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# ON DOMAIN-ROBUST PRECONDITIONERS FOR THE STOKES EQUATIONS *

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**Abstract.** It is well known that the LBB-constant of the Stokes equations and its discrete counterpart degenerate on domains with high aspect ratio. For the solution of the corresponding linear system we propose two preconditioners that are proven to be independent of the aspect ratio of the underlying domain. Both preconditioners are based on a refined approximation of the Schur complement using a coarser grid.

**Key words.** Stokes equations, LBB constant, finite elements, mixed methods.

**AMS subject classifications.** 65N15, 65N30, 35Q30, 76D07.

**1. Introduction.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain. We are concerned with linear systems arising from discretizations of the Stokes equations,

$$-\Delta u + Dp = f \ \text{ in } \Omega, \quad \operatorname{div} u = g \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega, \tag{1.1}$$

where $D_i = \frac{\partial}{\partial x_i}$, $D = (D_1, \ldots, D_n)^T$. The vector field $u = (u^1, \ldots, u^n)$ and the scalar variable $p$ can be regarded as the velocity and the pressure of a (very) viscous flow.

We use the standard Lebesgue and Sobolev spaces $L^2(\Omega)$, $H^1(\Omega)$ with norms

$$\|v\|^2 = \int_\Omega |v|^2\, dx, \quad \|v\|_1^2 = \|v\|^2 + \|Dv\|^2.$$

The inner product in $L^2(\Omega)$ is denoted by $(\cdot,\cdot)$. The space $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. Moreover, we set $X = H_0^1(\Omega)^n$ with norm $|v|_1 = \|Dv\|$ and

$$Y = L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q\, dx = 0\}.$$

The dual space $X'$ of $X$ is equipped with the norm

$$|f|_{-1} = \sup_{v \in X} \frac{f(v)}{|v|_1}.$$

For $f \in X'$ and $g \in Y' = Y$, the weak solution $(u, p) \in X \times Y$ of (1.1) is defined by

$$(Du, D\phi) - (\operatorname{div} \phi, p) = f(\phi) \quad \forall \phi \in X, \tag{1.2}$$

$$(\operatorname{div} u, \psi) = (g, \psi) \quad \forall \psi \in Y. \tag{1.3}$$

The existence proof for the Stokes equations requires the so-called inf-sup- or LBB-condition. For a large class of domains, it is proved in [3] that there exists a constant $L(\Omega) > 0$ which is the largest number such that

$$L(\Omega)\|q\| \leq \sup_{\phi \in X} \frac{-(\operatorname{div} \phi, q)}{|\phi|_1} = |Dq|_{-1} \quad \forall q \in Y. \tag{1.4}$$

For a more elaborated proof of this inequality, see [11], and for an extension to John domains; see [1]. An elementary proof which avoids the use of the theory of singular integrals is given

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in [5]. In [3] it is also proved that for domains with diameter $R$ that are star-shaped with respect to a ball of radius $r$,

$$L(\Omega) \geq c\Big(\frac{r}{R}\Big)^{n+1}. \tag{1.5}$$

In [7] it is shown that for stretched domains with aspect ratio $a$, we have the inequality $m/a \leq L(\Omega) \leq M/a$ with $m > 0$. In particular, this result is true for plates and channels. This result explains why most of the iterative methods for solving the linear system corresponding to (1.2), (1.3) behave poorly on domains with high aspect ratio.

The aim of this paper is to improve the standard preconditioner for problem (1.2), (1.3),

$$C^{-1} : X' \times Y' \to X \times Y, \quad C^{-1}(f,g) = \begin{bmatrix} Tf \\ g \end{bmatrix}, \tag{1.6}$$

where the inverse Laplacian $T : X' \to X$ is defined by

$$(DTf, D\phi) = f(\phi) \quad \forall \phi \in X. \tag{1.7}$$

By a simple energy estimate, it is easy to show that this preconditioner works in the continuous case and is $h$-independent for stable discretizations, but it depends on the LBB-constant $L(\Omega)$; see Section 6. In order to overcome this difficulty, we define preconditioners that use the LBB-condition only on the elements of a coarser mesh. Then the method does not depend on the domain $\Omega$ and the constant in (1.5) is moderate for non-degenerate elements. The first preconditioner described in Section 3 is the minimal modification of (1.6) and improves the pressure values by solving a discrete Laplacian on a (very) coarse mesh. An alternative is shown in Section 5 for the standard stabilized finite element method. On an arbitrary, not necessarily very coarse mesh, the Stokes equation is solved exactly or approximately with the aid of the same preconditioner.

Some modifications of the classical preconditioner (1.6) and the improved preconditioner of Section 5 are described and tested numerically in Section 6.

As we have mentioned above, we believe that most of the known numerical methods depend on the LBB-constant $L(\Omega)$, but it is difficult to prove this assertion since the methods are usually tested only on the unit square or unit cube. The assertion is true for all methods that are based on the Schur complement (see Section 6) of the Stokes operator. This is also numerically verified for the CG-method for the Schur complement in [8]. For other methods of this type, we refer to [2, 10, 16, 18] and the literature cited there.

The situation is unclear for multigrid methods. Clearly, the standard convergence proof is based on the LBB-constant. But in [13] it is explicitly stated that a multigrid method using smoothed aggregation and the smoother of [4] is domain-robust. Since this method does not fit into the framework of the present paper, further investigations of domain-robust multigrid methods would be desirable.

In the context of domain decomposition methods, domain-robust methods are well known; see [14] for the Stokes equations and [9] for almost incompressible elasticity.

**2. The negative norm of $Dq$.** Using integration by parts, it is easy to show for $\phi \in X$ that $\|\mathrm{div}\,\phi\|^2 + \|\mathrm{rot}\,\phi\|^2 = |\phi|_1^2$ and hence from (1.4) that

$$|Dq|_{-1} \leq \sup_{\phi\in X} \frac{\|q\|\,\|\mathrm{div}\,\phi\|}{|\phi|_1} \leq \|q\|. \tag{2.1}$$

From a simple variational analysis, we can conclude that the sup in (1.4) is attained at $w = T(Dq)$, where $T$ is the inverse Laplacian from (1.7), and

$$|Dq|_{-1} = \frac{-(\mathrm{div}\,w, q)}{|w|_1} = |w|_1. \tag{2.2}$$

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LEMMA 2.1. *If $\Omega_i$, $i = 1, \dots, I$, are disjoint open subsets of $\Omega$, then*

$$\sum_{i=1}^{I} \|Dq\|^2_{-1;\Omega_i} \le \|Dq\|^2_{-1;\Omega} \quad \forall q \in Y.$$

