega_1 \times (0,T)} |\mathbf{u}|^2 \mathrm{d}\mathbf{x}\, \mathrm{d}t, \tag{3.1}$$

subject to the state equations

$$\begin{array}{rcll} \sigma \frac{\partial \mathbf{y}}{\partial t} + \mathbf{curl}\,(\nu\, \mathbf{curl}\,\mathbf{y}) &=& \mathbf{u} & \text{in } \Omega_1 \times (0,T),\\ \mathbf{curl}\,(\mathbf{curl}\,\mathbf{y}) &=& \mathbf{0} & \text{in } \Omega_2 \times (0,T),\\ \mathrm{div}\,\mathbf{y} &=& 0 & \text{in } \Omega_2 \times (0,T),\\ \mathbf{y} &=& \mathcal{O}(|\mathbf{x}|^{-1}) & \text{for } |\mathbf{x}| \to \infty,\\ \mathbf{curl}\,\mathbf{y} &=& \mathcal{O}(|\mathbf{x}|^{-1}) & \text{for } |\mathbf{x}| \to \infty,\\ \mathbf{y}(0) &=& \mathbf{y}(T) & \text{in } \Omega_1,\\ \mathbf{y}|_{\Omega_1} \times \mathbf{n} &=& \mathbf{y}|_{\Omega_2} \times \mathbf{n} & \text{on } \Gamma \times (0,T),\\ \nu\, \mathbf{curl}\,\mathbf{y}|_{\Omega_1} \times \mathbf{n} &=& \mathbf{curl}\,\mathbf{y}|_{\Omega_2} \times \mathbf{n} & \text{on } \Gamma \times (0,T). \end{array} \tag{3.2}$$

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Here $\mathbf{y_d} \in L_2((0,T), \mathbf{L_2}(\Omega_1))$ is the given desired state and assumed to be multiharmonic. The regularization parameter λ is supposed to be positive. The computational domain $\Omega = \mathbb{R}^3$ is split into a conducting subdomain Ω_1 and its non-conducting complement Ω_2. The conducting domain Ω_1 is assumed to be a simply connected Lipschitz polyhedron, whereas the non-conducting domain Ω_2 is the complement of Ω_1 in $\mathbb{R}^3$, i.e., $\Omega_2 = \mathbb{R}^3 \backslash \overline{\Omega}_1$. Furthermore, we denote by Γ the interface of the two subdomains, $\Gamma = \overline{\Omega}_1 \cap \overline{\Omega}_2$. The exterior unit normal vector of Ω_1 on Γ is denoted by $\mathbf{n}$, i.e., the vector $\mathbf{n}$ points from Ω_1 into Ω_2; see Figure 3.1.

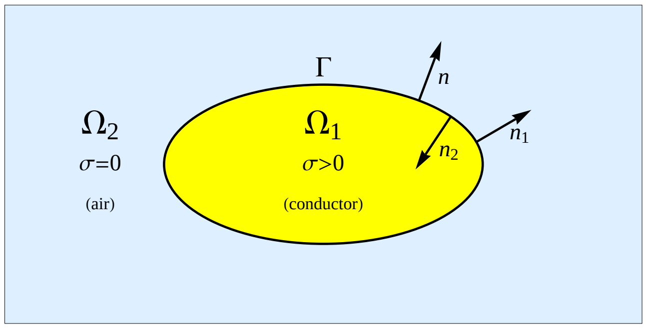

FIG. 3.1. *Decomposition of the computational domain* $\Omega = \mathbb{R}^3$.

The reluctivity $\nu = \nu(\mathbf{x})$ is supposed to be uniformly positive and independent of $|\mathbf{curl\, u}|$, i.e., we assume the eddy current problem (3.2) to be linear. Due to scaling arguments, it can always be achieved that $\nu = 1$ in Ω_2. The conductivity σ is zero in the non-conducting domain Ω_2 and piecewise constant and uniformly positive in the conductor Ω_1:

$$\begin{aligned} &\overline{\sigma} \geq \sigma(\mathbf{x}) \geq \underline{\sigma} > 0 \ \text{ a.e. in } \Omega_1 \quad \text{and} \quad \sigma(\mathbf{x}) = 0 \ \text{ a.e. in } \Omega_2, \\ &\overline{\nu} \geq \nu(\mathbf{x}) \geq \underline{\nu} > 0 \ \text{ a.e. in } \Omega_1 \quad \text{and} \quad \nu(\mathbf{x}) = 1 \ \text{ a.e. in } \Omega_2. \end{aligned} \tag{3.3}$$

Existence and uniqueness results for linear and non-linear eddy current problems in unbounded domains are provided in [29]. Therein the space of weakly divergence-free functions $\mathbf{V}$ is introduced as a subspace of $\mathbf{H}(\mathbf{curl}\,, \Omega_1)$. Furthermore, it is shown that the state equation (3.2) has a unique solution $\mathbf{y} \in L_2((0,T), \mathbf{V})$ with a weak derivative $\partial\mathbf{y}/\partial t \in L_2((0,T), \mathbf{V}^*)$. Another approach to prove existence and uniqueness is given by Arnold and Harrach [2]. Due to the unique solvability of the state equation (3.2), the existence of a solution operator $\mathbf{S}$ mapping $\mathbf{u}$ to $\mathbf{y}$ (i.e., $\mathbf{S}(\mathbf{u}) = \mathbf{y}$) is guaranteed. By standard arguments (see, e.g., [44]) it follows that the unconstrained minimization problem: find the control $\mathbf{u} \in L_2((0,T), \mathbf{L_2}(\Omega))$ that minimizes the cost functional

$$\frac{1}{2}\int_{\Omega_1 \times (0,T)} |\mathbf{S}(\mathbf{u}) - \mathbf{y_d}|^2 \mathrm{d}\mathbf{x}\, \mathrm{d}t + \frac{\lambda}{2}\int_{\Omega_1 \times (0,T)} |\mathbf{u}|^2 \mathrm{d}\mathbf{x}\, \mathrm{d}t$$

is also uniquely solvable.

4. The variational formulation. In order to solve our minimization problem, we formulate the optimality system, also called the Karush-Kuhn-Tucker system; see, e.g., [44]. Therefore, we formally consider the Lagrangian functional

$$\mathcal{L}(\mathbf{y}, \mathbf{u}, \mathbf{p}) := \mathcal{J}(\mathbf{y}, \mathbf{u}) + \int_{\Omega \times (0,T)} \left(\sigma \frac{\partial \mathbf{y}}{\partial t} + \mathbf{curl}\,(\nu\, \mathbf{curl}\, \mathbf{y}) - \mathbf{u} \right) \cdot \mathbf{p} \, \mathrm{d}\mathbf{x}\, \mathrm{d}t.$$

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Deriving the necessary optimality conditions

$$\text{Find } \mathbf{y},\mathbf{u},\mathbf{p}: \quad \begin{cases} \nabla_{\mathbf{p}}\mathcal{L}(\mathbf{y},\mathbf{u},\mathbf{p}) = \mathbf{0}, \\ \nabla_{\mathbf{y}}\mathcal{L}(\mathbf{y},\mathbf{u},\mathbf{p}) = \mathbf{0}, \\ \nabla_{\mathbf{u}}\mathcal{L}(\mathbf{y},\mathbf{u},\mathbf{p}) = \mathbf{0}, \end{cases}$$

yields a system of partial differential equations. We observe that $\mathbf{u} = \lambda^{-1}\mathbf{p}$ in $\Omega_1 \times (0,T)$, and hence we can eliminate the control. Therefore, we end up with the following reduced optimality system: find the state $\mathbf{y}$ and the co-state $\mathbf{p}$ such that

$$\begin{aligned} \sigma\frac{\partial \mathbf{y}}{\partial t} + \mathbf{curl}\,(\nu\,\mathbf{curl}\,\mathbf{y}) - \lambda^{-1}\mathbf{p} &= \mathbf{0} && \text{in } \Omega_1 \times (0,T), \\ \mathbf{curl}\,(\mathbf{curl}\,\mathbf{y}) &= \mathbf{0} && \text{in } \Omega_2 \times (0,T), \\ \operatorname{div}\mathbf{y} &= 0 && \text{in } \Omega_2 \times (0,T), \\ -\sigma\frac{\partial \mathbf{p}}{\partial t} + \mathbf{curl}\,(\nu\,\mathbf{curl}\,\mathbf{p}) + \mathbf{y} - \mathbf{y_d} &= \mathbf{0} && \text{in } \Omega_1 \times (0,T), \\ \mathbf{curl}\,(\mathbf{curl}\,\mathbf{p}) &= \mathbf{0} && \text{in } \Omega_2 \times (0,T), \\ \operatorname{div}\mathbf{p} &= 0 && \text{in } \Omega_2 \times (0,T), \\ \mathbf{p} = \mathcal{O}(|\mathbf{x}|^{-1}), \quad \mathbf{y} &= \mathcal{O}(|\mathbf{x}|^{-1}) && \text{for } |\mathbf{x}| \to \infty, \\ \mathbf{curl}\,\mathbf{p} = \mathcal{O}(|\mathbf{x}|^{-1}), \quad \mathbf{curl}\,\mathbf{y} &= \mathcal{O}(|\mathbf{x}|^{-1}) && \text{for } |\mathbf{x}| \to \infty, \\ \mathbf{p}(0) = \mathbf{p}(T), \quad \mathbf{y}(0) &= \mathbf{y}(T) && \text{in } \Omega_1. \end{aligned} \tag{4.1}$$

In the usual manner, we derive a space-time variational formulation. Multiplying (4.1) by space and time dependent test functions $(\mathbf{v},\mathbf{w}) = (\mathbf{v}(\mathbf{x},t),\mathbf{w}(\mathbf{x},t)) \in L_2((0,T),\mathbf{W_2})$ and integrating over the space-time domain $\Omega \times (0,T)$, we arrive at the following variational form: find $(\mathbf{y},\mathbf{p}) \in H^1((0,T),\mathbf{W_1})$ such that

$$\begin{aligned} &\int_0^T \left(\sigma\frac{\partial \mathbf{y}}{\partial t},\mathbf{v}\right)_{\mathbf{L_2}(\Omega_1)} \mathrm{d}t + \int_0^T (\nu\,\mathbf{curl}\,\mathbf{y},\mathbf{curl}\,\mathbf{v})_{\mathbf{L_2}(\Omega_1)}\,\mathrm{d}t \\ &\quad - \int_0^T (\nu\,\mathbf{curl}\,\mathbf{y},\mathbf{curl}\,\mathbf{v})_{\mathbf{L_2}(\Omega_2)}\,\mathrm{d}t - \frac{1}{\lambda}\int_0^T (\mathbf{p},\mathbf{v})_{\mathbf{L_2}(\Omega_1)}\,\mathrm{d}t \\ &\qquad = 0, \\ &-\int_0^T \left(\sigma\frac{\partial \mathbf{p}}{\partial t},\mathbf{w}\right)_{\mathbf{L_2}(\Omega_1)} \mathrm{d}t + \int_0^T (\nu\,\mathbf{curl}\,\mathbf{p},\mathbf{curl}\,\mathbf{w})_{\mathbf{L_2}(\Omega_1)}\,\mathrm{d}t \\ &\quad - \int_0^T (\nu\,\mathbf{curl}\,\mathbf{p},\mathbf{curl}\,\mathbf{w})_{\mathbf{L_2}(\Omega_2)}\,\mathrm{d}t + \int_0^T (\mathbf{y},\mathbf{w})_{\mathbf{L_2}(\Omega_1)}\,\mathrm{d}t \\ &\qquad = \int_0^T (\mathbf{y_d},\mathbf{w})_{\mathbf{L_2}(\Omega_1)}\,\mathrm{d}t, \end{aligned} \tag{4.2}$$

with the appropriate decay and periodicity conditions of (4.1). Here $\mathbf{W_1}$ and $\mathbf{W_2}$ are appropriate weighted Sobolev spaces on $\mathbb{R}^3$; cf. [24].

