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RAY SEQUENCES OF LAURENT-TYPE RATIONAL FUNCTIONS †

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Abstract. This paper is devoted to the study of asymptotic zero distribution of Laurent-type approximants under certain extremality conditions analogous to the condition of Grothmann [1], which can be traced back to Walsh's theory of exact harmonic majorants [8, 9]. We also prove results on the convergence of ray sequences of Laurent-type approximants to a function analytic on the closure of a finitely connected Jordan domain and on the zero distribution of optimal ray sequences. Some applications to the convergence and zero distribution of the best L_p approximants are given. The arising theory is similar to Walsh's theory of maximally convergent polynomials to a function in a simply connected domain [10].

Key words. Laurent-type rational functions, zero distributions, convergence, optimal ray sequences, best L_p approximants.

AMS subject classifications. 30E10, 30C15, 41A20, 31A15.

1. Majorization and zero distribution of Laurent-type rational functions. Let A be a bounded multiply connected domain whose boundary consists of a finite number of disjoint Jordan curves. We denote by $\overline{\mathbf{C}}$ the extended complex plane, by $\{G_l\}_{l=1}^n$ the set of bounded components of $\overline{\mathbf{C}} \setminus \overline{A}$ and by Ω the unbounded component. (It is clear that the G_l and Ω are Jordan domains and that $\overline{\mathbf{C}} \setminus \overline{A} = (\cup_{l=1}^n G_l) \cup \Omega$.) Finally, for each $l = 1, 2, \ldots, n$ we associate an arbitrary but fixed point $a_l \in G_l$.

We continue the study of the convergence and the limiting zero distribution of Laurent-type rationals of the form:

$$R_N(z) = \sum_{j=0}^{k} t_j^N z^j + \sum_{l=1}^{n} \sum_{j=1}^{m_l} s_{l,j}^N (z - a_l)^{-j}, \tag{1.1}$$

where the multi-index $N := (k, m_1, m_2, \ldots, m_n)$, which was started in [4]. A more detailed account on the subject can be found in [5]. In this paper, we shall consider different sufficient conditions that yield the same type of zero distributions as in [4]. Note that we do not require that $t_k^N \neq 0$ (in contrast with [4]), but only that the highest positive power $d_e(k)$ of z with nonzero coefficient in $R_N(z)$ satisfies

$$d_e(k) \leq k.$$

Similarly, we have for the highest degree $d_l(m_l)$ of the Laurent part of $R_N(z)$, associated with the pole a_l, that

$$d_l(m_l) \leq m_l, \quad l = 1, \ldots, n.$$

This paper is organized as follows. The rest of Section 1 deals with asymptotic zero distribution results for Laurent-type rational functions, that generalize certain results of [4]. In Section 2, we study the optimal choice of ray sequences of Laurent-type approximants to analytic functions on multiply connected domains, providing

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the asymptotically least error in approximation. The applications of general results from Sections 1 and 2 to the best Laurent-type approximants in $L_p(A)$, $1 \le p \le \infty$, are considered in Section 3. All proofs of the results stated in Sections 1-3 can be found in Section 4. For the convenience of the readers, we also include an Appendix in the end of paper, which contains some results from [4] referenced here.

By the Riemann mapping theorem there exists a unique conformal mapping $\phi_l : G_l \to D$ of G_l onto the open unit disk D, normalized by the conditions $\phi_l(a_l) = 0$ and $\phi_l^{'}(a_l) > 0$. The quantity $R_l := 1/\phi_l'(a_l)$ is called the *interior conformal radius* of G_l with respect to a_l. Similarly, there exists a conformal mapping $\Phi : \Omega \to D'$ of the unbounded component Ω onto the exterior of the unit circle $D' = \{z : |z| > 1\}$ normalized by $\Phi(\infty) = \infty$ and $\lim_{z\to\infty} \Phi(z)/z = 1/C$, where $C := \operatorname{cap} \overline{A}$ is the logarithmic capacity of $\overline{A}$ (cf. [7, p. 55]).

We shall keep the same notation $\phi_l(z)$ for the continuous extension of the conformal mapping $\phi_l : G_l \to D$ onto the boundary ∂G_l [7, p. 356]. Thus, for each $l = 1, 2, \ldots, n$, the mapping ϕ_l is defined on the closure $\overline{G_l}$, i.e. $\phi_l : \overline{G_l} \to \overline{D}$. Similarly, for the exterior mapping we take $\Phi : \overline{\Omega} \to \overline{D'}$.

Define the measures

$$\mu_e(B) := \omega(\infty, B, \Omega) \tag{1.2}$$

and

$$\mu_l(B) := \omega(a_l, B, G_l), \quad l = 1, \ldots, n, \tag{1.3}$$

for any Borel set $B \subset \mathbf{C}$, where $\omega(\infty, B, \Omega)$ is the harmonic measure of the set B at the point ∞ with respect to Ω, and $\omega(a_l, B, G_l)$ is the harmonic measure of B at the point a_l with respect to the domain G_l (cf. [2, 7]). It is well known that [2, p. 37]

$$\omega(\infty, B, \Omega) = m(\Phi(B \cap \partial\Omega)) \tag{1.4}$$

and

$$\omega(a_l, B, G_l) = m(\phi_l(B \cap \partial G_l)), \quad l = 1, \ldots, n, \tag{1.5}$$

where $dm = d\theta/2\pi$ on $\{z : |z| = 1\}$. Clearly, μ_e and μ_l, $l = 1, \ldots, n$, are compactly supported unit Borel measures, i.e.

$$\|\mu_e\| = \|\mu_l\| = 1, \quad l = 1, \ldots, n,$$

and $\operatorname{supp} \mu_e = \partial\Omega$, $\operatorname{supp} \mu_l = \partial G_l$.

Let us introduce the Green function $g_{G_l}(z, a_l)$ of the domain G_l with the pole at a_l, $l = 1, \ldots, n$, and the Green function $g_\Omega(z, \infty)$ of the domain Ω with the pole at ∞. Since ∂G_l, $l = 1, \ldots, n$, and $\partial\Omega$ are Jordan curves, then the above Green functions exist in the classical sense. Furthermore, we have

$$g_{G_l}(z, a_l) = \log \frac{1}{|\phi_l(z)|}, \quad z \in G_l, \ l = 1, \ \ldots, \ n, \tag{1.6}$$

and

$$g_\Omega(z, \infty) = \log |\Phi(z)|, \quad z \in \Omega, \tag{1.7}$$

(see [7, p. 18]).

Since

$$R_N(z) = \frac{t^N_{d_e(k)} P_N(z)}{\prod_{l=1}^n (z-a_l)^{d_l(m_l)}}, \quad t^N_{d_e(k)} \neq 0, \tag{1.8}$$

where $P_N(z)$ is a monic polynomial of degree $\sum_{l=1}^n d_l(m_l)+d_e(k)$ whose zeros coincide with those of $R_N(z)$, then $R_N(z)$ must have exactly $\sum_{l=1}^n d_l(m_l)+d_e(k)$ zeros.

Next we introduce the *normalized counting measure* in the zeros of $R_N(z)$:

$$\nu_N := \frac{1}{\sum_{l=1}^n d_l(m_l)+d_e(k)} \sum_{P_N(z_j)=0} \delta_{z_j}, \tag{1.9}$$

where δ_z is the unit point mass at z and where all zeros are counted according to their multiplicities.