*Proof.* Let $w_i \in X(\Omega_i)$ and $w \in X$ be the solutions of the problems

$$(Dw_i, D\phi)_{\Omega_i} = -(\operatorname{div}\phi, q)_{\Omega_i} \quad \forall \phi \in X(\Omega_i), \quad (Dw, D\phi) = -(\operatorname{div}\phi, q) \quad \forall \phi \in X.$$

Using (2.2) and extending the functions $w_i$ by $0$, we obtain

$$\sum_{i=1}^{I} |Dq|^2_{-1;\Omega_i} = \sum_{i=1}^{I} |w_i|^2_{1;\Omega_i} = -\sum_{i=1}^{I} \int_{\Omega_i} q \operatorname{div} w_i \, dx = \sum_{i=1}^{I} \int_{\Omega_i} DwDw_i \, dx$$
$$\le \frac{1}{2}|w|_1^2 + \frac{1}{2}\sum_{i=1}^{I} |w_i|^2_{1;\Omega_i} = \frac{1}{2}|Dq|^2_{-1} + \frac{1}{2}\sum_{i=1}^{I} |Dq|^2_{-1;\Omega_i}. \qquad \square$$

**3. A model preconditioner.** In this section, we construct a domain-robust preconditioner for the continuous Stokes equations. In view of the fact that only energy estimates are used, the preconditioner can easily be extended to conforming finite element approximations; see [12]. This will be explained in detail at the end of this section in Remark 3.6.

The preconditioner is described for dimension $n = 3$. Let $\Omega$ be a bounded polyhedral domain and let $\Pi_H$ be a subdivision of $\overline{\Omega}$ into closed polyhedral elements $\Lambda_H$ of diameter $H \sim 1$. In the following, the generic constant $c$ is allowed to depend on $H$. It is assumed that the intersection of two elements is void or contains a common point, edge or face. Let $Y_H \subset Y$ be the space of piecewise constant functions on the subdivision $\Pi_H$. The $L^2$-projection $Q_H^d : Y \to Y_H$ is defined by

$$Q_H^d q(x) = \mu(\Lambda_H)^{-1} \int_{\Lambda_H} q(y)\, dy, \quad x \in \operatorname{int}\Lambda_H.$$

Let $F_H = \{\Gamma_H\}$ be the set of interior faces. To each face $\Gamma_H$ we fix a normal direction $\nu = \nu(\Gamma_H)$ and denote the neighboring elements by $\Lambda_1, \Lambda_2$. We define the bilinear form $(\cdot,\cdot)_{1,H} : Y_H \times Y_H \to \mathbb{R}$ by

$$(q_H, \psi_H)_{1,H} = \sum_{\Gamma_H \in F_H} \mu(\Gamma_H)[q_H]_{\Gamma_H}[\psi_H]_{\Gamma_H},$$

where

$$[q_H]_{\Gamma_H} = q_H|_{\Lambda_2} - q_H|_{\Lambda_1}$$

denotes the “jump” of $q_H$ from $\Lambda_1$ to $\Lambda_2$. Moreover, let us define the corresponding norm $|\cdot|_{1,H}$ on $X_H$ and the operator $-\Delta_H : Y_H \to Y_H$ by

$$|q_H|^2_{1,H} = (q_H, q_H)_{1,H}, \quad (-\Delta_H q_H, \psi_H) = (q_H, \psi_H)_{1,H} \quad \forall \psi_H \in Y_H.$$

The inverse of $-\Delta_H$ is denoted by $T_H$.

Note that on a subdivision of $\Omega$ into unit cubes, the stencil of $\Delta_H$ coincides with the standard 7-point finite difference stencil of the discrete Laplacian.

Define the operator $C : X \times Y \to X' \times Y'$ by

$$C(v, q) = \begin{bmatrix} -\Delta v \\ q - Q_H^d q - \Delta_H Q_H^d q \end{bmatrix}$$

with inverse $C^{-1} : X' \times Y' \to X \times Y$

$$C^{-1}(f,g) = \begin{bmatrix} Tf \\ g - Q_H^d g + T_H Q_H^d g, \end{bmatrix},$$

which will be the desired domain-robust preconditioner for the Stokes equations. In the discrete case, the computational effort of evaluating this preconditioner is only slightly larger than the evaluation of (1.6) since $T_H$ is determined on a coarse grid.

The space $X \times Y$ is equipped with the norm

$$\|(v,q)\|_{X\times Y}^2 = |v|_1^2 + |Dq|_{-1}^2.$$

The Stokes equations in weak form (1.2), (1.3) define a continuous and bijective operator $L : X \times Y \to X' \times Y'$ which implies that $C^{-1}L : X \times Y \to X \times Y$ is also continuous and bijective.

THEOREM 3.1. *There are positive constants $c_1, c_2$ such that*

$$c_1 \le \|C^{-1}L\|_{X\times Y \to X\times Y} \le c_2,$$

*where the constants $c_1, c_2$ depend on the local shape of the subdivision $\Pi_H$ and on Poincaré's inequality*

$$\|v\| \le c_P |v|_1 \quad \forall v \in X.$$

We remark that the constant $c_P$ depends on the domain $\Omega$, but on stretched domains we can take a "small" direction $e$ in $\|v\| \le c_P \|D_e v\|$. For example, if $\Omega$ is contained in $Q = (0,a_1) \times (0,a_2) \times (0,a_3)$ with $a_1 \le a_2 \le a_3$ we have that $\|v\| \le a_1 \|D_1 v\|$. Thus, $c_P$ does not depend on the aspect ratio of the domain.

The dependence of the constants $c_1, c_2$ on the subdivision $\Pi_H$ will be explained after the proof in Remark 3.5. For the proof of the theorem, the following three technical lemmas are required.