5. Discretization scheme. The space-time variational formulation (4.2) is the starting point of our discretization in time and space. We discretize in time in terms of a multiharmonic approach. For the resulting system of frequency domain equations, a symmetric coupling method is applied to both the state variable and the co-state variable of each mode k. This coupling method allows us to reduce the unbounded exterior domain Ω_2 to the boundary Γ. The resulting variational formulation is discretized by standard finite and boundary elements.

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5.1. Reduction of the exterior domain to the boundary. Applying the integration by parts formula (2.1) in the exterior domain Ω_2 and using the fact that

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{y} = \mathbf{0} \quad \text{and} \quad \mathbf{curl}\,\mathbf{curl}\,\mathbf{p} = \mathbf{0} \quad \text{in } \Omega_2,$$

allows us to reduce the variational problem to one that is just living on the closure of the conductivity domain Ω_1:

$$\begin{aligned}
&\int_0^T \left(\sigma \frac{\partial \mathbf{y}}{\partial t}, \mathbf{v}\right)_{\mathbf{L_2}(\Omega_1)} \mathrm{d}t + \int_0^T (\nu\, \mathbf{curl}\,\mathbf{y}, \mathbf{curl}\,\mathbf{v})_{\mathbf{L_2}(\Omega_1)}\, \mathrm{d}t \\
&\qquad - \int_0^T \langle \gamma_N \mathbf{y}, \gamma_D \mathbf{v}\rangle_\tau \mathrm{d}t - \frac{1}{\lambda}\int_0^T (\mathbf{p}, \mathbf{v})_{\mathbf{L_2}(\Omega_1)}\, \mathrm{d}t = 0, \\
-&\int_0^T \left(\sigma \frac{\partial \mathbf{p}}{\partial t}, \mathbf{w}\right)_{\mathbf{L_2}(\Omega_1)} \mathrm{d}t + \int_0^T (\nu\, \mathbf{curl}\,\mathbf{p}, \mathbf{curl}\,\mathbf{w})_{\mathbf{L_2}(\Omega_1)}\, \mathrm{d}t \\
&\qquad - \int_0^T \langle \gamma_N \mathbf{p}, \gamma_D \mathbf{w}\rangle_\tau \mathrm{d}t + \int_0^T (\mathbf{y}, \mathbf{w})_{\mathbf{L_2}(\Omega_1)}\, \mathrm{d}t = \int_0^T (\mathbf{y_d}, \mathbf{w})_{\mathbf{L_2}(\Omega_1)}\, \mathrm{d}t.
\end{aligned} \tag{5.1}$$

Later, the expressions on the interface Γ are dealt with in terms of a symmetrical coupling method [13].

5.2. Multiharmonic discretization. Let us assume that the desired state $\mathbf{y_d}$ is multiharmonic, i.e., $\mathbf{y_d}$ has the form

$$\mathbf{y_d} = \sum_{k=0}^{N} \mathbf{y^c_{d,k}} \cos(k\omega t) + \mathbf{y^s_{d,k}} \sin(k\omega t), \tag{5.2}$$

where the Fourier coefficients are given by the formulas

$$\mathbf{y^c_{d,k}} = \frac{2}{T}\int_0^T \mathbf{y_d} \cos(k\omega t)\mathrm{d}t \quad \text{and} \quad \mathbf{y^s_{d,k}} = \frac{2}{T}\int_0^T \mathbf{y_d} \sin(k\omega t)\mathrm{d}t.$$

We mention that the multiharmonic representation (5.2) can also be seen as an approximation of a time-periodic desired state $\mathbf{y_d}$ by a truncated Fourier series. Due to the linearity of the optimality system (4.1), the state $\mathbf{y}$ and the co-state $\mathbf{p}$ are multiharmonic as well and therefore also have representations in terms of a truncated Fourier series, i.e.,

$$\mathbf{y} = \sum_{k=0}^{N} \mathbf{y^c_k} \cos(k\omega t) + \mathbf{y^s_k} \sin(k\omega t) \quad \text{and} \quad \mathbf{p} = \sum_{k=0}^{N} \mathbf{p^c_k} \cos(k\omega t) + \mathbf{p^s_k} \sin(k\omega t), \tag{5.3}$$

with unknown coefficients $(\mathbf{y^c_k}, \mathbf{y^s_k})$ and $(\mathbf{p^c_k}, \mathbf{p^s_k})$. Using the multiharmonic representation (5.3), we can state the optimality system (5.1) in the frequency domain. Consequently, the problem that we deal with reads as follows: for each mode $k = 0, 1, \ldots, N$, find the Fourier coefficients $(\mathbf{y^c_k}, \mathbf{y^s_k}, \mathbf{p^c_k}, \mathbf{p^s_k}) \in \mathbf{H}(\mathbf{curl}\,, \Omega_1)^4$ such that

$$\begin{aligned}
&k\omega(\sigma \mathbf{y^s_k}, \mathbf{v^c_k})_{\mathbf{L_2}(\Omega_1)} + (\nu\, \mathbf{curl}\,\mathbf{y^c_k}, \mathbf{curl}\,\mathbf{v^c_k})_{\mathbf{L_2}(\Omega_1)} \\
&\qquad -\langle \gamma_N \mathbf{y^c_k}, \gamma_D \mathbf{v^c_k}\rangle_\tau - \lambda^{-1}(\mathbf{p^c_k}, \mathbf{v^c_k})_{\mathbf{L_2}(\Omega_1)} = 0, \\
-&k\omega(\sigma \mathbf{y^c_k}, \mathbf{v^s_k})_{\mathbf{L_2}(\Omega_1)} + (\nu\, \mathbf{curl}\,\mathbf{y^s_k}, \mathbf{curl}\,\mathbf{v^s_k})_{\mathbf{L_2}(\Omega_1)} \\
&\qquad -\langle \gamma_N \mathbf{y^s_k}, \gamma_D \mathbf{v^s_k}\rangle_\tau - \lambda^{-1}(\mathbf{p^s_k}, \mathbf{v^s_k})_{\mathbf{L_2}(\Omega_1)} = 0, \\
-&k\omega(\sigma \mathbf{p^s_k}, \mathbf{w^c_k})_{\mathbf{L_2}(\Omega_1)} + (\nu\, \mathbf{curl}\,\mathbf{p^c_k}, \mathbf{curl}\,\mathbf{w^c_k})_{\mathbf{L_2}(\Omega_1)} \\
&\qquad -\langle \gamma_N \mathbf{p^c_k}, \gamma_D \mathbf{w^c_k}\rangle_\tau + (\mathbf{y^c_k}, \mathbf{w^c_k})_{\mathbf{L_2}(\Omega_1)} = (\mathbf{y^c_{d,k}}, \mathbf{w^c_k})_{\mathbf{L_2}(\Omega_1)}, \\
&k\omega(\sigma \mathbf{p^c_k}, \mathbf{w^s_k})_{\mathbf{L_2}(\Omega_1)} + (\nu\, \mathbf{curl}\,\mathbf{p^s_k}, \mathbf{curl}\,\mathbf{w^s_k})_{\mathbf{L_2}(\Omega_1)} \\
&\qquad -\langle \gamma_N \mathbf{p^s_k}, \gamma_D \mathbf{w^s_k}\rangle_\tau + (\mathbf{y^s_k}, \mathbf{w^s_k})_{\mathbf{L_2}(\Omega_1)} = (\mathbf{y^s_{d,k}}, \mathbf{w^s_k})_{\mathbf{L_2}(\Omega_1)},
\end{aligned} \tag{5.4}$$

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for all test functions $(\mathbf{v}_\mathbf{k}^\mathbf{c}, \mathbf{v}_\mathbf{k}^\mathbf{s}, \mathbf{w}_\mathbf{k}^\mathbf{c}, \mathbf{w}_\mathbf{k}^\mathbf{s}) \in \mathbf{H}(\mathbf{curl}\,, \Omega_1)^4$. Note that the mode $k = 0$ has to be treated separately. Clearly we do not have to solve for $\mathbf{p}_\mathbf{0}^\mathbf{s}$ and $\mathbf{y}_\mathbf{0}^\mathbf{s}$, since $\sin(0\omega t) = 0$, and therefore, for $k = 0$, (5.4) reduces to a 2×2 system for determining the Fourier coefficients $\mathbf{p}_\mathbf{0}^\mathbf{c}$ and $\mathbf{y}_\mathbf{0}^\mathbf{c}$. Due to the $L_2(0, T)$ orthogonality of the sine and cosine functions, we obtain a total decoupling of the Fourier coefficients with respect to the modes k. Therefore, for the purpose of solving, it is sufficient to have a look at a time-harmonic approximation, i.e., $\mathbf{y_d} = \mathbf{y}_\mathbf{d}^\mathbf{c} \cos(\omega t) + \mathbf{y}_\mathbf{d}^\mathbf{s} \sin(\omega t)$. Consequently, in the next sections, we analyze the following variational problem: find $(\mathbf{y}^\mathbf{c}, \mathbf{y}^\mathbf{s}, \mathbf{p}^\mathbf{c}, \mathbf{p}^\mathbf{s}) \in \mathbf{H}(\mathbf{curl}\,, \Omega_1)^4$ such that

$$
\begin{aligned}
\omega(\sigma \mathbf{y}^\mathbf{s}, \mathbf{v}^\mathbf{c})_{\mathbf{L_2}(\Omega_1)} &+ (\nu\, \mathbf{curl}\, \mathbf{y}^\mathbf{c}, \mathbf{curl}\, \mathbf{v}^\mathbf{c})_{\mathbf{L_2}(\Omega_1)} \\
&- \langle \gamma_N \mathbf{y}^\mathbf{c}, \gamma_D \mathbf{v}^\mathbf{c} \rangle_\tau - \lambda^{-1} (\mathbf{p}^\mathbf{c}, \mathbf{v}^\mathbf{c})_{\mathbf{L_2}(\Omega_1)} = 0, \\
-\omega(\sigma \mathbf{y}^\mathbf{c}, \mathbf{v}^\mathbf{s})_{\mathbf{L_2}(\Omega_1)} &+ (\nu\, \mathbf{curl}\, \mathbf{y}^\mathbf{s}, \mathbf{curl}\, \mathbf{v}^\mathbf{s})_{\mathbf{L_2}(\Omega_1)} \\
&- \langle \gamma_N \mathbf{y}^\mathbf{s}, \gamma_D \mathbf{v}^\mathbf{s} \rangle_\tau - \lambda^{-1} (\mathbf{p}^\mathbf{s}, \mathbf{v}^\mathbf{s})_{\mathbf{L_2}(\Omega_1)} = 0, \\
-\omega(\sigma \mathbf{p}^\mathbf{s}, \mathbf{w}^\mathbf{c})_{\mathbf{L_2}(\Omega_1)} &+ (\nu\, \mathbf{curl}\, \mathbf{p}^\mathbf{c}, \mathbf{curl}\, \mathbf{w}^\mathbf{c})_{\mathbf{L_2}(\Omega_1)} \\
&- \langle \gamma_N \mathbf{p}^\mathbf{c}, \gamma_D \mathbf{w}^\mathbf{c} \rangle_\tau + (\mathbf{y}^\mathbf{c}, \mathbf{w}^\mathbf{c})_{\mathbf{L_2}(\Omega_1)} = (\mathbf{y}_\mathbf{d}^\mathbf{c}, \mathbf{w}^\mathbf{c})_{\mathbf{L_2}(\Omega_1)}, \\
\omega(\sigma \mathbf{p}^\mathbf{c}, \mathbf{w}^\mathbf{s})_{\mathbf{L_2}(\Omega_1)} &+ (\nu\, \mathbf{curl}\, \mathbf{p}^\mathbf{s}, \mathbf{curl}\, \mathbf{w}^\mathbf{s})_{\mathbf{L_2}(\Omega_1)} \\
&- \langle \gamma_N \mathbf{p}^\mathbf{s}, \gamma_D \mathbf{w}^\mathbf{s} \rangle_\tau + (\mathbf{y}^\mathbf{s}, \mathbf{w}^\mathbf{s})_{\mathbf{L_2}(\Omega_1)} = (\mathbf{y}_\mathbf{d}^\mathbf{s}, \mathbf{w}^\mathbf{s})_{\mathbf{L_2}(\Omega_1)},
\end{aligned}
$$

for all test functions $(\mathbf{v}^\mathbf{c}, \mathbf{v}^\mathbf{s}, \mathbf{w}^\mathbf{c}, \mathbf{w}^\mathbf{s}) \in \mathbf{H}(\mathbf{curl}\,, \Omega_1)^4$.

