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CONFORMAL MAPPING OF CIRCULAR MULTIPLY CONNECTED DOMAINS ONTO SLIT DOMAINS*

ROMAN CZAPLA†, VLADIMIR MITYUSHEV†, AND NATALIA RYLKO‡

Abstract. The method of Riemann–Hilbert problems is used to unify and to simplify construction of conformal mappings of multiply connected domains. Conformal mappings of arbitrary circular multiply connected domains onto the complex plane with slits of prescribed inclinations are constructed. The mappings are derived in terms of uniformly convergent Poincaré series. In the proposed method, no restriction on the location of the boundary circles is assumed. Convergence and implementation of the numerical method are discussed.

Key words. Riemann–Hilbert problem, multiply connected domain, complex plane with slits

AMS subject classifications. 30C30, 30E25

1. Introduction.

Various numerical methods for conformal mappings of multiply connected domains were discussed in the recent book edited by Kühnau [10]. When studying conformal mappings between multiply connected domains, it is convenient to introduce the canonical domains and to study conformal mappings of arbitrary domains onto these canonical domains. Multiply connected domains in the extended complex plane whose boundaries consist of mutually disjoint circles form one of the most important classes of the canonical domains. Another class of the canonical domains consists of slit domains bounded by mutually disjoint parallel (concentric) slits. This class is well studied theoretically and numerically. Domains bounded by mutually disjoint arbitrarily oriented slits are important in fracture mechanics. Therefore, effective construction of the conformal mappings of such domains onto circular domains is important for both theoretical and practical applications. The Schwarz–Christoffel mappings include such mappings as special cases. At the same time, domains with polygonal boundaries can be considered as limit cases of the slit domains.

DeLillo and Kropf [7] (see also references therein) and DeLillo et al. [6] deduced computationally effective formulae for the Schwarz–Christoffel and canonical slit mappings in terms of the special infinite products for domains obeying some geometrical restrictions. Crowdy [3, 4, 5] expressed these infinite products in terms of the Schottky–Klein prime functions. Highly accurate numerical methods based on kernel methods were developed by Sanawi et al. [18, 19] by reduction to linear integral equations of the second kind.

Riemann–Hilbert problems for multiply connected domains explicitly or implicitly arise in the above investigations, since construction of conformal mappings can be reduced to the solution of Riemann–Hilbert problems [14, 21]. Hence, progress in constructive solution of Riemann–Hilbert problems yields numerical algorithms to construct conformal mappings. The review [21] contains some results following this line. The results presented in [21] are based on absolutely convergent series and corresponding algorithms which can be constructively applied to multiply connected domains obey geometrical restrictions; see also similar restrictions in [3, 4, 7]. In order to use analogous computational schemes in general cases, Riemann–Hilbert problems were stated in a form which includes additional polynomials with undetermined coefficients [21]. This complicates direct iteration schemes, since an additional system of linear algebraic equations arises. A similar method based on a Riemann–Hilbert

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†Dept. Computer Sciences and Computer Methods, Pedagogical University, ul. Podchorazych 2, Krakow 30-084, Poland ({romanczapla85,vmityushev}@gmail.com, mityu@up.krakow.pl).

‡Dept. Technology, Pedagogical University, ul. Podchorazych 2, Krakow 30-084, Poland (rylko@up.krakow.pl).

problem with negative winding number was used for conformal mappings of multiply connected domains close to circular. The computability of conformal mappings onto the canonical domains was discussed by Andreev and McNicholl [1].

In this paper, we use the results [12, 13, 14] to unify and to simplify applications of the method of Riemann–Hilbert problems to conformal mappings of multiply connected domains. The theoretical foundation of the method for general multiply connected domains was given in [12]. It is based on the generalized Schwarz method for non–overlapping domains [11] (Einzelschrittverfahren and Gesamtschrittverfahren in notations of [21]). In this paper, the general method is applied to construct conformal mappings of arbitrary circular multiply connected domains onto the complex plane with slits of prescribed inclinations. The mapping is derived in terms of the uniformly convergent Poincaré series (4.13). In the proposed method, no restriction on the location of the boundary circles is assumed. Undetermined constants are used but only in the right hand part of the boundary conditions, which does not complicate the explicit iterative method.

An implementation of the method to numerical solution of the Riemann–Hilbert problem is based on the method of functional equations. First, the Riemann-Hilbert problem is written as an $\mathbb{R}$–linear problem [12]. Next, the latter problem is reduced to a system of functional equations (without integral terms) with respect to functions analytic in the disks, the complement of the multiply connected domain to the complex plane. The method of successive approximations is justified for this system in a functional space in which convergence is uniform. Straightforward calculations of the successive approximations yields a Poincaré type series (4.13). The Poincaré series converges uniformly for any multiply connected domain without any geometrical restriction [14]. This allows the application of the algorithms of DeLillo et al. [6] and Wegmann [21] for arbitrary multiply connected domains in their simplest version. The main modification is the addition of terms with a fixed finite point w into (4.13). This simple correction resolves the problem of convergence outlined in [14].