LEMMA 3.2. *There is a constant $c$ with*

$$|Dq_H|_{-1} \le c|q_H|_{1,H}$$

*for all $q_H \in Y_H$.*

*Proof.* From integration by parts, we obtain for arbitrary $v \in X$

$$\begin{aligned} -(\mathrm{div}\, v, q_H) &= -\sum_{\Lambda_H} \int_{\Lambda_H} \mathrm{div}\, v\, q_H \, dx = -\sum_{\Gamma_H \in F_H} \int_{\Gamma_H} \nu \cdot v [q_H]_{\Gamma_H} \, d\sigma \\ &\le c \sum_{\Gamma_H \in F_H} \|v\|_{\Gamma_H} \|[q_H]_{\Gamma_H}\|_{\Gamma_H}, \end{aligned}$$

where $\|\cdot\|_{\Gamma_H}$ denotes the $L^2$-norm on $\Gamma_H$. By the trace theorem, we have $\|v\|_{\Gamma_H} \le c\|v\|_{1,2;\Lambda_1}$ and hence, by Poincaré's inequality

$$-(\mathrm{div}\, v, q_H) \le c|v|_1 |q_H|_{1,H}.$$

The result follows from the definition of the negative norm. ∎

LEMMA 3.3. *The following estimate holds for all $q \in Y$ :*

$$|Q_H^d q|_{1,H} \le c|Dq|_{-1}.$$

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*Proof*. Let $\Gamma_H \in F_H$ with neighboring elements $\Lambda_1, \Lambda_2$ with measures $\mu_1, \mu_2$. Denoting by $\overline{q}$ the mean value of $q$ over $\Lambda_1 \cup \Lambda_2$, we have

$$\overline{q} = \frac{q_1 + q_2}{\mu_1 + \mu_2}, \quad q_i = \int_{\Lambda_i} q\, dx \quad i = 1, 2.$$

From an elementary calculation it follows that

$$\begin{aligned}
\|q - \overline{q}\|^2_{\Lambda_1 \cup \Lambda_2} &= \int_{\Lambda_1 \cup \Lambda_2} \{|q|^2 - 2q\overline{q} + |\overline{q}|^2\}\, dx \\
&= \int_{\Lambda_1} |q|^2\, dx + \int_{\Lambda_2} |q|^2\, dx - \frac{|q_1 + q_2|^2}{\mu_1 + \mu_2} \\
&\geq \frac{|q_1|^2}{\mu_1} + \frac{|q_2|^2}{\mu_2} - \frac{|q_1 + q_2|^2}{\mu_1 + \mu_2} \\
&= \frac{\mu_2}{\mu_1(\mu_1 + \mu_2)} |q_1|^2 + \frac{\mu_1}{\mu_2(\mu_1 + \mu_2)} |q_2|^2 - \frac{2}{\mu_1 + \mu_2} q_1 q_2 \\
&= \frac{\mu_2 \mu_1}{\mu_1 + \mu_2} \left| \frac{q_2}{\mu_2} - \frac{q_1}{\mu_1} \right|^2 \geq m\mu(\Gamma_H)[Q^d_H q]^2_{\Gamma_H}.
\end{aligned}$$

Writing $\Omega(\Gamma_H) = \operatorname{int}(\Lambda_1 \cup \Lambda_2)$ and denoting the LBB-constant on $\Omega(\Gamma_H)$ by $L(\Gamma_H)$, we obtain that

$$\mu(\Gamma_H)[Q^d_H q]^2_{\Gamma_H} \leq \frac{1}{mL(\Gamma_H)^2} |Dq|^2_{-1;\Omega(\Gamma_H)}.$$

We sum this estimate over $\Gamma_H$. On the right-hand side, the domains $\Omega(\Gamma_H)$ overlap in a locally finite way. Hence we can write $\Omega(\Gamma_H) = \Omega_{ij}$ for $i = 1, \ldots, I_j$, $j = 1, \ldots, J$, such that the domains $\{\Omega_{ij}\}_{i=1,\ldots,I_j}$ are disjoint for every $1 \leq j \leq J$. We apply Lemma 2.1 to each $\cup_i \Omega_{ij}$ and end up with the estimate

$$|Q^d_H q|^2_{1,H} \leq \frac{J}{mL^2} |Dq|^2_{-1},$$

where $L$ is the minimum of the constants $L(\Gamma_H)$. $\square$

LEMMA 3.4. *The norms*

$$\left( \|q - Q^d_H q\|^2 + |Q^d_H q|^2_1 \right)^{1/2} \quad \textit{and} \quad |Dq|_{-1}$$

*are equivalent in $Y$ with constants that do only depend on local properties of the subdivision $\Pi_H$.*

*Proof*. From the LBB-condition on $\Lambda_H$ and Lemma 2.1, we obtain

$$\|q - Q^d_H q\|^2 \leq c \sum_{\Lambda_H} |Dq|^2_{-1;\Lambda_H} \leq c|Dq|^2_{-1}$$

and $|Q^d_H q|^2_{1,H} \leq c|Dq|^2_{-1}$ by Lemma 3.3. The other direction is simply proved by the triangle inequality, (2.1), and Lemma 3.2,

$$|Dq|_{-1} \leq |D(q - Q^d_H q)|_{-1} + |DQ^d_H q|_{-1} \leq \|q - Q^d_H q\| + |Q^d_H q|_1. \qquad \square$$

*Proof of Theorem 3.1:* For $(u, p) \in X \times Y$, let $v \in X$ and $q_H \in Y_H$ be the solutions of

$$-\Delta v = -\Delta u + Dp, \quad -\Delta_H q_H = Q^d_H \operatorname{div} u. \tag{3.1}$$

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Then

$$C^{-1}L(u,p) = \begin{bmatrix} v \\ q_\perp + q_H, \end{bmatrix}, \quad q_\perp = \operatorname{div} u - Q_H^d \operatorname{div} u. \tag{3.2}$$

In view of the fact that $Q_H^d q_\perp = 0$, the norms $\|q_\perp\| + |q_H|_{1,H}$ and $|D(q_\perp + q_H)|_{-1}$ are equivalent by Lemma 3.4.

For the estimate from above in Theorem 3.1, we have to show that

$$|v|_1^2 + \|q_\perp\|^2 + |q_H|_{1,H}^2 \leq c_2(|u|_1^2 + |Dp|_{-1}^2). \tag{3.3}$$

From the definition of $v$ and $q_\perp$ in (3.1), (3.2), we immediately obtain that

$$|v|_1 \leq |u|_1 + |Dp|_{-1}$$

and

$$\|q_\perp\| = \|(Id - Q_H^d)\operatorname{div} u\| \leq \|\operatorname{div} u\| \leq |u|_1.$$

For the other term, we have by (3.1)

$$|q_H|_{1,H}^2 = (\operatorname{div} u, q_H).$$

Treating the right-hand side exactly as in the proof of Lemma 3.2, we obtain

$$|q_H|_{1,H}^2 \leq c|u|_1|q_H|_{1,H},$$

which completes the proof of estimate (3.3).