We assume that $k=k(i),\ m_1=m_1(i),\ \ldots,\ m_n=m_n(i)$ (so that $N=N(i)$), for some increasing sequence Λ of integers i, and that $k(i)\to\infty,\ \ m_l(i)\to\infty,\ \ l=1,\ldots,n$, as $i\to\infty,\ \ i\in\Lambda$. Furthermore, we assume that the following limits exist:

$$\lim_{|N|\to\infty} \frac{m_l}{|N|} = \lim_{\substack{i\to\infty\\ i\in\Lambda}} \frac{m_l(i)}{|N(i)|} =: \alpha_l, \quad l=1,\ldots,n, \tag{1.10}$$

where

$$|N| = k + \sum_{l=1}^n m_l, \tag{1.11}$$

is the norm of the multi-index N. This normalization means that we deal with so-called "ray sequences" of rational functions. Clearly,

$$\alpha_l \geq 0, \quad l=1,\ldots,n, \tag{1.12}$$

$$\lim_{\substack{i\to\infty\\ i\in\Lambda}} \frac{k(i)}{|N(i)|} = 1 - \sum_{l=1}^n \alpha_l, \tag{1.13}$$

and

$$\sum_{l=1}^n \alpha_l \leq 1. \tag{1.14}$$

We say that a sequence of Borel measures $\{\mu_n\}_{n=1}^\infty$ converges to the measure μ, as $n\to\infty$, in the *weak* topology* (written $\mu_n \xrightarrow{*} \mu$) if

$$\lim_{n\to\infty} \int f d\mu_n = \int f d\mu$$

for any continuous function f on $\mathbf{C}$ having compact support.

THEOREM 1.1. *Suppose that $\{R_N(z)\}_{i\in\Lambda}$ converges to $f\not\equiv 0$ locally uniformly in A, as $i\to\infty$, $i\in\Lambda$, and there exist compact sets $B_l\subset G_l$, $l=1,\ \ldots,\ n$, and $B_e\subset\Omega$ such that*

$$\liminf_{\substack{i\to\infty\\ i\in\Lambda}} \sup_{z\in B_l} \left(\frac{1}{m_l}\log|R_N(z)| - g_{G_l}(z,a_l)\right) \geq 0 \tag{1.15}$$

and

$$\liminf_{\substack{i\to\infty\\ i\in\Lambda}} \sup_{z\in B_e} \left(\frac{1}{k}\log|R_N(z)| - g_\Omega(z,\infty)\right) \geq 0. \tag{1.16}$$

Then

$$\nu_N \xrightarrow{*} \mu := \left(1-\sum_{l=1}^{n}\alpha_l\right)\mu_e + \sum_{l=1}^{n}\alpha_l\mu_l, \ \text{as } i\to\infty,\ i\in\Lambda. \tag{1.17}$$

We remark that (1.15) and (1.16) are analogous to the condition introduced in [1], which goes back to Walsh's theory of exact harmonic majorants (cf. [8], [9]). Theorem 1.1 can be viewed as a generalization of Theorem 2.2 of [4] (see Theorem A in Appendix). We shall prove Theorem 1.1 in Section 4.

In our applications, conditions (1.15) and (1.16) may not hold along the same sequence Λ but rather may be satisfied for different subsequences. This leads to the following "one-sided" version of Theorem 1.1.

THEOREM 1.2. *Suppose that $\{R_N(z)\}_{i\in\Lambda}$ converges to $f \not\equiv 0$ locally uniformly in A, as $i\to\infty$, $i\in\Lambda$.*

If there exist compact sets $B_j\subset G_j$, $j=1,\ \ldots,\ n$, and the corresponding subsequences $\Lambda_j\subset\Lambda$, $j=1,\ \ldots,\ n$ such that

$$\liminf_{\substack{i\to\infty\\ i\in\Lambda_j}} \sup_{z\in B_j} \left(\frac{1}{m_l}\log|R_N(z)| - g_{G_j}(z,a_j)\right) \geq 0 \tag{1.18}$$

then for any weak limit measure ν_j of $\{\nu_N\}_{i\in\Lambda_j}$, as $i\to\infty$, we have*

$$\nu_j|_{\overline{\mathbf{C}}\setminus(\cup_{l\neq j}\overline{G_l}\cup\overline{\Omega})} = \alpha_j\mu_j,\ j=1,\ 2,\ \ldots,\ n. \tag{1.19}$$

If there exists a compact set $B_e\subset\Omega$ and the corresponding subsequence $\Lambda_e\subset\Lambda$, such that

$$\liminf_{\substack{i\to\infty\\ i\in\Lambda_e}} \sup_{z\in B_e} \left(\frac{1}{k}\log|R_N(z)| - g_\Omega(z,\infty)\right) \geq 0, \tag{1.20}$$

then for any weak limit measure ν_e of $\{\nu_N\}_{i\in\Lambda_e}$, as $i\to\infty$, we have*

$$\nu_e|_{\overline{\mathbf{C}}\setminus\cup_{l=1}^{n}\overline{G_l}} = \left(1-\sum_{l=1}^{n}\alpha_l\right)\mu_e. \tag{1.21}$$

We omit the proof of Theorem 1.2 because it is essentially contained in the proof of Theorem 1.1. In some cases, conditions (1.18) and (1.20) may be easier to verify and more convenient to use than the coefficient conditions introduced in [4], as is shown in the next section.

2. Optimal ray sequences of maximally convergent Laurent-type rational functions. We continue using the notation of the preceding section. Let f be a function analytic on $\overline{A}$ with the "nearest singularity" in G_l situated on the level curve $\Gamma_l := \{z : |\phi_l(z)| = r_l,\ 0<r_l<1\}$, $l=1,\ \ldots,\ n$, and the "nearest singularity" in Ω on the level curve $\Gamma_e := \{z : |\Phi(z)| = r_e,\ 1<r_e<\infty\}$. More precisely, f is analytic in the multiply connected region $A_{\rm an}$ bounded by Γ_l, $l = 1, \ldots, n$, and Γ_e, and has singularities on each boundary curve.

Our next theorem gives a lower bound for the rate of approximation of the function f in the uniform (Chebyshev) norm on $\overline{A}$ by a ray sequence of Laurent-type rationals (1.1). It is natural to investigate the behavior of the error in approximation in the $|N|$-th root sense, because $R_N(z)$ has $|N| + 1$ coefficients to be considered as free parameters in minimizing the error.

We assume that $N = N(i)$, where $i = 1, 2, \ldots$, and suppose that there is a constant $c > 0$ such that

$$|k(i+1) - k(i)| < c \quad \text{and} \quad |m_l(i+1) - m_l(i)| < c,\ l = 1, \ldots, n, \tag{2.1}$$

for every $i = 1, 2, \ldots$.

THEOREM 2.1. *Under assumptions* (2.1) *and* (1.10) *(with $\Lambda = \mathbf{N}$), we have*

$$\limsup_{i\to\infty} \|f - R_N\|_{\overline{A}}^{1/|N|} \geq \max\left((r_e)^{\sum_{l=1}^n \alpha_l - 1}, r_1^{\alpha_1}, \ldots, r_n^{\alpha_n}\right). \tag{2.2}$$

By the analogy to the Walsh's theory of maximally convergent polynomials [10, p. 79] we are led to the following

DEFINITION 1. *The ray sequence of Laurent-type rational functions* (1.1)*, satisfying* (1.10)*, converges maximally if* (2.1) *is valid and*

$$\limsup_{i\to\infty} \|f - R_N\|_{\overline{A}}^{1/|N|} = \max\left((r_e)^{\sum_{l=1}^n \alpha_l - 1}, r_1^{\alpha_1}, \ldots, r_n^{\alpha_n}\right). \tag{2.3}$$

Thus, a maximally convergent ray sequence approximates our function f in the uniform norm on $\overline{A}$ with the best possible geometric rate for the fixed numbers $\{\alpha_l\}_{l=1}^n$, $0 \leq \alpha_l \leq 1$, $l = 1, \ldots, n$.

Let us turn to the question of the best choice of $\{\alpha_l\}_{l=1}^n$ in the sense of convergence rate. If $\alpha_l = 0$ for some l, $1 \leq l \leq n$, or $\sum_{l=1}^n \alpha_l = 1$, then (2.2) indicates that this is not the best choice. Suppose now that for any $\{\alpha_l\}_{l=1}^n$, $0 < \alpha_l < 1$, $l = 1, \ldots, n$, we have a corresponding ray sequence of maximally convergent Laurent-type rational functions. What values $\{\alpha_l\}_{l=1}^n$ yield the least error in the $|N|$-th root sense? The answer is given in the following theorem.