A number of numerical methods have been developed to investigate multiple crack interactions in fracture mechanics. Most of these methods consider weakly interacting cracks or parallel cracks. There are a limited number of works related to higher order multiple crack interactions, since a huge computational effort is needed for the solution. Different inclinations of the cracks complicate numerical schemes. The higher order multiple crack interactions are the main point of the investigations, because the numerical analysis is simplified if cracks are sufficiently far away from each other and the number of cracks is small. Therefore, not all generally presented methods can be applied in practical computations for general locations of the cracks. Muravin and Turkel [15] modified the Element Free Galerkin method to get higher order multiple crack interactions. The results of Section 6 of the present paper can be viewed as analytical solutions to higher order multiple crack interaction problems for Laplace's equation. Such analytical approximate formulae were not known even in lower order multiple crack interactions.

2. Riemann–Hilbert and $\mathbb{R}$–linear problems. Let $z = x + iy$ denote a complex variable on the complex plane $\mathbb{C}$. Consider non–overlapping disks $\mathbb{D}_k = \{z \in \mathbb{C} : |z-a_k| < r_k\}$, $k = 1, 2, \ldots, n$. Let the boundary of $\mathbb{D}_k$, the circle $\partial\mathbb{D}_k$, be oriented in the counterclockwise direction and let D denote the complement of the closed disks $|z - a_k| \le r_k$ in the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Consider the second complex variable $\zeta = u + iv$ on the complex plane with slits Γ_k lying on the lines,

$$- \sin \alpha_k u + \cos \alpha_k v = c_k, \tag{2.1}$$

where c_k are real constants. Let D' denote the complement of all the segments Γ_k to $\widehat{\mathbb{C}}$. Let $\zeta = \varphi(z)$ be a conformal mapping of the circular multiply connected domain D onto D',

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which transforms the circle $|z-a_k|=r_k$ to the slit Γ_k. For definiteness, it is assumed that $\varphi(z)$ satisfies the hydrodynamic normalization at infinity,

$$\varphi(z)=z+\varphi_0+\frac{\varphi_1}{z}+\frac{\varphi_2}{z^2}+\dots. \tag{2.2}$$

Such a conformal mapping always exists and is unique up to an arbitrary additive constant for the given inclinations α_k [8]. It follows from (2.1) that $\varphi(z)$ satisfies the following Riemann–Hilbert problem [12],

$$\mathrm{Im}[e^{-i\alpha_k}\varphi(t)]=c_k,\quad |t-a_k|=r_k,\ k=1,2,\dots,n, \tag{2.3}$$

where c_k are undetermined constants, Im stands for the imaginary part. The problem (2.3) with $c_k=0$ in classes of meromorphic functions was investigated in [20].

LEMMA 2.1. *The problem* (2.2)–(2.3) *has a unique solution up to an arbitrary additive constant.*

Proof. One of the solutions $\varphi(z)$ exists as a conformal mapping. Let $\tilde{\varphi}(z)$ be another solution of (2.2)–(2.3). Then the function $\phi(z)=\varphi(z)-\tilde{\varphi}(z)$ is regular at infinity and satisfies (2.3),

$$\mathrm{Im}[e^{-i\alpha_k}\phi(t)]=c_k,\quad |t-a_k|=r_k,\ k=1,2,\dots,n, \tag{2.4}$$

with appropriate constants c_k. Equations (2.4) can be also written in the form

$$\mathrm{Re}[ie^{i\alpha_k}\overline{\phi(t)}-c_k]=0,\quad |t-a_k|=r_k,\ k=1,2,\dots,n, \tag{2.5}$$

In order to prove that $\phi(z)\equiv constant$, we follow Vekua's lines [20]. Let

$$t(s)=a_k+r_k\exp\left(\frac{is}{r_k}\right) \tag{2.6}$$

be the complex equation of the circle $|t-a_k|=r_k$ with a natural parameter s. Differentiate (2.4) along $|t-a_k|=r_k$ with respect to s,

$$\mathrm{Im}[e^{-i\alpha_k}\phi'(t)t'_s]=0,\quad |t-a_k|=r_k,\ k=1,2,\dots,n.$$

Hence, the function $e^{-i\alpha_k}\phi'(t)t'_s$ is real on $|t-a_k|=r_k$. Multiply (2.5) by this function,

$$\mathrm{Re}[i\overline{\phi(t)}\phi'(t)t'_s-c_ke^{-i\alpha_k}\phi'(t)t'_s]=0,\quad |t-a_k|=r_k,\ k=1,2,\dots,n. \tag{2.7}$$