For proving the estimate from below in Theorem 3.1, we have to show that

$$|u|_1^2 + |Dp|_{-1}^2 \leq c_1(|v|_1^2 + \|q_\perp\|^2 + |q_H|_{1,H}^2). \tag{3.4}$$

From (3.2) and (3.1), we obtain

$$\begin{aligned}(\operatorname{div} u, p) &= (\operatorname{div} u, p - Q_H^d p) + (\operatorname{div} u, Q_H^d p) \\ &= (q_\perp, p) + (q_H, Q_H^d p)_{1,H}.\end{aligned} \tag{3.5}$$

For the first term on the right-hand side, we use $q_\perp \perp Y_H$ and obtain from the local LBB-condition on the domains int $\Lambda_H$ and Lemma 2.1

$$\begin{aligned}(q_\perp, p) &= (q_\perp, p - Q_H^d p) \leq \|q_\perp\|\,\|p - Q_H^d p\| \\ &= \|q_\perp\|\Big(\sum_{\Lambda_H} \|p - Q_H^d p\|_{\Lambda_H}^2\Big)^{1/2} \leq c_1\|q_\perp\|\Big(\sum_{\Lambda_H} |Dp|_{-1;\Lambda_H}^2\Big)^{1/2} \\ &\leq \frac{c_1}{2\epsilon}\|q_\perp\|^2 + \frac{\epsilon}{2}|Dp|_{-1}^2,\end{aligned}$$

where we have used Young's inequality $ab \leq \epsilon^{-1}a^2/2 + \epsilon b^2/2$. For the second term in (3.5), we use Hölder's inequality and Lemma 3.3

$$(q_H, Q_H^d p)_{1,H} \leq |q_H|_{1,H}\,|Q_H^d p|_{1,H} \leq c_2|q_H|_{1,H}\,|Dp|_{-1} \leq \frac{c_2}{2\epsilon}|q_H|_{1,H}^2 + \frac{\epsilon}{2}|Dp|_{-1}^2.$$

Inserting the last two estimates into (3.5) gives

$$(\operatorname{div} u, p) \leq c(\epsilon)(\|q_\perp\|^2 + |q_H|_{1,H}^2) + \epsilon|Dp|_{-1}^2.$$

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From (3.1), it follows that

$$|u|_1^2 - (\mathrm{div}\, u, p) \le |v|_1 |u|_1$$

and hence, by the previous estimate,

$$|u|_1^2 \le |v|_1^2 + c(\epsilon)(\|q_\perp\|^2 + |q_H|_{1,H}^2) + 2\epsilon |Dp|_{-1}^2. \tag{3.6}$$

Again from (3.1), we have for $w = T(Dp)$ (see (2.2))

$$|Dp|_{-1}^2 = (Dp, w) = (Dv, Dw) - (Du, Dw) \le |Dp|_{-1}(|u|_1 + |v|_1). \tag{3.7}$$

Now (3.4) follows from (3.6) and (3.7) by choosing $\epsilon$ sufficiently small. ☐

REMARK 3.5. The constants $c_1, c_2$ depend on the LBB-constant on $\Lambda_H$ and $\Lambda_1 \cup \Lambda_2$ for neighboring elements, on the trace constant $\|u\|_{\Gamma_H} \le c\|u\|_{1;\Lambda_H}$, and on $\mu(\Lambda_H)/\mu(\Gamma_H)$ for $\Gamma_H \subset \Lambda_H$. All these constants are moderate for nondegenerate $\Lambda_H$ with diameter $H \sim 1$.

REMARK 3.6. The preconditioner of this section can also be used for conforming finite element approximations in the spaces $X_h \subset X$ and $Y_h \subset Y$. In this case, it is not required that $Y_h$ contains the space $Y_H$. The discrete negative norm of $Dq_h$ is

$$|Dq_h|_{-1,h} = \sup_{\phi_h \in X_h} \frac{-(\mathrm{div}\, \phi_h, q_h)}{|\phi_h|_1},$$

and it is assumed that the space $X_h$ is rich enough so that a discrete LBB-condition holds on each $\Lambda_H \in \Pi_H$,

$$L_H(\Lambda_H)\|q_h\|_{\Lambda_H} \le \sup_{\phi_h \in X_h \cap X(\Lambda_H)} \frac{-(\mathrm{div}\, \phi_h, q_h)}{|\phi_h|_1}$$

for all $q_h \in Y_h$ with $\int_{\Lambda_H} q_h\, dx = 0$.

**4. Finite elements.** Let $\Omega \subset \mathbb{R}^3$ be a bounded polyhedral domain and let $\Pi_H$ be a subdivision of $\Omega$ into simplices $\Lambda_H$ satisfying the usual regularity condition: each simplex contains a ball of radius $c_R^{-1}H$ and is contained in a ball of radius $c_R H$. The intersection of any two simplices is void or coincides with a common node, edge or face.

The subdivision $\Pi_H$ is divided further to a subdivision $\Pi_h$ of $\Omega$ satisfying the above regularity condition with $H$ replaced by $h$. Using continuous and piecewise linear shape functions on the subdivisions $\Pi_h, \Pi_H$ with or without zero boundary conditions, we obtain finite dimensional spaces $X_H \subset X_h \subset X$ and $Y_H \subset Y_h \subset Y$. If the subdivision $\Pi_H$ is very coarse it may happen that $X_H = \{0\}$.

The finite element functions defined above satisfy the well-known inverse estimates

$$|\phi_h|_1 \le ch^{-1}\|\phi_h\|, \quad |\phi_H|_1 \le cH^{-1}\|\phi_H\|.$$

By $R_h : X \to X_h$ we denote the local approximation operator from [17] which satisfies for $v \in X$

$$\|v - R_h v\| \le ch|v|_1, \quad |R_h v|_1 \le c|v|_1. \tag{4.1}$$

Analogous estimates hold for the operator $R_H : X \to X_H$. In the case that $X_H = \{0\}$ the first estimate coincides with Poincaré's inequality.

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Let $Q_h, Q_H$ be the $L^2$-projections into the corresponding spaces of continuous piecewise linear functions. In our notation, we do not distinguish between $Q_h : X \to X_h$ and $Q_h : Y \to Y_h$,

$$Q_h v \in X_h : \quad (Q_h v, \phi_h) = (v, \phi_h) \quad \forall \phi_h \in X_h,$$
$$Q_h q \in Y_h : \quad (Q_h q, \psi_h) = (q, \psi_h) \quad \forall \psi_h \in Y_h.$$

We have $\|v_h - Q_H v_h\| \le \|v_h\|$, $\|Q_H v_h\| \le \|v_h\|$ and

$$\|v_h - Q_H v_h\| \le cH|v_h|_1, \quad |Q_H v_h|_1 \le c|v_h|_1 \quad \forall v_h \in X_h. \tag{4.2}$$

The first estimate is proved by (4.1) for $h := H$, for the second we use the inverse estimate,

$$\begin{aligned} |Q_H v_h|_1 &\le |Q_H v_h - R_H v_h|_1 + |R_H v_h|_1 \le cH^{-1}\|Q_H v_h - R_H v_h\| + c|v_h|_1 \\ &\le cH^{-1}\big\{\|Q_H v_h - v_h\| + \|v_h - R_H v_h\|\big\} + c|v_h|_1 \le c|v_h|_1. \end{aligned}$$