5.3. Symmetric coupling method. We are now in a position to state the coupled variational problem, following the approach of Hiptmair in [24]. Using the Calderon map (2.3) and introducing the Neumann data as additional unknowns

$$
\boldsymbol{\lambda}^\mathbf{c} := \gamma_N \mathbf{y}^\mathbf{c}, \qquad \boldsymbol{\lambda}^\mathbf{s} := \gamma_N \mathbf{y}^\mathbf{s}, \qquad \boldsymbol{\eta}^\mathbf{c} := \gamma_N \mathbf{p}^\mathbf{c}, \qquad \boldsymbol{\eta}^\mathbf{s} := \gamma_N \mathbf{p}^\mathbf{s},
$$

allows us to state the eddy current problem in a framework that is suited for a FEM-BEM discretization. For simplicity, we introduce the abbreviation

$$
\begin{aligned}
\Upsilon &:= (\mathbf{y}^\mathbf{c}, \boldsymbol{\lambda}^\mathbf{c}, \mathbf{y}^\mathbf{s}, \boldsymbol{\lambda}^\mathbf{s}), & \Psi &:= (\mathbf{p}^\mathbf{c}, \boldsymbol{\eta}^\mathbf{c}, \mathbf{p}^\mathbf{s}, \boldsymbol{\eta}^\mathbf{s}), \\
\Phi &:= (\mathbf{w}^\mathbf{c}, \boldsymbol{\rho}^\mathbf{c}, \mathbf{w}^\mathbf{s}, \boldsymbol{\rho}^\mathbf{s}), & \Theta &:= (\mathbf{v}^\mathbf{c}, \boldsymbol{\mu}^\mathbf{c}, \mathbf{v}^\mathbf{s}, \boldsymbol{\mu}^\mathbf{s}).
\end{aligned}
$$

We mention that Υ represents the variables corresponding to the state $\mathbf{y}$, Ψ represents the variables corresponding to the adjoint state $\mathbf{p}$, and Φ and Θ are the corresponding test functions. According to the definition of Υ and Ψ, we introduce the appropriate product space

$$
\mathcal{W} := \mathbf{H}(\mathbf{curl}\,, \Omega_1) \times \mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0, \Gamma) \times \mathbf{H}(\mathbf{curl}\,, \Omega_1) \times \mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0, \Gamma).
$$

Therefore, we end up with the weak formulation of the reduced symmetric coupled optimality system: find $(\Upsilon, \Psi) \in \mathcal{W}^2$ such that

$$
\begin{aligned}
\omega(\sigma \mathbf{y^s}, \mathbf{v^c})_{\mathbf{L_2}(\Omega_1)} + (\nu\, \mathbf{curl}\, \mathbf{y^c}, \mathbf{curl}\, \mathbf{v^c})_{\mathbf{L_2}(\Omega_1)} - \lambda^{-1}(\mathbf{p^c}, \mathbf{v^c})_{\mathbf{L_2}(\Omega_1)}& \\
-\langle \mathbf{N}(\gamma_D \mathbf{y^c}), \gamma_D \mathbf{v^c}\rangle_\tau + \langle \mathbf{B}(\boldsymbol{\lambda}^\mathbf{c}), \gamma_D \mathbf{v^c}\rangle_\tau = 0&, \\
\langle \boldsymbol{\mu}^\mathbf{c}, (\mathbf{C} - \mathbf{Id})(\gamma_D \mathbf{y^c})\rangle_\tau - \langle \boldsymbol{\mu}^\mathbf{c}, \mathbf{A}(\boldsymbol{\lambda}^\mathbf{c})\rangle_\tau = 0&, \\
-\omega(\sigma \mathbf{y^c}, \mathbf{v^s})_{\mathbf{L_2}(\Omega_1)} + (\nu\, \mathbf{curl}\, \mathbf{y^s}, \mathbf{curl}\, \mathbf{v^s})_{\mathbf{L_2}(\Omega_1)} - \lambda^{-1}(\mathbf{p^s}, \mathbf{v^s})_{\mathbf{L_2}(\Omega_1)}& \\
-\langle \mathbf{N}(\gamma_D \mathbf{y^s}), \gamma_D \mathbf{v^s}\rangle_\tau + \langle \mathbf{B}(\boldsymbol{\lambda}^\mathbf{s}), \gamma_D \mathbf{v^s}\rangle_\tau = 0&, \\
\langle \boldsymbol{\mu}^\mathbf{s}, (\mathbf{C} - \mathbf{Id})(\gamma_D \mathbf{y^s})\rangle_\tau - \langle \boldsymbol{\mu}^\mathbf{s}, \mathbf{A}(\boldsymbol{\lambda}^\mathbf{s})\rangle_\tau = 0&, \\
-\omega(\sigma \mathbf{p^s}, \mathbf{w^c})_{\mathbf{L_2}(\Omega_1)} + (\nu\, \mathbf{curl}\, \mathbf{p^c}, \mathbf{curl}\, \mathbf{w^c})_{\mathbf{L_2}(\Omega_1)} + (\mathbf{y^c}, \mathbf{w^c})_{\mathbf{L_2}(\Omega_1)}& \\
-\langle \mathbf{N}(\gamma_D \mathbf{p^c}), \gamma_D \mathbf{w^c}\rangle_\tau + \langle \mathbf{B}(\boldsymbol{\eta}^\mathbf{c}), \gamma_D \mathbf{w^c}\rangle_\tau = (\mathbf{y^c_d}, \mathbf{w^c})_{\mathbf{L_2}(\Omega_1)}&, \\
\langle \boldsymbol{\rho}^\mathbf{c}, (\mathbf{C} - \mathbf{Id})(\gamma_D \mathbf{p^c})\rangle_\tau - \langle \boldsymbol{\rho}^\mathbf{c}, \mathbf{A}(\boldsymbol{\eta}^\mathbf{c})\rangle_\tau = 0&, \\
\omega(\sigma \mathbf{p^c}, \mathbf{w^s})_{\mathbf{L_2}(\Omega_1)} + (\nu\, \mathbf{curl}\, \mathbf{p^s}, \mathbf{curl}\, \mathbf{w^s})_{\mathbf{L_2}(\Omega_1)} + (\mathbf{y^s}, \mathbf{w^s})_{\mathbf{L_2}(\Omega_1)}& \\
-\langle \mathbf{N}(\gamma_D \mathbf{p^s}), \gamma_D \mathbf{w^s}\rangle_\tau + \langle \mathbf{B}(\boldsymbol{\eta}^\mathbf{s}), \gamma_D \mathbf{w^s}\rangle_\tau = (\mathbf{y^s_d}, \mathbf{w^s})_{\mathbf{L_2}(\Omega_1)}&, \\
\langle \boldsymbol{\rho}^\mathbf{s}, (\mathbf{C} - \mathbf{Id})(\gamma_D \mathbf{p^s})\rangle_\tau - \langle \boldsymbol{\rho}^\mathbf{s}, \mathbf{A}(\boldsymbol{\eta}^\mathbf{s})\rangle_\tau = 0&,
\end{aligned}
\tag{5.5}
$$

for all test functions $(\Phi, \Theta) \in \mathcal{W}^2$. For simplicity, we introduce the bilinear form $\mathcal{A}$ representing the latter variational problem:

$$\mathcal{A}((\Upsilon, \Psi), (\Phi, \Theta)) := a(\Upsilon, \Phi) + b(\Phi, \Psi) + b(\Upsilon, \Theta) - c(\Psi, \Theta),$$

where the bilinear forms a, b and c are given by

$$
\begin{aligned}
a(\Upsilon, \Phi) &= (\mathbf{y^c}, \mathbf{w^c})_{\mathbf{L_2}(\Omega_1)} + (\mathbf{y^s}, \mathbf{w^s})_{\mathbf{L_2}(\Omega_1)}, \\
b(\Upsilon, \Theta) &= \omega(\sigma \mathbf{y^s}, \mathbf{v^c})_{\mathbf{L_2}(\Omega_1)} - \omega(\sigma \mathbf{y^c}, \mathbf{v^s})_{\mathbf{L_2}(\Omega_1)} + \sum_{j \in \{c,s\}} (\nu\, \mathbf{curl}\, \mathbf{y^j}, \mathbf{curl}\, \mathbf{v^j})_{\mathbf{L_2}(\Omega_1)} \\
&\quad - \langle \mathbf{N}(\gamma_D \mathbf{y^j}), \gamma_D \mathbf{v^j}\rangle_\tau + \langle \mathbf{B}(\boldsymbol{\lambda}^\mathbf{j}), \gamma_D \mathbf{v^j}\rangle_\tau \\
&\quad + \left\langle \boldsymbol{\mu}^\mathbf{j}, (\mathbf{C} - \mathbf{Id})(\gamma_D \mathbf{y^j})\right\rangle_\tau - \left\langle \boldsymbol{\mu}^\mathbf{j}, \mathbf{A}(\boldsymbol{\lambda}^\mathbf{j})\right\rangle_\tau, \\
c(\Psi, \Theta) &= \lambda^{-1}(\mathbf{p^c}, \mathbf{v^c})_{\mathbf{L_2}(\Omega_1)} + \lambda^{-1}(\mathbf{p^s}, \mathbf{v^s})_{\mathbf{L_2}(\Omega_1)}.
\end{aligned}
$$

Using this notation, we can state (5.5) in the abstract form: Find $(\Upsilon, \Psi) \in \mathcal{W}^2$ such that

$$\mathcal{A}((\Upsilon, \Psi), (\Phi, \Theta)) = \sum_{j \in \{c,s\}} (\mathbf{y^j_d}, \mathbf{w^j})_{\mathbf{L_2}(\Omega_1)}, \tag{5.6}$$

for all test functions $(\Phi, \Theta) \in \mathcal{W}^2$. Indeed, the bilinear form $\mathcal{A}$ is symmetric and indefinite. Well-posedness of the variational problem (5.6) will be shown in the next section using the Theorem of Babuška-Aziz [3]. The variational formulation (5.6) is the starting point of the discretization in space.