THEOREM 2.2. *For the function f described above, a maximally convergent ray sequence is optimal in the sense of convergence rate if and only if*

$$\lim_{i\to\infty} \frac{m_j}{|N|} = \frac{(\log r_j)^{-1}}{\sum_{l=1}^n (\log r_l)^{-1} - (\log r_e)^{-1}} =: \alpha_j^*, \quad j = 1, \ldots, n. \tag{2.4}$$

In this case we have

$$\limsup_{i\to\infty} \|f - R_N\|_{\overline{A}}^{1/|N|} = (r_e)^{\sum_{l=1}^n \alpha_l^* - 1} = r_1^{\alpha_1^*} = \ldots = r_n^{\alpha_n^*}. \tag{2.5}$$

Furthermore, an optimal ray sequence converges to f locally uniformly in $A_{\rm an}$.

In addition to its approximation properties, an optimal ray sequence has a remarkable limiting zero distribution. Let us denote the exterior of Γ_e by Ω_{r_e} and the interior of Γ_l by G_{r_l}, $l = 1, \ldots, n$. We introduce measures

$$\mu_{r_e} := \omega(\infty, \cdot, \Omega_{r_e}), \tag{2.6}$$

where $\omega(\infty, \cdot, \Omega_{r_e})$ is the harmonic measure at ∞ with respect to Ω_{r_e}, and

$$\mu_{r_l} := \omega(a_l, \cdot, G_{r_l}), \quad l = 1, \ \ldots, \ n, \tag{2.7}$$

where $\omega(a_l, \cdot, G_{r_l})$ is the harmonic measure at a_l with respect to G_{r_l}.

THEOREM 2.3. *There exist subsequences of the optimal ray sequence of maximally convergent Laurent-type rational functions such that for the normalized counting measures* (1.9) *we have*

$$\nu_N \xrightarrow{*} \nu_e, \ \text{as } i \to \infty, \ i \in \Lambda_e \subset \mathbf{N}, \tag{2.8}$$

where

$$\nu_e|_{\overline{\mathbf{C}} \setminus \cup_{l=1}^n \overline{G}_{r_l}} = \left(1 - \sum_{l=1}^n \alpha_l^*\right) \mu_{r_e},$$

and

$$\nu_N \xrightarrow{*} \nu_j, \ \text{as } i \to \infty, \ i \in \Lambda_j \subset \mathbf{N}, \tag{2.9}$$

where

$$\nu_j|_{\overline{\mathbf{C}} \setminus (\cup_{l \neq j} \overline{G}_{r_l} \cup \overline{\Omega}_{r_e})} = \alpha_j^* \mu_{r_j}, \ j = 1, \ 2, \ \ldots, \ n.$$

This result shows that every boundary point of the domain A_{an} is a limit point for the zeros of the optimal ray sequence. Hence, the uniform convergence of the whole optimal ray sequence is impossible in any neighborhood of a boundary point.

If Λ_e and $\Lambda_l, \ l = 1, 2, \ldots, n$, have an infinite subsequence Λ' in common, then for the subsequence $\{R_N(z)\}_{i \in \Lambda'}$ of the optimal ray sequence of maximally convergent Laurent-type rational functions we have

$$\nu_N \xrightarrow{*} \left(1 - \sum_{l=1}^n \alpha_l^*\right) \mu_{r_e} + \sum_{l=1}^n \alpha_l^* \mu_{r_l}, \ \text{as } i \to \infty, \ i \in \Lambda'. \tag{2.10}$$

One might hope that (2.10) always holds for some subsequence of the optimal ray sequence. But this is not true in general, as we show by the example constructed with the help of Laurent series in Proposition 3.3.

3. Best Laurent-type approximants in $L_p(A), 1 \leq p \leq \infty$. We assume that all conditions imposed on the function f in Section 2 are valid. Let $L_p(A)$ be the linear normed space of all functions g such that $\|g\|_p < \infty$, where

$$\|g\|_p := \begin{cases} \left[\displaystyle\iint_A |g(x+iy)|^p \, dxdy\right]^{1/p}, & 1 \leq p < \infty, \\ \sup_{z \in A} |g(z)|, & p = \infty. \end{cases} \tag{3.1}$$

Since f is assumed to be analytic on $\overline{A}$, then it is obvious that $f \in L_p(A)$ for every $p, \ 1 \leq p \leq \infty$. We introduce the linear subspace $\mathcal{R}_N \subset L_p(A)$ of all Laurent-type rational functions of the form (1.1) having complex coefficients. A rational function $R_N^* \in \mathcal{R}_N$ is said to be a best approximant of the type N to f in $L_p(A), \ 1 \leq p \leq \infty$, out of $\mathcal{R}_N$, if

$$\|f - R_N^*\|_p = \inf_{R_N \in \mathcal{R}_N} \|f - R_N\|_p \, . \tag{3.2}$$

The existence of such best approximants follows by the linearity of $\mathcal{R}_N$.

All approximants $\{R_N^*\}_{k,m_1,\ldots,m_n=1}^{\infty}$, where $N=(k,m_1,\ldots,m_n)$, can be ordered in an infinite $(n+1)$-dimensional table according to their multi-indices, which is similar to Walsh's table [10]. For any $\{\alpha_l\}_{l=1}^n$, $0\le\alpha_l\le 1$, $l=1,\ \ldots,\ n$, we can consider a ray sequence in this table defined by

$$N:=N(i)=\left(\left[\left(1-\sum_{l=1}^n \alpha_l\right)i\right],[\alpha_1 i],\ldots,[\alpha_n i]\right), \tag{3.3}$$

where $[\cdot]$ denotes integer part and $i=1,\ 2,\ \ldots$.

Proposition 3.1. *Any ray sequence* (3.3) *of the best Laurent-type rational approximants to f in $L_p(A)$, $1\le p\le\infty$, is maximally convergent.*

Thus, choosing $\{\alpha_l^*\}_{l=1}^n$ to be as in (2.4) we obtain the optimal ray sequence $\{R_N\}_{i=1}^{\infty}$ defined by (3.3), which gives the best rate of convergence to f on $\overline{A}$ and overconverges to f locally uniformly in A_{an} according to Theorem 2.2.

As a direct consequence of Theorem 2.3 we have

Theorem 3.2. *There exist subsequences of the optimal ray sequence of best Laurent-type approximants to f in $L_p(A)$, $1\le p\le\infty$, defined by* (2.4) *and* (3.3), *such that for the normalized counting measures we have*

$$\nu_N \xrightarrow{*} \nu_e,\ \textit{as}\ i\to\infty,\ i\in\Lambda_e\subset\mathbf{N}, \tag{3.4}$$

where

$$\nu_e|_{\overline{\mathbf{C}}\setminus\cup_{l=1}^n\overline{G}_{r_l}}=\left(1-\sum_{l=1}^n\alpha_l^*\right)\mu_{r_e},$$

and

$$\nu_N \xrightarrow{*} \nu_j,\ \textit{as}\ i\to\infty,\ i\in\Lambda_j\subset\mathbf{N}, \tag{3.5}$$

where

$$\nu_j|_{\overline{\mathbf{C}}\setminus(\cup_{l\ne j}\overline{G}_{r_l}\cup\overline{\Omega}_{r_e})}=\alpha_j^*\mu_{r_j},\ j=1,\ 2,\ \ldots,\ n.$$

As we mentioned after Theorem 2.3, (2.10) may not hold for any subsequence of the optimal ray sequence. We give an example of this kind for the best $L_2(A)$ approximants on an annulus A.

Proposition 3.3. *Consider the Laurent series*

$$f(z):=\sum_{k=1}^{\infty}\frac{(z(1+z))^{4^k}}{C_{4^k}^{4^k/2}}+\sum_{k=1}^{\infty}\left(\frac{1}{2z}\left(1+\frac{1}{2z}\right)\right)^{2\cdot 4^k}\frac{1}{C_{2\cdot 4^k}^{4^k}} \tag{3.6}$$

with the exact annulus of convergence $A_{\mathrm{an}}=\{z:1/2<|z|<1\}$. For any sequence

$$R_{m(i),n(i)}=\sum_{k=-m(i)}^{n(i)}a_k z^k,\ i\in\Lambda',$$

of the partial sums of this Laurent series satisfying

$$\lim_{\substack{i\to\infty\\ i\in\Lambda'}}\frac{m(i)}{m(i)+n(i)}=\frac{1}{2}, \tag{3.7}$$

it is impossible that zeros accumulate at both points $z = -1$ and $z = -1/2$ simultaneously as $i \to \infty,\ i \in \Lambda'$.