LEMMA 4.1. *For each $q_h \in Y_h$ there exists $w_h \in X_h$, $|w_h|_1 = 1$, such that*

$$|Dq_h|_{-1} \le ch|q_h|_1 + c(\operatorname{div} w_h, q_h).$$

*Proof*. Let $w = T(Dq_h) \in X$ (see (2.2)). From (4.1), we obtain

$$\begin{aligned} |Dq_h|_{-1} = \frac{-(\operatorname{div} w, q_h)}{|w|_1} &\le \frac{-(\operatorname{div}(w - R_h w), q_h)}{|w|_1} + \frac{-(\operatorname{div} R_h w, q_h)}{|w|_1} \\ &\le ch|q_h|_1 + c\frac{|(\operatorname{div} R_h w, q_h)|}{|R_h w|_1}. \end{aligned}$$

The lemma is proved by setting $w_h = \pm R_h w / |R_h w|_1$. ∎

**5. A preconditioner for a stabilized finite element method.** We adopt the finite dimensional spaces $X_H \subset X_h \subset X$ and $Y_H \subset Y_h \subset Y$ from the previous section.

We consider a stabilized finite element approximation of the Stokes equations with bilinear form $a(\cdot,\cdot) : (X_h \times Y_h)^2 \to \mathbb{R}$ defined by

$$a((u_h, p_h), (\phi_h, \psi_h)) = (Du_h, D\phi_h) - (\operatorname{div} \phi_h, p_h) + (\operatorname{div} u_h, \psi_h) + \omega h^2 (Dp_h, D\psi_h),$$

where the coefficient $\omega > 0$ is a stabilization parameter; see [6]. Then the corresponding operator $L_h : X_h \times Y_h \to X_h \times Y_h$, $L_h = (L_{h,X}, L_{h,Y})$ is

$$\begin{aligned} a((u_h, p_h), (\phi_h, \psi_h)) &= (L_h(u_h, p_h), (\phi_h, \psi_h)) \\ &= (L_{h,X}(u_h, p_h), \phi_h) + (L_{h,Y}(u_h, p_h), \psi_h) \quad \forall \phi_h \in X_h \; \forall \psi_h \in Y_h. \end{aligned}$$

For $(u_h, p_h) \in X_h \times Y_h$, we define $(v_h, q_h) \in X_h \times Y_h$ by

$$(Dv_h, D\phi_h) = (Du_h, D(\phi_h - Q_H\phi_h)) - (\operatorname{div}(\phi_h - Q_H\phi_h), p_h) \quad \forall \phi_h \in X_h, \tag{5.1}$$

$$\begin{aligned} (q_h, \psi_h) = &(\operatorname{div} u_h, \psi_h - Q_H\psi_h) \\ &+ \omega h^2 (Dp_h, D(\psi_h - Q_H\psi_h)) \quad \forall \psi_h \in Y_h, \end{aligned} \tag{5.2}$$

or

$$v_h = T_h(Id - Q_H)L_{h,X}(u_h, p_h), \quad q_h = (Id - Q_H)L_{h,Y}(u_h, p_h),$$

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where $T_h : X_h \to X_h$ denotes the inverse of the discrete Laplacian. Furthermore, define $(v_H, q_H) \in X_H \times Y_H$ by

$$\begin{aligned}
(5.3)\qquad & (Dv_H, D\phi_H) - (\operatorname{div}\phi_H, q_H) = (Du_h, D\phi_H) - (\operatorname{div}\phi_H, p_h) \quad \forall \phi_H \in X_H,\\
(5.4)\qquad & (\operatorname{div} v_H, \psi_H) + \omega H^2(Dq_H, D\psi_H) = (\operatorname{div} u_h, \psi_H)\\
& \qquad\qquad + \omega h^2(Dp_h, D\psi_H) \quad \forall \psi_H \in Y_H,
\end{aligned}$$

or

$$(v_H, q_H) = L_H^{-1} Q_H L_h(u_h, p_h).$$

Then the preconditioner for the stabilized finite element method is defined by

$$\begin{aligned}
C^{-1}L_h(u_h, p_h) &= \begin{bmatrix} v_h + v_H \\ q_h + q_H \end{bmatrix}\\
&= \begin{bmatrix} T_h(Id - Q_H)L_{h,X}(u_h, p_h) \\ (Id - Q_H)L_{h,Y}(u_h, p_h) \end{bmatrix} + L_H^{-1} \begin{bmatrix} Q_H L_{h,X}(u_h, p_) \\ Q_H L_{h,Y}(u_h, p_h) \end{bmatrix}.
\end{aligned}$$

Clearly it holds that $C^{-1}L_h : X_h \times Y_h \to X_h \times Y_h$.

In the following, the constants $c, c_1, c_2, \ldots$ may depend on the constant $c_R$ from Section 4 and on the constant $c_L$ in the inequality

$$(5.5)\qquad \|q_h - Q_H q_h\| \le c_L |Dq_h|_{-1} \quad \forall q_h \in Y_h.$$

The constants do not depend on $h, H$ and the LBB-constant $L(\Omega)$.

The space $X_h \times Y_h$ is equipped with the norm

$$\|(v_h, q_h)\|^2_{X_h \times Y_h} = |v_h|_1^2 + |Dq_h|_{-1}^2.$$

THEOREM 5.1. *There exist positive constants $c_1, c_2$ such that*

$$c_1 \le \|C^{-1}L_h\|_{X_h \times Y_h \to X_h \times Y_h} \le c_2.$$

We use the following technical lemmas.