REMARK 5.1. In the multiharmonic setting, the variational problem reads as follows: find $(\boldsymbol{\Upsilon}, \boldsymbol{\Psi}) \in \mathcal{W}^{2N+1}$, with $\boldsymbol{\Upsilon} = (\Upsilon_0, \dots, \Upsilon_N)$ and $\boldsymbol{\Psi} = (\Psi_0, \dots, \Psi_N)$ such that

$$\mathcal{A}^N((\boldsymbol{\Upsilon}, \boldsymbol{\Psi}), (\boldsymbol{\Phi}, \boldsymbol{\Theta})) = \sum_{k=0}^{N} \sum_{j \in \{c,s\}} (\mathbf{y^j_{d,k}}, \mathbf{w^j})_{\mathbf{L_2}(\Omega_1)}, \tag{5.7}$$

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for all test functions $(\mathbf{\Phi}, \mathbf{\Theta}) \in \mathcal{W}^{2N+1}$. Here the big bilinear form $\mathcal{A}^N$ is given by

$$\mathcal{A}^N((\mathbf{\Upsilon}, \mathbf{\Psi}), (\mathbf{\Phi}, \mathbf{\Theta})) := \sum_{k=0}^{N} \mathcal{A}_k((\Upsilon_k, \Psi_k), (\Phi_k, \Theta_k)),$$

where $\mathcal{A}_k$ denotes $\mathcal{A}$, with ω formally replaced by $k\omega$.

5.4. Discretization in space. We now use a quasi-uniform and shape-regular triangulation $\mathcal{T}_h$ of the domain Ω_1 with mesh size $h > 0$ with tetrahedral elements. $\mathcal{T}_h$ induces a mesh $\mathcal{K}_h$ of triangles on the boundary $\Gamma = \partial\Omega_1$. On these meshes we consider $\mathcal{ND}_1(\mathcal{T}_h)$, the Nédélec basis functions of order 1 [34, 35], a conforming finite element subspace of $\mathbf{H}(\mathbf{curl}\,, \Omega_1)$. Moreover, we use the space of divergence-free Raviart-Thomas [38] basis functions $\mathcal{RT}_0^0(\mathcal{K}_h) := \{\boldsymbol{\lambda}_\mathbf{h} \in \mathcal{RT}_0(\mathcal{K}_h), \mathrm{div}_\Gamma \boldsymbol{\lambda}_\mathbf{h} = 0\}$, a conforming finite element subspace of $\mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0, \Gamma)$. Furthermore, the discrete FE-BE subspace $\mathcal{W}_h$ of $\mathcal{W}$ is given by

$$\mathcal{W}_h := \mathcal{ND}_1(\mathcal{T}_h) \times \mathcal{RT}_0^0(\mathcal{K}_h) \times \mathcal{ND}_1(\mathcal{T}_h) \times \mathcal{RT}_0^0(\mathcal{K}_h).$$

The corresponding discrete variational problem is stated as: find $(\Upsilon_h, \Psi_h) \in \mathcal{W}_h^2$ such that

$$\mathcal{A}((\Upsilon_h, \Psi_h), (\Phi_h, \Theta_h)) = \sum_{j \in \{c,s\}} (\mathbf{y_d^j}, \mathbf{w_h^j})_{\mathbf{L_2}(\Omega_1)}, \tag{5.8}$$

for all test functions $(\Phi_h, \Theta_h) \in \mathcal{W}_h^2$.

6. Preconditioning and implementation. This section is devoted to the fast solution of the variational problem (5.8). After recalling an abstract well-posedness and preconditioning result [47], we use this theory to construct a parameter-robust preconditioner for our problem. Additionally, we address the practical realization of this theoretical preconditioner.

6.1. Abstract preconditioning theory. In this subsection we briefly recall an abstract result of Zulehner [47]. Let V and Q be Hilbert spaces with the inner products $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_Q$. The associated norms are given by $\|\cdot\|_V = \sqrt{(\cdot,\cdot)_V}$ and $\|\cdot\|_Q = \sqrt{(\cdot,\cdot)_Q}$. Furthermore, let X be the product space $X = V \times Q$, equipped with the inner product

$$((v,q),(w,r))_X = (v,w)_V + (q,r)_Q,$$

and the associated norm

$$\|(v,q)\|_X = \sqrt{((v,q),(v,q))_X}.$$

Consider a mixed variational problem in the product space $X = V \times Q$: find $z = (w,r) \in X$ such that

$$\mathcal{A}(z,y) = \mathcal{F}(y), \quad \text{for all } y \in X,$$

with

$$\mathcal{A}(z,y) = a(w,v) + b(v,r) + b(w,q) - c(r,q) \quad \text{and} \quad \mathcal{F}(y) = f(v) + g(q),$$

for $y = (v,q)$ and $z = (w,r)$. We introduce $B \in L(V,Q^*)$ and its adjoint $B^* \in L(Q,V^*)$ by

$$\langle Bw, q\rangle = b(w,q) \quad \text{and} \quad \langle B^*r, v\rangle = \langle Bv, r\rangle.$$

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Furthermore, we denote by $\mathcal{A} \in L(X, X^*)$ the operator induced by

$$\langle \mathcal{A}x, y\rangle = \mathcal{A}(x, y).$$

The next theorem provides necessary and sufficient conditions for parameter-independent bounds and can be found in Zulehner [47].

THEOREM 6.1 ([47, Theorem 2.6]). *If there are constants $\underline{c}_w$, $\underline{c}_r$, $\overline{c}_w$, $\overline{c}_r > 0$ such that*

$$\underline{c}_w \|w\|_V^2 \leq a(w, w) + \|Bw\|_{Q^*}^2 \leq \overline{c}_w \|w\|_V^2, \quad \textit{for all } w \in V$$

and

$$\underline{c}_r \|r\|_Q^2 \leq c(r, r) + \|B^* r\|_{V^*}^2 \leq \overline{c}_r \|r\|_Q^2, \quad \textit{for all } r \in Q,$$

then

$$\underline{c}\|z\|_X \leq \|\mathcal{A}x\|_{X^*} \leq \overline{c}\|z\|_X, \quad \textit{for all } z \in X \tag{6.1}$$

is satisfied with constants $\underline{c}$, $\overline{c} > 0$ that depend only on $\underline{c}_w$, $\overline{c}_w$, $\underline{c}_r$, $\overline{c}_r$.

Indeed, in addition to the qualitative result for $\overline{c}$ and $\underline{c}$, Theorem 6.1 also provides a quantitative estimate of $\overline{c}$ and $\underline{c}$ in terms of $\underline{c}_w$, $\overline{c}_w$, $\underline{c}_r$, $\overline{c}_r$. Tracking the proof of the previous theorem in [47], the constants $\underline{c}$ and $\overline{c}$ fulfill the rough estimate

$$\begin{aligned}
\underline{c} &\geq -\frac{(-3+\sqrt{5})\left(\underline{c}_r^2 \min\left(\frac{1}{2}, \underline{c}_r\right)^2 + \underline{c}_w^2 \min\left(\frac{1}{2}, \underline{c}_w\right)^2\right)}{4 \max\left(\sqrt{\overline{c}_r \max(1, \overline{c}_r)}, \sqrt{\overline{c}_w \max(1, \overline{c}_w)}\right)}, \\
\overline{c} &\leq \sqrt{2} \max\left(\sqrt{\overline{c}_r \max(1, \overline{c}_r)}, \sqrt{\overline{c}_w \max(1, \overline{w}_r)}\right).
\end{aligned} \tag{6.2}$$

We mention that these estimates are not sharp. As exposed in [47], an immediate consequence of (6.1) is an estimate of the condition number $\kappa(\mathcal{A})$:

$$\kappa(\mathcal{A}) = \|\mathcal{A}\|_{L(X,X^*)} \|\mathcal{A}^{-1}\|_{L(X^*,X)} \leq \frac{\overline{c}}{\underline{c}}.$$

Therefore, robust estimates of the form (6.1) imply a robust estimate for the condition number. More precisely, (6.1) means that solving the discrete variational problem connected with the inner product in X will supply a good preconditioner for $\mathcal{A}$.

6.2. Well-posedness in some non-standard norms: a constructive approach. In [31], Kolmbauer and Langer state a parameter-robust well-posedness result for the FEM discretization of the eddy current optimal control problem in a bounded domain. Using the technique of interpolation spaces, they introduce a parameter-dependent non-standard norm in $\mathbf{H}(\mathbf{curl}\,, \Omega_1)^4$ and show that the inf-sup and sup-sup conditions that appear in the theorem of Babuška-Aziz are fulfilled with constants independent of any discretization and model parameters. Indeed we re-use this result for the FEM-discretized domain Ω_1. Furthermore, we have to take into account the different parameter settings in the conducting domain Ω_1 and the non-conducting domain Ω_2; cf. (3.3). Since the exterior domain Ω_2 is reduced to the boundary, we incorporate the boundary integral operators in terms of a Schur complement approach. Consequently, for the Ω_1-part we define the non-standard norm

$$\begin{aligned}
\|\mathbf{y}\|_{\mathcal{F}_I}^2 := (\nu\, \mathbf{curl}\, \mathbf{y}, \mathbf{curl}\, \mathbf{y})_{\mathbf{L_2}(\Omega_1)} + \omega(\sigma \mathbf{y}, \mathbf{y})_{\mathbf{L_2}(\Omega_1)} + \frac{1}{\sqrt{\lambda}} (\mathbf{y}, \mathbf{y})_{\mathbf{L_2}(\Omega_1)} \\
- \langle \mathbf{N}\gamma_D \mathbf{y}, \gamma_D \mathbf{y}\rangle_\tau + \sup_{\boldsymbol{\lambda} \in \mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0, \Gamma)} \frac{\langle \mathbf{B}\boldsymbol{\lambda}, \gamma_D \mathbf{y}\rangle_\tau^2}{\langle \mathbf{A}\boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle_\tau}.
\end{aligned}$$

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For the interface part, we just use the single layer potential $\mathbf{A}$ that induces a norm on $\mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0, \Gamma)$:

$$\|\boldsymbol{\lambda}\|_{\mathcal{B}}^2 := \langle \mathbf{A}\boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle_\tau .$$

These definitions give rise to a norm in the product space $\mathcal{W}^2$

$$\|(\Upsilon,\Psi)\|_{\mathcal{C}_I}^2 := \sum_{j\in\{c,s\}} \left(\sqrt{\lambda}\left[\|\mathbf{y}^j\|_{\mathcal{F}_I}^2 + \|\boldsymbol{\lambda}^{\mathbf{j}}\|_{\mathcal{B}}^2\right] + \frac{1}{\sqrt{\lambda}}\left[\|\mathbf{p}^j\|_{\mathcal{F}_I}^2 + \|\boldsymbol{\eta}^{\mathbf{j}}\|_{\mathcal{B}}^2\right]\right). \tag{6.3}$$

The main result is summarized in the following lemma that claims that an inf-sup condition and a sup-sup condition are fulfilled with the parameter-independent constants $\frac{1}{\sqrt{5}}$ and 2.

LEMMA 6.2. *We have*

$$\frac{1}{\sqrt{5}}\|(\Upsilon,\Psi)\|_{\mathcal{C}_I} \le \sup_{(\Phi,\Theta)\in\mathcal{W}^2} \frac{\mathcal{A}((\Upsilon,\Psi),(\Phi,\Theta))}{\|(\Phi,\Theta)\|_{\mathcal{C}_I}} \le 2\|(\Upsilon,\Psi)\|_{\mathcal{C}_I},$$

for all $(\Upsilon,\Psi)\in\mathcal{W}^2$.