Observe that the partial sum $R_{m,n}$ of the Laurent series (3.6) is the best L_2 approximant to f on A_{an} and, at the same time, on any subannulus $A \subset A_{\mathrm{an}}$, among the Laurent-type rational functions of the form

$$r_{m,n}(z) = \sum_{k=-m}^{n} a_k z^k.$$

Clearly, we can choose a subannulus A such that the optimal ray sequence of $R_{m,n}$'s for A will be defined by (3.7). Since (2.10) means that zeros of some subsequence of the optimal ray sequence accumulate at every point of both circles $|z| = 1$ and $|z| = 1/2$ in this case, then Proposition 3.3 is, indeed, a counterexample.

REMARK 1. *One can consider the best Laurent-type approximants to f in the spaces defined by the contour integral over ∂A, provided that ∂A is rectifiable. It is possible to deduce similar results in this case and the argument remains very close to the given one.*

4. Proofs.

4.1. Proof of Theorem 1.1. We need to state several auxiliary results before we proceed with the proof.

LEMMA 4.1. *Under the assumptions of* Theorem 1.1 *we have*

$$\lim_{\substack{i\to\infty \\ i\in\Lambda}} \frac{d_l(m_l)}{m_l} = 1 \tag{4.1}$$

and

$$\lim_{\substack{i\to\infty \\ i\in\Lambda}} \frac{d_e(k)}{k} = 1. \tag{4.2}$$

Proof. Since the proofs of both statements are similar, we prove only (4.2). Consider

$$\begin{aligned}
&\frac{1}{k}\log|R_N(z)| - g_\Omega(z,\infty) \\
&= \frac{1}{k}\left(\log|R_N(z)| - d_e(k) g_\Omega(z,\infty)\right) + \left(\frac{d_e(k)}{k} - 1\right) g_\Omega(z,\infty) \\
&\le \frac{1}{k}\log\|R_N\|_{\partial\Omega} + \left(\frac{d_e(k)}{k} - 1\right) g_\Omega(z,\infty),
\end{aligned}$$

where we applied the maximum principle to the function $\log|R_N(z)| - d_e(k) g_\Omega(z,\infty)$, which is subharmonic in Ω (even at ∞). We know from Lemma 5.2 of [4] (cf. Lemma C in Appendix) that

$$\lim_{i\to\infty} \|R_N\|_{\partial\Omega}^{1/k} = 1. \tag{4.3}$$

Thus, (1.16) implies

$$\liminf_{i\to\infty} \left(\frac{d_e(k)}{k} - 1\right) \inf_{z\in B_e} g_\Omega(z,\infty) \ge 0$$

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and (4.2) follows. □

LEMMA 4.2. *If the conditions of* Theorem 1.1 *are satisfied, then* (1.15) *holds with* B_l *replaced by any closed disk contained in* $G_l \setminus B_l$, $l = 1,\ 2,\ \ldots,\ n$. *Analogously, we can replace* B_e *in* (1.16) *by any closed disk in* $\Omega \setminus B_e$.

Proof. Let D be any closed disk in $G_l \setminus B_l$ for some fixed l, $1 \le l \le n$, and suppose that

$$\liminf_{\substack{i\to\infty\\ i\in\Lambda}} \left(\sup_{z\in D} h_N(z)\right) =: c < 0,$$

where $h_N(z) := \frac{1}{m_l}\log|R_N(z)| - g_{G_l}(z, a_l)$ is subharmonic in G_l for any $i \in \Lambda$. Then we consider a harmonic function h in $G_l \setminus D$ with the boundary values

$$h(z) = \begin{cases} 0, & z \in \partial G_l, \\ c, & z \in \partial D. \end{cases} \tag{4.4}$$

By Lemma 5.2 of [4] (cf. Lemma C in Appendix) and the properties of a harmonic majorant to a subharmonic function we obtain

$$\liminf_{\substack{i\to\infty\\ i\in\Lambda}} \left(\sup_{z\in B_l} h_N(z)\right) \le \liminf_{\substack{i\to\infty\\ i\in\Lambda}} \left(\sup_{z\in B_l} h(z)\right) < 0,$$

which contradicts (1.15).

Using an identical argument, we can show that

$$\liminf_{\substack{i\to\infty\\ i\in\Lambda}} \sup_{z\in D} \left(\frac{1}{k}\log|R_N(z)| - g_\Omega(z,\infty)\right) < 0$$

is impossible for any $D \subset \Omega \setminus B_e$. □

LEMMA 4.3. *If the conditions of* Theorem 1.1 *are valid, then for* ν_N *defined by* (1.9) *we have*

$$\nu_N(B) \to 0, \ \text{as } i \to \infty, \ i \in \Lambda, \tag{4.5}$$

for any closed set $B \subset (\cup_{l=1}^{n} G_l) \cup \Omega$.

Proof. We can assume that $B \subset \Omega$, because the proof of (4.5) for G_l is the same. Consider

$$v_N(z) := \frac{1}{k}\log|R_N(z)| - g_\Omega(z,\infty) + \frac{1}{k}\sum_j g_\Omega(z, z_j),$$

where $g_\Omega(z, z_j)$ is the Green function of Ω with the pole at z_j and by z_j's we denote all the zeros of $R_N(z)$ in B (counted according to their multiplicities). Note that $v_N(z)$ is subharmonic in Ω. Let D be a disk in Ω such that $D \cap B = \emptyset$. By the maximum principle for $v_N(z)$ in Ω we obtain from (4.3)

$$\limsup_{\substack{i\to\infty\\ i\in\Lambda}} \left(\sup_{z\in D} v_N(z)\right) \le 0.$$

Since

$$\frac{1}{k}\log|R_N(z)| - g_\Omega(z,\infty) \le v_N(z), \quad z \in \Omega,$$

we obtain by Lemma 4.2 that

$$\lim_{\substack{i\to\infty \\ i\in\Lambda}} \inf_{z\in D} \left(\frac{1}{k} \sum_j g_\Omega(z, z_j) \right) = 0.$$

Let $a := \inf_{\substack{z\in D \\ \xi\in B}} g_\Omega(z,\xi) > 0$, where positivity follows from $B \cap D = \emptyset$ and the properties of Green functions. Thus,

$$\lim_{\substack{i\to\infty \\ i\in\Lambda}} \nu_N(B) \le \lim_{\substack{i\to\infty \\ i\in\Lambda}} \frac{\inf_{z\in D}\left(\frac{1}{k}\sum_j g_\Omega(z,z_j)\right)}{a} = 0.$$

□

Proof of Theorem 1.1.

Proof. Let $R_0 > 0$ be such that $\overline{A} \subset \{z : |z| < R_0/2\}$. We denote all zeros of $R_N(z)$ outside of $\{z : |z| < R_0\}$ by z_j^N's. It follows from Lemma 4.3 that there are only $o(|N|)$ of them as $i \to \infty,\ i \in \Lambda'$. Then, we introduce

$$q_N(z) := t^N_{d_e(k)} \prod_{j=1}^{o(|N|)} \left(z - z_j^N\right) \tag{4.6}$$

and write by (1.8)

$$R_N(z) := \frac{q_N(z)p_N(z)}{\prod_{l=1}^n (z-a_l)^{d_l(m_l)}}, \tag{4.7}$$

where p_N is a monic polynomial that absorbs the rest of zeros of R_N.

It follows from (4.6) that

$$|q_N(z)| = |t^N_{d_e(k)}| \prod_{j=1}^{o(|N|)} \left| 1 - \frac{z}{z_j^N} \right| |z_j^N|$$

and

$$\left(\frac{1}{2}\right)^{o(|N|)} |t^N_{d_e(k)}| \prod_{j=1}^{o(|N|)} |z_j^N| \le |q_N(z)| \le \left(\frac{3}{2}\right)^{o(|N|)} |t^N_{d_e(k)}| \prod_{j=1}^{o(|N|)} |z_j^N| \tag{4.8}$$

for any $z \in \{|z| \le R_0/2\}$.