LEMMA 5.2. *The following estimates hold*

$$\begin{aligned}
(5.6)\qquad & \|q_h\| \le c|Dq_h|_{-1} && \forall q_h \in Y_h,\ q_h \perp Y_H,\\
(5.7)\qquad & |DQ_H q_h|_{-1} \le c|Dq_h|_{-1} && \forall q_h \in Y_h,\\
(5.8)\qquad & |Q_H q_h|_1 \le cH^{-1}|Dq_h|_{-1} && \forall q_h \in Y_h,\\
(5.9)\qquad & |q_h|_1 \le ch^{-1}|Dq_h|_{-1} && \forall q_h \in Y_h.
\end{aligned}$$

*Proof*. The inequality (5.6) is simply proved by using $q_h \perp Y_H$ and (5.5),

$$\|q_h\|^2 = (q_h, q_h - Q_H q_h) \le \|q_h\|\,\|q_h - Q_H q_h\| \le \|q_h\|\, c_L |Dq_h|_{-1}.$$

The second estimate follows from the triangle inequality, $|D\psi|_{-1} \le \|\psi\|$ (see (2.1)), and (5.5)

$$\begin{aligned}
|DQ_H q_h|_{-1} &\le |D(Q_H q_h - q_h)|_{-1} + |Dq_h|_{-1}\\
&\le \|Q_H q_h - q_h\| + |Dq_h|_{-1} \le (c_L + 1)|Dq_h|_{-1}.
\end{aligned}$$

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For the proof of the estimate (5.8), we use the operator $Q_H^d$ from Section 3 and obtain from the inverse estimate

$$|Q_H q_h|_1^2 = \sum_{\Lambda_H} |Q_H q_h - Q_H^d q_h|_{1;\Lambda_H}^2 \le cH^{-2} \sum_{\Lambda_H} \|Q_H q_h - Q_H^d q_h\|_{\Lambda_H}^2$$
$$\le cH^{-2} \sum_{\Lambda_H} (\|Q_H q_h - q_h\|_{\Lambda_H}^2 + \|q_h - Q_H^d q_h\|_{\Lambda_H}^2).$$

The first term can be bounded by (5.5), the second by the local LBB-condition on $\Lambda_H$ and Lemma 2.1

$$\sum_{\Lambda_H} \|q_h - Q_H^d q_h\|_{\Lambda_H}^2 \le c \sum_{\Lambda_H} |Dq_h|_{-1;\Lambda_H}^2 \le c|Dq_h|_{-1}^2.$$

For the last estimate we use the $L^2$-projection $Q_h^d$ into the piecewise constant functions on $\Pi_h$ and Lemma 2.1

$$|q_h|_1^2 = \sum_{\Lambda_h} |q_h - Q_h^d q_h|_{1;\Lambda_h}^2 \le ch^{-2}\|q_h - Q_h^d q_h\|^2 \le ch^{-2}|Dq_h|_{-1}^2. \qquad \square$$

LEMMA 5.3. *The norms*

$$\left(\|q_h - Q_H q_h\|^2 + |DQ_H q_h|_{-1}^2\right)^{1/2} \ \textit{and}\ \ |Dq_h|_{-1}$$

*are equivalent on $Y_h$.*

*Proof.* This is a consequence of (5.5), (5.7) and

$$|Dq_h|_{-1} \le |D(q_h - Q_H q_h)|_{-1} + |DQ_H q_h|_{-1} \le \|q_h - Q_H q_h\| + |DQ_H q_h|_{-1}. \qquad \square$$

*Proof of Theorem 5.1:* From (5.1) and (5.2), it follows that $(Dv_h, Dv_H) = 0$ and $Q_H q_h = 0$. From Lemma 5.3, we obtain that $\|(v_h + v_H, q_h + q_H)\|_{X_h \times Y_h}$ is equivalent to

$$\left(|v_h|_1^2 + |v_H|_1^2 + \|q_h\|^2 + |Dq_H|_{-1}^2\right)^{1/2}.$$

For the estimate from above in Theorem 5.1, it is sufficient to prove that

$$|v_h|_1^2 + |v_H|_1^2 + \|q_h\|^2 + |Dq_H|_{-1}^2 + \omega H^2 |q_H|_1^2 \le c_2\left(|u_h|_1^2 + |Dp_h|_{-1}^2\right). \tag{5.10}$$

The proof of this estimate is very similar to the proof of the corresponding estimate in Theorem 3.1. Inserting $\phi_h = v_h$ in (5.1) and using (4.2) yield

$$|v_h|_1^2 \le (|u_h|_1 + |Dp_h|_{-1})|v_h - Q_H v_h|_1 \le c(|u_h|_1 + |Dp_h|_{-1})|v_h|_1.$$

From inserting $\psi_h = q_h$ in (5.2) and using the inverse estimate, it follows that

$$\|q_h\|^2 \le \|\operatorname{div} u\|\, \|q_h - Q_H q_h\| + \omega h^2 |p_h|_1 |q_h - Q_H q_h|_1$$
$$\le |u_h|_1 \|q_h\| + ch|p_h|_1 \|q_h\|.$$

Now (5.9) applied to $p_h$ completes the estimate for $\|q_h\|$.

For estimating $|v_H|_1^2$ and $|q_H|_1^2$, we set $\phi_H = v_H$ in (5.3), $\psi_H = q_H$ in (5.4) and add the resulting equalities

$$|v_H|_1^2 + \omega H^2 |q_H|_1^2 = (Du_h, Dv_H) - (\operatorname{div} v_H, p_h) + (\operatorname{div} u_h, q_H) + \omega h^2 (Dp_h, Dq_H)$$
$$\le c(\epsilon)|u_h|_1^2 + c|Dp_h|_{-1}^2 + ch^2|p_h|_1^2 + \epsilon|Dq_H|_{-1}^2 + \frac{1}{2}|v_H|_1^2 + \frac{\omega}{2}H^2|q_H|_1^2.$$

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Using (5.9) for $p_h$, we conclude that

$$|v_H|_1^2 + \omega H^2 |q_H|_1^2 \le c(\epsilon)|u_h|_1^2 + c|Dp_h|_{-1}^2 + 2\epsilon |Dq_H|_{-1}^2. \tag{5.11}$$

For estimating $|Dq_H|_{-1}$, we use Lemma 4.1 for $h := H$

$$|Dq_H|_{-1} \le cH|q_H|_1 + c(\operatorname{div} w_H, q_H) \tag{5.12}$$

for $w_H \in X_H$ with $|w_H|_1 = 1$. By (5.3) for $\phi_H = w_H$,

$$\begin{aligned}(\operatorname{div} w_H, q_H) &= -(Du_h, Dw_H) + (\operatorname{div} w_H, p_h) + (Dv_H, Dw_H)\\ &\le |u_h|_1 + |Dp_h|_{-1} + |v_H|_1\end{aligned}$$

and hence, by (5.12)

$$|Dq_H|_{-1} \le cH|q_H|_1 + c|v_H|_1 + c|u_h|_1 + c|Dp_h|_{-1}.$$

We square this estimate and multiply it with a sufficiently small constant $\eta > 0$,

$$\eta |Dq_H|_{-1}^2 \le \frac{\omega}{2} H^2 |q_H|_1^2 + \frac{1}{2}|v_H|_1^2 + c|u_h|_1^2 + c|Dp_h|_{-1}^2.$$

The proof of (5.10) is completed by adding this estimate to (5.11) and choosing $\epsilon$ sufficiently small.