Integrate (2.7) on s over ∂D,

$$\mathrm{Re}\left[i\int_{\partial D}\overline{\phi(t)}\phi'(t)dt+\sum_{k=1}^{n}c_ke^{-i\alpha_k}\int_{\partial\mathbb{D}_k}\phi'(t)dt\right]=0,\quad |t-a_k|=r_k,\ k=1,2,\dots,n. \tag{2.8}$$

Here, the relation $\partial D=-\cup_{k=1}^{n}\partial\mathbb{D}_k$ is used. Each integral $\int_{\partial\mathbb{D}_k}\phi'(t)dt$ in (2.8) is equal to zero, since the increment of $\phi(t)$ along every circle $\partial\mathbb{D}_k$ vanishes. Application of Green's formula,

$$\int_D w_{\overline{z}}dxdy=\frac{1}{2i}\int_{\partial D}wdz,$$

to the first integral in (2.8) yields

$$\int_D|\phi'(z)|^2dxdy=0.$$

Therefore, $\phi(z)$ is a constant. □

REMARK 2.2. One can see that Lemma 2.1 is valid for an arbitrary multiply connected D with smooth boundary.

The problem (2.3) can be reduced to the $\mathbb{R}$–linear problem [14],

$$\varphi(t) = \varphi_k(t) + e^{2i\alpha_k}\overline{\varphi_k(t)} + ie^{i\alpha_k}c_k, \quad |t-a_k| = r_k,\ k = 1,2,\ldots,n, \tag{2.9}$$

where $\varphi_k(z)$ is analytic in $|z-a_k| < r_k$ and continuously differentiable in $|z-a_k| \le r_k$. Differentiate (2.9) with respect to s along the circles $|t-a_k| = r_k$ and divide the results by $t'(s) = i\frac{t-a_k}{r_k}$ calculated using (2.6),

$$\psi(t) = \psi_k(t) - e^{2i\alpha_k}\left(\frac{r_k}{t-a_k}\right)^2\overline{\psi_k(t)}, \quad |t-a_k| = r_k,\ k = 1,2,\ldots,n, \tag{2.10}$$

where $\psi(z) = \varphi'(z)$ and $\psi_k(z) = \varphi_k'(z)$.

3. Functional equations. The $\mathbb{R}$–linear problem (2.10) can be reduced to functional equations. Let

$$z^*_{(m)} = \frac{r_m^2}{\overline{z-a_m}} + a_m$$

denote the inversion with respect to the circle $|t-a_m| = r_m$. Following [12, 14] introduce the function,

$$\Phi(z) := \begin{cases} \psi_k(z) + \sum_{m\neq k} e^{2i\alpha_m}\left(\frac{r_m}{z-a_m}\right)^2\overline{\psi_m\left(z^*_{(m)}\right)}, & |z-a_k| \le r_k, \\ & k = 1,2,\ldots,n, \\ \psi(z) + \sum_{m=1}^{n} e^{2i\alpha_m}\left(\frac{r_m}{z-a_m}\right)^2\overline{\psi_m\left(z^*_{(m)}\right)}, & z \in D, \end{cases}$$

analytic in the domains $\mathbb{D}_k$ $(k = 1,2,\ldots,n)$ and D. Calculate the jump across the circle $|t-a_k| = r_k$,

$$\Delta_k := \Phi^+(t) - \Phi^-(t),\ |t-a_k| = r_k,$$

where $\Phi^+(t) := \lim_{z\to t\ z\in D}\Phi(z)$, $\Phi^-(t) := \lim_{z\to t\ z\in D_k}\Phi(z)$. Using (2.10) we get $\Delta_k = 0$. It follows from the principle of analytic continuation that $\Phi(z)$ is analytic in the extended complex plane. Moreover, $\psi(\infty) = \varphi'(\infty) = 1$ yields $\Phi(\infty) = 1$. Then Liouville's theorem implies that $\Phi(z) \equiv 1$. The definition of $\Phi(z)$ in $|z-a_k| \le r_k$ yields the following system of functional equations,

$$\psi_k(z) = -\sum_{m\neq k} e^{2i\alpha_m}\left(\frac{r_m}{z-a_m}\right)^2\overline{\psi_m\left(z^*_{(m)}\right)} + 1, \quad |z-a_k| \le r_k,\ k = 1,2,\ldots,n. \tag{3.1}$$

Let $\psi_k(z), k = 1,2,\ldots,n$, be a solution of (3.1). Then the function $\psi(z)$ can be found from the definition of $\Phi(z)$ in D,

$$\psi(z) = -\sum_{m=1}^{n} e^{2i\alpha_m}\left(\frac{r_m}{z-a_m}\right)^2\overline{\psi_m\left(z^*_{(m)}\right)} + 1, \quad z \in D\cup\partial D. \tag{3.2}$$

4. Method of successive approximations. Solution to the functional equations (3.1) is based on the following general result [9].