Proof. This proof follows the same strategy as the proof in [31] for the Ω_1 part. We directly verify the inf-sup and sup-sup condition. By an appropriate distribution of the regularization parameter λ and applying Cauchy's inequality several times, the sup-sup condition follows with constant 2. For the special choice of the test function

$$(\Phi,\Theta) = (\Phi_1,\Theta_1) + 2(\Phi_2,\Theta_2) + (\Phi_3,\Theta_3) + (\Phi_4,\Theta_4),$$

given by

$$\begin{aligned}
&(\Phi_1,\Theta_1) := (\Upsilon,-\Psi),\\
&(\Phi_2,\Theta_2) := \left(\frac{1}{\sqrt{\lambda}}\mathbf{p}^{\mathbf{c}}, -\frac{1}{\sqrt{\lambda}}\boldsymbol{\eta}^{\mathbf{c}}, \frac{1}{\sqrt{\lambda}}\mathbf{p}^{\mathbf{s}}, -\frac{1}{\sqrt{\lambda}}\boldsymbol{\eta}^{\mathbf{s}}, \sqrt{\lambda}\mathbf{y}^{\mathbf{c}}, -\sqrt{\lambda}\boldsymbol{\lambda}^{\mathbf{c}}, \sqrt{\lambda}\mathbf{y}^{\mathbf{s}}, -\sqrt{\lambda}\boldsymbol{\lambda}^{\mathbf{s}}\right),\\
&(\Phi_3,\Theta_3) := \left(-\frac{1}{\sqrt{\lambda}}\mathbf{p}^{\mathbf{s}}, -\frac{1}{\sqrt{\lambda}}\boldsymbol{\eta}^{\mathbf{s}}, \frac{1}{\sqrt{\lambda}}\mathbf{p}^{\mathbf{c}}, \frac{1}{\sqrt{\lambda}}\boldsymbol{\eta}^{\mathbf{c}}, \sqrt{\lambda}\mathbf{y}^{\mathbf{s}}, \sqrt{\lambda}\boldsymbol{\lambda}^{\mathbf{s}}, -\sqrt{\lambda}\mathbf{y}^{\mathbf{c}}, -\sqrt{\lambda}\boldsymbol{\lambda}^{\mathbf{c}}\right),\\
&(\Phi_4,\Theta_4) := \left(\mathbf{0}, \frac{1}{\sqrt{\lambda}}\boldsymbol{\lambda}^{\mathbf{c}}, \mathbf{0}, \frac{1}{\sqrt{\lambda}}\boldsymbol{\lambda}^{\mathbf{s}}, \mathbf{0}, \sqrt{\lambda}\boldsymbol{\eta}^{\mathbf{c}}, \mathbf{0}, \sqrt{\lambda}\boldsymbol{\eta}^{\mathbf{s}}\right),
\end{aligned}$$

the inf-sup condition follows with constant $\frac{1}{\sqrt{5}}$. $\square$

So far, we obtained a well-posedness result for our problem with the nice parameter-independent constants $\frac{1}{\sqrt{5}}$ and 2. For applications, one has to know how to deal with the individual parts of the norm $\|\cdot\|_{\mathcal{C}_I}$. Especially, the contribution from the interface

$$-\langle \mathbf{N}\gamma_D\mathbf{y}, \gamma_D\mathbf{y}\rangle_\tau + \sup_{\boldsymbol{\lambda}\in\mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0,\Gamma)} \frac{\langle \mathbf{B}\boldsymbol{\lambda}, \gamma_D\mathbf{y}\rangle_\tau^2}{\langle \mathbf{A}\boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle_\tau},$$

is difficult to deal with. In the next subsections, we investigate how to avoid the contributions from the interface terms to the norm $\|\cdot\|_{\mathcal{F}_I}$ and to preserve good constants in the well-posedness results at the same time. Therefore, we introduce a slightly modified norm that still involves the contribution from the boundary and preserves parameter-robust constants in the well-posedness result. The important point is that this slight modification allows us to get rid of the boundary contribution later on.

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6.3. Well-posedness in non-standard norms: a useful generalization. A simple observation yields that for fixed parameters ω, σ, and ν, the problem (5.8) is well conditioned for $\lambda \geq 1$. Therefore, the additional λ-scaling in (6.3) is not necessary for this parameter set. This is taken into account by introducing the shortcut $\tilde{\lambda} = \min(1, \lambda)$ and the definition of a new norm for the Ω_1-part.

$$\|\mathbf{y}\|_{\mathcal{F}}^2 := (\nu\, \mathbf{curl}\,\mathbf{y}, \mathbf{curl}\,\mathbf{y})_{\mathbf{L_2}(\Omega_1)} + \omega(\sigma \mathbf{y}, \mathbf{y})_{\mathbf{L_2}(\Omega_1)} + \frac{1}{\sqrt{\tilde{\lambda}}}(\mathbf{y}, \mathbf{y})_{\mathbf{L_2}(\Omega_1)} \\ - \langle \mathbf{N}\gamma_D \mathbf{y}, \gamma_D \mathbf{y}\rangle_\tau + \sup_{\boldsymbol{\lambda} \in \mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0, \Gamma)} \frac{\langle \mathbf{B}\boldsymbol{\lambda}, \gamma_D \mathbf{y}\rangle_\tau^2}{\langle \mathbf{A}\boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle_\tau}.$$

Indeed, this means that for the case $0 < \lambda \leq 1$ we re-use the parameter-robust norm $\|\cdot\|_{\mathcal{F}_I}$ and for the case $\lambda \geq 1$ we just drop the λ-scaling. This small modification will be essential to derive an easy computable preconditioner in the next section. Consequently, we can define a new norm in the product space $\mathcal{W}^2$:

$$\|(\Upsilon, \Psi)\|_{\mathcal{C}}^2 := \sum_{j \in \{c,s\}} \left(\sqrt{\tilde{\lambda}} \left[\|\mathbf{y}^j\|_{\mathcal{F}}^2 + \|\boldsymbol{\lambda}^{\mathbf{j}}\|_{\mathcal{B}}^2 \right] + \frac{1}{\sqrt{\tilde{\lambda}}} \left[\|\mathbf{p}^j\|_{\mathcal{F}}^2 + \|\boldsymbol{\eta}^{\mathbf{j}}\|_{\mathcal{B}}^2 \right] \right).$$

Furthermore, this decomposition directly gives rise to the splitting

$$\|(\Upsilon, \Psi)\|_{\mathcal{C}}^2 =: \|\Upsilon\|_{\mathcal{C}_1}^2 + \frac{1}{\tilde{\lambda}} \|\Psi\|_{\mathcal{C}_1}^2,$$

with

$$\|\Upsilon\|_{\mathcal{C}_1}^2 := \sum_{j \in \{c,s\}} \left(\sqrt{\tilde{\lambda}} \left[\|\mathbf{y}^j\|_{\mathcal{F}}^2 + \|\boldsymbol{\lambda}^{\mathbf{j}}\|_{\mathcal{B}}^2 \right] \right).$$

Indeed, this splitting is of importance since according to the notation of Theorem 6.1 we have the following correspondence: $\|\cdot\|_V^2 = \|\cdot\|_{\mathcal{C}_1}^2$ and $\|\cdot\|_Q^2 = \frac{1}{\tilde{\lambda}}\|\cdot\|_{\mathcal{C}_1}^2$. The main result is summarized in the following lemma that states that an inf-sup condition and a sup-sup condition are fulfilled with parameter-independent constants.

LEMMA 6.3. *We have*

$$\underline{c}\|(\Upsilon, \Psi)\|_{\mathcal{C}} \leq \sup_{(\Phi,\Theta) \in \mathcal{W}^2} \frac{\mathcal{A}((\Upsilon, \Psi), (\Phi, \Theta))}{\|(\Phi, \Theta)\|_{\mathcal{C}}} \leq \overline{c}\|(\Upsilon, \Psi)\|_{\mathcal{C}},$$

for all $(\Upsilon, \Psi) \in \mathcal{W}^2$, where $\underline{c}$ and $\overline{c}$ are generic constants independent of any involved discretization or model parameters.

Proof. In order to show the inf-sup and sup-sup condition for $\mathcal{A}$, we use Theorem 6.1. We start by showing an inf-sup and a sup-sup condition for $b(\cdot, \cdot)$. Using Cauchy's inequality several times immediately yields

$$\sup_{\Theta \in \mathcal{W}} \frac{b(\Upsilon, \Theta)^2}{\frac{1}{\tilde{\lambda}}\|\Theta\|_{\mathcal{C}_1}^2} \leq 4\|\Upsilon\|_{\mathcal{C}_1}^2.$$

For the special choice of the test function $\Theta = \Theta_1 + 2\Theta_2 + \Theta_3$ given by

$$\Theta_1 = (-\mathbf{y}^{\mathbf{s}}, -\boldsymbol{\lambda}^{\mathbf{s}}, \mathbf{y}^{\mathbf{c}}, \boldsymbol{\lambda}^{\mathbf{c}}), \quad \Theta_2 = (\mathbf{y}^{\mathbf{c}}, -\boldsymbol{\lambda}^{\mathbf{c}}, \mathbf{y}^{\mathbf{s}}, -\boldsymbol{\lambda}^{\mathbf{s}}), \quad \Theta_3 = (\mathbf{0}, \boldsymbol{\mu}^{\mathbf{c}}, \mathbf{0}, \boldsymbol{\mu}^{\mathbf{s}}),$$

we obtain

$$\frac{\left(\|\Upsilon\|_{\mathcal{C}_1}^2 - \sum_{j\in\{c,s\}} \|\mathbf{y^j}\|_{\mathbf{L_2}(\Omega)}^2\right)^2}{4\|\Upsilon\|_{\mathcal{C}_1}^2} \leq \sup_{\Theta\in\mathcal{W}} \frac{b(\Upsilon,\Theta)^2}{\frac{1}{\tilde{\lambda}}\|\Theta\|_{\mathcal{C}_1}^2}.$$

By definition, we have

$$a(\Upsilon,\Upsilon) = \sum_{j\in\{c,s\}} \|\mathbf{y^j}\|_{\mathbf{L_2}(\Omega_1)}^2.$$

Using the trivial inequality $a^2 + \frac{1}{4}b^2 \geq \frac{1}{4}(a^2+b^2) \geq \frac{1}{8}(a+b)^2$, we obtain the inf-sup bound

$$\begin{aligned} a(\Upsilon,\Upsilon) + \sup_{\Theta\in\mathcal{W}} \frac{b(\Upsilon,\Theta)^2}{\frac{1}{\tilde{\lambda}}\|\Theta\|_{\mathcal{C}_1}^2} \\ \geq \frac{\left(\sum_{j\in\{c,s\}} \|\mathbf{y^j}\|_{\mathbf{L_2}(\Omega_1)}^2\right)^2}{\|\Upsilon\|_{\mathcal{C}_1}^2} + \frac{\left(\|\Upsilon\|_{\mathcal{C}_1}^2 - \sum_{j\in\{c,s\}} \|\mathbf{y^j}\|_{\mathbf{L_2}(\Omega)}^2\right)^2}{4\|\Upsilon\|_{\mathcal{C}_1}^2} \geq \frac{1}{8}\|\Upsilon\|_{\mathcal{C}_1}^2, \end{aligned}$$

and the sup-sup bound

$$a(\Upsilon,\Upsilon) + \sup_{\Theta\in\mathcal{W}} \frac{b(\Upsilon,\Theta)^2}{\frac{1}{\tilde{\lambda}}\|\Theta\|_{\mathcal{C}_1}^2} \leq 4\|\Upsilon\|_{\mathcal{C}_1}^2, \quad \forall \Upsilon \in \mathcal{W}.$$