By Theorem I.3.6 of [6] and Corollary 4.3 of [4] we obtain

$$\sup_{z\in\overline{A}} \frac{|p_N(z)|^{1/\deg p_N}}{\prod_{l=1}^n |z-a_l|^{\alpha_l}} \ge C^{1-\sum_{l=1}^n \alpha_l}.$$

Taking in account (1.10) we have

$$\liminf_{\substack{i\to\infty \\ i\in\Lambda}} \left\| \frac{p_N(z)}{\prod_{l=1}^n (z-a_l)^{m_l}} \right\|_{\overline{A}}^{\frac{1}{|N|}} \ge C^{1-\sum_{l=1}^n \alpha_l}. \tag{4.9}$$

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Thus,

$$
\begin{aligned}
C^{1-\sum_{l=1}^{n}\alpha_l} &\leq \liminf_{\substack{i\to\infty\\ i\in\Lambda}} \left\| \frac{p_N(z)}{\prod_{l=1}^{n}(z-a_l)^{m_l}} \right\|_A^{\frac{1}{|N|}} \\
&\leq \limsup_{\substack{i\to\infty\\ i\in\Lambda}} \|R_N\|_A^{\frac{1}{|N|}} \liminf_{\substack{i\to\infty\\ i\in\Lambda}} \left\| \frac{1}{q_N} \right\|_A^{\frac{1}{|N|}} \leq \frac{1}{\limsup_{\substack{i\to\infty\\ i\in\Lambda}} \left(|t^N_{d_e(k)}| \prod_{j=1}^{o(|N|)} |z_j^N| \right)^{1/|N|}},
\end{aligned}
$$

where we used Lemma 5.2 of [4] (cf. Lemma 5.3 in Appendix) and (4.8) on the last step. Comparing the first and the last terms in the above chain of inequalities yields

$$
\limsup_{\substack{i\to\infty\\ i\in\Lambda}} \left(|t^N_{d_e(k)}| \prod_{j=1}^{o(|N|)} |z_j^N| \right)^{\frac{1}{|N|}} \leq C^{\sum_{l=1}^{n}\alpha_l - 1}. \tag{4.10}
$$

Our next goal is to show that the inequality in (4.10) can be replaced by the equality and that $\limsup$ can be replaced by $\lim$. Suppose to the contrary that there exists a subsequence of indices $\Lambda' \subset \Lambda$ such that

$$
\lim_{\substack{i\to\infty\\ i\in\Lambda'}} \left(|t^N_{d_e(k)}| \prod_{j=1}^{o(|N|)} |z_j^N| \right)^{\frac{1}{|N|}} < C^{\sum_{l=1}^{n}\alpha_l - 1}. \tag{4.11}
$$

Consider a subharmonic function

$$
\omega_N(z) := \frac{1}{|N|} \left(\log|R_N(z)| - k g_\Omega(z,\infty) \right), \quad z \in \Omega.
$$

For $|z| = R$ with $R > R_0$ large enough we estimate

$$
\begin{aligned}
\omega_N(z) &= \log|q_N(z)|^{\frac{1}{|N|}} + \frac{k}{|N|} \left(\frac{1}{k} \log \left| \frac{p_N(z)}{\prod_{l=1}^{n}(z-a_l)^{d_l(m_l)}} \right| - g_\Omega(z,\infty) \right) \\
&\leq \log|q_N(z)|^{\frac{1}{|N|}} + \frac{k}{|N|} \left(\log|z| - g_\Omega(z,\infty) + \frac{1}{k} \log \left(\frac{R+R_0}{R-R_0} \right)^{|N|} \right) \\
&= \log|q_N(z)|^{\frac{1}{|N|}} + \log \left(\frac{R+R_0}{R-R_0} \right) + \frac{k}{|N|} \left(\log|z| - g_\Omega(z,\infty) \right).
\end{aligned} \tag{4.12}
$$

We observe that $q_N(z)$ is a polynomial of degree $o(|N|)$, therefore by (4.8) and the Bernstein-Walsh lemma [10, p. 77] we have

$$
\limsup_{\substack{i\to\infty\\ i\in\Lambda'}} \|q_N\|_{|z|=R}^{\frac{1}{|N|}} \leq \limsup_{\substack{i\to\infty\\ i\in\Lambda'}} \left(\left| t^N_{d_e(k)} \right| \prod_{j=1}^{o(|N|)} \left| z_j^N \right| \right)^{1/|N|}.
$$

Since

$$
\lim_{\substack{i\to\infty\\ i\in\Lambda}} \lim_{|z|\to\infty} \frac{k}{|N|} \left(\log|z| - g_\Omega(z,\infty) \right) = \log C^{1-\sum_{l=1}^{n}\alpha_l},
$$

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then we can choose $R > 0$ to be sufficiently large so that (4.12) and (4.11) implies

$$\limsup_{\substack{i\to\infty \\ i\in\Lambda'}} \sup_{|z|=R} \omega_N(z) < 0. \tag{4.13}$$

Using the same argument as in the proof of Lemma 4.2 we get

$$\limsup_{\substack{i\to\infty \\ i\in\Lambda'}} \sup_{z\in B_e} \omega_N(z) < 0,$$

which contradicts to

$$\liminf_{\substack{i\to\infty \\ i\in\Lambda}} \sup_{z\in B_e} \omega_N(z) = \liminf_{\substack{i\to\infty \\ i\in\Lambda}} \sup_{z\in B_e} \frac{k}{|N|}\left(\frac{1}{k}\log|R_N(z)| - g_\Omega(z,\infty)\right) =$$
$$(1-\sum_{l=1}^{n}\alpha_l) \liminf_{\substack{i\to\infty \\ i\in\Lambda}} \sup_{z\in B_e} \left(\frac{1}{k}\log|R_N(z)| - g_\Omega(z,\infty)\right) \ge 0,$$

where we used (1.10) and (1.16).

Thus we have by (4.8)

$$\lim_{\substack{i\to\infty \\ i\in\Lambda}} |q_N(z)|^{\frac{1}{|N|}} = \lim_{\substack{i\to\infty \\ i\in\Lambda}} \left(|t^N_{d_e(k)}| \prod_{j=1}^{o(|N|)} |z_j^N|\right)^{\frac{1}{|N|}} = C^{\sum_{l=1}^n \alpha_l - 1}, \tag{4.14}$$

$z \in \{z : |z| \le R_0/2\}$.

Recall that the logarithmic potential of a Borel measure σ with compact support is given by

$$U^\sigma(z) = \int \log\frac{1}{|t-z|}d\sigma(t),\ \ z\in\mathbf{C}.$$

Let ν be any weak* limit of the normalized counting measures ν_N defined by (1.9). We know from Lemma 4.3 that the measures $\tilde{\nu}_N$ associated with the zeros of p_N will converge to ν in the weak* topology along the same subsequence. Without loss of generality we assume that this subsequence coincides with Λ. Note that $\operatorname{supp}\nu \subset \partial A$ and $\|\nu\| = 1$ by Lemma 4.1. Since all measures $\tilde{\nu}_N$ are compactly supported (with support in $\{|z| \le R_0\}$), then we can apply Theorem I.6.9 of [6] to obtain

$$\begin{aligned} U^\nu(z) &= \liminf_{\substack{i\to\infty \\ i\in\Lambda}} U^{\tilde{\nu}_N}(z) \\ &= \liminf_{\substack{i\to\infty \\ i\in\Lambda}} \frac{1}{|N|}\log\frac{1}{|p_N(z)|} = \liminf_{\substack{i\to\infty \\ i\in\Lambda}} \frac{1}{|N|}\log\frac{|q_N(z)|}{|R_N(z)\prod_{l=1}^n (z-a_l)^{d_l(m_l)}|}, \end{aligned} \tag{4.15}$$

q.e. in $\mathbf{C}$.