THEOREM 4.1. *Let A be a compact operator in a Banach space $\mathcal{B}$ and let $f \in \mathcal{B}$. If, for any complex number ν satisfying the inequality $|\nu| \leq 1$, the equation,*

$$x = \nu A x,$$

has only the zero solution, then the unique solution of the equation,

$$x = Ax + f,$$

can be found by the method of successive approximations. The approximations converge in $\mathcal{B}$ to the solution

$$x = \sum_{k=0}^{\infty} A^k f.$$

Introduce a space $\mathcal{H}(D^+)$ consisting of functions analytic in $D^+ = \cup_{k=1}^{n} \mathbb{D}_k$ and Hölder continuous in the closure of D^+ endowed with the norm,

$$||\omega|| = \sup_{t \in \partial D^+} |\omega(t)| + \sup_{t_{1,2} \in \partial D^+} \frac{|\omega(t_1)| - \omega(t_2)|}{|t_1 - t_2|^{\alpha}}, \tag{4.1}$$

where $0 < \alpha \leq 1$, $\partial D^+ = \cup_{k=1}^{n} \partial \mathbb{D}_k = -\partial D$ is the boundary of D^+. The space $\mathcal{H}(D^+)$ is Banach, since the norm in $\mathcal{H}(D^+)$ coincides with the norm of functions Hölder continuous on ∂D^+ (sup on $D^+ \cup \partial D^+$ in (4.1) is equal to sup on ∂D^+). It follows from Harnack's theorem that convergence in the space $\mathcal{H}(D^+)$ implies the uniform convergence in the closure of D^+.

THEOREM 4.2. *The system (3.1) has a unique solution for any circular multiply connected domain D. This solution can be found by the method of successive approximations convergent in the space $\mathcal{H}(D^+)$, i.e., uniformly convergent in every disk $|z - a_k| \leq r_k$.*

Proof. Let $|\nu| \leq 1$. Consider equations in $\mathcal{H}(D^+)$ with a compact operator on the right side [12],

$$\psi_k(z) = -\nu \sum_{m \neq k} e^{2i\alpha_m} \left(\frac{r_m}{z - a_m} \right)^2 \overline{\psi_m \left(z^*_{(m)} \right)}, \quad |z - a_k| \leq r_k, \ k = 1, 2, \ldots, n. \tag{4.2}$$

Let $\psi_k(z)$ be a solution of (4.2). Introduce the function $\phi(z)$ analytic in D and Hölder continuous in its closure as follows,

$$\psi(z) = -\nu \sum_{m=1}^{n} e^{2i\alpha_m} \left(\frac{r_m}{z - a_m} \right)^2 \overline{\psi_m \left(z^*_{(m)} \right)}, \quad z \in D \cup \partial D. \tag{4.3}$$

Calculating the difference $\psi(t) - \psi_k(t)$ on each $|t - a_k| = r_k$, we arrive at the $\mathbb{R}$–linear conjugation relations,

$$\psi(t) = \psi_k(t) - \nu e^{2i\alpha_k} \left(\frac{r_k}{t - a_k} \right)^2 \overline{\psi_k(t)}, \quad |t - a_k| = r_k. \tag{4.4}$$

Moreover, (4.3) implies that $\psi(\infty) = 0$. Let $|\nu| < 1$. According to [2] the $\mathbb{R}$–linear problem (4.4) has only zero solutions. Hence, the system (4.2) also has only zero solutions. Consider now the case $|\nu| = 1$, where $\nu = e^{2i\theta}$ for some θ. Then (4.4) can be written in the form

$$e^{-i(\theta+\alpha_k)} \psi(t) = e^{-i(\theta+\alpha_k)} \psi_k(t) - e^{i(\theta+\alpha_k)} \left(\frac{r_k}{t - a_k} \right)^2 \overline{\psi_k(t)}, \quad |t - a_k| = r_k. \tag{4.5}$$

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Integration of (4.5) with respect to s yields

$$e^{-i\alpha_k}\phi(t) = e^{-i\alpha_k}\phi_k(t) + e^{i\alpha_k}\overline{\phi_k(t)} + d_k, \quad |t-a_k| = r_k, \tag{4.6}$$

where $\phi'(z) = e^{-i\theta}\psi(z)$, $\phi_k'(z) = e^{-i\theta}\psi_k(z)$, d_k are constants of integration and $\phi(z)$ is analytic in D. The imaginary part of (4.6) gives the problem,

$$\mathrm{Im}\; e^{-i\alpha_k}\phi(t) = \mathrm{Im}\; d_k, \quad |t-a_k| = r_k,$$

which has only constant solutions in accordance with Lemma 2.1. Then, (4.6) yields

$$\mathrm{Re}\; e^{-i\alpha_k}\phi_k(t) = h_k, \quad |t-a_k| = r_k,$$

for some constant h_k. Therefore, each $\phi_k(z)$ is also a constant. Then $\psi(z) \equiv 0$ and $\psi_k(z) \equiv 0$.