For the second estimate, again an inf-sup and a sup-sup condition for b can be derived in the same manner. The following estimates are the second ingredient:

$$\frac{\tilde{\lambda}}{\lambda}\left(\frac{1}{\tilde{\lambda}} \sum_{j\in\{c,s\}} \|\mathbf{p^j}\|_{\mathbf{L_2}(\Omega_1)}^2\right) \leq c(\Psi,\Psi) \leq \frac{1}{\tilde{\lambda}} \sum_{j\in\{c,s\}} \|\mathbf{p^j}\|_{\mathbf{L_2}(\Omega_1)}^2.$$

Thus, we have the inf-sup bound

$$\begin{aligned} c(\Psi,\Psi) + \sup_{\Theta\in\mathcal{W}} \frac{b(\Phi,\Psi)^2}{\|\Phi\|_{\mathcal{C}_1}^2} \\ &\geq \frac{\tilde{\lambda}}{\lambda} \frac{\left(\sum_{j\in\{c,s\}} \frac{1}{\tilde{\lambda}}\|\mathbf{p^j}\|_{\mathbf{L_2}(\Omega_1)}^2\right)^2}{\frac{1}{\tilde{\lambda}}\|\Psi\|_{\mathcal{C}_1}^2} + \frac{\left(\frac{1}{\tilde{\lambda}}\|\Psi\|_{\mathcal{C}_1}^2 - \sum_{j\in\{c,s\}} \frac{1}{\tilde{\lambda}}\|\mathbf{y^j}\|_{\mathbf{L_2}(\Omega)}^2\right)^2}{4\frac{1}{\tilde{\lambda}}\|\Psi\|_{\mathcal{C}_1}^2} \\ &\geq \frac{1}{2}\min\left(\frac{\tilde{\lambda}}{\lambda}, \frac{1}{4}\right)\frac{1}{\tilde{\lambda}}\|\Psi\|_{\mathcal{C}_1}^2, \end{aligned}$$

and the sup-sup bound

$$c(\Psi,\Psi) + \sup_{\Theta\in\mathcal{W}} \frac{b(\Phi,\Psi)^2}{\|\Phi\|_{\mathcal{C}_1}^2} \leq 4\frac{1}{\tilde{\lambda}}\|\Psi\|_{\mathcal{C}_1}^2, \quad \forall \Psi \in \mathcal{W}.$$

Summarizing, we have $\underline{c}_w = 1/8$, $\overline{c}_w = 4$, $\underline{c}_r = \frac{1}{2}\min\left(\frac{\tilde{\lambda}}{\lambda}, \frac{1}{4}\right)$, and $\overline{c}_r = 4$. Combining these estimates according to (6.2), we obtain the final estimate

$$\underline{c} > \begin{cases} \frac{3-\sqrt{5}}{32768} & \lambda \leq 4 \\ \frac{(3-\sqrt{5})(256+\lambda^4)}{65536\lambda^4} & \lambda > 4 \end{cases} \quad \text{and} \quad \overline{c} \leq 2\sqrt{4}.$$

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It is easy to verify that $\underline{c}$ is uniformly bounded from below by a constant independent of λ. Consequently, the lower and upper bound are independent of any involved parameters. ∎

In general, an inf-sup bound for $\mathcal{W}^2$ does not imply such a lower bound on a subspace. However, in this case the same result holds indeed for the finite element subspace $\mathcal{W}_h^2 \subset \mathcal{W}^2$ since the proof can be repeated for the finite element functions step by step.

LEMMA 6.4. *We have*

$$\underline{c}\|(\Upsilon_h, \Psi_h)\|_{\mathcal{C}} \leq \sup_{(\Phi_h,\Theta_h)\in\mathcal{W}_h^2} \frac{\mathcal{A}((\Upsilon_h, \Psi_h), (\Phi_h, \Theta_h))}{\|(\Phi_h, \Theta_h)\|_{\mathcal{C}}} \leq \overline{c}\|(\Upsilon_h, \Psi_h)\|_{\mathcal{C}},$$

for all $(\Upsilon_h, \Psi_h) \in \mathcal{W}_h^2$, where $\underline{c}$ and $\overline{c}$ are generic constants independent of any involved discretization or model parameters.

From Lemma 6.3 and Lemma 6.4 in combination with the Theorem of Babuška-Aziz, we immediately conclude that there exists a unique solution of the corresponding variational problems (5.6) and (5.8), and that the solution continuously depends on the data uniformly in all involved parameters.

6.4. A canonical preconditioner.

6.4.1. Ω_1-part. For practical applications, we want to get rid of the interface Schur complement contribution to the $\|\cdot\|_{\mathcal{F}}$ norm. Here the $\tilde{\lambda}$-scaling is essential to show an equivalence to a simpler norm, where the equivalence constants are independent of ω, λ, and σ. Consequently, we can get rid of the additional expression involving the boundary integral operators and can use

$$\|\mathbf{y}\|_{\tilde{\mathcal{F}}}^2 := (\nu\,\mathbf{curl}\,\mathbf{y}, \mathbf{curl}\,\mathbf{y})_{\mathbf{L_2}(\Omega_1)} + \omega(\sigma\mathbf{y}, \mathbf{y})_{\mathbf{L_2}(\Omega_1)} + \frac{1}{\sqrt{\min(1,\lambda)}}(\mathbf{y}, \mathbf{y})_{\mathbf{L_2}(\Omega_1)}.$$

In order to obtain this simpler norm $\|\mathbf{y}\|_{\tilde{\mathcal{F}}}$, we have to pay the price that the norm equivalence depends on the minimal value of the reluctivity ν, i.e.,

$$\|\mathbf{y}\|_{\tilde{\mathcal{F}}}^2 \leq \|\mathbf{y}\|_{\mathcal{F}}^2 \leq c\ \max(1, \underline{\nu}^{-1})\|\mathbf{y}\|_{\tilde{\mathcal{F}}}^2. \tag{6.4}$$

Here c depends on the norm bounds for the boundary integral operators $\mathbf{B}$ and $\mathbf{N}$, the ellipticity constant for $\mathbf{A}$, and the constant in the trace theorem (2.1) but not on h, N, λ, ω, σ, and ν. Note that this type of equivalence (6.4) is not available for the $\|\cdot\|_{\mathcal{F}_I}$ norm.

6.4.2. Interface part. In the following, we also need the finite element space $\mathcal{S}_1(\mathcal{K}_h)$, the space of scalar, continuous, and piecewise linear finite element functions on the interface $\mathcal{K}_h$. Using the identity (e.g., [14])

$$\langle \mathbf{A}\,\mathbf{curl}_\Gamma\phi_h, \mathbf{curl}_\Gamma\psi_h\rangle_\tau = \langle D\phi_h, \psi_h\rangle_{H^{1/2}(\Gamma)}, \quad \forall \phi_h, \psi_h \in \mathcal{S}_1(\mathcal{K}_h),$$

where $D : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is the hyper-singular operator for the Laplacian, allows us to use tools from the Galerkin boundary element methods for Laplace problems. In order to construct a basis for the finite element space $\mathcal{RT}_0^0(\mathcal{K}_h)$, we use the identity

$$\mathcal{RT}_0^0(\mathcal{K}_h) = \mathbf{curl}_\Gamma\mathcal{S}_1(\mathcal{K}_h),$$

which holds true for a simply connected interface $\mathcal{K}_h$. Indeed, in the following we use the semi-norm

$$\|\phi\|_{\tilde{\mathcal{B}}}^2 := \langle D\phi, \phi\rangle_{H^{1/2}(\Gamma)}$$

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for the boundary element part. In fact, this semi-norm is a norm in the finite element space $\mathcal{S}_1^0(\mathcal{K}_h) = \mathcal{S}_1(\mathcal{K}_h)\backslash\mathbb{R}$, characterized by

$$\mathcal{S}_1^0(\mathcal{K}_h) := \{\phi_h \in \mathcal{S}_1(\mathcal{K}_h) : \int_{\mathcal{K}_h} \phi_h(\mathbf{x})\mathrm{d}S_{\mathbf{x}} = 0\}.$$

We enforce the zero average by adding the equation

$$\mathcal{P}(\phi_h, \psi_h) := \int_{\mathcal{K}_h} \phi_h(\mathbf{x})\mathrm{d}S_{\mathbf{x}} \int_{\mathcal{K}_h} \psi_h(\mathbf{x})\mathrm{d}S_{\mathbf{x}} = 0$$

to our variational problem for all relevant functions ϕ_h^c, ϕ_h^s, ψ_h^c and ψ_h^s.

6.4.3. Final estimate. The fact that $\mathbf{curl}_\Gamma : \mathcal{S}_1^0(\mathcal{K}_h) \to \mathcal{RT}_0^0(\mathcal{K}_h)$ is an isomorphism allows us to introduce new variables

$$\begin{aligned}
\lambda_h^j &= \mathbf{curl}_\Gamma \phi_h^j \qquad \text{and} \qquad \eta_h^j = \mathbf{curl}_\Gamma \psi_h^j \qquad \text{with} \quad \phi_h^j, \psi_h^j \in \mathcal{S}_1^0(\mathcal{K}_h),\\
\rho_h^j &= \mathbf{curl}_\Gamma \zeta_h^j \qquad \text{and} \qquad \mu_h^j = \mathbf{curl}_\Gamma \xi_h^j \qquad \text{with} \quad \zeta_h^j, \xi_h^j \in \mathcal{S}_1^0(\mathcal{K}_h),
\end{aligned}$$

for $j \in \{c, s\}$. Using these new variables gives rise to the following definition:

$$\begin{aligned}
\tilde{\Upsilon}_h &:= (\mathbf{y_h^c}, \phi_h^c, \mathbf{y_h^s}, \phi_h^s), \qquad & \tilde{\Phi}_h &:= (\mathbf{w_h^c}, \zeta_h^c, \mathbf{w_h^s}, \zeta_h^s),\\
\tilde{\Psi}_h &:= (\mathbf{p_h^c}, \psi_h^c, \mathbf{p_h^s}, \psi_h^s), \qquad & \tilde{\Theta}_h &:= (\mathbf{v_h^c}, \xi_h^c, \mathbf{v_h^s}, \xi_h^s).
\end{aligned}$$

Therefore, the bilinear form of the new variational problem related to (5.7) is given by $\tilde{\mathcal{A}}$, defined by

$$\tilde{\mathcal{A}}((\tilde{\Upsilon}_h, \tilde{\Psi}_h), (\tilde{\Phi}_h, \tilde{\Theta}_h) := \mathcal{A}((\Upsilon_h, \Psi_h), (\Phi_h, \Theta_h)) - \sum_{j\in\{c,s\}} \left[\mathcal{P}(\phi_h^j, \zeta_h^j) + \mathcal{P}(\psi_h^j, \xi_h^j)\right].$$

Using the new finite element product space

$$\mathcal{U}_h := \mathcal{ND}_1(\mathcal{T}_h) \times \mathcal{S}_1(\mathcal{K}_h) \times \mathcal{ND}_1(\mathcal{T}_h) \times \mathcal{S}_1(\mathcal{K}_h),$$

equipped with the norm

$$\|(\tilde{\Upsilon}, \tilde{\Psi})\|_{\tilde{\mathcal{C}}}^2 := \sum_{j\in\{c,s\}} \left(\sqrt{\tilde{\lambda}}\left[\|\mathbf{y}^j\|_{\tilde{\mathcal{F}}}^2 + \|\phi^j\|_{\tilde{\mathcal{B}}}^2\right] + \frac{1}{\sqrt{\tilde{\lambda}}}\left[\|\mathbf{p}^j\|_{\tilde{\mathcal{F}}}^2 + \|\psi^j\|_{\tilde{\mathcal{B}}}^2\right]\right),$$

gives rise to the following result:

THEOREM 6.5. *We have*

$$\underline{c}\|(\tilde{\Upsilon}_h, \tilde{\Psi}_h)\|_{\tilde{\mathcal{C}}} \leq \sup_{(\tilde{\Phi}_h, \tilde{\Theta}_h)\in\mathcal{U}_h^2} \frac{\tilde{\mathcal{A}}((\tilde{\Upsilon}_h, \tilde{\Psi}_h), (\tilde{\Phi}_h, \tilde{\Theta}_h))}{\|(\tilde{\Phi}_h, \tilde{\Theta}_h)\|_{\tilde{\mathcal{C}}}} \leq \bar{c}\sqrt{\max(1, \underline{\nu}^{-1})}\|(\tilde{\Upsilon}_h, \tilde{\Psi}_h)\|_{\tilde{\mathcal{C}}},$$

for all $(\tilde{\Upsilon}_h, \tilde{\Psi}_h) \in \mathcal{U}_h^2$, *where* $\underline{c}$ *and* $\bar{c}$ *are generic constants independent of any involved discretization or model parameters.*

Proof. The proof follows from Lemma 6.4, the norm equivalence (6.4), and the change of the variables described in this subsection. ∎

6.5. A practical preconditioner. We have to solve the discrete variational problems connected with the norm $\tilde{\mathcal{C}}$. The solution of the variational problem connected with $\|\cdot\|_{\tilde{\mathcal{C}}}$ supplies a good preconditioner for the variational problem associated with the bilinear form $\tilde{\mathcal{A}}$. In large-scale computations, the individual parts of the norm and/or preconditioner $\tilde{\mathcal{C}}$ have to be replaced by easy "invertible" and robust symmetric and positive definite norms and/or preconditioners such that the spectral equivalence inequalities

$$\underline{c}_{\tilde{\mathcal{F}}}\|\mathbf{y_h}\|^2_{\tilde{\mathcal{F}}_p} \leq \|\mathbf{y_h}\|^2_{\tilde{\mathcal{F}}} \leq \overline{c}_{\tilde{\mathcal{F}}}\|\mathbf{y_h}\|^2_{\tilde{\mathcal{F}}_p} \quad \text{and} \quad \underline{c}_{\tilde{\mathcal{B}}}\|\phi_h\|^2_{\tilde{\mathcal{B}}_p} \leq \|\phi_h\|^2_{\tilde{\mathcal{B}}} \leq \overline{c}_{\tilde{\mathcal{B}}}\|\phi_h\|^2_{\tilde{\mathcal{B}}_p},$$

are valid with positive constant $\underline{c}_{\tilde{\mathcal{F}}}$, $\overline{c}_{\tilde{\mathcal{F}}}$, $\underline{c}_{\tilde{\mathcal{B}}}$, and $\overline{c}_{\tilde{\mathcal{B}}}$, which should be independent of the involved parameters σ, ω, and λ and may only depend polylogarithmically on the space discretization parameter h.

The finite element part corresponding to the $\tilde{\mathcal{F}}$-norm requires the solution of the variational problem

$$(\nu\,\mathbf{curl}\,\mathbf{y_h}, \mathbf{curl}\,\mathbf{v_h})_{\mathbf{L_2}(\Omega_1)} + \omega(\sigma\mathbf{y_h}, \mathbf{v_h})_{\mathbf{L_2}(\Omega_1)} + \frac{1}{\sqrt{\tilde{\lambda}}}(\mathbf{y_h}, \mathbf{v_h})_{\mathbf{L_2}(\Omega_1)} = (\mathbf{f}, \mathbf{v_h})_{\mathbf{L_2}(\Omega_1)}.$$

Depending on the parameter setting $(\nu, \sigma, \omega, \lambda)$, candidates for robust and (almost) optimal preconditioners or solvers are multigrid preconditioners [1, 23], auxiliary space preconditioners [25, 45], and domain decomposition preconditioners [26, 40, 41, 42].

The boundary element part corresponding to the $\tilde{\mathcal{B}}$-norm requires the solution of the variational problem

$$\langle D\phi_h, \psi_h\rangle_{H^{1/2}(\Gamma)} = \langle \rho, \psi_h\rangle_{H^{1/2}(\Gamma)}, \quad \forall \psi_h \in \mathcal{S}_1^0(\mathcal{K}_h).$$

This problem can be tackled by domain decomposition or multilevel methods [21, 43], purely algebraic approaches like $\mathcal{H}$-matrices approximations [18, 19] and ACA-methods [7], or alternative techniques like those in [39]. These practical preconditioners can be used to accelerate the Minimal Residual method [36] applied to the symmetric and indefinite linear system with system matrix

$$\tilde{\mathcal{A}}_h = \begin{bmatrix} \mathbf{M} & \cdot & \cdot & \cdot & \mathbf{K}_\nu - \mathbf{N} & \mathbf{B} & -\mathbf{M}_{\sigma,\omega} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \mathbf{B}^T & -(\mathbf{A}+\mathbf{P}) & \cdot & \cdot \\ \cdot & \cdot & \mathbf{M} & \cdot & \mathbf{M}_{\sigma,\omega} & \cdot & \mathbf{K}_\nu - \mathbf{N} & \mathbf{B} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mathbf{B}^T & -(\mathbf{A}+\mathbf{P}) \\ \mathbf{K}_\nu - \mathbf{N} & \mathbf{B} & \mathbf{M}_{\sigma,\omega} & \cdot & -\lambda^{-1}\mathbf{M} & \cdot & \cdot & \cdot \\ \mathbf{B}^T & -(\mathbf{A}+\mathbf{P}) & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -\mathbf{M}_{\sigma,\omega} & \cdot & \mathbf{K}_\nu - \mathbf{N} & \mathbf{B} & \cdot & \cdot & -\lambda^{-1}\mathbf{M} & \cdot \\ \cdot & \cdot & \mathbf{B}^T & -(\mathbf{A}+\mathbf{P}) & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

The finite element matrices $\mathbf{M}$, $\mathbf{M}_{\sigma,\omega}$, and $\mathbf{K}_\nu$ and the boundary element matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{N}$ arise from the discretization of $\tilde{\mathcal{A}}$ in a straightforward manner. The corresponding

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block-diagonal preconditioner $\tilde{\mathcal{C}}_h$ is given by

$$\tilde{\mathcal{C}}_h = \begin{bmatrix} \sqrt{\tilde{\lambda}}\tilde{\mathbf{F}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \sqrt{\tilde{\lambda}}(\mathbf{A}+\mathbf{P}) & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \sqrt{\tilde{\lambda}}\tilde{\mathbf{F}} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \sqrt{\tilde{\lambda}}(\mathbf{A}+\mathbf{P}) & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \frac{1}{\sqrt{\tilde{\lambda}}}\tilde{\mathbf{F}} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{\sqrt{\tilde{\lambda}}}(\mathbf{A}+\mathbf{P}) & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{\sqrt{\tilde{\lambda}}}\tilde{\mathbf{F}} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{\sqrt{\tilde{\lambda}}}(\mathbf{A}+\mathbf{P}) \end{bmatrix},$$

where $\tilde{\mathbf{F}} = \mathbf{K}_{\boldsymbol{\nu}} + \mathbf{M}_{\boldsymbol{\sigma},\boldsymbol{\omega}} + 1/\sqrt{\tilde{\lambda}}\,\mathbf{M}$ is the discretization of $\|\cdot\|_{\tilde{\mathcal{F}}}$. Combining the previous results, we obtain that the condition number of the preconditioned system can be estimated by a constant c_ν that is independent of the meshsize h and all involved parameters λ, k, ω, and σ, i.e.,

$$\kappa_{\tilde{\mathcal{C}}_h}(\tilde{\mathcal{C}}_h^{-1}\tilde{\mathcal{A}}_h) := \|\tilde{\mathcal{C}}_h^{-1}\tilde{\mathcal{A}}_h\|_{\tilde{\mathcal{C}}_h}\|\tilde{\mathcal{A}}_h^{-1}\tilde{\mathcal{C}}_h\|_{\tilde{\mathcal{C}}_h} \le c\max(1,\underline{\nu}^{-1}) = c_\nu.$$

Combining the canonical preconditioner with special choices of the practical preconditioners yields the preconditioner $\tilde{\mathcal{C}}_{h,p}$, and we obtain the final bound for the condition number of the preconditioned system

$$\kappa_{\tilde{\mathcal{C}}_{h,p}}(\tilde{\mathcal{C}}_{h,p}^{-1}\tilde{\mathcal{A}}_h) \le c_\nu \frac{\max(\overline{c}_{\tilde{\mathcal{F}}},\overline{c}_{\tilde{\mathcal{B}}})}{\min(\underline{c}_{\tilde{\mathcal{F}}},\underline{c}_{\tilde{\mathcal{B}}})}.$$

Using the convergence rate estimate of the MinRes method (e.g., [16]), we finally arrive at the following theorem.

THEOREM 6.6. *The MinRes method applied to the preconditioned system converges. At the m-th iteration, the preconditioned residual $\mathbf{r^{2m}} = \tilde{\mathcal{C}}_{h,p}^{-1}\mathbf{f_h} - \tilde{\mathcal{C}}_{h,p}^{-1}\tilde{\mathcal{A}}_h\mathbf{w^{2m}}$ is bounded by*

$$\left\|\mathbf{r^{2m}}\right\|_{\tilde{\mathcal{C}}_{h,p}} \le \frac{2q^m}{1+q^{2m}}\left\|\mathbf{r^0}\right\|_{\tilde{\mathcal{C}}_{h,p}}, \quad \textit{where} \quad q = \frac{\kappa_{\tilde{\mathcal{C}}_{h,p}}(\tilde{\mathcal{C}}_{h,p}^{-1}\tilde{\mathcal{A}}_h) - 1}{\kappa_{\tilde{\mathcal{C}}_{h,p}}(\tilde{\mathcal{C}}_{h,p}^{-1}\tilde{\mathcal{A}}_h) + 1}.$$

REMARK 6.7. So far, from Lemma 6.5, we obtain a qualitative estimate for the condition number $\kappa_{\tilde{\mathcal{C}}_h}(\tilde{\mathcal{C}}_h^{-1}\tilde{\mathcal{A}}_h)$, where the value c_ν is overestimated and can be very large. Anyhow, from Lemma 6.2 we obtain that for the practical relevant case $0 < \lambda < 1$ we have the following quantitative estimate of the condition number

$$\kappa_{\tilde{\mathcal{C}}_{h,p}}(\tilde{\mathcal{C}}_{h,p}^{-1}\tilde{\mathcal{A}}_h) \le 2\sqrt{5}\max(1,\underline{\nu}^{-1})\frac{\max(\overline{c}_{\tilde{\mathcal{F}}},\overline{c}_{\tilde{\mathcal{B}}})}{\min(\underline{c}_{\tilde{\mathcal{F}}},\underline{c}_{\tilde{\mathcal{B}}})}.$$

Therefore, we also expect for the “well-conditioned“ case $\lambda \ge 1$ that the constant c_ν is of acceptable size.