By the Bernstein-Walsh lemma, we have from (4.14)

$$\limsup_{\substack{i\to\infty \\ i\in\Lambda}} \sup_{E} |q_N(z)|^{\frac{1}{|N|}} \le C^{\sum_{l=1}^n \alpha_l - 1},$$

for any compact set $E \subset \mathbf{C}$. Suppose that $z_0 \in \{|z| > R_0\}$ and take $r > 0$ to be sufficiently small to satisfy $D_r(z_0) := \{|z_0 - z| \le r\} \subset \{|z| > R_0\}$. It follows from

Lemma 4.2 and the continuity of Green's function that for any $\varepsilon > 0$ we can choose r such that

$$\liminf_{\substack{i\to\infty\\ i\in\Lambda}} \sup_{D_r(z_0)} \frac{1}{|N|}\log|R_N(z)| \geq (1-\sum_{l=1}^{n}\alpha_l)g_\Omega(z_0,\infty)+\varepsilon.$$

Note that the convergence in (4.15) is uniform on the compact subsets of $\{|z| > R_0\}$. Hence, with μ as defined by (1.17),

$$\begin{aligned}\inf_{D_r(z_0)} U^\nu(z) &\leq \limsup_{\substack{i\to\infty\\ i\in\Lambda}} \sup_{D_r(z_0)} \log|q_N(z)|^{\frac{1}{|N|}} - \liminf_{\substack{i\to\infty\\ i\in\Lambda}} \sup_{D_r(z_0)} \frac{1}{|N|}\log|R_N(z)| \\ &\qquad + \lim_{\substack{i\to\infty\\ i\in\Lambda}} \sup_{D_r(z_0)} \sum_{l=1}^{n}\frac{d_l(m_l)}{|N|}\log\frac{1}{|z-a_l|} \\ &\leq (\sum_{l=1}^{n}\alpha_l - 1)\log C - (1-\sum_{l=1}^{n}\alpha_l)g_\Omega(z_0,\infty) + \sum_{l=1}^{n}\alpha_l\log\frac{1}{|z_0-a_l|} + 2\varepsilon \\ &= (1-\sum_{l=1}^{n}\alpha_l)\left(\log\frac{1}{C} - g_\Omega(z_0,\infty)\right) + \sum_{l=1}^{n}\alpha_l\log\frac{1}{|z_0-a_l|} + 2\varepsilon = U^\mu(z_0)+2\varepsilon.\end{aligned}$$

Since both potentials are continuous in $\{|z| > R_0\}$, letting $\varepsilon \to 0$ we obtain

$$U^\nu(z_0) \leq U^\mu(z_0), \ \ \forall \ \ z_0 \in \{|z| > R_0\}.$$

Considering the harmonic function $u(z) := U^\nu(z) - U^\mu(z)$, $|z| > R_0$, such that $u(z) \leq 0$ in $\{|z| > R_0\}$ and $u(\infty) = 0$, we conclude by the maximum principle that

$$U^\nu(z) \equiv U^\mu(z), \ \ \forall \ \ z \in \{|z| > R_0\}.$$

But $U^\nu(z)$ and $U^\mu(z)$ are harmonic in Ω, therefore

$$U^\nu(z) \equiv U^\mu(z), \ \ \forall \ \ z \in \Omega. \tag{4.16}$$

Suppose now that $z \in A$ and $f(z) \neq 0$. There is at most a countable number of zeros of f in A. Thus, we produce by (4.15) for quasi every $z \in A$:

$$\begin{aligned} U^\nu(z) &= \liminf_{\substack{i\to\infty\\ i\in\Lambda}} \frac{1}{|N|}\log\frac{|q_N(z)|}{|R_N(z)\prod_{l=1}^{n}(z-a_l)^{d_l(m_l)}|} \\ &= \lim_{\substack{i\to\infty\\ i\in\Lambda}} \log|q_N(z)|^{\frac{1}{|N|}} - \lim_{\substack{i\to\infty\\ i\in\Lambda}} \log|R_N(z)|^{\frac{1}{|N|}} + \sum_{l=1}^{n}\alpha_l\log\frac{1}{|z-a_l|} \\ &= (1-\sum_{l=1}^{n}\alpha_l)\log\frac{1}{C} + \sum_{l=1}^{n}\alpha_l\log\frac{1}{|z-a_l|} = U^\mu(z), \end{aligned}$$

where we used (4.14), (4.1) and

$$\lim_{\substack{i\to\infty\\ i\in\Lambda}} R_N(z) = f(z) \neq 0.$$

Observe, that both potentials are harmonic and continuous in A, therefore

$$U^\nu(z) = U^\mu(z), \ \ z \in A.$$

Since potentials are continuous in the *fine topology* (see Section I.5 of [6]) and since the boundary of A in the fine topology is the same as the Euclidean boundary (see Corollary I.5.6 of [6]), then we have by the above equality and (4.16):

$$u(z) = U^{\nu}(z) - U^{\mu}(z) = 0, \quad z \in \overline{A} \cup \Omega. \tag{4.17}$$

Note, that $u(z)$ is harmonic in each G_l and that $u(z) \equiv 0$ on $\partial G_l,\ l = 1, \dots, n$. Therefore,

$$u(z) \equiv 0, \quad z \in \mathbf{C}, \tag{4.18}$$

by the minimum-maximum principle for harmonic functions and the continuity of $u(z)$ in the fine topology. It follows now from Theorem II.2.1 of [6] that

$$\nu \equiv \mu.$$

□

4.2. Proofs of Theorem 2.1, Theorem 2.2 and Theorem 2.3. Proof of Theorem 2.1.

Proof. Suppose to the contrary that

$$\limsup_{i\to\infty} \|f - R_N\|_{\overline{A}}^{\frac{1}{|N|}} < \max\left((r_e)^{\sum_{l=1}^{n} \alpha_l - 1}, r_1^{\alpha_1}, \dots, r_n^{\alpha_n}\right). \tag{4.19}$$

First, we assume that the max in (4.19) is equal to $r_j^{\alpha_j}, 1 \le j \le n$. It follows from (1.10) that

$$\limsup_{i\to\infty} \|f - R_N\|_{\overline{A}}^{\frac{1}{m_j}} < r_j. \tag{4.20}$$

In the rest of proof we follow the usual scheme for converse-type theorems (see [10, pp. 78-81], for example). Let the value of lim sup in (4.20) be equal to $q < r_j$ and let $\varepsilon > 0$ be such that $q + \varepsilon < r_j$. Then the series

$$\sum_{i=1}^{\infty} \left(R_{N(i+1)}(z) - R_{N(i)}(z)\right) + R_{N(1)}(z) \tag{4.21}$$

converges uniformly on $\{z : |\phi_j(z)| = q+\varepsilon\}$. Indeed, by the analogue of the Bernstein-Walsh lemma for R_N stated in Lemma 5.1 of [4] (cf. Lemma B in Appendix) we have that series (4.21) can be estimated from above as follows:

$$\begin{aligned}
&\sum_{i=1}^{\infty} |R_{N(i+1)}(z) - R_{N(i)}(z)| + |R_{N(1)}(z)| \\
&\le M_1 \sum_{i=1}^{\infty} \|R_{N(i+1)} - R_{N(i)}\|_{\overline{A}} (q+\varepsilon)^{-\max(m_j(i), m_j(i+1))} \\
&\le M_1 \sum_{i=1}^{\infty} (\|f - R_{N(i)}\|_{\overline{A}} + \|f - R_{N(i+1)}\|_{\overline{A}})(q+\varepsilon)^{-\max(m_j(i), m_j(i+1))} \\
&\le M_2 \sum_{i=1}^{\infty} \left(q + \frac{\varepsilon}{2}\right)^{\min(m_j(i), m_j(i+1))} (q+\varepsilon)^{-\max(m_j(i), m_j(i+1))} \\
&\le M_3 \sum_{i=1}^{\infty} \left(\frac{q + \frac{\varepsilon}{2}}{q+\varepsilon}\right)^{m_j(i)} < \infty.
\end{aligned}$$

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Note, that we used (2.1) in the above argument. Since the series (4.21) converges uniformly to f on $\overline{A}$, by (4.19), and also on $\{z : |\phi_j(z)| = q+\varepsilon\}$, then this implies the uniform convergence between ∂G_j and $\{z : |\phi_j(z)| = q+\varepsilon\}$ to an analytic continuation of f through Γ_j, which is a contradiction. A similar argument can be used in the case when the max in (4.19) is equal to $(r_e)^{\sum_{l=1}^n \alpha_l - 1}$. □

Proof of Theorem 2.2.