Theorem 4.1 yields the convergence of the method of successive approximations applied to the system (3.1). □

Let $\psi_k(z)$ be a solution to the system of functional equations (3.1). Let $w \in D$ be a fixed point not equal to infinity. Introduce the functions

$$\phi_m(z) = \int_{w^*_{(m)}}^{z} \psi_m(t)dt + \phi_m(w^*_{(m)}), \quad |z-a_m| \le r_m, \quad m = 1, 2, \ldots, n, \tag{4.7}$$

and

$$\omega(z) = \sum_{m=1}^{n} e^{2i\alpha_m}\left[\overline{\phi_m\left(z^*_{(m)}\right)} - \overline{\phi_m\left(w^*_{(m)}\right)}\right], \quad z \in D. \tag{4.8}$$

The functions $\omega(z)$ and $\phi_m(z)$ analytic in D and in D_m, respectively, and continuously differentiable in the closures of the domains considered. One can see from (4.7) that the function $\phi_m(z)$ is determined by $\psi_m(z)$ up to an additive constant which vanishes in (4.8). The function $\omega(z)$ vanishes at $z = w$. Investigate the function $\omega(z)$ on the boundary of D. It follows from (4.8) and $t = t^*_{(k)}$ ($|t-a_k| = r_k$) for each fixed k that

$$\omega(t) = e^{2i\alpha_k}\left[\overline{\phi_k\left(t\right)} - \overline{\phi_k\left(w^*_{(k)}\right)}\right] + \Psi_k(t), \tag{4.9}$$

where

$$\Psi_k(z) = \sum_{m \ne k} e^{2i\alpha_m}\left[\overline{\phi_m\left(z^*_{(m)}\right)} - \overline{\phi_m\left(w^*_{(m)}\right)}\right].$$

Using the relation [12]

$$\frac{d}{dz}\left[\overline{\phi_m\left(z^*_{(m)}\right)}\right] = -\left(\frac{r_m}{z-a_m}\right)^2 \overline{\phi_m'\left(z^*_{(m)}\right)}, \; |z-a_m| > r_m,$$

calculate the derivative

$$\Psi_k'(z) = -\sum_{m \ne k} e^{2i\alpha_m}\left(\frac{r_m}{z-a_m}\right)^2 \overline{\psi_m\left(z^*_{(m)}\right)}.$$

Application of (3.1) yields

$$\Psi_k'(z) = \psi_k(z) - 1, \quad |z - a_k| \leq r_k.$$

Then (4.9) and (4.7) imply that

$$e^{-i\alpha_k}\omega(t) = e^{i\alpha_k}\left[\overline{\phi_k(t)} - \overline{\phi_k\left(w_{(k)}^*\right)}\right] + e^{-i\alpha_k}[\phi_k(t) - t + d_k], \quad |t - a_k| = r_k,$$

where d_k is a constant of integration. Calculating the imaginary part of the relation gives

$$\operatorname{Im}[e^{-i\alpha_k}(\omega(t) + t)] = p_k, \quad |t - a_k| = r_k, \tag{4.10}$$

where p_k is a constant. Comparing (4.10) and (2.3) and using Lemma 2.1 we conclude that the required conformal mapping has the form

$$\varphi(z) = z + \omega(z) + constant, \tag{4.11}$$

where $\omega(z)$ is calculated by (4.8).

Application of the method of successive approximations to (3.1) and term-by-term integration of the obtained uniformly convergent series yields the exact formula,

$$\begin{aligned}\varphi_k(z) = q_k + z + \sum_{k_1 \neq k} e^{2i\alpha_{k_1}}\left(\overline{z_{(k_1)}^* - w_{(k_1)}^*}\right) + \sum_{k_1 \neq k}\sum_{k_2 \neq k_1} e^{2i(\alpha_{k_1} - \alpha_{k_2})}\left(z_{(k_2k_1)}^* - w_{(k_2k_1)}^*\right) \\ + \sum_{k_1 \neq k}\sum_{k_2 \neq k_1}\sum_{k_3 \neq k_2} e^{2i(\alpha_{k_1} - \alpha_{k_2} + \alpha_{k_3})}\left(\overline{z_{(k_3k_2k_1)}^* - w_{(k_3k_2k_1)}^*}\right) + \ldots, \quad |z - a_k| \leq r_k.\end{aligned} \tag{4.12}$$

Using (4.8) and (4.12), we write the function (4.11) up to an arbitrary additive constant in the form

$$\begin{aligned}\varphi(z) = z + \sum_{k=1}^{n} e^{2i\alpha_k}\left(\overline{z_{(k)}^* - w_{(k)}^*}\right) + \sum_{k=1}^{n}\sum_{k_1 \neq k} e^{2i(\alpha_k - \alpha_{k_1})}\left(z_{(k_1k)}^* - w_{(k_1k)}^*\right) \\ + \sum_{k=1}^{n}\sum_{k_1 \neq k}\sum_{k_2 \neq k_1} e^{2i(\alpha_k - \alpha_{k_1} + \alpha_{k_2})}\left(\overline{z_{(k_2k_1k)}^* - w_{(k_2k_1k)}^*}\right) + \ldots.\end{aligned} \tag{4.13}$$