7. Discretization error analysis. In this section, we give a complete estimate of the error depending on the discretization parameter h. Since we assume the desired state to have a multiharmonic representation, we do not introduce a discretization error in time. Furthermore, the discretization error is analyzed for the time-harmonic case since for the multiharmonic case the same estimates are valid by summing over all modes $k = 0, \ldots, N$. We

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obtain an optimal estimation of the discretization error in terms of the approximation error in the non-standard norm $\|\cdot\|_{\mathcal{C}}$

$$\|(\Upsilon,\Psi)-(\Upsilon_h,\Psi_h)\|_{\mathcal{C}} \le c \inf_{(\Phi_h,\Theta_h)\in\mathcal{W}_h^2} \|(\Upsilon,\Psi)-(\Phi_h,\Theta_h)\|_{\mathcal{C}},$$

with the constant c only depending on the geometry but not on the mesh width h, the involved parameter ω, k, σ, ν, and λ, and the solution (Υ,Ψ). Due to the norm equivalence of the standard graph norm $\|\cdot\|_{\mathcal{W}^2}$ of the product space $\mathcal{W}^2$, given by

$$\|(\Upsilon,\Psi)\|_{\mathcal{W}^2}^2 := \sum_{j\in\{c,s\}} \Bigg[\|\mathbf{y^j}\|^2_{\mathbf{H}(\mathbf{curl},\Omega_1)} + \|\boldsymbol{\lambda}^{\mathbf{j}}\|^2_{\mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)} + \|\mathbf{p^j}\|^2_{\mathbf{H}(\mathbf{curl},\Omega_1)} + \|\boldsymbol{\eta}^{\mathbf{j}}\|^2_{\mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)} \Bigg],$$

to the non-standard norm $\|\cdot\|_{\mathcal{C}}$, i.e.,

$$\underline{C}\|(\Upsilon,\Psi)\|_{\mathcal{W}^2} \le \|(\Upsilon,\Psi)\|_{\mathcal{C}} \le \overline{C}\|(\Upsilon,\Psi)\|_{\mathcal{W}^2},$$

we obtain the Cea-type estimate in the norm of the product space $\mathcal{W}^2$ as well, i.e.,

$$\|(\Upsilon,\Psi)-(\Upsilon_h,\Psi_h)\|_{\mathcal{W}^2} \le C \inf_{(\Phi_h,\Theta_h)\in\mathcal{W}_h^2} \|(\Upsilon,\Psi)-(\Phi_h,\Theta_h)\|_{\mathcal{W}^2}, \tag{7.1}$$

with a constant C that is independent of the mesh width h and the solution (Υ,Ψ). Therefore, it remains to estimate the approximation error for both the Ω_1-part and the interface part. We start by recalling a well-known result for estimating the approximation error in terms of the interpolation error. Let $\mathbf{\Pi}$ be the canonical interpolation operator for the finite element space $\mathcal{ND}_1(\mathcal{T}_h)$. Then the following interpolation error estimate is valid.

LEMMA 7.1. *For $\mathbf{y}\in\mathbf{H^s}(\mathbf{curl},\Omega_1)$, $s>\frac{1}{2}$, the interpolation error can be estimated by*

$$\|\mathbf{y}-\mathbf{\Pi}\mathbf{y}\|_{\mathbf{H}(\mathbf{curl},\Omega_1)} \le Ch^{\min(1,s)}\left(\|\mathbf{y}\|_{\mathbf{H^s}(\Omega_1)} + \|\mathbf{curl}\,\mathbf{y}\|_{\mathbf{H^s}(\Omega_1)}\right),$$

where the constant C is independent of the mesh size h.

Proof. See [10]. □

In order to give a bound for the approximation error on the boundary, we use the fact that we are estimating Neumann traces of the interior functions.

LEMMA 7.2. *For $\boldsymbol{\lambda}=\gamma_N\mathbf{y}\in\mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0,\Gamma)$, the approximation error can be estimated by*

$$\inf_{\boldsymbol{\lambda}_{\mathbf{h}}\in\mathcal{RT}_0^0(\mathcal{K}_h)} \|\boldsymbol{\lambda}-\boldsymbol{\lambda}_{\mathbf{h}}\|_{\mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)} \le C\|\mathbf{curl}\,\mathbf{y}-\mathbf{\Pi}\mathbf{curl}\,\mathbf{y}\|_{\mathbf{H}(\mathbf{curl},\Omega_1)},$$

where the constant C is independent of the mesh size h.

Proof. See [24, Theorem 8.1]. □

The main result of the space discretization error analysis for the time-harmonic eddy current optimal control problem is summarized in the next theorem.

THEOREM 7.3. *Let the solution $(\mathbf{y^c},\mathbf{y^s},\mathbf{p^c},\mathbf{p^s})$ of the eddy current optimal control problem be as regular as*

$$\mathbf{y^j}\in\mathbf{H^s}(\Omega_1), \mathbf{curl}\,\mathbf{y^j}\in\mathbf{H^s}(\Omega_1), \mathbf{curl}\,\mathbf{curl}\,\mathbf{y^j}\in\mathbf{H^s}(\Omega_1), \quad j\in\{c,s\},$$
$$\mathbf{p^j}\in\mathbf{H^s}(\Omega_1), \mathbf{curl}\,\mathbf{p^j}\in\mathbf{H^s}(\Omega_1), \mathbf{curl}\,\mathbf{curl}\,\mathbf{p^j}\in\mathbf{H^s}(\Omega_1), \quad j\in\{c,s\},$$

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for some $s > \frac{1}{2}$. Then the following estimate holds:

$$\|(\Upsilon, \Psi) - (\Upsilon_h, \Psi_h)\|_{\mathcal{W}^2} \leq Ch^{\min(1,s)} \Bigg(\sum_{j \in \{c,s\}} \|\mathbf{y}^{\mathbf{j}}\|_{\mathbf{H}^s(\Omega_1)} + \|\mathbf{curl}\,\mathbf{y}^{\mathbf{j}}\|_{\mathbf{H}^s(\Omega_1)}$$

$$+ \|\mathbf{curl}\,\mathbf{curl}\,\mathbf{y}^{\mathbf{j}}\|_{\mathbf{H}^s(\Omega_1)} + \|\mathbf{p}^{\mathbf{j}}\|_{\mathbf{H}^s(\Omega_1)} + \|\mathbf{curl}\,\mathbf{p}^{\mathbf{j}}\|_{\mathbf{H}^s(\Omega_1)} + \|\mathbf{curl}\,\mathbf{curl}\,\mathbf{p}^{\mathbf{j}}\|_{\mathbf{H}^s(\Omega_1)} \Bigg),$$

where the constant C is independent of the mesh size h.

Proof. The key tools of this proof are the the Cea-type estimate (7.1) in combination with the approximation properties in Lemma 7.1 and Lemma 7.2. Indeed, we have

$$\begin{aligned}\inf_{\lambda_h \in \mathcal{RT}_0^0(\mathcal{K}_h)} \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_{\mathbf{h}}\|_{\mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)} &\leq C\|\mathbf{curl}\,\mathbf{y} - \boldsymbol{\Pi}\mathbf{curl}\,\mathbf{y}\|_{\mathbf{H}(\mathbf{curl}\,,\Omega_1)} \\ &\leq Ch^{\min(1,s)} \left(\|\mathbf{curl}\,\mathbf{y}\|_{\mathbf{H}^s(\Omega_1)} + \|\mathbf{curl}\,\mathbf{curl}\,\mathbf{y}\|_{\mathbf{H}^s(\Omega_1)}\right),\end{aligned}$$

and

$$\begin{aligned}\inf_{\mathbf{y_h} \in \mathcal{ND}_1(\mathcal{T}_h)} \|\mathbf{y} - \mathbf{y_h}\|_{\mathbf{H}(\mathbf{curl}\,,\Omega_1)} &\leq \|\mathbf{y} - \boldsymbol{\Pi}\mathbf{y}\|_{\mathbf{H}(\mathbf{curl}\,,\Omega_1)} \\ &\leq Ch^{\min(1,s)} \left(\|\mathbf{y}\|_{\mathbf{H}^s(\Omega_1)} + \|\mathbf{curl}\,\mathbf{y}\|_{\mathbf{H}^s(\Omega_1)}\right).\end{aligned}$$

By applying the previous two estimates to each component of the product space $\mathcal{W}^2$, the desired result follows. □

Of course, the previous result also holds for all modes $k = 0, \ldots, N$ and therefore also for the multiharmonic case by summing over all modes $k = 0, \ldots, N$.

8. Conclusion. The method developed in this work shows great potential for solving distributed optimal control problems for multiharmonic eddy current problems in an efficient and optimal way. The key points of our method are the usage of a non-standard time discretization technique in terms of a truncated Fourier series, a space discretization in terms of a symmetric FEM-BEM coupling method, and the construction of parameter-independent solvers for the resulting system of equations in the frequency domain. The theory developed in this paper establishes a theoretical estimate of the convergence rate of MinRes as a solver when our proposed preconditioner is applied. Due to the natural decoupling of the frequency domain equations, an efficient parallel implementation of the solution procedure is straightforward.

Indeed, the theory developed in this paper shows two possibilities to construct efficient and parameter-robust solvers:

- If the theoretical preconditioner corresponding to the norm $\|\cdot\|_{\mathcal{C}_I}$ can be replaced by an efficient and parameter-robust practical preconditioner, we obtain a fully parameter-robust solver. This issue is subject to future research.
- Otherwise, we can use the canonical preconditioner corresponding to the simpler norm $\|\cdot\|_{\tilde{\mathcal{F}}}$. This preconditioner can be realized by standard preconditioners, but we have to pay the price that we loose robustness with respect to the reluctivity ν.

In some applications, it is reasonable to add so-called box constraints in the conducting domain Ω_1 for the control $\mathbf{u}$ or/and the state $\mathbf{y}$ to an optimal control problem like (3.1)-(3.2). In the standard approach, these constraints can be handled by a simple projection to the box [33], leading to a non-linear optimality system that can be solved by superlinearly convergent semi-smooth Newton methods [22, 27]. Unfortunately, in the multiharmonic approach, box constraints for $\mathbf{u}$ or/and $\mathbf{y}$ cannot be handled in such an easy way. However, box constraints for their Fourier coefficients can be treated by such a projection. Indeed, using the framework of [20] and the preconditioners constructed in our work, efficient solvers for the Jacobi-systems, which arise at each step of the semi-smooth Newton method applied to the latter mentioned constrained optimization problems, can be constructed. The resulting solvers are at least robust in the discretization parameters h and N; cf. [28].

A general time-periodic desired state $\mathbf{y_d}$ can be approximated in terms of a truncated Fourier series, i.e., a multiharmonic representation. Therefore, we introduce a time-discretization error due to the truncation of the Fourier series. Let us assume that the solution of the interior problem be as regular as $(\mathbf{y}, \mathbf{p}) \in H^r((0,T), \mathbf{H}(\mathbf{curl}\,, \Omega_1)^2) \cap H^{2r}((0,T), \mathbf{L_2}(\Omega_1)^2)$ for some $r \geq \frac{1}{2}$ and $(\mathbf{y}(\cdot,t), \mathbf{p}(\cdot,t)) \in \mathbf{H^s}(\mathbf{curl}\,\mathbf{curl}\,, \Omega_1)^2$ for some $s > \frac{1}{2}$. Then an a-priori error estimate for the space and time discretization error of order $\mathcal{O}(h^{\min(1,s)} + N^{-r})$ can be shown. Therefore, for smooth desired states, we obtain a higher order of convergence.

Anyway, the preconditioners proposed and analyzed in this paper can be useful for all these cases, too. The application of our solver to practical problems, including different control and observation domains or the presence of control or/and state constraints, will be presented in a subsequent paper.

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