For the proof of the estimate from below in Theorem 5.1, we show that

$$|u_h|_1^2 + |Dp_h|_{-1}^2 + \omega h^2 |p_h|_1^2 \le c_1\big(|v_h|_1^2 + |v_H|_1^2 + \|q_h\|^2 + |Dq_H|_{-1}^2\big), \tag{5.13}$$

From (5.1) and (5.3), we obtain by choosing $\phi_h = u_h$ and $\phi_H = Q_H u$ and using (4.2)

$$\begin{aligned}|u_h|_1^2 - (\operatorname{div} u_h, p_h) =&(Du_h, D(u_h - Q_H u_h)) - (\operatorname{div}(u_h - Q_H u_h), p_h)\\ &+ (Du_h, Q_H u_h) - (\operatorname{div} Q_H u_h, p_h)\\ =&(Dv_h, Du_h) + (Dv_H, DQ_H u) - (\operatorname{div} Q_H u_h, q_H)\\ \le&\frac{1}{2}|u_h|_1^2 + c|v_h|_1^2 + c|v_H|_1^2 + c|Dq_H|_{-1}.\end{aligned} \tag{5.14}$$

Similarly, by (5.2) and (5.4)

$$\begin{aligned}(\operatorname{div} u_h, p_h) + \omega h^2 |p_h|_1^2 =&(\operatorname{div} u_h, p_h - Q_H p_h) + \omega h^2 (Dp_h, D(p_h - Q_H p_h))\\ &+ (\operatorname{div} u_h, Q_H p_h) + \omega h^2 (Dp_h, DQ_H p_h)\\ =&(q_h, p_h) + (\operatorname{div} v_H, Q_H p_h) + \omega H^2 (Dq_H, DQ_H p_h)\\ =&A + B + C.\end{aligned} \tag{5.15}$$

From $q_h \perp Y_H$ and (5.5), we obtain

$$A = (q_h, p_h - Q_H p_h) \le \|q_h\|\,\|p_h - Q_H p_h\| \le c_L \|q_h\|\,|Dp_h|_{-1}.$$

For the second term, we use (5.7)

$$B \le |v_H|_1 |DQ_H p_h|_{-1} \le c|v_H|_1 |Dp_h|_{-1}.$$

The last term is bounded by (5.8),

$$\omega H^2 (Dq_H, DQ_H p_h) \le \omega H^2 |q_H|_1 |Q_H p_h|_1 \le c\omega |Dq_H|_{-1} |Dp_h|_{-1}.$$

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Collecting these estimates, adding (5.15) and (5.14), and using Young's inequality $ab \le \epsilon a^2/2 + \epsilon^{-1} b^2/2$, yield for every $\epsilon > 0$

$$|u_h|_1^2 + \omega h^2 |p_h|_1^2 \le c(\epsilon)\big(|v_h|_1^2 + |v_H|_1^2 + \|q_h\|^2 + |Dq_H|_{-1}^2\big) + \epsilon |Dp_h|_{-1}^2. \tag{5.16}$$

For estimating $|Dp_h|_{-1}$, we use Lemma 4.1

$$|Dp_h|_{-1} \le ch|p_h|_1 + c(\operatorname{div} w_h, p_h) \tag{5.17}$$

for $w_h \in X_h$ with $|w_h|_1 = 1$. From (5.1) and (5.3), we conclude

$$\begin{aligned}
(\operatorname{div} w_h, p_h) =& (\operatorname{div} (w_h - Q_H w_h, p_h)) + (\operatorname{div} Q_H w_h, p_h) \\
=& (Du_h, D(w_h - Q_H w_h)) - (Dv_h, Dw_h) + (Du_h, DQ_H w_h) \\
&- (Dv_H, DQ_H w_h) + (q_H, \operatorname{div} Q_H w_h) \\
\le& c(|u_h|_1 + |v_h|_1 + |v_H|_1| + |Dq_H|_{-1}).
\end{aligned}$$

Combining this estimate with (5.17) gives

$$|Dp_h|_{-1} \le c(h|p_h|_1 + |u_h|_1 + |v_h|_1 + |v_H|_1 + |Dq_H|_{-1}).$$

(5.13) follows from this estimate and (5.16). □

**6. Modifications of the preconditioners and numerical results.** We start with a slight modification of the standard preconditioner (1.6). The analysis is carried out for the continuous problem, but remains true for all conforming discretizations of the Stokes equations.

The eigenvalue problem for the Schur complement $S = -\operatorname{div} TD$, where $T : X' \to X$ is the inverse Laplacian from (1.7), is defined by

$$Sq = \mu q \quad q \in Y. \tag{6.1}$$

From [15] we know that in the case of a domain with smooth boundary all spectral values of $S$ are eigenvalues with $\mu \in \mathbb{R}$ and $\mu_{min} = L(\Omega)^2$, $\mu_{max} = 1$.

For a parameter $a > 0$ we set

$$C = \begin{bmatrix} -\Delta & 0 \\ 0 & 1/a \end{bmatrix}$$

such that the preconditioned problem is

$$C^{-1}L = \begin{bmatrix} T & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} -\Delta & D \\ \operatorname{div} & 0 \end{bmatrix} = \begin{bmatrix} 1 & TD \\ a \operatorname{div} & 0 \end{bmatrix}.$$

There is a one-to-one corresponding between the eigenvalue problem for the Schur complement in (6.1) and the eigenvalue problem for the preconditioned operator

$$C^{-1}L(v, q) = \lambda(v, q). \tag{6.2}$$

We take the negative divergence of the first equation in (6.2)

$$-\operatorname{div} v + Sq = -\lambda \operatorname{div} v$$

and use the second equation for expressing $\operatorname{div} v$

$$Sq = (1 - \lambda)\operatorname{div} v = (1 - \lambda)\frac{\lambda}{a} q,$$

which gives the relation

$$(1-\lambda)\lambda = a\mu. \tag{6.3}$$

On the other hand, assume that we have a solution $q$ of the eigenvalue problem (6.1) with eigenvalue $\mu$. For every $\lambda \in \mathbb{C}$ satisfying (6.3), we set $v = -TDq/(1-\lambda)$ and

$$C^{-1}L(v,q) = \begin{bmatrix} 1 & TD \\ a\,\mathrm{div} & 0 \end{bmatrix} \begin{bmatrix} v \\ q \end{bmatrix} = \begin{bmatrix} -\frac{1}{1-\lambda}TDq + TDq \\ -\frac{a}{1-\lambda}\mathrm{div}\,TDq \end{bmatrix} = \begin{bmatrix} -\frac{\lambda}{1-\lambda}TDq \\ \frac{a\mu}{1-\lambda}q \end{bmatrix} = \lambda(v,q).$$

The best choice in (6.3) seems to be $a = \frac{1}{4}$, which is also confirmed by numerical experiments. In this case, all $\lambda$ are real with

$$\lambda = \frac{1}{2} \pm \frac{1}{2}\sqrt{1-\mu}$$

such that for small $\mu_{min}$

$$\lambda_{min} \sim \frac{\mu_{min}}{4}, \quad \lambda_{max} \sim 1, \quad \frac{\lambda_{max}}{\lambda_{min}} \sim \frac{4}{\mu_{min}}.$$

Thus, the condition number of this preconditioned method is larger than the condition number of the Schur complement, which is $\mu_{min}^{-1}$. Nevertheless, the preconditioned method should be preferred in view of the fact that in the CG-method for the Schur complement the system $Tf$ must be solved exactly.