Proof. In view of (2.3), we only need to verify that the right hand side of (2.3) takes its minimal value for $\{\alpha_l\}_{l=1}^n$ given by (2.4), in order to prove that this ray sequence is optimal. It is a simple exercise to check that (2.5) holds for the ray sequence defined by (2.4). Next, assume that for some choice of $\{\alpha_l\}_{l=1}^n$ we have

$$\max\left((r_e)^{\sum_{l=1}^n \alpha_l - 1}, r_1^{\alpha_1}, \dots, r_n^{\alpha_n}\right) < \max\left((r_e)^{\sum_{l=1}^n \alpha_l^* - 1}, r_1^{\alpha_1^*}, \dots, r_n^{\alpha_n^*}\right). \tag{4.22}$$

Then we obtain by (2.5) that $r_l^{\alpha_l} < r_l^{\alpha_l^*}$, $l = 1, \dots, n$. Consequently, $\alpha_l > \alpha_l^*$, $l = 1, \dots, n$, and

$$(r_e)^{\sum_{l=1}^n \alpha_l - 1} > (r_e)^{\sum_{l=1}^n \alpha_l^* - 1}. \tag{4.23}$$

It is clear that (4.23) contradicts (4.22) because of (2.5).

To show that the optimal ray sequence converges to f locally uniformly in A_{an}, we essentially repeat the proof of Theorem 2.1. Indeed, for any sufficiently small $\varepsilon > 0$, we can estimate the series (4.21) on $\{z : |\Phi(z)| = r_e - \varepsilon\}$ as follows:

$$\begin{aligned}
&\sum_{i=1}^{\infty} |R_{N(i+1)}(z) - R_{N(i)}(z)| + |R_{N(1)}(z)| \\
&\le M_1 \sum_{i=1}^{\infty} \|R_{N(i+1)} - R_{N(i)}\|_{\overline{A}} (r_e - \varepsilon)^{\max(k(i),k(i+1))} \\
&\le M_1 \sum_{i=1}^{\infty} (\|f - R_{N(i)}\|_{\overline{A}} + \|f - R_{N(i+1)}\|_{\overline{A}})(r_e - \varepsilon)^{\max(k(i),k(i+1))} \\
&\le M_2 \sum_{i=1}^{\infty} \left(r_e - \frac{\varepsilon}{2}\right)^{-\min(k(i),k(i+1))} (r_e - \varepsilon)^{\max(k(i),k(i+1))} \\
&\le M_3 \sum_{i=1}^{\infty} \left(\frac{r_e - \varepsilon}{r_e - \varepsilon/2}\right)^{k(i)} < \infty.
\end{aligned}$$

Applying the same argument to $\{z : |\phi_j(z)| = r_j + \varepsilon\}$, $j = 1, 2, \dots, n$, and letting $\varepsilon \to 0$, we complete the proof. □

Proof of Theorem 2.3.

Proof. Since we know by Theorem 2.2 that the optimal ray sequence of $\{R_N(z)\}_{i\in\Lambda}$ defined by (2.4) converges to $f \not\equiv 0$ locally uniformly in A_{an}, then Theorem 2.3 follows from Theorem 1.2 if we show the existence of compact sets $B_l \subset G_{r_l}$, $l = 1, \dots, n$, and $B_e \subset \Omega_{r_e}$ such that

$$\liminf_{\substack{i\to\infty \\ i\in\Lambda_l}} \sup_{z\in B_l} \left(\frac{1}{m_l} \log |R_N(z)| - g_{G_{r_l}}(z, a_l)\right) \ge 0, \quad l = 1, \dots, n, \tag{4.24}$$

and

$$\liminf_{\substack{i\to\infty\\ i\in\Lambda_e}} \sup_{z\in B_e} \left(\frac{1}{k}\log|R_N(z)| - g_{\Omega_{r_e}}(z,\infty)\right) \geq 0 \tag{4.25}$$

for some $\Lambda_e \subset \Lambda$ and $\Lambda_l \subset \Lambda,\ l = 1,\ 2,\ \ldots,\ n.$

The proofs of (4.24) and (4.25) are similar, therefore we only give the proof of (4.24) for some fixed $j,\ 1 \leq j \leq n$. Assume that (4.24) does not hold for $l = j$, i.e.

$$\limsup_{\substack{i\to\infty\\ i\in\Lambda}} \sup_{z\in B_j} \left(\frac{1}{m_j}\log|R_N(z)| - g_{G_{r_j}}(z,a_j)\right) = b < 0, \tag{4.26}$$

where $B_j \subset G_{r_j}$ is a closed disk. It follows from (2.5) that

$$\limsup_{\substack{i\to\infty\\ i\in\Lambda}} \|R_{N(i+1)} - R_{N(i)}\|_{\partial G_j}^{\frac{1}{m_j(i)}} \leq r_j. \tag{4.27}$$

Observe, that $g_{G_{r_j}}(z,a_j) = g_{G_j}(z,a_j) + \log r_j,\ z \in G_{r_j}$. For the function

$$h_i(z) := \frac{1}{m_j(i)}\log|R_{N(i+1)}(z) - R_{N(i)}(z)| - g_{G_j}(z,a_j),$$

which is subharmonic in G_j for any $i \in \Lambda$, we obtain by (4.26) and (4.27)

$$\limsup_{\substack{i\to\infty\\ i\in\Lambda}} h_i(z) \leq \begin{cases} \log r_j, & z \in \partial G_j, \\ b + \log r_j, & z \in \partial B_j. \end{cases} \tag{4.28}$$

Let us consider a harmonic majorant of $h_i(z)$ in $G_j \setminus B_j$, with the boundary values given by the right hand side of (4.28). Then, by (4.28),

$$\limsup_{\substack{i\to\infty\\ i\in\Lambda}} (h_i(z) - \log r_j) < 0,\ z \in \partial G_{r_j},$$

which implies

$$\limsup_{\substack{i\to\infty\\ i\in\Lambda}} \|R_{N(i+1)} - R_{N(i)}\|_{\partial G_{r_j}}^{\frac{1}{m_j(i)}} < 1. \tag{4.29}$$

By an argument analogous to that of the proof of Theorem 2.1, it follows that the sequence $\{R_N(z)\}_{i\in\Lambda}$ converges uniformly to an analytic continuation of f through ∂G_{r_j}, contradicting to our assumptions about f.
$\square$

4.3. Proof of Proposition 3.1.. *Proof.* Since (2.1) is obviously satisfied for the ray sequence $\{R_N^*(z)\}_{i=1}^{\infty}$ of the best Laurent-type rational approximants defined by multi-index (3.3), we only need to show that (2.3) holds. Using a standard argument based on the Cauchy formula, we can represent f by its additive splitting

$$f(z) = f_e(z) + \sum_{l=1}^{n} f_l(z), \quad z \in A_{\text{an}}, \tag{4.30}$$

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where f_e is analytic inside Γ_e and f_l is analytic outside Γ_l (even at ∞), $f_l(\infty) = 0,\ \ l = 1,\ \ldots,\ n$. By the results of Walsh [10, pp. 75-80] we can find a sequence of polynomials $\{p_k(z)\}_{k=1}^{\infty}$, $\deg p_k \le k$, such that

$$\limsup_{k\to\infty} \|f_e - p_k\|_{\partial\Omega}^{\frac{1}{k}} \le \frac{1}{r_e}, \tag{4.31}$$

where we use the uniform norm on $\partial\Omega$. With the help of the transforms $t = 1/(z-a_l)$, we obtain in the same way that there exist sequences of polynomials $\{q_{l,m_l}(t)\}_{m_l=1}^{\infty}$, $\deg q_{l,m_l} \le m_l$, such that

$$\limsup_{m_l\to\infty} \|f_l(z) - q_{l,m_l}\left(\frac{1}{z-a_l}\right)\|_{\partial G_l}^{\frac{1}{m_l}} \le r_l,\ \ l = 1,\ \ldots,\ n. \tag{4.32}$$