5. Numerical examples. Following [6, 7] one can use formula (4.13) in computations. We use another implementation based on the functional equations (3.1). The method of successive approximations is applied to (3.1). We start with the initial guess, $\psi_k^{(0)}(z) = 1$, $k = 1, \ldots, n$. The iteration is then given by

$$\psi_k^{(\mathrm{it}+1)}(z) = -\sum_{m \neq k} e^{2i\alpha_m}\left(\frac{r_m}{z - a_m}\right)^2 \overline{\psi_m^{(\mathrm{it})}\left(z_{(m)}^*\right)} + 1, \quad |z - a_k| \leq r_k, k = 1, \ldots, n.$$

The approximations converge uniformly for any location of non–overlapping disks. Further, the function $\psi(z)$ is constructed by (3.2). The conformal mapping $\varphi(z)$ is constructed by integration of $\psi(z)$. The results reported here are from a Mathematica© implementation. The functions $\psi(z)$ and $\varphi(z)$ are calculated in a symbolic form that has advantages in applications. For instance, this method can yield analytical formulae to describe the macroscopic

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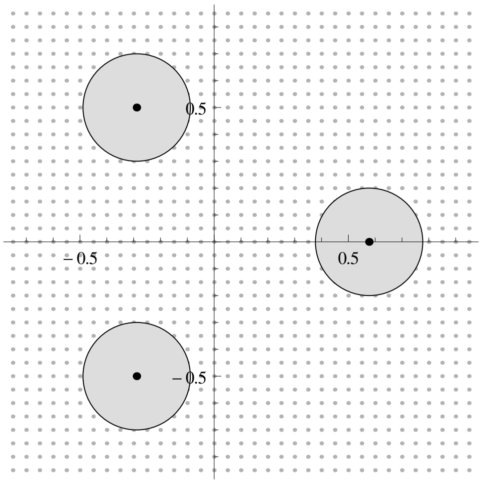
properties of fractured media; see the next section. Examples are presented in Figs. 5.1–5.2. Details of the numerics are given in Tables 5.1 and 5.2. Note that in this symbolic implementation, the cost of each iteration increases rapidly as the number of iterations increases; see Table 5.1. Also, note that the number of iterations needed to achieve a given level of accuracy increases as the circles become closer to touching; see Table 5.2. (A preliminary numerical method, similar to [21, Sec. 12.4], which only updates values of the $\psi_k^{(\mathrm{it})}(z)$'s on the circles, $|z-a_k| = r_k$, using Fourier series, has been implemented in MATLAB. It is much faster than the symbolic calculation, but does not yield analytic formulae. We plan to report on tests of this method in a future paper.)

The speed at which (4.13) converges to its limit (the rate of convergence) depends on the choice of the point w. The point $w = \infty$ is the unique exceptional point in D for which the series (4.13) can diverge [14]. This unlucky infinite point tacitly was taken in previous numerical methods [7, 11] which led to geometrical restrictions to get absolutely convergent algorithms. If $w \in D\backslash\{\infty\}$, the series (4.13) always uniformly converges. However, convergence can be slow because of the eventual conditional convergence. Our computations show that the rate of convergence is high when the point w is close to all the centers a_k and simultaneously is far away from the surrounding centers. For instance, in Fig. 5.2, $w = \frac{i}{6}$ is the geometrical center of D, but $w = 0$ and $w = \frac{i}{3}$ are equidistant from four surrounding points. For this reason we put $w = 0$ at "the middle of D" in the computations presented in Figs. 5.1–5.2.



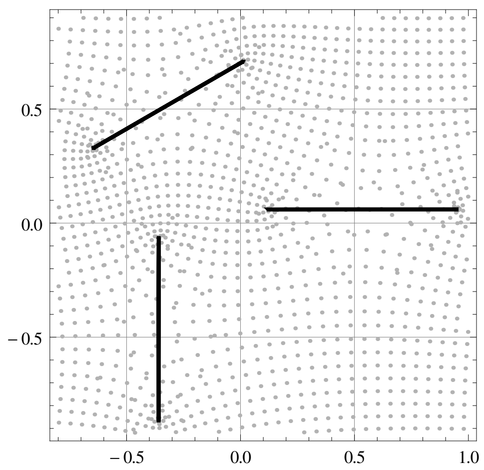


FIG. 5.1. *Conformal mapping of the exterior of 3 disks with the centers at* $a_1 = \frac{1}{\sqrt{3}}$, $a_2 = \frac{1}{\sqrt{3}}\, e^{\frac{2}{3}\pi i}$, $a_3 = \frac{1}{\sqrt{3}}\, e^{-\frac{2}{3}\pi i}$ *of the radii,* $r_1 = r_2 = r_3 = 0.2$, *onto the plane with slits of the inclinations,* $\alpha_1 = 0$, $\alpha_2 = \frac{\pi}{6}$, $\alpha_3 = \frac{\pi}{2}$, *respectively.*