For our numerical experiments we use the stabilized finite element method described in Section 5 using piecewise linear elements on plane domains $\Omega \subset \mathbb{R}^2$,

$$(Du_h, D\phi_h) - (\mathrm{div}\,\phi_h, p_h) = (f_h, \phi_h)_h \quad \forall \phi_h \in X_h, \tag{6.4}$$

$$(\mathrm{div}\,u_h, \psi_h) + \omega h^2(Dp_h, D\psi_h) = (g_h, \psi_h)_h \quad \forall \psi_h \in Y_h, \tag{6.5}$$

for $(f_h, g_h) \in X_h \times Y_h$. The inner product $(\cdot,\cdot)_h$ is formed by the standard cubature formula using the nodes of each element (= lumped mass matrix). Let $\{\phi_{h,j}\}$, $\{\psi_{h,l}\}$ be the nodal bases of the spaces $X_h$ and $Y_h$, respectively. Define the matrices

$$\begin{aligned} A &= (a_{ij}), & a_{ij} &= (D\phi_{h,j}, D\phi_{h.i}), \\ B &= (b_{kj}), & b_{kj} &= (\mathrm{div}\,\phi_{h,j}, \psi_{h,k}), \\ C &= (c_{kl}), & c_{kl} &= \omega h^2(D\psi_{h,l}, D\psi_{h,k}), \\ D &= (d_{ij}), & d_{ij} &= (\phi_{h,j}, \phi_{h,i})_h, \\ D' &= (d'_{kl}), & d'_{kl} &= (\psi_{h,l}, \psi_{h,k})_h. \end{aligned}$$

By the lumped mass matrix technique, the matrices $D$ and $D'$ are diagonal with entries $d_{ii} = O(h^2)$. Now the system (6.4), (6.5) is equivalent to the linear system

$$\begin{bmatrix} A & -B^T \\ B & C \end{bmatrix} \begin{bmatrix} \underline{u} \\ \underline{p} \end{bmatrix} = \begin{bmatrix} D\underline{f} \\ D'\underline{g} \end{bmatrix}, \tag{6.6}$$

where the coefficient vectors are underlined.

Method I is the standard preconditioner (1.6), which is now

$$C_I^{-1} = \begin{bmatrix} \tilde{A}^{-1} & 0 \\ 0 & \frac{1}{4}D'^{-1} \end{bmatrix},$$

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where $\tilde{A}^{-1}$ is an approximation of $A^{-1}$ consisting of one step of a multigrid $V$-cycle with three smoothing steps with the standard lexicographic Gauss-Seidel method for solving the system $A\underline{u} = \underline{f}$.

Method II is the preconditioner described in Section 5. According to Section 4, we assume that the triangulation $\Pi_h$ is constructed by refining a coarse triangulation $\Pi_H$ $s$ times. Moreover, assume that $X_H = \{0\}$. Let $N_h = \dim Y_h$, $N_H = \dim Y_H$. Each nodal basis function $\psi_{H,k}$ of $Y_H$ can be represented in the form

$$\psi_{H,k} = \sum_{l=1}^{N_h} r_{kl}\psi_{h,l}.$$

The matrix $R \in \mathbb{R}^{N_H \times N_h}$ with $R = (r_{kl})$ satisfies for $q \in Y_h$

$$(q, \psi_{H,k}) = \sum_{l} r_{kl}(q, \psi_{h,l}) \ \Rightarrow \ \underline{q}_H = R\underline{q}$$

and, apart from a factor, coincides with the restriction operator of the underlying finite difference method. In order to avoid the solution of a linear system for the computation of the $L^2$-projection, we replace $Q_H$ by the operator $\tilde{Q}_H : Y_h \to Y_H$ defined by

$$\tilde{Q}_H q(P_{H,k}) = \frac{(q, \psi_{H,k})}{(1, \psi_{H,k})}.$$

Since $\tilde{Q}_H$ reproduces locally linear functions, it is a consistent approximation of $Q_H$. The matrix representation of $\tilde{Q}_H$ is $4^{-s}R^T R$ $(n = 2!)$. Denoting the matrix corresponding to $\omega H^2(D\psi_{H,l}, D\psi_{H,k})$ by $C_H$, the preconditioner can be represented in the case $X_H = \{0\}$ as

$$C_{II}^{-1} = \begin{bmatrix} \tilde{A}^{-1} & 0 \\ 0 & \frac{1}{4}D'^{-1}(I_{N_h} - 4^{-s}R^T R) + \frac{1}{10}R^T C_H^{-1} R \end{bmatrix},$$

where the second factor $1/10$ was determined by experiment.

In the stabilized method we use $\omega = 0.1$. The domains are $\Omega_N = (0, N) \times (0, 1)$ and the mesh parameters are $h = 1/64$ and $H = 1$ implying $X_H = \{0\}$. We start with a random vector $(\underline{u}_0, \underline{p}_0)$ and determine the initial residual $r_0$ of the system (6.6) (not the residual of the preconditioned system!). Then 20 steps of the preconditioned GMRES-algorithm are performed with residual $r_{20}$. The number

$$\rho = \sqrt[20]{\frac{r_{20}}{r_0}}$$

can be regarded as the convergence factor of the method. Using the methods I and II the following convergence factors $\rho$ are obtained on the domains $\Omega_N$:

| $N$ | 1 | 4 | 16 | 64 |
|---|---|---|---|---|
| I | 0.646 | 0.808 | 0.962 | 0.989 |
| II | 0.633 | 0.702 | 0.737 | 0.730 |

These convergence factors are stable with respect to the mesh size, which is demonstrated by the results for $h = 1/128$ and again $H = 1$:

| $N$ | 1 | 4 | 16 | 64 |
|---|---|---|---|---|
| I | 0.650 | 0.807 | 0.958 | 0.988 |
| II | 0.641 | 0.716 | 0.724 | 0.730 |

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