Consider

$$\begin{aligned}
\limsup_{i\to\infty} \|f - R_N^*\|_p^{\frac{1}{|N|}} &\le \limsup_{i\to\infty} \left\| f - \left(p_k(z) + \sum_{l=1}^{n} q_{l,m_l}\left(\frac{1}{z-a_l}\right)\right)\right\|_p^{\frac{1}{|N|}} \\
&\le \limsup_{i\to\infty} \left(\|f_e - p_k\|_\infty + \sum_{l=1}^{n} \left\| f_l - q_{l,m_l}\left(\frac{1}{z-a_l}\right)\right\|_\infty \right)^{\frac{1}{|N|}} \\
&\le \max\left((r_e)^{\sum_{l=1}^{n}\alpha_l - 1}, r_1^{\alpha_1}, \ldots, r_n^{\alpha_n}\right), \qquad (4.33)
\end{aligned}$$

where we used (4.31), (4.32) and (3.3) in the last step. Using Lemma 5.1 of [4] (cf. Lemma B in Appendix) and the estimate (cf. [10, p. 96])

$$|(f - R_N^*)(z)| \le \frac{1}{[\pi\,(\mathrm{dist}(z,\partial A))^2]^{\frac{1}{p}}} \|f - R_N^*\|_p, \quad z \in A,$$

we can show with the help of series (4.21) that

$$\limsup_{i\to\infty} \|f - R_N^*\|_\infty^{\frac{1}{|N|}} \le \limsup_{i\to\infty} \|f - R_N^*\|_p^{\frac{1}{|N|}}.$$

Taking into account (4.33) and Theorem 2.1, we obtain that $\{R_N^*(z)\}_{i=1}^{\infty}$ converges to f maximally. ☐

4.4. Proof of Proposition 3.3.. *Proof.* First, we consider the part of Laurent series (3.6) containing positive powers:

$$f^+(z) := \sum_{k=1}^{\infty} \frac{(z(1+z))^{4^k}}{C_{4^k}^{4^k/2}}. \tag{4.34}$$

Observe that $C_{4^k}^{4^k/2}$ is the largest coefficient of the polynomial $p_k(z) := (z(1+z))^{4^k}$. Since the powers of z in the polynomials p_k, with different k's, do not overlap and since every coefficient is at most 1, then the series (4.34) converges in $|z| < 1$. It cannot converge in any bigger disk centered at $z = 0$ because infinitely many coefficients in (4.34) are equal to 1. However, the subsequence of partial sums

$$s_{2\cdot 4^k}(z) = \sum_{j=1}^{k} \frac{(z(1+z))^{4^j}}{C_{4^j}^{4^j/2}}$$

of this power series is also convergent in $|z(z+1)| < 1$, which contains some neighborhood of $z = -1$, i.e., the series is overconvergent in the sense of Ostrowski [3]. Obviously, $s_n \equiv s_{2\cdot 4^k}$ for any n such that $2 \cdot 4^k \le n < 4^{(k+1)}$, $k = 1, 2, \ldots,$ and so the $s_n(z)$'s, with n in this range, also converge in some neighborhood of $z = -1$, as $n \to \infty$.

We would like to show that convergence near $z = -1$ holds even for $4^{k+1} \le n \le 5 \cdot 4^k$. For this purpose, we estimate with the help of Stirling's formula:

$$|s_n(z) - s_{2\cdot 4^k}(z)| = |\sum_{j=1}^{n} C^j_{4^{k+1}} z^{4^{k+1}+j}|/C^{2\cdot 4^k}_{4^{k+1}} \le$$

$$4^k|z|^{5\cdot 4^k} C^{4^k}_{4^{k+1}}/C^{2\cdot 4^k}_{4^{k+1}} \le 2\cdot 4^k|z|^{5\cdot 4^k}(16/27)^{4^k} \to 0, \tag{4.35}$$

as $k \to \infty$ and $|z| < (27/16)^{1/5}$. Thus, we have shown that the subsequence of partial sums $s_n(z)$ for $2\cdot 4^k \le n \le 5\cdot 4^k$, $k = 1, 2, \ldots,$ converges in some neighborhood of $z = -1$.

Applying a similar argument to the series

$$\sum_{k=1}^{\infty} \frac{(z(1+z))^{2\cdot 4^k}}{C^{4^k}_{2\cdot 4^k}}, \tag{4.36}$$

we deduce that it is convergent in $|z| < 1$ and that the subsequence of partial sums $s_n(z)$ for $4^k \le n \le 5\cdot 4^k/2$, $k = 1, 2, \ldots,$ converges in some neighborhood of $z = -1$. After the transformation $z \to 1/(2z)$, series (4.36) becomes the Laurent part of (3.6). Hence, the subsequence of partial sums of the Laurent part of (3.6), with $4^k \le m \le 5\cdot 4^k/2$, $k = 1, 2, \ldots,$ converges in some neighborhood of $z = -1/2$.

Note that the intervals $2\cdot 4^k \le n \le 5\cdot 4^k$, $k = 1, 2, \ldots,$ and $4^k \le m \le 5\cdot 4^k/2$, $k = 1, 2, \ldots,$ cover the whole set of natural numbers with some overlap. If a subsequence $\{R_{m(i),n(i)}\}_{i\in\Lambda'}$ of the partial sums of (3.6) has zeros accumulating at $z = -1$ and $z = -1/2$ simultaneously, then, by Hurwitz's theorem, it must contain an infinite subsequence Λ'' such that $m(i)$ and $n(i)$ for $i \in \Lambda''$ lie outside of the corresponding intervals above. But in this case relation (3.7) cannot be satisfied. $\square$

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5. Appendix. As [4] has not yet appeared, we include the statements of three results from [4], for the convenience of the readers. Theorem 5.1 corresponds to Theorem 2.2 of [4]. Lemma 5.2 is Lemma 5.1 of [4] and Lemma 5.3 is Lemma 5.2 of [4].

We continue using the notation of Section 1, and, in addition, we require that $d_e(k) = k$ and $d_l(m_l) = m_l$, $\quad l = 1, \ldots, n$.

THEOREM 5.1. . *Suppose that the sequence $\{R_N(z)\}_{i\in\Lambda}$ (cf. (1.1)) converges locally uniformly in A to $f(z)(\not\equiv 0)$ and (1.10) holds.*

If

(i) $\lim\limits_{\substack{i\to\infty\\ i\in\Lambda}} |t^N_k|^{1/k} = \dfrac{1}{C}$

and

(ii) $\lim\limits_{\substack{i\to\infty\\ i\in\Lambda}} |s^N_{l,m_l}|^{1/m_l} = R_l, \quad l = 1, \ldots, n,$

then the normalized zero counting measures ν_N for R_N satisfy

(iii) $\nu_N \stackrel{*}{\to} \mu_w$ *in the weak* sense as* $i \to \infty$, $i \in \Lambda$, *where*

$$\mu_w := \left(1 - \sum_{l=1}^{n} \alpha_l\right) \mu_e + \sum_{l=1}^{n} \alpha_l \mu_l.$$

Conversely, suppose that $\alpha_l > 0$, $l = 1, \ldots, n$, *with* $\sum_{l=1}^{n} \alpha_l \neq 1$. *If each* a_l *has some neighborhood free of zeros of* $\{R_N(z)\}_{i\in\Lambda}$, *then* (iii) *implies* (i) *and* (ii).

LEMMA 5.2. *For the rational function* $R_N(z)$ *defined by* (1.1) *we have that*

$$|R_N(z)| \leq \|R_N\|_{\partial\Omega} \left|\Phi(z)\right|^k, \quad z \in \Omega,$$

and

$$|R_N(z)| \leq \frac{\|R_N\|_{\partial G_l}}{\left|\phi_l(z)\right|^{m_l}}, \quad z \in G_l, \quad l = 1, \ldots, n,$$

where the norms are Chebyshev norms.

LEMMA 5.3. *Assume that the sequence* $\{R_N(z)\}_{i\in\Lambda}$ *converges locally uniformly in* A *to* $f(z) (\not\equiv 0)$. *Then*

$$\lim_{\substack{i\to\infty \\ i\in\Lambda}} \|R_N\|_{\partial\Omega}^{1/k} = 1$$

and

$$\lim_{\substack{i\to\infty \\ i\in\Lambda}} \|R_N\|_{\partial G_l}^{1/m_l} = 1, \quad l = 1, \ldots, n.$$

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