6. Application to multiple crack interaction. The present section is based on the results [16] where the dipole matrix $\mathcal{M}$ for circular inclusions were analytically calculated. The effective conductivity tensor Λ of the dilute composites can by obtained through the dipole matrix $\mathcal{M}$ [17],

$$\Lambda = I - \frac{\nu}{\pi}\mathcal{M}\left(I + \frac{\nu}{2\pi}\mathcal{M}\right)^{-1} + O(\nu^3), \tag{6.1}$$

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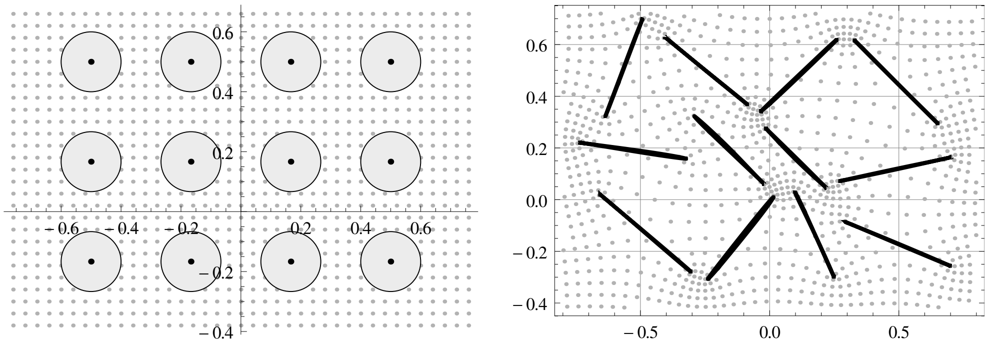

FIG. 5.2. *Conformal mapping of the exterior of 12 disks with the centers at* $\frac{1}{3}[m_1 - \frac{3}{2} + i(m_2 - \frac{3}{2})]$ *(*$m_1 = 0, 1, 2, 3$*,* $m_2 = 1, 2, 3$*) of the radii* 0.1 *onto the plane with slits of the inclinations* $2.45, 3.00, 1.21, 0.88, 2.37361, 2.45, 1.98, 2.37, 0.75, 2.74, 0.23, 2.36$ *randomly chosen on* $(0, \pi)$.

TABLE 5.1
Dependence of the CPU time spent in the Mathematica© *kernel on the number of iterations for three disks.*

it	time(s)	it	times(s)
0	0.08	6	10.3
1	0.20	7	29.3
2	2.44	8	107.4
3	6.17	9	436.2
4	7.28	10	1902.8
5	5.13		

where ν is the concentration of inclusions and I is the identity matrix. Before computation of the dipole matrix, we note that it is invariant under conformal mapping. Therefore, formula (6.1) and the results [16] for circular holes can be applied to fracture materials by use of the conformal mapping (4.13).

Let G be an arbitrary multiply connected domain bounded by piece-wise smooth curves $\partial G_k, k = 1, 2, \ldots, n$. Let $n(\tau)$ denote the unit outward normal vector to ∂G_k at the point τ written as a complex value. Following [16] consider the $\mathbb{R}$–linear condition,

$$\psi^{(\xi)}(\tau) = \psi_k^{(\xi)}(\tau) - [\overline{n(\tau)}]^2 \overline{\psi_k^{(\xi)}(\tau)} + \xi, \ \tau \in \partial G_k, \ k = 1, 2, \ldots, n,$$

where $\xi = 1$ or $\xi = -i$. The function $\psi^{(\xi)}(\tau)$ at infinity has the asymptotic behavior

$$\psi^{(\xi)}(\tau) \sim \frac{m^{(\xi)}}{2\pi} \frac{1}{\tau^2} + O\left(\frac{1}{|\tau|^3}\right).$$

The matrix

$$\mathcal{M} = \begin{pmatrix} \mathrm{Re}\ m^{(1)} & \mathrm{Im}\ m^{(1)} \\ \mathrm{Re}\ m^{(-i)} & \mathrm{Im}\ m^{(-i)} \end{pmatrix} \tag{6.2}$$

is called the dipole matrix.

The dipole matrix (6.2) has the same form for the potentials $\psi^{(\xi)}(z)$ and $\Psi^{(\xi)}(\zeta)$ related by the conformal mapping $\zeta = \varphi(z)$ given by (4.13). Two iterations of the scheme described

TABLE 5.2

Details of the numerics are given for the domains with the same centers, a_k, and inclinations, α_k, as the domain in Fig. 5.1, but with increasing radii, $r_1 = r_2 = r_3 = r$. Data for each radius (r) are the computed values of the inclinations (α), the maximal deviation of the computed points on the slits from the line approximations (d), and the lengths of the slits (ℓ) at each step of the iteration (it). Note that the number of iterations increases as r approaches 0.5, where for $r = 0.5$ the circles would touch.

	slit $\alpha = 0$			**slit $\alpha = \frac{\pi}{6}$**			**slit $\alpha = \frac{\pi}{2}$**		
it	α	d	ℓ	α	d	ℓ	α	d	ℓ
	$r = 0.3$								
0	0.0722	0.0525	1.2644	0.4854	0.0737	1.1286	1.4933	0.0199	1.2084
1	0.0028	0.0060	1.3209	0.5246	0.0043	1.0659	1.5682	0.0145	1.1961
2	0.0014	0.0011	1.3108	0.5236	0.0019	1.0749	1.5706	0.0008	1.1761
3	0.0002	0.0002	1.3104	0.5237	0.0002	1.0760	1.5706	0.0003	1.1777
4	0.0000	0.0000	1.3105	0.5236	0.0000	1.0757	1.5708	0.0000	1.1780
	$r = 0.4$								
0	0.1176	0.1398	1.7706	0.4616	0.2054	1.4479	1.4369	0.0672	1.6384
1	0.0035	0.0457	1.8999	0.5285	0.0315	1.3405	1.5643	0.0782	1.5653
2	0.0108	0.0169	1.8384	0.5204	0.0220	1.3862	1.5682	0.0093	1.4678
3	0.0023	0.0038	1.8314	0.524	0.0061	1.3894	1.5676	0.0087	1.4874
4	0.0009	0.0017	1.8358	0.5242	0.0011	1.3851	1.5700	0.0022	1.4935
5	0.0003	0.0004	1.8351	0.5235	0.0007	1.3855	1.5704	0.0005	1.4914
6	0.0000	0.0002	1.8350	0.5236	0.0001	1.3857	1.5707	0.0002	1.4913
	$r = 0.49$								
0	0.1555	0.2729	2.3062	0.4417	0.4297	1.7130	1.3804	0.1564	2.0700
1	0.0219	0.2686	2.5205	0.5378	0.1542	1.6022	1.4208	0.4744	1.8432
2	0.0459	0.1350	2.2575	0.4947	0.1665	1.7339	1.5415	0.0769	1.6625
3	0.0189	0.0807	2.3054	0.5209	0.1046	1.7020	1.5534	0.1635	1.6509
4	0.0253	0.0907	2.2539	0.5383	0.0544	1.6644	1.5466	0.0872	1.7374
5	0.0063	0.0424	2.2659	0.5164	0.0383	1.6728	1.5496	0.0852	1.6580
6	0.0013	0.0294	2.2079	0.5254	0.0373	1.6672	1.5587	0.0507	1.6707
7	0.0049	0.0219	2.2160	0.5250	0.0124	1.6554	1.5638	0.0279	1.6549
8	0.0064	0.0193	2.2184	0.5222	0.0149	1.6567	1.5642	0.0195	1.6547
9	0.0013	0.0075	2.2164	0.5241	0.0045	1.6597	1.5667	0.0102	1.6455
10	0.0005	0.0057	2.2088	0.5239	0.0043	1.6603	1.5690	0.0078	1.6451

in [16, formulae (2.8), Thm 2.1] yield the following approximate formulae,

$$m^{(1)} = -2\pi \left[\sum_{k=1}^{n} r_k^2 - \sum_{k=1}^{n} \sum_{m \neq k} \left(\frac{r_k r_m}{a_k - a_m} \right)^2 \right],$$

$$m^{(-i)} = -2\pi \left[\sum_{k=1}^{n} r_k^2 + \sum_{k=1}^{n} \sum_{m \neq k} \left(\frac{r_k r_m}{a_k - a_m} \right)^2 \right].$$

The above formulae are obtained in the circular domain D on the plane z. The same formulae hold for the conformally equivalent domain D' with slits on the plane ζ. In order to get formulae in terms of the geometrical parameters of the domain D', the inverse mapping to (4.13) has to be investigated. This study and higher order formulae for the dipole matrix will be obtained by applications of the iterative scheme [16] in a separate paper.

7. Discussion. A numerical method for conformal mappings of circular multiply connected domains onto the plane with slits of prescribed inclinations is constructed in the present paper. First, the problem is reduced to the Riemann–Hilbert problem (2.3) which can be written in the form of the $\mathbb{R}$–linear problem (2.10). The latter one is reduced to the system of functional equations (3.2). These functional equations have the following two advantages. They contain only compositions of the functions (not any integrals) easily realized in symbolic computations. Fewer iterations are needed to yield a useful approximation. Moreover, uniform convergence takes place for any locations of the disks. These features yield a unified method based on direct iterations to construct conformal mappings without geometrical restrictions imposed in the previous works.

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